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EQUICONVERGENCE THEOREMS FOR FOURIER-BESSEL EXPANSIONS WITH APPLICATIONS TO THE HARMONIC ANALYSIS OF RADIAL FUNCTIONS IN EUCLIDEAN AND NONEUCLIDEAN SPACES

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ABSTRACT. We shall prove an equiconvergence theorem between Fourier-Bessel expansions of functions in certain weighted Lebesgue spaces and the classical cosine Fourier expansions of suitable related functions. These weighted Lebesgue spaces arise naturally in the harmonic analysis of radial functions on euclidean spaces and we shall use the equiconvergence result to deduce sharp results for the pointwise almost everywhere convergence of Fourier integrals of radial functions in the Lorentz spaces $L^{p,q}(\mathbf{R}^n)$. Also we shall briefly apply the above approach to the study of the harmonic analysis of radial functions on noneuclidean hyperbolic spaces.

In 1869 Hermann Hankel proved what is by now known as Hankel's inversion formula for Fourier-Bessel expansions:

$$f(x) = \int_0^{+\infty} \left\{ \int_0^{+\infty} f(t) \frac{J_\alpha(yt)}{(yt)^\alpha} t^{2\alpha+1} dt \right\} \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1} dy.$$

Revisiting [Hankel] we shall prove an equiconvergence theorem between Fourier-Bessel expansions of functions in certain weighted Lebesgue spaces and the classical cosine Fourier expansions of suitable related functions. These weighted Lebesgue spaces arise naturally in the harmonic analysis of radial functions on euclidean spaces and we shall use the equiconvergence result to deduce sharp results for the pointwise almost everywhere convergence of Fourier integrals of radial functions in the Lorentz spaces $L^{p,q}(\mathbf{R}^n)$. In particular we shall prove that the partial sums of the Fourier integrals of radial functions in $L^{\frac{2n}{n+1},1}(\mathbf{R}^n) + L^{\frac{2n}{n-1},1}(\mathbf{R}^n)$ converge almost everywhere, while for the other Lorentz spaces either the partial sums cannot be defined, or they may diverge at every point. Finally we shall briefly study the equiconvergence between Fourier-Jacobi and cosine expansions. This is related to the harmonic analysis of radial functions on noneuclidean hyperbolic spaces. It is noteworthy that in this case pointwise convergence holds only for $L^{p,q}$ spaces with p in a nonsymmetric range around 2.

The plan of the paper is the following:

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The first section contains some preliminary material and a short and elementary proof of the above Hankel's inversion formula for test functions. The second section contains the equiconvergence theorems. In the third section we study the harmonic analysis of radial functions on euclidean spaces, and in the fourth section we construct functions with Fourier-Bessel integrals diverging everywhere. The fifth section is devoted to the harmonic analysis of radial functions on noneuclidean spaces.

1. Hankel's inversion formula for test functions

Let $\alpha > -\frac{1}{2}$. An eigenfunction of the problem

$$\frac{d^2}{dx^2}f(x) + \frac{2\alpha+1}{x}\frac{d}{dx}f(x) = -y^2 f(x)$$

is given by $(xy)^{-\alpha}J_\alpha(xy)$, where $J_\alpha(t)$ is the Bessel function of the first kind of order α,

$$J_\alpha(z) = \sum_{k=0}^{+\infty} \frac{(-)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)}\left(\frac{z}{2}\right)^{\alpha+2k}.$$

Denote by $\mathscr{S}(\mathbf{R}_+)$ the space of indefinitely differentiable even functions on $\mathbf{R}$ with rapidly decreasing derivatives, and if f is in $\mathscr{S}(\mathbf{R}_+)$ denote by $\hat{f}$ the Hankel, or Fourier-Bessel transform,

$$\hat{f}(y) = \int_0^{+\infty} f(x)\frac{J_\alpha(xy)}{(xy)^\alpha}x^{2\alpha+1}\,dx.$$

Then one can easily verify that if $\Delta_x = -x^{-2\alpha-1}(d/dx)\{x^{2\alpha+1}d/dx\}$,

$$\int_0^{+\infty} \Delta_x f(x)g(x)x^{2\alpha+1}dx = \int_0^{+\infty} f(x)\Delta_x g(x)x^{2\alpha+1}\,dx,$$

and since $\Delta_x[J_\alpha(xy)/(xy)^\alpha] = y^2 J_\alpha(xy)/(xy)^\alpha$,

$$(\Delta_x f)^\wedge(y) = y^2\hat{f}(y), \quad (x^2 f(x))^\wedge(y) = \Delta_y \hat{f}(y).$$

In particular the Hankel transform maps $\mathscr{S}(\mathbf{R}_+)$ onto itself.

Theorem (1.1). *If f is in $\mathscr{S}(\mathbf{R}_+)$, then*

$$f(x) = \int_0^{+\infty}\left\{\int_0^{+\infty} f(t)\frac{J_\alpha(yt)}{(yt)^\alpha}t^{2\alpha+1}dt\right\}\frac{J_\alpha(xy)}{(xy)^\alpha}y^{2\alpha+1}dy.$$

This result is well known, however the following proof is very elementary and perhaps new.

Proof. First step. The inversion formula holds if $f(x) = \exp(-x^2/2)$. Indeed a term by term integration of the series defining the Bessel functions yields

$$\begin{aligned}\hat{f}(y) &= \sum_{k=0}^{+\infty}\frac{(-)^k 2^{-\alpha-2k}y^{2k}}{\Gamma(k+1)\Gamma(k+\alpha+1)}\int_0^{+\infty} x^{2k+2\alpha+1}e^{-x^2/2}dx \\ &= \sum_{k=0}^{+\infty}\frac{(-y^2/2)^k}{\Gamma(k+1)} = f(y).\end{aligned}$$

Second step. Let $g(t) = f(t) - f(x)\exp(x^2 - t^2)/2$. To prove the inversion formula for the function f it is enough to show that

$$\int_0^{+\infty} \hat{g}(y) \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1} dy = g(x) = 0.$$

Third step. Write $g(t) = (t^2 - x^2)h(t)$. Then

$$\begin{aligned}
&\int_0^{+\infty} \hat{g}(y) \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1} dy \\
&= \int_0^{+\infty} \{\Delta_y \hat{h}(y) - x^2 \hat{h}(y)\} \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1} dy \\
&= \int_0^{+\infty} \hat{h}(y) \left\{ \Delta_y \left[\frac{J_\alpha(xy)}{(xy)^\alpha} \right] - x^2 \frac{J_\alpha(xy)}{(xy)^\alpha} \right\} y^{2\alpha+1} dy \\
&= 0. \quad \square
\end{aligned}$$

2. Equiconvergence theorems

In this section we study certain extensions of Hankel's inversion formula to functions not necessarily in the space $\mathscr{S}(\mathbf{R}_+)$. In particular we shall consider the Rth partial sums of the Fourier-Bessel integrals:

$$S_R f(x) = \int_0^R \hat{f}(y) \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1} dy.$$

Definition. For a function f in $L^1(\mathbf{R}_+, \frac{x^{\alpha+1/2}}{1+x} dx)$ we define

$$S_R f(x) = \int_0^{+\infty} S_R(x, y) f(y)\, dy,$$

where

$$\begin{aligned}
&S_R(x, y) \\
(2.1)\qquad &= x^{-\alpha} y^{\alpha+1} \int_0^R t J_\alpha(xt) J_\alpha(yt)\, dt \\
(2.2)\qquad &= R x^{-\alpha} y^{\alpha+1} \frac{x J_{\alpha+1}(Rx) J_\alpha(Ry) - y J_{\alpha+1}(Ry) J_\alpha(Rx)}{x^2 - y^2}.
\end{aligned}$$

$S_R f$ is the Rth partial sum of the Fourier-Bessel expansion of f. See [Watson].

Definition. For a function g in $L^1(\mathbf{R}_+, \frac{dx}{1+x})$ we define

$$F_R g(x) = \frac{1}{\pi} \int_0^{+\infty} \frac{\sin R(x-y)}{x-y} g(y)\, dy.$$

$F_R g$ is the Rth partial sum of the Fourier cosine expansion of g. See [Zygmund].

Definition. For a function f in $L^1(\mathbf{R}_+, \frac{x^{\alpha+1/2}}{1+x} dx)$ we define

$$T_R f(x) = x^{-\alpha-1/2} F_R[y^{\alpha+1/2} f(y)](x) = \int_0^{+\infty} T_R(x, y) f(y) dy,$$

where

$$T_R(x, y) = \pi^{-1} \left(\frac{y}{x}\right)^{\alpha+1/2} \frac{\sin R(x-y)}{x-y}.$$

For the meaning of the measure $\frac{x^{\alpha+1/2}}{1+x}dx$ see Lemma (2.4) below. In particular the integrals defining the operators S_R, F_R and T_R are well defined.

Theorem (2.3). *Let f be in $L^1(\mathbf{R}_+, \frac{x^{\alpha+1/2}}{1+x}dx)$ and let $0 < x < +\infty$. Then*

$$\lim_{R\to+\infty} |S_R f(x) - T_R f(x)| = 0.$$

The convergence is uniform in every interval $0 < \varepsilon < x < \eta < +\infty$.

The proof of this theorem is quite close to the original proof of Hankel's inversion formula, however while the classical proofs we know consider only functions in the space $L^1(\mathbf{R}_+, x^{\alpha+1/2}dx)$, we deal with the larger space $L^1(\mathbf{R}_+, \frac{x^{\alpha+1/2}}{1+x}dx)$. This space is in some sense optimal, and we shall see that the theorem implies sharp results on the almost everywhere convergence of Fourier integrals of radial functions in euclidean spaces.

The key to the proof is to compare the kernels associated to the operators $\{S_R\}$ with the ones associated to the $\{T_R\}$'s.

Lemma (2.4). *Let ε and η be arbitrary positive numbers.*

(i) *If $0 < y < \varepsilon < x$ then*

$$|S_R(x, y)| \le C y^{\alpha+1/2}.$$

The constant C depends on ε and x, but it is independent of y and R.

(ii) *If $0 < x < \eta < y$ then*

$$|S_R(x, y)| \le C y^{\alpha-1/2}.$$

The constant C depends on η and x, but it is independent of y and R.

(iii) *If $\varepsilon < x, y < \eta$ then*

$$\left| S_R(x, y) - \pi^{-1}\left(\frac{y}{x}\right)^{\alpha+1/2} \frac{\sin R(x-y)}{x-y} \right| \le C.$$

The constant C depends on ε and η, but it is independent of x, y and R.

Proof. The proof of the lemma is based on the classical estimates for Bessel functions

$$|J_\alpha(t)| \le \begin{cases} C & \text{if } 0 \le t \le 1, \\ C t^{-1/2} & \text{if } 1 < t < +\infty, \end{cases} \tag{2.5}$$

and, if $t \to +\infty$,

$$J_\alpha(t) = \sqrt{\frac{2}{\pi t}} \left[\cos\left(t - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{1-4\alpha^2}{8t} \sin\left(t - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + O(t^{-2}) \right]. \tag{2.6}$$

Similar estimates hold even for t complex, and $|\operatorname{Arg} t| < \pi - \varepsilon$. Again see [Watson].

Proof of (i). If $0 < y < \varepsilon < x$, by (2.2) and (2.5),

$$\begin{aligned} |S_R(x, y)| &\le C R x^{-\alpha} y^{\alpha+1} \frac{x(Rx)^{-1/2}(Ry)^{-1/2} + y(Ry)^{-1/2}(Rx)^{-1/2}}{x^2 - y^2} \\ &\le C x^{-\alpha-1/2}(x-\varepsilon)^{-1} y^{\alpha+1/2}. \end{aligned}$$

Proof of (ii). If $0 < x < \eta < y$,

$$|S_R(x, y)| \le CRx^{-\alpha}y^{\alpha+1}\frac{x(Rx)^{-1/2}(Ry)^{-1/2} + y(Ry)^{-1/2}(Rx)^{-1/2}}{y^2 - x^2}$$
$$\le Cx^{-\alpha-1/2}\frac{\eta}{\eta - x}y^{\alpha-1/2}.$$

Proof of (iii). If t is small, by (2.5) we have

$$tJ_\alpha(xt)J_\alpha(yt) = O(1), \tag{2.7}$$

while if t, x, y are away from 0, by (2.6) we have

(2.8)

$$\begin{aligned} tJ_\alpha(xt)J_\alpha(yt) &= \frac{2}{\pi}x^{-1/2}y^{-1/2}\left[\cos\left(xt - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\cos\left(yt - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\right.\\ &\quad + \frac{1-4\alpha^2}{8yt}\cos\left(xt - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\sin\left(yt - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\\ &\quad \left.+ \frac{1-4\alpha^2}{8xt}\cos\left(yt - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\sin\left(xt - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + O(t^{-2})\right]\\ &= \pi^{-1}x^{-1/2}y^{-1/2}\left[\cos((x-y)t) + \sin((x+y)t - \alpha\pi)\right.\\ &\quad - \frac{1-4\alpha^2}{8t}\frac{x-y}{xy}\sin((x-y)t)\\ &\quad \left.- \frac{1-4\alpha^2}{8t}\frac{x+y}{xy}\cos((x+y)t - \alpha\pi) + O(t^{-2})\right]. \end{aligned}$$

Integrating (2.7) between 0 and 1 and (2.8) between 1 and R it is easy to check that

$$x^{-\alpha}y^{\alpha+1}\int_0^R tJ_\alpha(xt)J_\alpha(yt)\,dt = \pi^{-1}\left(\frac{y}{x}\right)^{\alpha+1/2}\int_0^R \cos((x-y)t)\,dt + O(1),$$

and (iii) follows. □

Lemma (2.9). (i) *Let* $\operatorname{Supp} f \subseteq [0, \varepsilon]$. *Then if* $x > \varepsilon$

$$|S_Rf(x)| + |T_Rf(x)| \le C\int_0^\varepsilon |f(y)|y^{\alpha+1/2}\,dy.$$

The constant C *depends on* ε *and* x, *but it is independent of* f *and* R.

(ii) *Let* $\operatorname{Supp} f \subseteq (\eta, +\infty)$. *Then if* $x < \eta$

$$|S_Rf(x)| + |T_Rf(x)| \le C\int_\eta^{+\infty} |f(y)|y^{\alpha-1/2}dy.$$

The constant C *depends on* η *and* x, *but it is independent of* f *and* R.

Proof. The proof is an immediate consequence of the previous lemma. □

The above lemma is nothing but a weak form of the Riemann localization principle for the operators $\{S_R\}$ and $\{T_R\}$. We want to recall here that the classical Riemann localization principle for Fourier cosine expansions holds not only in the space $L^1(\mathbf{R}_+, dx)$, but also in the larger space $L^1(\mathbf{R}_+, \frac{dx}{1+x})$.

Proof of the theorem. The theorem is obviously true if f is indefinitely differentiable and has compact support contained in $(0, +\infty)$. Since the set of these functions is dense in $L^1(\mathbf{R}_+, \frac{dx}{1+x})$, to prove the theorem it is enough to show that, if $\varepsilon < x < \eta$,

$$|S_R f(x) - T_R f(x)| \leq C \int_0^{+\infty} |f(y)| \frac{y^{\alpha+1/2}}{1+y}\, dy.$$

By the previous localization lemma it is not a restriction to assume that f is supported in $[\varepsilon/2, 2\eta]$, and under this assumption the above inequality is an immediate consequence of Lemma (2.4)(iii). □

For functions supported in the interval $[0, 1]$, beside the Fourier-Bessel integrals one can also define the Fourier-Bessel series

$$\sum_{k=1}^{+\infty} \hat{f}(\beta_k) \frac{2\beta_k^\alpha J_\alpha(\beta_k x)}{x^\alpha J_{\alpha+1}^2(\beta_k)},$$

where $0 < \beta_1 < \beta_2 < \cdots$ are the positive zeroes of J_α and

$$\hat{f}(y) = \int_0^1 f(x) \frac{J_\alpha(xy)}{(xy)^\alpha} x^{2\alpha+1}\, dx.$$

Combining Theorem 2.3 with the equiconvergence between Fourier-Bessel and trigonometric Fourier series in [Stone] one gets the following.

Theorem (2.10). *Let f be in $L^1([0, 1], x^{2\alpha+1}\, dx)$ and let $0 < x < 1$. Then, for $\beta_n < R < \beta_{n+1}$,*

$$\lim_{n\to+\infty} \left| \sum_{k=1}^{n} \hat{f}(\beta_k) \frac{2\beta_k^\alpha J_\alpha(\beta_k x)}{x^\alpha J_{\alpha+1}^2(\beta_k)} - \int_0^R \hat{f}(y) \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1}\, dy \right| = 0.$$

We sketch a direct proof of this fact.

It is immediate to extend the Fourier-Bessel transform and its partial sums to the complex domain, and using the estimates (2.5) and (2.6) one easily obtains the following.

Lemma (2.11). *Let f be in $L^1([0, 1], x^{2\alpha+1} dx)$. Then*

(i) *$\hat{f}$ is even, entire, and*

$$|\hat{f}(z)| \leq C\|f\|_1 e^{|\operatorname{Im} z|} |z|^{-\alpha-\frac{1}{2}}.$$

(ii) *If $S_z f(x) = \int_0^z \hat{f}(y) \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1}\, dy$, with $|\operatorname{Arg} z| < \pi$, then*

$$|S_z f(x)| \leq C\|f\|_1 x^{-\alpha-\frac{1}{2}} |z| e^{(1+x)|\operatorname{Im} z|}.$$

Lemma (2.12). (i) *In a neighborhood of each β_k,*

$$\frac{S_z f(x)}{z J_\alpha^2(z)} = \frac{S_{\beta_k} f(x)}{\beta_k J_{\alpha+1}^2(\beta_k)} (z - \beta_k)^{-2} + \frac{\hat{f}(\beta_k) \beta_k^\alpha J_\alpha(\beta_k x)}{x^\alpha J_{\alpha+1}^2(\beta_k)} (z - \beta_k)^{-1} + O(1).$$

(ii) *If $\beta_n < R < \beta_{n+1}$ and $0 < x < 1$,*

$$\sum_{k=1}^{n} \hat{f}(\beta_k) \frac{2\beta_k^\alpha J_\alpha(\beta_k x)}{x^\alpha J_{\alpha+1}^2(\beta_k)} = \pi^{-1} \int_{-\infty}^{+\infty} \frac{S_{R+it} f(x)}{(R+it) J_\alpha^2(R+it)}\, dt.$$

This lemma is essentially due to L. Schläfli. See [Watson, Chapter 18].

Lemma (2.13). *If* $\beta_n < R < \beta_{n+1}$,

$$\int_{-\infty}^{+\infty} \frac{dt}{(R+it)J_\alpha^2(R+it)} = \pi.$$

This follows from the identity

$$\frac{d}{dz}\left\{\frac{J_{-\alpha}(z)}{J_\alpha(z)}\right\} = -\frac{2\sin\pi\alpha}{\pi z J_\alpha^2(z)}.$$

Proof of the theorem. By the Lemmas 2.12 and 2.13,

$$\sum_{k=1}^{n} \hat{f}(\beta_k)\frac{2\beta_k^\alpha J_\alpha(\beta_k x)}{x^\alpha J_{\alpha+1}^2(\beta_k)} - S_R f(x) = \pi^{-1}\int_{-\infty}^{+\infty} \frac{S_{R+it}f(x) - S_R f(x)}{(R+it)J_\alpha^2(R+it)}\,dt.$$

By (2.6)

$$|(R+it)J_\alpha^2(R+it)| \geq Ce^{2|t|},$$

and, as in Lemma 2.11,

$$|S_{R+it}f(x) - S_R f(x)| \leq C\|f\|_1 x^{-\alpha-1/2}|t|e^{(1+x)|t|}.$$

Hence the above integral is dominated by $C\|f\|_1$, and the theorem follows. □

3. Applications to the harmonic analysis on Euclidean spaces

To every function f defined on $\mathbf{R}_+$ we associate a radial function $f(|w|)$, $w \in \mathbf{R}^N$, and formally

$$\int_{|\xi|\leq R} e^{iw\cdot\xi}\int_{\mathbf{R}^N} e^{-iz\cdot\xi} f(|z|)\,dz\,d\xi$$
$$= \int_0^R \frac{J_{\frac{N-2}{2}}(|w|t)}{(|w|t)^{\frac{N-2}{2}}}\int_0^{+\infty}\frac{J_{\frac{N-2}{2}}(st)}{(st)^{\frac{N-2}{2}}} f(s)s^{N-1}ds\, t^{N-1}\,dt.$$

See e.g. [Stein-Weiss, Chapter IV]. It is therefore natural to study Fourier-Bessel expansions of functions in Lorentz spaces on $\mathbf{R}_+$ with respect to the measure $x^{2\alpha+1}\,dx$.

We recall that the Lorentz space $L^{p,q}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$, $1 < p < +\infty$, $1 \leq q \leq +\infty$, is the set of all measurable functions f on $\mathbf{R}_+$ with the quasi-norm

$$\|f\|_{p,q} = \left\{\frac{q}{p}\int_0^{+\infty} [t^{1/p} f^*(t)]^q \frac{dt}{t}\right\}^{1/q} < +\infty,$$

where f^* denotes the nonincreasing rearrangement of f. Again see [Stein-Weiss, Chapter V].

Theorem (3.1). *If* f *is in* $L^{\frac{4\alpha+4}{2\alpha+3},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx) + L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}dx)$, *then for almost every* x *in* $\mathbf{R}_+$,

$$\lim_{R\to+\infty} S_R f(x) = f(x).$$

Since the space $L^{\frac{4\alpha+4}{2\alpha+3},1}(\mathbf{R}_+, x^{2\alpha+1}dx) + L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ contains all the spaces $L^p(\mathbf{R}_+, x^{2\alpha+1}dx)$ with $\frac{4\alpha+4}{2\alpha+3} < p < \frac{4\alpha+4}{2\alpha+1}$, we have the following corollary, also proved in [Kanjin] and [Prestini].

Corollary (3.2). *If* f *is in* $L^p(\mathbf{R}_+, x^{2\alpha+1}dx)$, $\frac{4\alpha+4}{2\alpha+3} < p < \frac{4\alpha+4}{2\alpha+1}$, *then for almost every* x *in* $\mathbf{R}_+$

$$\lim_{R\to+\infty} S_R f(x) = f(x).$$

The theorem easily follows from the equiconvergence theorem and the following lemma.

Lemma (3.3). *If* f *is in* $L^{\frac{4\alpha+4}{2\alpha+3},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx) + L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}dx)$ *then*
(i)

$$\int_0^{+\infty} |f(x)| \frac{x^{\alpha+1/2}}{1+x}\,dx < +\infty,$$

and
(ii) *for some* $p > 1$ *and every* $0 < \varepsilon < \eta < +\infty$

$$\int_\varepsilon^\eta |x^{\alpha+1/2} f(x)|^p\,dx < +\infty.$$

Proof. If f is in $L^{\frac{4\alpha+4}{2\alpha+3},1}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$, by Hölder inequality for Lorentz spaces,

$$\begin{aligned}\int_0^{+\infty} |f(x)| \frac{x^{\alpha+1/2}}{1+x}\,dx &\le \int_0^{+\infty} |f(x)| x^{-\alpha-1/2} x^{2\alpha+1} dx \\ &\le C\|f\|_{\frac{4\alpha+4}{2\alpha+3},1} \|x^{-\alpha-1/2}\|_{\frac{4\alpha+4}{2\alpha+1},\infty}.\end{aligned}$$

Similarly, if f is in $L^{\frac{4\alpha+4}{2\alpha+1},1}(\mathbf{R}_+, x^{2\alpha+1}dx)$,

$$\begin{aligned}\int_0^{+\infty} |f(x)| \frac{x^{\alpha+1/2}}{1+x}\,dx &\le \int_0^{+\infty} |f(x)| x^{-\alpha-3/2} x^{2\alpha+1} dx \\ &\le C\|f\|_{\frac{4\alpha+4}{2\alpha+1},1} \|x^{-\alpha-3/2}\|_{\frac{4\alpha+4}{2\alpha+3},\infty}.\end{aligned}$$

This proves (i). The proof of (ii) is immediate. □

Proof of the theorem. By the equiconvergence theorem $\{S_R f(x)\}$ is equiconvergent with $\{T_R f(x)\} = \{x^{-\alpha-1/2} F_R[y^{\alpha+1/2} f(y)](x)\}$. By the classical Riemann localization principle the convergence of $F_R[y^{\alpha+1/2}f(y)](x)$ depends only on the behaviour of the function $y^{\alpha+1/2}f(y)$ in a neighbourhood of x. But this function is locally in $L^p(\mathbf{R}_+, dy)$ for some $p > 1$, so that by the Carleson-Hunt theorem $F_R[y^{\alpha+1/2}f(y)](x) \to x^{\alpha+1/2}f(x)$ for almost every x if $R \to +\infty$. □

A natural way to prove the almost everywhere convergence of the means $\{S_R f\}$ to the function f is to study the boundedness of the maximal operator S^* defined by

$$S^* f(x) = \sup_{R>0} |S_R f(x)|.$$

For the operator S^* we have the following theorem which extends previous results of S. Chanillo and E. Prestini.

Theorem (3.4). *If* f *is in the Lorentz space* $L^{p,1}(\mathbf{R}_+, x^{2\alpha+1}dx)$, $p = \frac{4\alpha+4}{2\alpha+3}$ *or* $p = \frac{4\alpha+4}{2\alpha+1}$, *then*

$$\|S^* f\|_{p,\infty} \le C\|f\|_{p,1},$$

i.e. the maximal operator S^* *is of restricted weak type* (p, p).

This theorem of course implies Theorem (3.1) and Corollary (3.2). The proof we have is based on ideas in [Chanillo] and [Prestini]. Recently F. Soria communicated to us that this result has been also obtained in [Romera-Soria]. We therefore omit our proof, which can be found in [Crespi].

4. Divergence results

In this section we show that the results we have obtained on the pointwise convergence of Fourier-Bessel integrals of functions in Lorentz spaces are sharp. We start by repeating a basic remark due to J. L. Rubio de Francia, which shows that we cannot even define the partial sum operators $\{S_R\}$ on the Lorentz spaces $L^{p,r}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ when $p = \frac{4\alpha+4}{2\alpha+1}$ and $1 < r \leq +\infty$, or $p > \frac{4\alpha+4}{2\alpha+1}$.

Theorem (4.1). *Let $p = \frac{4\alpha+4}{2\alpha+1}$ and $1 < r \leq +\infty$, or let $p > \frac{4\alpha+4}{2\alpha+1}$. Then S_R does not exist as an operator from $L^{p,r}(\mathbf{R}_+, x^{2\alpha+1}dx)$ into the space of tempered distributions $\mathcal{S}^*(\mathbf{R}_+)$.*

Proof. Suppose the contrary. Then, by duality, S_R maps the space of test functions $\mathcal{S}(\mathbf{R}_+)$ into $L^{q,s}(\mathbf{R}_+, x^{2\alpha+1}dx)$, $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}+\frac{1}{s} = 1$. To see that this is false it is enough to consider a test function f with $\hat{f}(y) = 1$ if $0 \leq y \leq R$. For such f,

$$S_R f(x) = \int_0^R \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1} dy = (Rx)^{-\alpha-1} J_{\alpha+1}\left(\frac{x}{R}\right),$$

and this function belongs only to $L^{\frac{4\alpha+4}{2\alpha+3},\infty}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$. □

The case $p \leq \frac{4\alpha+4}{2\alpha+3}$ is different. In this case we are able to exhibit functions with Fourier-Bessel integrals diverging at every point. This contains a result in [Kanjin] on almost everywhere divergence, and indeed extends Kolmogorov construction of an everywhere divergent trigonometric Fourier series.

Theorem (4.2). *For any $r > 1$ there exist functions f in $L^{\frac{4\alpha+4}{2\alpha+3},r}(\mathbf{R}_+, x^{2\alpha+1}dx)$ supported in the interval $[0, 1]$ with $\{S_R f(x)\}$ diverging at every x in $\mathbf{R}_+$.*

Observe that every function in $L^{\frac{4\alpha+4}{2\alpha+3},r}(\mathbf{R}_+, x^{2\alpha+1}\,dx)$ with compact support also belongs to every $L^{p,s}(\mathbf{R}_+, x^{2\alpha+1}dx)$, $p < \frac{4\alpha+4}{2\alpha+3}$.

Proof. The asymptotic estimates for Bessel functions imply, if $y \to +\infty$,

$$\left\| \frac{J_\alpha(y\cdot)}{(y\cdot)^\alpha} \chi_{[0,1]}(\cdot) \right\|_{\frac{4\alpha+4}{2\alpha+1},s} \approx y^{-\alpha-1/2}(\ln y)^{1/s}, \tag{4.3}$$

and if $|y - t| < 1$,

$$\left\| \left[\frac{J_\alpha(y\cdot)}{(y\cdot)^\alpha} - \frac{J_\alpha(t\cdot)}{(t\cdot)^\alpha} \right] \chi_{[0,1]}(\cdot) \right\|_{\frac{4\alpha+4}{2\alpha+1},s} \approx y^{-\alpha-1/2}. \tag{4.4}$$

(The proof of the above estimates is not hard, and indeed easy when $s = \frac{4\alpha+4}{2\alpha+1}$.)

The estimate (4.3), Hölder inequality for Lorentz spaces and the Banach-Steinhaus theorem, imply that there exists a sequence $\{y_n\}$ tending to $+\infty$, and a function f in $L^{\frac{4\alpha+4}{2\alpha+3},r}(\mathbf{R}_+, x^{2\alpha+1}dx)$ with support in $[0, 1]$ such that

$$|y_n^{\alpha+1/2} \hat{f}(y_n)| \to +\infty.$$

But by (4.4) we also have that for every t_n, with $|y_n - t_n| < 1$,

$$|y_n^{\alpha+1/2}\hat{f}(y_n) - t_n^{\alpha+1/2}\hat{f}(t_n)| < C < +\infty.$$

Let x be a fixed point in $\mathbf{R}_+$. To the sequence $\{y_n\}$ we can associate a sequence $\{t_n\}$ such that in the intervals with extremes y_n and t_n the function $y \to J_\alpha(xy)$ is of constant signum, and $c_0 < |y_n - t_n| < c_1$, c_0 and c_1 depending only on x. Because of the previous estimates, the function $y^{\alpha+1/2}\hat{f}(y)$ is large and of constant signum on the intervals with extremes y_n and t_n, and this is enough to conclude the proof. Indeed if $n \to +\infty$,

$$|S_{y_n}f(x) - S_{t_n}f(x)| = \left| x^{-\alpha-1/2} \int_{t_n}^{y_n} |(xy)^{1/2} J_\alpha(xy)|\, |y^{\alpha+1/2}\hat{f}(y)| dy \right| \to +\infty. \quad \square$$

5. Applications to the harmonic analysis on noneuclidean spaces

Let $\alpha \geq \beta \geq -\frac{1}{2}$. The eigenfunctions of the problem

$$(\sinh x)^{-2\alpha-1}(\cosh x)^{-2\beta-1} \frac{d}{dx}\left\{(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}\frac{d}{dx}f(x)\right\}$$
$$= -((\alpha+\beta+1)^2 + \lambda^2)f(x)$$

which are even and equal to 1 at 0 are the Jacobi functions $\{\varphi_\lambda\}$,

$$\varphi_\lambda(x) = {}_2\mathbf{F}_1\left(\frac{\alpha+\beta+1-i\lambda}{2}, \frac{\alpha+\beta+1+i\lambda}{2}; \alpha+1; -\sinh^2 x\right).$$

For f good enough one can define a Fourier-Jacobi transform

$$\hat{f}(\lambda) = 2^{2(\alpha+\beta+1)} \int_0^{+\infty} f(x)\varphi_\lambda(x)(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1} dx,$$

and one has an inversion formula

$$f(x) = \frac{1}{2\pi}\int_0^{+\infty} \hat{f}(\lambda)\varphi_\lambda(x)|C(\lambda)|^{-2}\, d\lambda,$$

where

$$C(\lambda) = 2^{\alpha+\beta}\pi^{-1/2}\Gamma(\alpha+1)\frac{\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(\frac{\alpha+\beta+1+i\lambda}{2}\right)\Gamma\left(\frac{\alpha-\beta+1+i\lambda}{2}\right)}.$$

For $\alpha = \beta = -\frac{1}{2}$ this is the classical cosine Fourier transform, and in general for special α and β the Jacobi functions can be interpreted as spherical functions on noncompact Riemannian symmetric spaces of rank one. See the survey [Koornwinder].

Define the Rth partial sums of the Fourier-Jacobi transform by

$$S_R f(x) = \int_0^{+\infty} S_R(x, y) f(y)\, dy,$$

where

$$S_R(x, y) = \frac{2^{2\alpha+2\beta+1}}{\pi}(\sinh y)^{2\alpha+1}(\cosh y)^{2\beta+1}\int_0^R \varphi_\lambda(x)\varphi_\lambda(y)|C(\lambda)|^{-2}\, d\lambda.$$

It is possible to obtain asymptotic expansions of Jacobi functions in terms of Bessel functions. See [Stanton-Tomas] and [Trimèche].

In particular

$$\varphi_\lambda(x) \approx \sqrt{\frac{2}{\pi}} 2^\alpha \Gamma(\alpha+1) \lambda^{-\alpha-1/2} (\sinh x)^{-\alpha-1/2} (\cosh x)^{-\beta-1/2} \cos\left(\lambda x - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right),$$

$$C(\lambda) \approx \sqrt{\frac{2}{\pi}} 2^{2\alpha+\beta} \Gamma(\alpha+1)(i\lambda)^{-\alpha-1/2},$$

and one can conjecture that

$$\begin{aligned} S_R(x, y) &\approx \frac{2}{\pi}\left(\frac{\sinh y}{\sinh x}\right)^{\alpha+1/2}\left(\frac{\cosh y}{\cosh x}\right)^{\beta+1/2} \\ &\quad \times \int_0^R \cos\left(\lambda x - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)\cos\left(\lambda y - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) d\lambda \\ &\approx \frac{1}{\pi}\left(\frac{\sinh y}{\sinh x}\right)^{\alpha+1/2}\left(\frac{\cosh y}{\cosh x}\right)^{\beta+1/2} \frac{\sin R(x-y)}{x-y}. \end{aligned}$$

If this is correct one immediately obtains an equiconvergence theorem between Jacobi and cosine expansions.

In the sequel we shall consider in some detail the case $\alpha = \beta = \frac{1}{2}$, which corresponds to the harmonic analysis on the 3-dimensional hyperbolic space. If $\alpha = \beta = \frac{1}{2}$

$$\varphi_\lambda(x) = \frac{\sin \lambda x}{\lambda \sinh x \cosh x}, \quad C(\lambda) = \frac{2}{i\lambda},$$

and one gets an explicit expression for $S_R(x, y)$,

$$\begin{aligned} S_R(x, y) &= \frac{2}{\pi}\left(\frac{\sinh y \cosh y}{\sinh x \cosh x}\right) \int_0^R \sin \lambda x \sin \lambda y \, d\lambda \\ &= \frac{1}{\pi}\left(\frac{\sinh y \cosh y}{\sinh x \cosh x}\right) \frac{\sin R(x-y)}{x-y} - \frac{1}{\pi}\left(\frac{\sinh y \cosh y}{\sinh x \cosh x}\right) \frac{\sin R(x+y)}{x+y}. \end{aligned}$$

From this we easily obtain the following.

Theorem (5.1). *If*

$$\int_0^{+\infty} |f(x)| \frac{\sinh x \cosh x}{1+x} dx < +\infty,$$

then for every x, $0 < x < +\infty$,

$$\lim_{R\to+\infty} \left| S_R f(x) - \frac{1}{\pi \sinh x \cosh x} \int_0^{+\infty} \frac{\sin R(x-y)}{x-y} f(y) \sinh y \cosh y \, dy \right| = 0.$$

Proof. By the Riemann-Lebesgue lemma, if $x > 0$ and $R \to +\infty$,

$$\frac{1}{\pi \sinh x \cosh x} \int_0^{+\infty} \frac{\sin R(x+y)}{x+y} f(y) \sinh y \cosh y \, dy \to 0. \quad \square$$

We now consider Fourier-Jacobi expansions of functions in Lorentz spaces on $\mathbf{R}_+$ with respect to the measure $(\sinh x \cosh x)^2 \, dx$.

Theorem (5.2). (i) *If* f *is in*

$$L^{\frac{3}{2}, 1}(\mathbf{R}_+, (\sinh x \cosh x)^2 dx) + \bigcup_{s<+\infty} L^{2, s}(\mathbf{R}_+, (\sinh x \cosh x)^2 \, dx),$$

then for almost every x *in* $\mathbf{R}_+$,

$$\lim_{R\to+\infty} |S_R f(x) - f(x)| = 0.$$

(ii) *If* $p = 2$ *and* $s = +\infty$, *or if* $p > 2$, *the operators* $\{S_R\}$ *do not exist as operators from* $L^{p,s}(\mathbf{R}_+, (\sinh x \cosh x)^2 dx)$ *into the space of tempered distributions.*

(iii) *If* $r > 1$ *there exist functions* f *in* $L^{\frac{3}{2},r}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$, *supported in the interval* $[0, 1]$, *with* $\{S_R f(x)\}$ *diverging at every* x *in* $\mathbf{R}_+$.

Proof. The proof of (i), (ii) and (iii) is similar to the proof of Theorems (3.1), (4.1) and (4.2) respectively.

Proof of (i). The function $(\sinh x \cosh x)^{-1}/(1+x)$ is in the Lorentz space $L^{3,\infty}(\mathbf{R}_+, (\sinh x \cosh x)^2 dx)$ and also in every $L^{2,s}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$ if $s > 1$.

Therefore by the Hölder inequality for Lorentz spaces

$$\int_0^{+\infty} |f(x)| \frac{\sinh x \cosh x}{1+x}\, dx < +\infty.$$

Also, by the Carleson-Hunt theorem and the Riemann localization principle,

$$\lim_{R\to+\infty} \frac{1}{\pi} \int_0^{+\infty} \frac{\sin R(x-y)}{x-y} f(y) \sinh y \cosh y\, dy = f(x) \sinh x \cosh x.$$

(i) then follows by Theorem (5.1).

Proof of (ii). It is enough to observe that the Jacobi functions $\{\varphi_\lambda\}$ when $\lambda > 0$ are in $L^{2,\infty}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$, so that the Fourier-Jacobi transforms of functions in $L^{2,1}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$ are continuous for $\lambda > 0$.

The Fourier-Jacobi transforms of functions in $L^{q,s}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$, $q < 2$, are analytic. Therefore the operators S_R cannot map a test function into $L^{2,1}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$ or into $L^{q,r}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$, $q < 2$. By duality the operators S_R cannot be defined on the space

$$L^{2,\infty}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$$

or on the spaces $L^{p,s}(\mathbf{R}_+, (\sinh x \cosh x)^2\, dx)$ if $p > 2$.

Proof of (iii). For y in $[0, 1]$,

$$(\sinh y \cosh y)^2\, dy \approx y^2\, dy, \quad \varphi_\lambda(y) \approx \frac{\sin \lambda y}{\lambda y},$$

$$\varphi_\lambda(y) - \varphi_\mu(y) \approx \frac{\sin \lambda y}{\lambda y} - \frac{\sin \mu y}{\mu y},$$

so that as in (4.3) and (4.4), if $\lambda \to +\infty$ and $|\lambda - \mu| < 1$,

$$\|\varphi_\lambda \chi_{[0,1]}\|_{3,s} \approx \lambda^{-1} (\log \lambda)^{1/s}, \quad \|(\varphi_\lambda - \varphi_\mu)\chi_{[0,1]}\|_{3,s} \approx \lambda^{-1}.$$

There exists a sequence $\{\lambda_n\} \to +\infty$ and a function f in

$$L^{3/2,r}(\mathbf{R}_+, (\sinh x \cosh x)^2 dx),$$

$r > 1$, with support in $[0, 1]$ such that $|\lambda_n \hat{f}(\lambda_n)| \to +\infty$, and if $|\lambda_n - \mu_n| < 1$,

$$|\lambda_n \hat{f}(\lambda_n) - \mu_n \hat{f}(\mu_n)| \le C.$$

Let x be a fixed point in $\mathbf{R}_+$. To the sequence $\{\lambda_n\}$ we can associate a sequence $\{\mu_n\}$ so that the function $\lambda \to \sin \lambda x$ is of constant signum on the intervals with extremes λ_n and μ_n, and $c_0 < |\lambda_n - \mu_n| < c_1$. Therefore, in the above intervals, the function $\lambda \hat{f}(\lambda)$ is large and of constant signum, and if $n \to +\infty$,

$$|S_{\lambda_n} f(x) - S_{\mu_n} f(x)| = \left| \frac{1}{8\pi} (\sinh x \cosh x)^{-1} \int_{\mu_n}^{\lambda_n} |\lambda \hat{f}(\lambda)| \, |\sin \lambda x| \, d\lambda \right| \to +\infty. \quad \square$$

We end by referring to [Colzani] for analogous results on expansions in Jacobi polynomials, i.e. the harmonic analysis of radial functions on elliptic noneuclidean spaces.

Added in proof. In *The Hilbert with exponential weights*, Proc. Amer. Math. Soc. **114** (1992), 451–457, the authors have studied the convergence of the Fourier expansions of radial functions on hyperbolic spaces. In *Bochner-Riesz means of functions in weak-L^p*, to appear in Mh. Math., the authors have considered the problem of convergence of Bochner-Riesz means of radial and nonradial functions in weak-$L^p(\mathbf{R}^n)$.

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