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EQUIDISTRIBUTION OF KRONECKER SEQUENCES ALONG CLOSED HOROCYCLES

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Abstract

It is well known that (i) for every irrational number α the Kronecker sequence $m\alpha$ ($m = 1, \dots, M$) is equidistributed modulo one in the limit $M \to \infty$, and (ii) closed horocycles of length ℓ become equidistributed in the unit tangent bundle $T_1\mathcal{M}$ of a hyperbolic surface $\mathcal{M}$ of finite area, as $\ell \to \infty$. In the present paper both equidistribution problems are studied simultaneously: we prove that for any constant $\nu > 0$ the Kronecker sequence embedded in $T_1\mathcal{M}$ along a long closed horocycle becomes equidistributed in $T_1\mathcal{M}$ for almost all α, provided that $\ell = M^\nu \to \infty$. This equidistribution result holds in fact under explicit diophantine conditions on α (e.g. for $\alpha = \sqrt{2}$) provided that $\nu < 1$, or $\nu < 2$ with additional assumptions on the Fourier coefficients of certain automorphic forms. Finally, we show that for $\nu = 2$, our equidistribution theorem implies a recent result of Rudnick and Sarnak on the uniformity of the pair correlation density of the sequence $n^2\alpha$ modulo one.

1 Introduction

New developments in the ergodic theory of unipotent flows have, in the past, led to the solution of important problems in number theory. A famous example is Margulis' proof of the Oppenheim conjecture on values of quadratic forms, and the subsequent proof of a quantitative version [EMM]. The latter crucially uses Ratner's theorem [R1,2], which provides a complete description of all invariant ergodic measures of a unipotent flow.

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The present work is motivated by recent studies of the local spacing distributions of the sequence $n^2\alpha$ modulo one ($n = 1, 2, 3, \dots$), which are conjectured to coincide – for generic values of α – with the spacing distribution of independent random variables from a Poisson process [RuS], [RuSZ]. We will show that the convergence of the pair correlation density of $n^2\alpha$ mod one to the Poisson answer is implied by the equidistribution of Kronecker sequences along closed unipotent orbits (horocycles) in the unit tangent bundle $T_1\mathcal{M}$ of a non-compact hyperbolic surface $\mathcal{M}$ with finite area. The major difficulty in proving equidistribution for the unipotent cascades considered here is that it is unknown if all possible limit measures are necessarily invariant under a unipotent element, and hence Ratner's theory cannot be applied.

To be more precise, let us realize the hyperbolic surface $\mathcal{M}$ as a quotient $\mathcal{M} = \Gamma \setminus \mathcal{H}$, where Γ is a cofinite Fuchsian group acting on the Poincaré upper half-plane

$$\mathcal{H} = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}\,,\ y > 0\}$$

with metric $ds = |dz|/y$, where $dz = dx + idy$ is the complex line element. We assume that $\mathcal{M}$ is non-compact, i.e. has at least one cusp. After a suitable coordinate transformation we may assume that one of the cusps lies at infinity, and that the corresponding isotropy subgroup $\Gamma_\infty \subset \Gamma$ is generated by the translation $z \mapsto z + 1$.

Let $T_1\mathcal{H}$ be the unit tangent bundle of $\mathcal{H}$, and denote its elements by (z, θ), with $z \in \mathcal{H}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, where θ is an angular variable measured from the vertical counterclockwise. The action of an element $\gamma \in \Gamma$ on $T_1\mathcal{H}$ is then given by

$$(z, \theta) \mapsto \big(\gamma z, \theta - 2\beta_\gamma(z)\big) \tag{1.1}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma z = \frac{az + b}{cz + d}, \quad \beta_\gamma(z) = \arg(cz + d)\,.$$

We have in particular $T_1\mathcal{M} = \Gamma \setminus T_1\mathcal{H}$.

For any $y > 0$, the curve $\{(x + iy, 0) \mid x \in \mathbb{R}\}$ is an example of an orbit of the horocycle flow on $T_1\mathcal{H}$. By our assumption on Γ_∞, the above orbit will be closed in $T_1\mathcal{M}$, with length y^{-1}. In the limit $y \to 0$, the orbit in fact becomes *equidistributed* on $T_1\mathcal{M}$ with respect to the Poincaré area $d\mu = dxdy/y^2$ times the uniform measure in the phase variable θ:

Theorem 1 (Sarnak [S]). *For any bounded continuous function f : $T_1\mathcal{M} \to \mathbb{C}$ we have*

$$\lim_{y \to 0} \int_0^1 f\,(x + iy, 0)\, dx = \langle f \rangle\,, \tag{1.2}$$

were $\langle f \rangle$ denotes the average of f over $T_1\mathcal{M}$,

$$\langle f \rangle = \frac{1}{2\pi\mu(\mathcal{M})} \int_{\mathcal{M}} \int_0^{2\pi} f(z,\theta) d\theta \, d\mu(z) \, .$$

The uniform measure on the closed orbit in Theorem 1 may in fact be replaced by uniform measures supported on comparatively small sub-arcs of the orbit [St].

Our investigation is concerned with the equidistribution of the point set

$$\Gamma\{(m\alpha + iy, 0) \mid m = 1, \dots, M\} \tag{1.3}$$

as $M \to \infty$ and $y \to 0$. Clearly, it can be expected to be easier to establish equidistribution the faster the number of points M grows relative to the hyperbolic length of the orbit, y^{-1}. In particular, when $M \gg y^{-1/\nu}$ for some $0 < \nu < 1$, we will see that equidistribution is a direct consequence of the equidistribution of $m\alpha$ mod one and Theorem 1 (provided α is badly approximable, cf. Remark 1.6 below). The harder case is when M is small compared with y^{-1}, and especially, as we shall see, when $M \ll y^{-1/2}$.

Our main result is the following.

Theorem 2. *Fix $\nu > 0$. Then there is a set $P = P(\Gamma, \nu) \subset \mathbb{R}$ of full Lebesgue measure such that for any $\alpha \in P$, any bounded continuous function $f : T_1\mathcal{M} \to \mathbb{C}$, and any constants $0 < C_1 < C_2$, we have*

$$\frac{1}{M} \sum_{m=1}^{M} f\,(m\alpha + iy, 0) \to \langle f \rangle \, , \tag{1.4}$$

uniformly as $M \to \infty$ and $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$.

In fact, we will prove a slightly stronger result than Theorem 2 (cf. Theorem 2$'$ in section 5), wherein the test function f is allowed to be unbounded, satisfying a certain growth condition in the cusps. This will be important for the application to the distribution of $n^2\alpha$ mod one. As we will show in section 8, if relation (1.4) holds for a specific α, then the pair correlation density of $n^2\alpha$ mod one is uniform, i.e. coincides with the correlation density of independent random variables from a Poisson process. Theorem 2$'$ therefore implies the result by Rudnick and Sarnak [RuS] on $n^2\alpha$.

Remark 1.1. For any given fixed $\delta \in \mathbb{R}$, Theorem 2 remains true if we replace $f(m\alpha + iy, 0)$ by $f(\delta + m\alpha + iy, 0)$ in (1.4).

Remark 1.2. The conclusion in Theorem 2 is certainly *false* for all *rational* α. For if $\alpha = p/q$ with $p, q \in \mathbb{Z}$, $q > 0$, then the set of points $\Gamma\{(m\alpha + iy, 0) \mid m = 1, \dots, M\}$ on $\Gamma \setminus T_1\mathcal{H}$ has cardinality $\leqq q$, for *all* M, y. In section 7 we will give a much larger set of counter-examples in the case where Γ is a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$.

We will also prove a theorem which gives an explicit diophantine condition on α ensuring (1.4) to hold. An irrational number $\alpha \in \mathbb{R}$ is said to be of *type* K if there exists a constant $C > 0$ such that

$$\left|\alpha - \tfrac{p}{q}\right| > \tfrac{C}{q^K}$$

for all $p, q \in \mathbb{Z}$, $q > 0$. The smallest possible value of K is $K = 2$, and it is well known that for any given $K > 2$, the set of α's of type K is of full Lebesgue measure in $\mathbb{R}$. Cf. e.g. [Sc, §1].

We recall that under the above normalization of the cusp at ∞, any Maass waveform $\phi(z)$ (of weight 0) on $\Gamma \setminus \mathcal{H}$ has a Fourier expansion involving the Macdonald–Bessel function:

$$\phi(z) = c_0 y^{\frac{1}{2}-ir} + \sum_{n\neq 0} c_n \sqrt{y} K_{ir}\big(2\pi|n|y\big) e(nx)\,, \tag{1.5}$$

where $-\frac{1}{4} - r^2 \leqq 0$ is the eigenvalue of ϕ. Furthermore, if $\eta_1 = \infty, \eta_2, ..., \eta_\kappa$ are the inequivalent cusps of Γ, then we have for the Eisenstein series $E_k(z,s)$ ($k \in \{1, \ldots, \kappa\}$) associated to the cusp η_k,

$$E_k(z,s) = \delta_{k1} y^s + \varphi_{k1}(s) y^{1-s} + \sum_{n\neq 0} \psi_{n,k}(s) \sqrt{y} K_{s-\frac{1}{2}}\big(2\pi|n|y\big) e(nx)\,. \tag{1.6}$$

We fix some $s_0 \in (1/2, 1)$ such that $s = s_0$ is not a pole of any of the Eisenstein series $E_k(z,s)$, $k = 1, \ldots, \kappa$. We will assume that $\beta \geqq 0$ is a number such that the following holds, for each $\varepsilon > 0$:

For each fixed Maass waveform as in (1.5), we have $c_n = O_\varepsilon(|n|^{\beta+\varepsilon})$ as $|n| \to \infty$. (1.7a)

For each holomorphic cusp form $\phi(z) = \sum_{n=1}^\infty c_n e(nz)$ of even integer weight $m \geqq 2$ on $\Gamma \setminus \mathcal{H}$, we have $c_n = O_\varepsilon(|n|^{\frac{m-1}{2}+\beta+\varepsilon})$ as $|n| \to \infty$. (1.7b)

For each fixed $s \in \big(\frac{1}{2} + i[0,\infty)\big) \cup \{s_0\}$, and all $k \in \{1, \ldots, \kappa\}$, we have $\psi_{n,k}(s) = O_\varepsilon(|n|^{\beta+\varepsilon})$ as $|n| \to \infty$. (1.7c)

Thus, if Γ is a congruence subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, and we assume the Ramanujan conjecture for Maass waveforms to be true, then any $\beta \geqq s_0 - \frac{1}{2}$ will work. In particular, since s_0 can be taken arbitrarily close to $1/2$, we may then take $\beta > 0$ arbitrarily small. *Unconditionally*, for Γ a congruence subgroup, we know that (1.7a)–(1.7c) hold for $\beta \geqq \max\big(\frac{7}{64}, s_0 - \frac{1}{2}\big)$, by a recent result of Kim and Sarnak [KS]. (Recall here that the Ramanujan conjecture has been proved in the holomorphic case [D], i.e. (1.7b) is known to be true with $\beta = 0$. Also, on congruence subgroups, (1.7c) can be shown to hold with $\beta = s_0 - \frac{1}{2}$.)

For *general* Γ the situation is very different: It follows from elementary bounds that (1.7a)–(1.7c) hold for $\beta = s_0$ (cf. Lemma 2.7 below regarding (1.7c)). Furthermore, according to Bernstein and Reznikov [BR], (1.7a) holds for $\beta = 1/3$ if we restrict ourselves to the case of Maass waveforms which are cusp forms. Also, by Good [G], (1.7b) holds with $\beta = 1/3$ as long as the weight m is > 2. It can be expected that (1.7b) with $\beta = 1/3$ should also be provable for $m = 2$, and that it should be possible to extend the methods in [BR] to the case of the Eisenstein series and non-cuspidal Maass waveforms, so as to prove (1.7a) with $\beta = 1/3$ and (1.7c) with some $\beta = \beta(s_0) > 1/3$ such that $\beta(s_0) \to 1/3$ if $s_0 \to 1/2$. (We are grateful to P. Sarnak for discussions on these matters.)

Theorem 3. *Let $s_0 \in (1/2, 1)$ and $\beta > 0$ be as in (1.7a)–(1.7c). Let $\alpha \in \mathbb{R}$ be of type $K \geqq 2$, and let ν be a positive number satisfying*

$$\nu < \begin{cases} \frac{2}{1+2\beta} & \text{if } \beta < \frac{3-K}{2(K-1)} \\ \frac{2}{2K\beta+K-2} & \text{if } \frac{3-K}{2(K-1)} \leqq \beta < \frac{1}{2} \\ \frac{2}{2K+2\beta-3} & \text{if } \frac{1}{2} \leqq \beta. \end{cases} \tag{1.8}$$

Then for any bounded continuous function $f : T_1\mathcal{M} \to \mathbb{C}$, and any constant $C_1 > 0$, we have

$$\frac{1}{M}\sum_{m=1}^{M} f(m\alpha + iy, 0) \to \langle f \rangle, \tag{1.9}$$

uniformly as $M \to \infty$, $y \to 0^+$ so long as $y \geqq C_1 M^{-\nu}$.

Notice that the right-hand side in (1.8) is a continuous function of β and K, for $\beta \geqq 0$, $K \geqq 2$. The restriction on ν in (1.8) is the best possible which can be obtained by our method of using the absolute bounds in (1.7a)–(1.7c), cf. Remark 6.4 below.

Corollary 1.3. *Let Γ be a congruence subgroup of $PSL(2, \mathbb{Z})$ and assume the Ramanujan conjecture for Maass waveforms on $\Gamma \setminus \mathcal{H}$ to hold. Then, if α is of type $K \geqq 2$, (1.9) holds for any positive number*

$$\nu < \begin{cases} 2 & \text{if } K < 3 \\ \frac{2}{K-2} & \text{if } K \geqq 3. \end{cases}$$

Remark 1.4. In the case $K \geqq 3$, the bound $\nu < \frac{2}{K-2}$ in Corollary 1.3 is actually the best possible restriction on ν, as follows from Proposition 7.1 below. To be precise, let α be any irrational number and let K_0 be the infimum of all numbers K such that α is of type K. Then if $2 < K_0 < \infty$, (1.9) is *false* for each $\nu > \frac{2}{K_0-2}$. Furthermore, if $\liminf_{q\to\infty}(\inf_{p\in\mathbb{Z}} q^{K_0}|\alpha - p/q|)$ is finite and sufficiently small, then (1.9) is also false for $\nu = \frac{2}{K_0-2}$.

REMARK 1.5. Unconditionally, if Γ is a congruence subgroup, it follows from Theorem 3 and [KS] that if $2 \leqq K \leqq 103/39$, then (1.9) holds for any $\nu < 64/39$.

REMARK 1.6. As we have pointed out, for any group Γ, (1.7a)–(1.7c) hold for $\beta = s_0$. Hence, Theorem 3 implies that (1.9) holds whenever $\nu < (K-1)^{-1}$. However, this fact can also be derived more straightforwardly from Sarnak's theorem 1, since we can prove directly that under the above conditions the set of points $\{(m\alpha + iy, 0) \mid m = 1, \ldots, M\}$ tends to become more and more equidistributed *along the closed horocycle* $\{(x+iy, 0) \mid x \in [0,1]\}$ (in the hyperbolic metric). We give an outline of this argument at the end of section 6.

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2 Spectral Preliminaries

In this section we recall some basic facts concerning the spectral expansion of functions on $\Gamma \setminus T_1\mathcal{H}$, and collect some useful bounds and formulas. We start by introducing some notation which will be in force throughout this paper.

We let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a cofinite Fuchsian group such that $\Gamma \setminus \mathcal{H}$ has at least one cusp.

Concerning the cusps and the fundamental domain, we will use the same notation as in [H2, p. 268]. Specifically: we let $\mathcal{F} \subset \mathcal{H}$ be a canonical (closed) fundamental domain for $\Gamma \setminus \mathcal{H}$, and let $\eta_1, \ldots, \eta_\kappa$ (where $\kappa \geqq 1$) be the vertices of $\mathcal{F}$ along $\partial\mathcal{H} = \mathbb{R} \cup \{\infty\}$. Since $\mathcal{F}$ is canonical, $\eta_1, \ldots, \eta_\kappa$ are $\Gamma-$inequivalent.

For each $k \in \{1, \ldots, \kappa\}$ we choose $N_k \in \mathrm{PSL}(2,\mathbb{R})$ such that $N_k(\eta_k) = \infty$ and such that the stabilizer Γ_{η_k} is $[T_k]$, where $T_k := N_k^{-1}\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) N_k$. Since $\mathcal{F}$ is canonical, by modifying N_k we can *also* ensure that

$$N_k(\mathcal{F}) \bigcap \{z \in \mathcal{H} \mid \ \mathrm{Im}\, z \geqq B\} = \{z \in \mathcal{H} \mid 0 \leqq \mathrm{Re}\, z \leqq 1, \ \ \mathrm{Im}\, z \geqq B\}, \quad (2.1)$$

for all B large enough. We *fix*, once and for all, a constant $B_0 > 1$ such that (2.1) holds for all $B \geqq B_0$ and all $k \in \{1, \ldots, \kappa\}$.

In line with what we stated in the introduction, we will make the assumption that Γ is normalized so that

$$\eta_1 = \infty\,, \qquad N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

This means that Γ has one cusp located at ∞ of standard width $z \mapsto z+1$.

We recall the definition of the *invariant height function,* $\mathcal{Y}_\Gamma(z)$:

$$\mathcal{Y}_\Gamma(z) = \sup\bigl\{\text{Im } N_k W(z) \bigm| k \in \{1, \ldots, \kappa\}\,,\ W \in \Gamma\bigr\}\,. \tag{2.3}$$

(Cf. [I1, (3.8)].) This definition does not depend on the choice of $\mathcal{F}$ or of the maps N_j. In fact, we have

$$\mathcal{Y}_\Gamma(z) = \sup\bigl\{\text{Im } NW(z) \bigm| N \in \mathcal{S}_\Gamma\,,\ W \in \Gamma\bigr\}\,, \tag{2.4}$$

where $\mathcal{S}_\Gamma$ is the set of all $N \in \text{PSL}(2,\mathbb{R})$ such that $\eta = N^{-1}\infty$ is a cusp and $\Gamma_\eta = N^{-1}\left[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right] N$. Relation (2.4) follows directly from the fact that each $N \in \mathcal{S}_\Gamma$ has a factorization $N = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) N_j A$, for some $x \in \mathbb{R}$, $A \in \Gamma$.

The function $\mathcal{Y}_\Gamma(z)$ is well known to be continuous and Γ-invariant, and bounded from below by a positive constant which only depends on the group Γ. Notice that we have $\mathcal{Y}_\Gamma(z) \to \infty$ as $z \in \mathcal{F}$ approaches any of the cusps. We also remark that, for any constant $0 \leqq c < 1$:

$$\int_{\mathcal{F}} \mathcal{Y}_\Gamma(z)^c d\mu(z) < \infty\,. \tag{2.5}$$

This is easily seen by splitting the region $\mathcal{F}$ into κ cuspidal regions $\mathcal{C}_k = N_k^{-1}\bigl([0,1] \times [B,\infty)\bigr)$ with B large – cf. (2.1) – and a remaining compact region, and then using the fact that $\mathcal{Y}_\Gamma(z) =$ Im $N_k(z)$ for all $z \in \mathcal{C}_k$.

When proving Theorem 2 we will first assume $f \in C_c^\infty(\Gamma \setminus T_1\mathcal{H})$, i.e. that f is infinitely differentiable and has compact support as a function on $\Gamma \setminus T_1\mathcal{H}$. Following [S, p. 725], we start by applying Fourier expansion in the variable θ:

$$f(z,\theta) = \sum_{v \in \mathbb{Z}} \widehat{f}_v(z) e^{iv\theta}, \tag{2.6}$$

$$\text{where} \quad \widehat{f}_v(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z,\theta) e^{-iv\theta} d\theta\,. \tag{2.7}$$

A simple computation then shows that

$$\widehat{f}_v\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)^{2v}}{|cz+d|^{2v}}\, \widehat{f}_v(z)\,, \quad \text{for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\,,\ z \in \mathcal{H}\,. \tag{2.8}$$

We will call any function on $\mathcal{H}$ satisfying the automorphy relation in (2.8) *a function of weight* $2v$ *on* $\Gamma \setminus \mathcal{H}$, and we will use $C(\Gamma \setminus \mathcal{H}, 2v)$, $C_c(\Gamma \setminus \mathcal{H}, 2v)$, $L_2(\Gamma \setminus \mathcal{H}, 2v)$, etc. to denote the corresponding function spaces. Hence in our case we have

$$\widehat{f}_v \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H}, 2v)\,.$$

Let us fix $v \in \mathbb{Z}$ temporarily. The function $\widehat{f_v}$ has a spectral expansion with respect to the "weight-$2v$-Laplacian",

$$\Delta_{2v} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - 2ivy\frac{\partial}{\partial x} \, .$$

This operator Δ_{2v} acts on functions of weight $2v$ on $\Gamma \backslash \mathcal{H}$. We let $\phi_0, \phi_1, \ldots$ be the discrete eigenfunctions of $-\Delta_{2v}$, taken to be orthonormal and to have increasing eigenvalues $\lambda_0 \leqq \lambda_1 \leqq \ldots$. Cf. [H2, p. 370 (items 1–3)]; in particular, one knows that $|v|(1-|v|) \leqq \lambda_0$. We let $L_2(\lambda, 2v)$ denote the subspace of $L_2(\Gamma \backslash \mathcal{H}, 2v)$ generated by the ϕ_j's with $\lambda_j = \lambda$. The functions in $L_2(\lambda, 2v)$ are usually called *Maass waveforms of weight* $2v$, and they all belong to $C^\infty(\mathcal{H}) \cap L_2(\Gamma \setminus \mathcal{H}, 2v)$. According to [H2, pp. 317 (Prop. 5.3), 414 (lines 12–16)], we now have the following spectral expansion, for any given $f_v \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H}, 2v)$:

$$f_v(z) = \sum_{n \geq 0} d_n \phi_n(z) + \sum_{k=1}^{\kappa} \int_0^\infty g_k(t) E_k \left(z, \tfrac{1}{2} + it, 2v \right) dt \, , \tag{2.9}$$

with uniform and absolute convergence over $z \in \mathcal{H}$-compacta. Here $E_k(z, s, 2v)$ is the Eisenstein series of weight $2v$ associated to the cusp η_k, cf. [H2, pp. 355 (Def. 5.3), 368 (5.19)]. The coefficients d_n and $g_k(t)$ are given by $d_n = \langle f_v, \phi_n \rangle$ and $g_k(t) = \frac{1}{2\pi} \int_{\mathcal{F}} f_v(z) \overline{E_k\left(z, \frac{1}{2} + it, 2v\right)} d\mu(z)$ (cf. [H2, p. 243 (Remark 2.4)]). In particular, each $g_k(t)$ is a continuous function on $[0, \infty)$.

The proof in [H2] of the beautiful convergence in (2.9) starts by considering the spectral expansion (in L_2-sense) of the function $\Delta_{2v} f_v + a(1-a) f_v \in L_2(\Gamma \setminus \mathcal{H}, 2v)$, for some fixed number $a > |v| + 1$; this is then integrated against the Green's function $G_a(z, w, 2v)$. It is seen in this proof that

$$\sum_{n \geq 0} |d_n|^2 \big(a(1-a) - \lambda_n \big)^2 + 2\pi \sum_{k=1}^{\kappa} \int_0^\infty \left| g_k(t) \right|^2 \left(a(1-a) - \tfrac{1}{4} - t^2 \right)^2 dt$$
$$= \int_{\mathcal{F}} \left| \Delta_{2v} f_v(z) + a(1-a) f_v(z) \right|^2 d\mu(z) < \infty \, . \tag{2.10}$$

Cf. [H2, pp. 91 (9.36), 244–245].

We will need a bound on the rate of convergence in (2.9) which is *uniform* over all $z \in \mathcal{H}$. This is obtained in the next two lemmas, the first of which is a generalization of [I1, Prop. 7.2].

LEMMA 2.1. *Given $v \in \mathbb{Z}$ and $\phi_0, \phi_1, \ldots$ as above, we have, for all $z \in \mathcal{H}$ and $T \geqq 1$,*

$$\sum_{\lambda_n \leqq \frac{1}{4}+T^2} \left|\phi_n(z)\right|^2 + \sum_{k=1}^{\kappa} \int_0^T \left|E_k\left(z, \tfrac{1}{2}+it, 2v\right)\right|^2 dt$$

$$= O\big((T+|v|)\mathcal{Y}_\Gamma(z) + (T+|v|)^2\big)\,. \quad (2.11)$$

The implied constant depends only on Γ, and not on v, T, z.

The uniformity in v in the above bound will not be essential in the proofs of the main results in the present paper.

Proof. For $v = 0$, this is Proposition 7.2 in [I1]. We will now assume $v \in \mathbb{Z}$, $v \neq 0$, and will show how to carry over the proof in [I1] to this case.

We let χ_δ be the characteristic function of the interval $[0, \delta]$, where δ is a (small) positive constant to be specified later. We define, for $z, w \in \mathcal{H}$,

$$u(z,w) = \frac{|z-w|^2}{4 \operatorname{Im} z \operatorname{Im} w} \qquad \text{(cf. [I1, p. 8(1.4)])}\,;$$

$$k(z,w) = (-1)^v \frac{(w-\overline{z})^{2v}}{|w-\overline{z}|^{2v}}\ \chi_\delta\big(u(z,w)\big)\,;$$

$$K(z,w) = \sum_{T\in\Gamma} k(z,Tw)\frac{(cw+d)^{2v}}{|cw+d|^{2v}} \quad \left(\text{wherein } T = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)\right).$$

This agrees with [H1, pp. 359–360, (2.7)] (for trivial character and "$\Phi{=}\chi_{4\delta}$"). As in [I1, pp. 107–109], we fix $w \in \mathcal{H}$ and consider $K(z,w)$ as a function of z; this function belongs to $L_2(\Gamma \backslash \mathcal{H}, 2v)$ and has compact support in $\Gamma \backslash \mathcal{H}$. We may now apply Bessel's inequality to obtain

$$\sum_{n\geqq 0} \left|h(t_n)\phi_n(w)\right|^2 + \frac{1}{2\pi}\sum_{k=1}^{\kappa}\int_0^\infty \left|h(t)E_k(w, \tfrac{1}{2}+it, 2v)\right|^2 dt$$

$$\leqq \int_{\mathcal{F}} \left|K(z,w)\right|^2 d\mu(z)\,. \quad (2.12)$$

Here

$$h(t) = \int_{\mathcal{H}} k(i,z) y^{\frac{1}{2}+it} d\mu(z)\,, \quad (2.13)$$

and the t_n's are defined through $\lambda_n = s_n(1-s_n)$, $s_n = \frac{1}{2} + it_n$, with $s_n \in \left(\frac{1}{2} + i[0,\infty)\right) \cup \left(\frac{1}{2}, |v|\right]$ (recall here that $\lambda_n \geqq |v|(1-|v|)$). The proof of (2.12) and (2.13) uses [H2, pp. 291 (3.23), 373 (item 12)] and unfolding of the integral $\langle K(\cdot,w), \phi_n\rangle = \overline{\int_{\Gamma\backslash\mathcal{H}} K(w,z)\phi_n(z)d\mu(z)}$ and the analogous Eisenstein integral, together with an application of [H1, p. 364 (Prop. 2.14)]. Notice here that the proof in [H1] of Prop. 2.14 remains valid if the assumption in [H1, p. 359 (Def. 2.10)] is replaced by the weaker assumption that Φ is piecewise continuous and of compact support.

By an argument exactly as in [I1, pp. 109–110] we have the following upper bound:

$$\int_{\mathcal{F}} \left|K(z,w)\right|^2 d\mu(z) = O\left(\delta^{3/2}\mathcal{Y}_\Gamma(w)+\delta\right). \tag{2.14}$$

Next, we want to bound $h(t)$ from below. Let us assume

$$\delta \leqq (100|v|)^{-2} \qquad \text{(hence in particular, } \delta \leqq 10^{-4}\text{)}. \tag{2.15}$$

Let D be the hyperbolic disc defined by $u(i,z) \leqq \delta$. One easily checks that $|z-i| < 3\sqrt{\delta}$ holds for all $z \in D$, and hence

$$\left|\operatorname{Arg}(z+i) - \tfrac{\pi}{2}\right| \leqq \arcsin\left(3\sqrt{\delta}/2\right) \leqq 3\sqrt{\delta}\,.$$

Hence, for all $z \in D$, we have $|\operatorname{Arg} k(i,z)| \leqq 6|v|\sqrt{\delta} < \pi/3$ (cf. (2.15)), and thus $\operatorname{Re} k(i,z) \geqq 1/2$, since $|k(i,z)| = 1$. Using the fact that the hyperbolic area of D is $4\pi\delta$, we obtain

$$\left|h\left(\tfrac{i}{2}\right)\right| \geqq \int_{\mathcal{H}} \operatorname{Re} k(i,z)\, d\mu(z) \geqq 2\pi\delta\,.$$

Now take any $t \in \mathbb{C}$ such that $s = \frac{1}{2} + it \in \left(\frac{1}{2} + i[0,\infty)\right) \cup \left(\frac{1}{2}, |v|\right]$. Notice that $z = x+iy \in D$ implies $|y-1| < 3\sqrt{\delta} < |2v|^{-1}$, and in this y-interval we have

$$\begin{aligned}\left|\tfrac{d}{dy}(y^s)\right| &= |s| y^{\operatorname{Re} s - 1} \leqq |s| \max\left((1-|2v|^{-1})^{-1/2}, (1+|2v|^{-1})^{|v|-1}\right)\\ &< |s| \max\left(2, \exp\{(|v|-1)|2v|^{-1}\}\right) < 3|s|\,.\end{aligned}$$

(since $\log(1+|2v|^{-1}) < |2v|^{-1}$). It follows that $|y^s - 1| \leqq 3|s||y-1| \leqq 9\sqrt{\delta}|s|$ for all $z \in D$. We now keep $T \geqq |v|$, and let $\delta = (100T)^{-2}$ (notice that (2.15) is then fulfilled). We then have $9\sqrt{\delta}|s| < 1/4$ for all $s = \frac{1}{2} + it \in \frac{1}{2} + i[0,T]$, and also for all $s \in [1/2, |v|]$. Hence, for all these s, we have by (2.13), $\left|h(t) - h\left(\frac{i}{2}\right)\right| \leqq \frac{1}{4}\mu(D) \leqq \pi\delta$, and thus

$$|h(t)| \geqq \pi\delta\,.$$

Combined with (2.12) and (2.14), this gives

$$\begin{aligned}\sum_{\lambda_n \leqq \frac{1}{4}+T^2} \left|\phi_n(w)\right|^2 + \sum_{k=1}^{\kappa} \int_0^T \left|E_k\left(w, \tfrac{1}{2}+it, 2v\right)\right|^2 dt \\ = O\left(\delta^{-1/2}\mathcal{Y}_\Gamma(w) + \delta^{-1}\right) = O\left(T\mathcal{Y}_\Gamma(w) + T^2\right).\end{aligned}$$

This holds for all $T \geqq |v|$. The desired inequality (2.11) now follows, using the fact that the left-hand side in (2.11) is an increasing function of T. □

LEMMA 2.2. *Let $v \in \mathbb{Z}$ and $f_v \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H}, 2v)$ be given. We then have in* (2.9), *for all $T \geqq 1$, $z \in \mathcal{H}$,*

$$\Big|f_v(z) - \sum_{\lambda_n \leqq 1/4+T^2} d_n\phi_n(z) - \sum_{k=1}^{\kappa}\int_0^T g_k(t)E_k\big(z,\tfrac12+it,2v\big)\,dt\Big|$$
$$\leqq O\big(T^{-1} + T^{-3/2}\sqrt{\mathcal{Y}_\Gamma(z)}\big)\,. \quad (2.16)$$

The implied constant depends on Γ, v *and* f_v, *but* not *on* T, z.

Proof. By (2.9), the difference in the left-hand side of (2.16) equals

$$\sum_{\lambda_n > 1/4+T^2} d_n\phi_n(z) + \sum_{k=1}^{\kappa}\int_T^\infty g_k(t)E_k\left(z,\tfrac12+it,2v\right)dt\,. \quad (2.17)$$

By (2.10), the sum and the κ integrals

$$\sum_{\lambda_n>1/4+1} |d_n|^2\lambda_n^2; \qquad \int_1^\infty \big|g_k(t)\big|^2 t^4 dt \quad (k=1,2,\dots,\kappa)$$

are all bounded from above by some finite constant (which depends on v, f_v, Γ). Hence by writing $d_n\phi_n(z) = (d_n\lambda_n)\cdot(\phi_n(z)/\lambda_n)$ and $g_k(t)E_k(\dots) = (g_k(t)t^2)\cdot(E_k(\dots)/t^2)$ in (2.17), and then applying Cauchy's inequality, we find that the modulus of (2.17) is bounded from above by

$$O(1)\sqrt{\sum_{\lambda_n>1/4+T^2}\lambda_n^{-2}|\phi_n(z)|^2} + O(1)\sum_{k=1}^{\kappa}\sqrt{\int_T^\infty t^{-4}\left|E_k\left(z,\tfrac12+it,2v\right)\right|^2 dt}$$
$$= O(1)\sqrt{\sum_{t_n>T} t_n^{-4}\big|\phi_n(z)\big|^2 + \sum_{k=1}^{\kappa}\int_T^\infty t^{-4}\left|E_k\left(z,\tfrac12+it,2v\right)\right|^2 dt}\,,$$

where the t_n's are as in the proof of Lemma 2.1. Now (2.16) follows from Lemma 2.1 (applied for our *fixed* v, so that the right-hand side in (2.11) is $O(T\mathcal{Y}_\Gamma(z) + T^2)$), and partial summation. □

Next, we will review some facts about the Fourier expansion of Maass waveforms of even integer weight. We will use the standard notation $W_{k,m}(z)$ for the Whittaker function as in (e.g.) [O1, Ch. 7 (10.04), (11.03)]. $W_{k,m}(z)$ is a holomorphic function for $k,m \in \mathbb{C}$, $|\mathrm{Arg}(z)| < \pi$.

LEMMA 2.3. *Let* $\phi \in L_2(\lambda, 2v)$, $\phi \not\equiv 0$, $v \in \mathbb{Z}$; *we can then write* $\lambda = s(1-s)$ *for a unique* $s \in \frac12 + i[0,\infty)$ *or* $s \in \left(\frac12, \max(1,|v|)\right]$. *We have a Fourier expansion*

$$\phi(z) = c_0y^{1-s} + \sum_{n\neq 0}\frac{c_n}{\sqrt{|n|}}W_{v\cdot\mathrm{sgn}(n),s-\frac12}\big(4\pi|n|y\big)e(nx)\,. \quad (2.18)$$

From this it follows that, as $y\to\infty$,

$$\phi(z) = c_0y^{1-s} + O(y^{|v|}e^{-2\pi y})\,. \quad (2.19)$$

If $\mathrm{Re}\, s = 1/2$, *then we necessarily have* $c_0 = 0$.

Proof. This is a restatement of [H2, pp. 370-371 (items 1,3,5)] and [H2, pp. 26 (Prop 4.12), 349]. Concerning the translation of [H2, p. 370 (5.28)] into (2.18), cf. [H2, pp. 347 (Lemma 4.5), 348 (lines 3-5), 355 (Def. 5.4)] and [O1, p. 256 (footnote ‡)]. □

We remark that the Whittaker function is always real valued in (2.18); in fact, we have $W_{k,m}(z) \in \mathbb{R}$ for all $k \in \mathbb{R}$, $m \in \mathbb{R} \cup i\mathbb{R}$, $z > 0$, as follows from [O1, Ch. 7 (9.03), (11.02), Ex. 11.1].

Each Maass waveform of even integer weight allows an explicit description in terms of either a Maass waveform of weight zero or a holomorphic cusp form, and the coefficients in the Fourier expansions of the two forms are proportional. We will state this fact in a precise form in the next two lemmas, thus slightly generalizing formulas given earlier in [J1,2]. For the proofs of the main results in the present paper it will not be essential to know the exact formulas for the proportionality factors involved. However, these formulas, as well as the uniformity in v obtained in Lemma 2.1, might be of importance in future applications, e.g. to obtain results on *the rate of convergence* in various asymptotic geometric problems. In this vein, note that the (uniform) asymptotic properties of $W_{k,m}(z)$ are known for all relevant ranges of k, m, z; cf. [O2], [Du1,2].

We use the following standard notation:

$$\mathrm{K}_{2v} = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v\,, \qquad \Lambda_{2v} = iy\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + v\,,$$

$$(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha) = \alpha(\alpha+1)\dots(\alpha+n-1) \qquad \text{for } \alpha \in \mathbb{C}\,,\ n \geqq 0\,.$$

LEMMA 2.4. *Let ϕ be as in Lemma 2.3, and assume $\operatorname{Re} s < 1$. Then there exists a Maass waveform ϕ_0 of weight 0 on $\Gamma \setminus \mathcal{H}$ satisfying $\Delta_0\phi_0 + \lambda\phi_0 = 0$ and $\|\phi_0\|_{L_2} = \|\phi\|_{L_2}$, such that*

$$\phi = \big\{(s)_{|v|}(1-s)_{|v|}\big\}^{-\frac12} \begin{cases} \mathrm{K}_{2v-2}\mathrm{K}_{2v-4}\dots\mathrm{K}_0(\phi_0) & \text{if } v \geqq 1 \\ \phi_0 & \text{if } v = 0 \\ \Lambda_{2v+2}\Lambda_{2v+4}\dots\Lambda_0(\phi_0) & \text{if } v \leqq -1\,. \end{cases}$$

Let us, furthermore, assume ϕ_0 to have the following Fourier expansion in the standard format involving the Macdonald–Bessel function $K_\mu(z)$:

$$\phi_0(z) = d_0 y^{1-s} + \sum_{n\neq 0} d_n \sqrt{y} K_{s-\frac12}\big(2\pi|n|y\big)e(nx)$$

(*cf.* [H2, Ch. 6, §4]). *We then have in* (2.18)*:*

$$\begin{aligned} c_0 &= \begin{cases} \operatorname{sgn}(v)^v\big\{(1-s)_{|v|}/(s)_{|v|}\big\}^{1/2}d_0 & \text{if } v\neq 0 \text{ and } \frac12 < s < 1 \\ d_0 & \text{otherwise} \end{cases} \\ c_n &= \tfrac12\big\{-\operatorname{sgn}(n)\big\}^v\big\{(s)_{|v|}(1-s)_{|v|}\big\}^{-\frac12\operatorname{sgn}(nv)}d_n \quad \text{for } n\neq 0\,. \end{aligned} \tag{2.20}$$

Notice that all square roots appearing in the lemma are well defined, since clearly $(s)_{|v|}(1-s)_{|v|} > 0$ for all $s \in \frac{1}{2} + i[0,\infty)$ and for all $s \in \left(\frac{1}{2}, 1\right)$, and $(1-s)_{|v|}/(s)_{|v|} > 0$ for all $s \in (1/2, 1)$. We also remark that the above formulas are consistent with [J1, (1.8)], [J2, (6),(7)] in the case $\mathrm{Re}\, s = 1/2$, $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ (since the $\varphi_{j,k}$-expansions in [J1,2] agree with our formulas above after multiplication by a certain constant $c = c(s,v) \in \mathbb{C}$ with $|c| = 1$).

Proof. The first assertion follows from [H2, p. 382(d),(e),(f),(h)], used repeatedly.

It remains to prove the formulas for the c_n's. The case $v = 0$ is trivial since $K_{s-\frac{1}{2}}(y) = (\pi/2y)^{1/2} W_{0,s-\frac{1}{2}}(2y)$ (cf. e.g. [O1, Ch. 7, Ex. 10.1, (11.03)]). The cases $v > 0$ and $v < 0$ are now treated by first noticing that we may differentiate term-by-term any number of times in (2.18) (cf. [H2, pp. 25 (Remark 4.11), 349 (line 1)]), and then making repeated use of [J1, Lemma 1.1]. (Notice here that [J1, Lemma 1.1] can be extended to *complex* t. Notice also that the right-hand side in the second formula in [J1, Lemma 1.1] should be corrected by interchanging "$k-1$" and "$-(k-1)$" in the arguments of the Whittaker functions.) □

Before stating the next lemma, we recall that if $m \in \{2,4,6,\dots\}$, and if ϕ_0 is a holomorphic cusp form of weight m on $\Gamma \setminus \mathcal{H}$, then $y^{m/2}\phi_0$ and $y^{m/2}\overline{\phi_0}$ are Maass waveforms of weight m and $-m$,

$$y^{m/2}\phi_0 \in L_2(\lambda, m)\,, \quad y^{m/2}\overline{\phi_0} \in L_2(\lambda, -m), \quad \text{where } \lambda = \tfrac{m}{2}\left(1 - \tfrac{m}{2}\right)\,.$$

In particular, $\|y^{m/2}\phi_0\|_{L_2} = \|y^{m/2}\overline{\phi_0}\|_{L_2} = \sqrt{\int_{\mathcal{F}} |\phi_0(z)|^2 y^m d\mu(z)}$. Cf. [H2, p. 382 (item 22), Prop. 5.14].

Lemma 2.5. *Let ϕ be as in Lemma 2.3, and assume $s \geqq 1$. Then, if $v = 0$, we must have $s = 1$, and ϕ is a constant function. Now assume $v \neq 0$. We then have $s \in \{1, 2, \dots, |v|\}$, and there exists a holomorphic cusp form ϕ_0 on $\Gamma \setminus \mathcal{H}$ of weight $2s$ with $\|y^s\phi_0\|_{L_2} = \|\phi\|_{L_2}$, such that*

$$\phi = \left\{(|v|-s)!(2s)_{|v|-s}\right\}^{-\frac{1}{2}} \begin{cases} \mathrm{K}_{2v-2}\mathrm{K}_{2v-4}\dots\mathrm{K}_{2s}\big(y^s\phi_0\big) & \text{if } v \geqq 1 \\ \Lambda_{2v+2}\Lambda_{2v+4}\dots\Lambda_{-2s}\big(y^s\overline{\phi_0}\big) & \text{if } v \leqq -1\,. \end{cases}$$

(If $s = |v|$, this should be interpreted as $\phi = y^s\phi_0$ if $v \geqq 1$, $\phi = y^s\overline{\phi_0}$ if $v \leqq -1$.) Furthermore, if ϕ_0 has Fourier expansion

$$\phi_0(z) = \sum_{n=1}^{\infty} d_n e(nz)\,,$$

then we have in (2.18)

$$c_n = \big\{(|v|-s)!(2s)_{|v|-s}\big\}^{-\frac{1}{2}}(4\pi)^{-s}|n|^{\frac{1}{2}-s}\begin{cases}(-1)^{v-s}d_n & \text{if } v>0\\ \overline{d_{-n}} & \text{if } v<0\end{cases}$$

for all n with $nv>0$; also, $c_0=0$, and $c_n=0$ for all n with $nv<0$.

Proof. The case $v=0$ is trivial, cf. [H2, p. 71 (Claim 9.2)]. If $v\neq 0$, then we have $s\in\{1,2,\ldots,|v|\}$ by [H2, pp. 350 (5.4), 427 (line 1)]. The rest of the proof is very similar to the proof of Lemma 2.4, except that we also use [H2, p. 383 (a),(d),(g)], and the fact that

$$W_{s,s-\frac{1}{2}}(y) = e^{-y/2}y^s\,, \tag{2.21}$$

which follows easily from (e.g.) [O1, Ch. 7 (9.03),(9.04),(10.09),(11.03)]. □

The above Lemmas 2.3, 2.4, 2.5 deal with the Fourier expansion of $\phi(z)$ at the cusp $\eta_1=\infty$. Of course, there are analogous results for the Fourier expansions corresponding to the other cusps $\eta_2,\ldots,\eta_\kappa$ of Γ (these results may be proved e.g. by applying the above lemmas to the Fuchsian groups $N_k\Gamma N_k^{-1}$, $k=2,\ldots,\kappa$). One consequence of this is the following: For any $\phi\in L_2(\lambda,2v)$ with $\lambda=s(1-s)$ as in Lemma 2.3, there exists a constant $C>0$ such that

$$|\phi(z)| \leqq C\mathcal{Y}_\Gamma(z)^{1-\operatorname{Re}s}\,, \qquad \forall z\in\mathcal{H}\,. \tag{2.22}$$

(To see this, notice that for each $k\in\{1,\ldots,\kappa\}$, the η_k-analog of (2.19) implies that $\phi(z)=O((\operatorname{Im} N_kz)^{1-\operatorname{Re}s})$ as $z\to\eta_k$ inside $\mathcal{F}$. Hence there is a $C>0$ such that (2.22) holds for all $z\in\mathcal{F}$. By Γ-invariance, (2.22) now holds for all $z\in\mathcal{H}$.)

One main ingredient in our proof of Theorem 2 will be the well-known Rankin–Selberg type bound on the sums $\sum_{|n|\leqq N}|c_n|^2$ of the Fourier coefficients of the Maass waveforms (cf. e.g. [I1, Thm. 3.2], [I2, Thm. 5.1]). We will also need a similar bound on the Fourier coefficients of the Eisenstein series, which we will prove in Lemma 2.7 below.

We first recall the following explicit formula for the Eisenstein series of even integer weight in terms of the Eisenstein series of weight zero:

Lemma 2.6. *Consider the Fourier expansion of the Eisenstein series of weight zero,*

$$E_k(z,s,0) = \delta_{k1}y^s + \varphi_{k1}(s)y^{1-s} + \sum_{n\neq 0}\psi_{n,k}(s)\sqrt{y}K_{s-\frac{1}{2}}\big(2\pi|n|y\big)e(nx)\,.$$

We then have the following expansion of the Eisenstein series of weight $2v$:

$$E_k(z,s,2v) = \delta_{k1}y^s + (-1)^v \frac{\Gamma(s)^2}{\Gamma(s+v)\Gamma(s-v)}\varphi_{k1}(s)y^{1-s}$$
$$+ \sum_{n\neq 0} \frac{(-1)^v\Gamma(s)}{2\Gamma(s+v\cdot \operatorname{sgn}(n))}\frac{\psi_{n,k}(s)}{\sqrt{|n|}} W_{v\cdot\operatorname{sgn}(n),s-\frac12}\big(4\pi|n|y\big)e(nx)\,.$$

Proof. For $\operatorname{Re} s > 1$ this follows from the explicit formulas in [H2, pp. 368 (5.22), 369 (5.23)]. (Cf. also our remarks in the proof of Lemma 2.3, as well as [H2, p. 280 (Prop. 3.7)].) For general s the result now follows by meromorphic continuation, since it is known that, for any fixed $k \in \{1,\dots,\kappa\}$ and $v \in \mathbb{Z}$, $E_k(z,s,2v)$ has a Fourier expansion of the above type, with each coefficient being a meromorphic function in s. Cf. [H2, p. 374 (items 15-17)]. □

Lemma 2.7. *Let $v \in \mathbb{Z}$, $k \in \{1,\dots,\kappa\}$ and $s = \sigma + it \in \mathbb{C}$, $\sigma \geqq 1/2$. We assume that the Eisenstein series $E_k(z,s,2v)$ does not have a pole at s, and we take the Fourier expansion to be*

$$E_k(z,s,2v) = \delta_{k1}y^s + c_0y^{1-s} + \sum_{n\neq 0}\frac{c_n}{\sqrt{|n|}}W_{v\cdot\operatorname{sgn}(n),s-\frac12}\big(4\pi|n|y\big)e(nx)\,. \quad (2.23)$$

We then have, for $N \geqq 1$,

$$\sum_{1\leqq|n|\leqq N}|c_n|^2 = O(1)\begin{cases} N\log 2N & \text{if } \sigma = \frac12\,,\\ N^{2\sigma} & \text{if } \sigma > \frac12\,.\end{cases} \quad (2.24)$$

(*The implied constant depends only on Γ, v, s.*)

Proof. In the case $\sigma = 1/2$, $v = 0$, the desired result was proved in [St, Prop. 4.1]. We will review the proof from [St], and show that it generalizes to $\sigma \geqq 1/2$, $v \in \mathbb{Z}$.

We recall that there are only a finite number of poles of $E_k(z,s,0)$ in the half plane $\sigma \geqq 1/2$, and all these poles belong to the interval $(1/2,1]$. It now follows from the formulas in Lemma 2.6 that if $s \neq 1$, $\operatorname{Re} s \geqq 1/2$, and s is not a pole of $E_k(z,s,2v)$, then s cannot be a pole of $E_k(z,s,0)$; these formulas also imply (for $s \neq 1$) that it suffices to prove (2.24) for $v = 0$. However, as we will see, it is just as easy to prove Lemma 2.7 directly for general $v \in \mathbb{Z}$, and this also covers the case $s = 1$ ($v \neq 0$).

We keep $0 < Y < H$ and study the following integral:

$$J = \int_{\mathcal{D}} \big|E_k(z,s,2v)\big|^2 d\mu(z)\,, \qquad \text{where} \quad \mathcal{D} = (0,1)\times(Y,H)\,. \quad (2.25)$$

We tessellate $\mathcal{D}$ by translates of the fundamental region, i.e. we write $\mathcal{D} = \cup_{T\in\Gamma}(\mathcal{D}\cap T(\mathcal{F}))$, an essentially disjoint union. It then turns out that $\mathcal{D}$ is fully covered by the translates of a *truncated* region

$$\mathcal{F}_B = \mathcal{F} - \bigcup_{j=1}^{\kappa} N_j^{-1}\big([0,1] \times [B,\infty)\big)\,, \quad \text{where } B = \max\big(B_0, H, \tfrac{1}{Y}\big)\,.$$

(Cf. (2.1). For details, see the proof in [St].) The upshot of this is that

$$J = \int_{\mathcal{F}_B} \#\big\{T \in \Gamma \bigm| T(z) \in \mathcal{D}\big\} \cdot \big|E_k(z,s,2v)\big|^2 d\mu(z)\,.$$

But we have, by [I1, Lemma 2.10],

$$\begin{aligned}\#\big\{T \in \Gamma \bigm| T(z) \in \mathcal{D}\big\} &\leqq \#\big\{W_0 \in [S] \setminus \Gamma \bigm| \operatorname{Im} W_0(z) > Y\big\}\\ &= 1 + O(Y^{-1})\,,\end{aligned}$$

where the implied constant depends only on Γ, i.e. the bound is uniform over all $z \in \mathcal{H}$ and $Y > 0$. Hence,

$$J = O(1+Y^{-1}) \int_{\mathcal{F}_B} \big|E_k(z,s,2v)\big|^2 d\mu(z)\,. \tag{2.26}$$

We now decompose $\mathcal{F}_B$ as a union of the compact region $\mathcal{F}_{B_0}$ and the cuspidal regions $N_j^{-1}([0,1] \times [B_0,B))$, for $j = 1,\ldots,\kappa$. From the Fourier expansion of $E_k(z,s,2v)$ in the cusp η_j it follows that for all points $z = N_j^{-1}(x'+y'i)$, $y' \geqq B_0$ we have $E_k(z,s,2v) = O((y')^\sigma)$ (cf. Lemma 2.6 and the proof of (2.22)); of course, the implied constant depends on s and v. Using this we easily obtain

$$J = O(1+Y^{-1}) \begin{cases} \log(2B) & \text{if } \sigma = \frac{1}{2}\,,\\ B^{2\sigma-1} & \text{if } \sigma > \frac{1}{2}\,. \end{cases} \tag{2.27}$$

On the other hand, substituting (2.23) directly in the definition of J, (2.25), and then applying Parseval's formula, we get

$$J \geqq 4\pi \sum_{n\neq 0} |c_n|^2 \int_{4\pi|n|Y}^{4\pi|n|H} W_{v\cdot\operatorname{sgn}(n),s-\frac{1}{2}}(u)^2\,\frac{du}{u^2}.$$

We now take $H = 1$ and $Y = (4\pi N)^{-1}$. With this choice we have $[1,4\pi] \subset [4\pi|n|Y, 4\pi|n|H]$ whenever $1 \leqq |n| \leqq N$, and hence the last inequality implies

$$\sum_{1\leqq|n|\leqq N} |c_n|^2 \leqq O(J)\,.$$

Using this fact and (2.27), we obtain (2.24). □

3 Averages of Eigenfunctions for Generic α

In this section we will study the sum $\frac{1}{M}\sum_{m=1}^{M} f(m\alpha+iy)$ in the case when f is an eigenfunction on $\Gamma\backslash\mathcal{H}$, i.e. either a Maass waveform or an Eisenstein series.

PROPOSITION 3.1. *Let $\varepsilon > 0$, and let ϕ be a non-constant Maass waveform of even integer weight, say $\phi \in L_2(\lambda, 2v)$ and $\lambda = s(1-s)$ as in Lemma 2.3. We write*

$$S = S(M, y, \alpha) = \frac{1}{M}\sum_{m=1}^{M} \phi(m\alpha + iy)\,.$$

We then have, for all $M \geqq 1$ and $0 < y \leqq 1$,

$$\int_0^1 |S|^2 d\alpha = O(y^{-\varepsilon}M^{\varepsilon-1}) + \big[\text{if } s \in \big(\tfrac{1}{2}, 1\big) \ : O(y^{2-2s})\big]\,. \tag{3.1}$$

(The implied constant depends on $\Gamma, \phi, \varepsilon$, but not on M, y.)

Proof. As in Lemma 2.3 we have a Fourier expansion

$$\phi(z) = c_0 y^{1-s} + \sum_{n\neq 0} \frac{c_n}{\sqrt{|n|}} W_{v\cdot\mathrm{sgn}(n), s-\frac{1}{2}}\big(4\pi|n|y\big)e(nx)\,. \tag{3.2}$$

It follows that

$$\begin{aligned} S &= c_0 y^{1-s} + \frac{1}{M}\sum_{n\neq 0}\sum_{m=1}^{M} \frac{c_n}{\sqrt{|n|}} W_{v\cdot\mathrm{sgn}(n), s-\frac{1}{2}}\big(4\pi|n|y\big)e(nm\alpha) \\ &= c_0 y^{1-s} + \frac{1}{M}\sum_{l\neq 0}\Bigg(\sum_{\substack{m|l\\ 1\leqq m\leqq M}} \frac{c_{l/m}}{\sqrt{|l/m|}} W_{v\cdot\mathrm{sgn}(l), s-\frac{1}{2}}\big(4\pi|l/m|y\big)\Bigg)e(l\alpha)\,. \end{aligned}$$

We will write $s = \sigma + it$, and use $W(Y)$ as a shorthand for $W_{v\cdot\mathrm{sgn}(l), s-\frac{1}{2}}(Y)$. Applying Parseval's formula and then Cauchy's inequality, we get

$$\begin{aligned} \int_0^1 |S|^2\, d\alpha &= |c_0|^2 y^{2-2\sigma} + \frac{1}{M^2}\sum_{l\neq 0}\Bigg|\sum_{\substack{m|l\\ 1\leqq m\leqq M}} \frac{c_{l/m}}{\sqrt{|l/m|}} W\big(4\pi|l/m|y\big)\Bigg|^2 \\ &\leqq |c_0|^2 y^{2-2\sigma} + \frac{1}{M^2}\sum_{l\neq 0}\tau(|l|)\sum_{\substack{m|l\\ 1\leqq m\leqq M}} \frac{|c_{l/m}|^2}{|l/m|} W(4\pi|l/m|y)^2 \end{aligned}$$

(where $\tau(|l|)$ is the number of divisors of $|l|$)

$$= |c_0|^2 y^{2-2\sigma} + \frac{1}{M^2}\sum_{n\neq 0}\frac{|c_n|^2}{|n|} W_{v\cdot\mathrm{sgn}(n), s-\frac{1}{2}}\big(4\pi|n|y\big)^2 \cdot \sum_{m=1}^{M}\tau(|mn|).$$

In this sum we will use the following crude bound, which applies whenever $n \neq 0$ and $c_n \neq 0$:

$$W_{v\cdot\mathrm{sgn}(n),s-\frac{1}{2}}(Y) = O(e^{-Y/4}), \quad \forall Y > 0. \tag{3.3}$$

(The implied constant depends on v, s.) To prove (3.3), first recall that $W_{k,m}(Y) \sim e^{-Y/2}Y^k$ as $Y \to \infty$ (cf. [O1, Ch. 7 (11.05)]). It now remains to treat the case $0 < Y \leqq 1$. If $s \in \frac{1}{2} + i[0,\infty)$ or $s \in \left(\frac{1}{2}, 1\right)$, then we have, as $Y \to 0^+$:

$$W_{v\cdot\mathrm{sgn}(n),s-\frac{1}{2}}(Y) = O(1)\begin{cases} Y^{1/2}\log(1/Y) & \text{if } s = \frac{1}{2} \\ Y^{1-\sigma} & \text{otherwise}. \end{cases} \tag{3.4}$$

Cf. [O1, Ch. 7 (11.04), Ex. 11.1]. To prove (3.4) for $s = 1/2$ one may use [O1, Ch. 7 (9.03), (11.02), Ex. 11.1] to show $W_{\pm v,c}(Y) = O(|c|^{-1}Y^{1/2-|c|})$, uniformly for $0 < Y < 1/4$ and complex c with $0 < |c| < 1/4$ (and fixed $v \in \mathbb{Z}$); then apply the maximum modulus principle in the c-variable along $|c| = -(10\log Y)^{-1}$.

Clearly, when $s \in \frac{1}{2} + i[0,\infty)$ or $s \in \left(\frac{1}{2}, 1\right)$, (3.3) follows from (3.4).

By Lemma 2.5, the only remaining possibility is $s \in \{1, 2, \dots, |v|\}$, and in this case we know that $c_n \neq 0$ can only hold if $nv > 0$. Using (2.21) and the recursion relation $W_{k+1,m}(Y) = (Y/2 - k - Y(\partial/\partial Y))W_{k,m}(Y)$ (cf. [AS, p. 507 (13.4.33)]), we now find that

$$W_{v\cdot\mathrm{sgn}(n),s-\frac{1}{2}}(Y) = W_{|v|,s-\frac{1}{2}}(Y) = O(Y^s) \qquad \text{as } Y \to 0^+.$$

Now (3.3) is completely proved.

Using (3.3) together with the well-known estimate $\tau(|l|) = O(|l|^\varepsilon)$, we obtain

$$\int_0^1 |S|^2 d\alpha \leqq |c_0|^2 y^{2-2\sigma} + O(M^{\varepsilon-1}) \sum_{n\neq 0} |c_n|^2 |n|^{\varepsilon-1} e^{-2\pi|n|y}. \tag{3.5}$$

Recall here that by Lemma 2.3 and Lemma 2.5, $c_0 \neq 0$ is possible only if $s \in (1/2, 1)$. We have the following mean square bound on the coefficients c_n:

$$\sum_{1\leqq|n|\leqq N} |c_n|^2 = O(N). \tag{3.6}$$

(The implied constant depends on ϕ.) In the case $\mathrm{Re}\, s < 1$, (3.6) follows from Lemma 2.4 and [I1, Thm. 3.2]; in the remaining case, i.e. $s \in \{1, 2, \dots, |v|\}$, (3.6) follows from Lemma 2.5 and [I2, Thm. 5.1], and partial summation.

Now (3.1) follows from (3.5) and (3.6), by partial summation. □

Corollary 3.2. *Let ϕ, λ, s be as in Proposition 3.1, and let $\nu > 0$ and $c > 0$. Let $A > 1$ be an arbitrary integer. If $1/2 < s < 1$ then we also* *assume $A > \nu^{-1}(2-2s)^{-1}$. Then there is a set $P = P(\Gamma, \phi, \nu, c, A) \subset \mathbb{R}$ of full Lebesgue measure such that for each $\alpha \in P$, we have*

$$\lim_{M\to\infty} \frac{1}{M^A} \sum_{m=1}^{M^A} \phi\big(m\alpha + icM^{-A\nu}\big) = 0\,. \tag{3.7}$$

Proof. Let us fix $\varepsilon > 0$ so small that $A(1-(\nu+1)\varepsilon) > 1$. Applying Proposition 3.1 with $y = cM^{-\nu}$ we obtain

$$\int_0^1 \left| \frac{1}{M} \sum_{m=1}^{M} \phi(m\alpha + icM^{-\nu}) \right|^2 d\alpha = O(M^{-b})\,, \qquad \text{as } M \to \infty\,. \tag{3.8}$$

Here $b = \min(1-(\nu+1)\varepsilon, \nu(2-2s))$ if $1/2 < s < 1$, otherwise $b = 1-(\nu+1)\varepsilon$. It follows from our assumptions that $Ab > 1$, and hence

$$\int_0^1 \sum_{M=1}^{\infty} \left| \frac{1}{M^A} \sum_{m=1}^{M^A} \phi\big(m\alpha + ic(M^A)^{-\nu}\big) \right|^2 d\alpha = O(1) \sum_{M=1}^{\infty} (M^A)^{-b} < \infty\,.$$

It follows that the integrand in the left-hand side is finite for almost all α, and hence, a fortiori, there is a set $P \subset [0,1]$ of full Lebesgue measure such that (3.7) holds for all $\alpha \in P$.

Finally, we notice that the sum $\sum_m \phi(m\alpha + iy)$ is invariant under $\alpha \mapsto \alpha + 1$, since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. The desired result follows from this. □

PROPOSITION 3.3. *Let $\varepsilon > 0$, $k \in \{1, \ldots, \kappa\}$, $v \in \mathbb{Z}$, and take $s = \sigma + it$ either on $\frac{1}{2} + i[0,\infty)$ or on $(1/2, 1)$, such that $E_k(z, s, 2v)$ does not have a pole at s. We write*

$$S = S(M, y, \alpha) = \frac{1}{M} \sum_{m=1}^{M} E_k(m\alpha + iy, s, 2v)\,.$$

We then have, for all $M \geqq 1$ and $0 < y \leqq 1$,

$$\int_0^1 |S|^2 d\alpha = O(y^{1-2\sigma-\varepsilon} M^{\varepsilon-1} + y^{2-2\sigma})$$

(The implied constant depends only on $\Gamma, v, s, \varepsilon$.)

Proof. Mimicking the proof of Proposition 3.1, we obtain (3.5) with the term $|c_0|^2 y^{2-2\sigma}$ replaced by $|\delta_{k1} y^s + c_0 y^{1-s}|^2$ – the c_n's are now the Fourier coefficients in the expansion of $E_k(z, s, 2v)$ as in (2.23) in Lemma 2.7. Using the bound from Lemma 2.7 in the form $\sum_{1 \leqq |n| \leqq N} |c_n|^2 = O(N^{2\sigma+\varepsilon})$, together with partial summation, we obtain $\int_0^1 |S|^2 d\alpha \leqq O(M^{\varepsilon-1} y^{1-2\sigma-2\varepsilon} + y^{2-2\sigma})$. This proves the desired bound, with 2ε in place of ε. □

COROLLARY 3.4. *Let $k, v, s = \sigma + it$ be as in Proposition 3.3 (in particular, $1/2 \leqq \sigma < 1$), and let $\nu > 0$ and $c > 0$. We make the* *assumption that* $\sigma < 1/2(1+1/\nu)$*, and we let* A *be an arbitrary integer greater than* $\max((1+\nu-2\nu\sigma)^{-1},(2\nu-2\nu\sigma)^{-1})$*. Then there is a set* $P = P(\Gamma,k,v,s,\nu,c,A) \subset \mathbb{R}$ *of full Lebesgue measure such that for each* $\alpha \in P$*, we have*

$$\lim_{M\to\infty} \frac{1}{M^A} \sum_{m=1}^{M^A} E_k(m\alpha + icM^{-A\nu}, s, 2v) = 0\,.$$

Proof. This is similar to the proof of Corollary 3.2, using Proposition 3.3 instead of Proposition 3.1. □

4 Averages of Eigenfunctions: Conditional Results

In this section we will show that, in Corollaries 3.2 and 3.4, we may under certain conditions replace the sequence $1^A, 2^A, 3^A, \ldots$ by the full sequence $1, 2, 3, \ldots$, and allow the constant c to vary over any fixed compact interval. The main reason for this is that the hyperbolic distance between $iM^{-A\nu}$ and $i(M+1)^{-A\nu}$ tends to 0 as $M \to \infty$. We will start with the simplest case; that of ϕ being a cusp form.

Throughout section 4 and 5, we will let $\mathbf{D}$ denote a fixed countable dense subset of $\mathbb{R}^+$.

LEMMA 4.1. *Let* ϕ *be a Maass waveform of even integer weight, and assume that* ϕ *is a cusp form. Let* $\nu > 0$*,* $\alpha \in \mathbb{R}$ *and* $A \in \mathbb{Z}^+$*. We assume that, for all* $c \in \mathbf{D}$*,*

$$\lim_{M\to\infty} \frac{1}{M^A} \sum_{m=1}^{M^A} \phi(m\alpha + icM^{-A\nu}) = 0\,. \tag{4.1}$$

Then, for any fixed $0 < C_1 < C_2$*, we have*

$$\frac{1}{M} \sum_{m=1}^{M} \phi(m\alpha + iy) \to 0\,,$$

uniformly as $M \to \infty$ *and* $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$*.*

Proof. Let $0 < C_1 < C_2$ be given, and fix some $\varepsilon > 0$. Since ϕ is a cusp form, it is bounded and uniformly continuous over all of $\mathcal{H}$. We fix positive numbers B and δ_0 such that $|\phi(z)| \leqq B$ holds for all $z \in \mathcal{H}$, and such that $|\phi(z) - \phi(w)| < \varepsilon$ holds whenever $\delta(z,w) < \delta_0$, where $\delta(\,\cdot\,,\cdot\,)$ denotes hyperbolic distance.

Let us also fix a finite subset $S \subset [C_1, C_2] \cap \mathbf{D}$ such that for each $c \in [C_1, C_2]$ there is at least one $c_1 \in S$ with $|\log(c/c_1)| \leqq \delta_0/2$. Because of our assumption (4.1) and the fact that S is finite, we know that for all sufficiently large numbers M_1,

$$\left|\frac{1}{M_1^A}\sum_{m=1}^{M_1^A}\phi(m\alpha+ic_1M_1^{-A\nu})\right| \leqq \varepsilon\,, \qquad \forall c_1\in S\,. \tag{4.2}$$

We now let an arbitrary number $M \in \mathbb{Z}^+$ be given and define $M_1 \in \{2,3,\dots\}$ through $(M_1-1)^A \leqq M < M_1^A$. We make the assumption that M is so large that (4.2) holds, and so that

$$\tfrac{M}{M_1^A} > \max\left(1-\tfrac{\varepsilon}{2B}, \exp\left(-\tfrac{\delta_0}{2\nu}\right)\right)\,. \tag{4.3}$$

Given any $c \in [C_1, C_2]$ we pick $c_1 \in S$ such that $|\log(c/c_1)| \leqq \delta_0/2$. We now have

$$\begin{aligned}\left|\frac{1}{M}\sum_{m=1}^{M}\phi(m\alpha+icM^{-\nu})-\frac{1}{M_1^A}\sum_{m=1}^{M_1^A}\phi(m\alpha+ic_1M_1^{-A\nu})\right| \\ \leqq \frac{1}{M}\sum_{m=1}^{M}\left|\phi(m\alpha+icM^{-\nu})-\phi(m\alpha+ic_1M_1^{-A\nu})\right| \\ +\left(\tfrac{1}{M}-\tfrac{1}{M_1^A}\right)MB+\tfrac{1}{M_1^A}(M_1^A-M)B\,.\end{aligned} \tag{4.4}$$

Here the last two terms are equal, and are each less than $\varepsilon/2$, by (4.3). Furthermore,

$$\begin{aligned}\delta(m\alpha+icM^{-\nu}, m\alpha+ic_1M_1^{-A\nu}) &= \left|\log\left(\frac{cM^{-\nu}}{c_1M_1^{-A\nu}}\right)\right| \\ &\leqq \left|\log(c/c_1)\right|+\nu\left|\log(M/M_1^A)\right| < \tfrac{\delta_0}{2}+\tfrac{\delta_0}{2}=\delta_0\,,\end{aligned}$$

again by (4.3). It follows that $|\phi(m\alpha+icM^{-\nu})-\phi(m\alpha+ic_1M_1^{-A\nu})| < \varepsilon$ holds for each m. Hence the difference in (4.4) is less than 2ε, and using (4.2) we conclude that

$$\left|\frac{1}{M}\sum_{m=1}^{M}\phi(m\alpha+icM^{-\nu})\right| < 3\varepsilon\,.$$

This holds for all sufficiently large M, and all $c \in [C_1, C_2]$. This concludes the proof. □

In order to extend the above lemma to the case when ϕ is not a cusp form, or when ϕ is replaced by the Eisenstein series, we need a good bound on those contributions to our sum which come from points lying far out in the cusps of $\Gamma \setminus \mathcal{H}$. This is provided by the following lemma.

Lemma 4.2. *Let $\nu > 0$, $\alpha \in \mathbb{R}$, $A \in \mathbb{Z}^+$ and $s_0 \in (1/2, 1)$ be given, such that $s = s_0$ is not a pole of any of the Eisenstein series $E_k(z, s, 0)$, $k = 1, \dots, \kappa$. We assume that, for some constant $K_1 > 0$,*

$$\frac{1}{M^A}\left|\sum_{m=1}^{M^A} E_k(m\alpha + iM^{-A\nu}, s_0, 0)\right| \leqq K_1\,,\ \forall k \in \{1, \dots, \kappa\}\,,\ M \in \mathbb{Z}^+. \tag{4.5}$$

We let $0 < C_1 < C_2$ be given. Then there exists a constant $K_2 > 0$ such that

$$\frac{1}{M}\sum_{m=1}^{M} \mathcal{Y}_\Gamma(m\alpha + icM^{-\nu})^{s_0} \leqq K_2\,, \tag{4.6}$$

for all $M \in \mathbb{Z}^+$, $c \in [C_1, C_2]$.

Proof. We let $G(z) = \sum_{k=1}^{\kappa} E_k(z, s_0, 0)$; this is a continuous, real-valued and Γ-invariant function on $\mathcal{H}$. It follows from [H2, pp. 280 (Prop. 3.7), 297(F)] that $G(z) \sim \mathcal{Y}_\Gamma(z)^{s_0}$ as $z \in \mathcal{F}$ approaches any of the cusps. Hence $B = \inf_{z\in\mathcal{F}} G(z)$ is finite, i.e. $B > -\infty$, and there is a positive constant C_3 such that, for the function $G_1(z) = G(z) + |B| + 1$,

$$G_1(z) \geqq C_3 \mathcal{Y}_\Gamma(z)^{s_0}, \qquad \forall z \in \mathcal{F}\,. \tag{4.7}$$

Then this inequality actually holds for all $z \in \mathcal{H}$, since both sides are Γ-invariant functions.

It follows from (4.5) that

$$\frac{1}{M^A}\sum_{m=1}^{M^A} G_1(m\alpha + iM^{-A\nu}) \leqq |B| + 1 + \kappa K_1\,, \qquad \forall M \in \mathbb{Z}^+,$$

and hence there is a positive constant $C_4 > 0$ such that

$$\frac{1}{M^A}\sum_{m=1}^{M^A} \mathcal{Y}_\Gamma(m\alpha + iM^{-A\nu})^{s_0} \leqq C_4\,, \qquad \forall M \in \mathbb{Z}^+. \tag{4.8}$$

But the invariant height function $\mathcal{Y}_\Gamma(z)$ satisfies the following elementary inequality:

$$\mathcal{Y}_\Gamma(z_1) \leqq e^{\delta(z_1,z_2)} \mathcal{Y}_\Gamma(z_2)\,. \tag{4.9}$$

This inequality follows directly from the definition (2.3), if we write $\delta_0 = \delta(z_1, z_2)$ and notice that we have $\delta(N_k W(z_1), N_k W(z_2)) = \delta_0$ for all $k \in \{1, \dots, \kappa\}$, $W \in \Gamma$, and hence

$$\begin{aligned}\operatorname{Im} N_k W(z_1) &\leqq \sup\bigl\{\operatorname{Im} z \bigm| z \in \mathcal{H}\,,\ \delta(z, N_k W(z_2)) \leqq \delta_0\bigr\}\\ &= e^{\delta_0} \operatorname{Im} N_k W(z_2) \leqq e^{\delta_0} \mathcal{Y}_\Gamma(z_2)\,.\end{aligned}$$

Now let $M \in \mathbb{Z}^+$ and let c be any positive real number. We choose $M_0 \in \{2, 3, 4, \dots\}$ such that $(M_0 - 1)^A \leqq M < M_0^A$. We then have

$1 < M_0^A/M \leqq M_0^A/(M_0-1)^A \leqq 2^A$, and, for any $x \in \mathbb{R}$,

$$\delta(x+icM^{-\nu}, x+iM_0^{-A\nu}) = \big|\log(cM^{-\nu}/M_0^{-A\nu})\big| \leqq |\log c| + A\nu\log 2\,.$$

Hence, by (4.8) and (4.9),

$$\frac{1}{M}\sum_{m=1}^{M}\mathcal{Y}_\Gamma\big(m\alpha+icM^{-\nu}\big)^{s_0} \leqq \frac{M_0^A}{M}\cdot\frac{1}{M_0^A}\sum_{m=1}^{M_0^A}\big[e^{|\log c|+A\nu\log 2}\mathcal{Y}_\Gamma(m\alpha+iM_0^{-A\nu})\big]^{s_0}$$
$$< 2^{(1+s_0\nu)A}e^{s_0|\log c|}C_4\,.$$

The lemma follows from this. □

Using Lemma 4.2, we are now able to extend Lemma 4.1 to the case of arbitrary Maass waveforms.

Proposition 4.3. *Let $\nu > 0$, $\alpha \in \mathbb{R}$, $A \in \mathbb{Z}^+$, $s_0 \in (1/2, 1)$, and assume that* (4.5) *in Lemma 4.2 holds; also let ϕ be a nonconstant Maass waveform of even integer weight, and assume that, for all $c \in \mathbf{D}$,*

$$\lim_{M\to\infty}\frac{1}{M^A}\sum_{m=1}^{M^A}\phi(m\alpha+icM^{-A\nu}) = 0\,. \tag{4.10}$$

We then have, for any fixed constants $0 < C_1 < C_2$,

$$\frac{1}{M}\sum_{m=1}^{M}\phi(m\alpha+iy) \to 0\,, \tag{4.11}$$

uniformly as $M \to \infty$ and $C_1M^{-\nu} \leqq y \leqq C_2M^{-\nu}$.

Proof. If ϕ is a cusp form then this was proved in Lemma 4.1, and the assumption (4.5) is not needed. We now assume that ϕ is not a cusp form. We write $\phi \in L_2(\lambda, 2v)$, $\lambda = s(1-s)$ as in Lemma 2.3. We then have $1/2 < s < 1$ by Lemma 2.3 and Lemma 2.5, and by (2.22) there is a constant $C > 0$ such that $|\phi(z)| \leqq C\mathcal{Y}_\Gamma(z)^{1-s}$ for all $z \in \mathcal{H}$.

We let $0 < C_1 < C_2$ be given, and fix some $\varepsilon > 0$. We take $K_2 > 0$ so that (4.6) in Lemma 4.2 holds, and we fix a constant $Y > 0$ so large that

$$K_2CY^{1-s-s_0} < \varepsilon\,. \tag{4.12}$$

We let $H : \mathbb{R}^+ \to [0,1]$ be a continuous function such that $H(y) = 1$ for $0 < y \leqq Y$ and $H(y) = 0$ for $Y+1 \leqq y$, and then let

$$f(z) = H\big(\mathcal{Y}_\Gamma(z)\big)\cdot\phi(z)\,. \tag{4.13}$$

Now, if $\mathcal{Y}_\Gamma(z) > Y$, we have $|\phi(z)-f(z)| \leqq |\phi(z)| \leqq C\mathcal{Y}_\Gamma(z)^{1-s} \leqq CY^{1-s-s_0}\mathcal{Y}_\Gamma(z)^{s_0}$. If $\mathcal{Y}_\Gamma(z) \leqq Y$, the same difference is $|\phi(z)-f(z)| = 0$. Hence,

$$\big|\phi(z)-f(z)\big| \leqq CY^{1-s-s_0}\mathcal{Y}_\Gamma(z)^{s_0} \qquad \text{for } all\ z \in \mathcal{H}\,.$$

Using this together with (4.6) and (4.12), we obtain

$$\left|\frac{1}{M}\sum_{m=1}^{M}\phi(m\alpha+icM^{-\nu})-\frac{1}{M}\sum_{m=1}^{M}f(m\alpha+icM^{-\nu})\right|<\varepsilon \tag{4.14}$$

for all $M\in\mathbb{Z}^+$, $c\in[C_1,C_2]$.

In particular, for each $c\in\mathbf{D}$, (4.10) and (4.14) imply

$$\limsup_{M\to\infty}\left|\frac{1}{M^A}\sum_{m=1}^{M^A}f(m\alpha+icM^{-A\nu})\right|\leqq\varepsilon\,. \tag{4.15}$$

But it follows from our definition in (4.13) that f is a function of weight $2v$ (cf. (2.8)) which has compact support on $\Gamma\setminus\mathcal{H}$; in particular, f is bounded and uniformly continuous on all of $\mathcal{H}$. Hence, by arguing as in the proof of Lemma 4.1, using (4.15) in place of (4.1), we find that for all sufficiently large M,

$$\left|\frac{1}{M}\sum_{m=1}^{M}f(m\alpha+iy)\right|<4\varepsilon\,,\qquad\forall y\in[C_1M^{-\nu},C_2M^{-\nu}]\,.$$

Hence for these M,y we also have, by (4.14),

$$\left|\frac{1}{M}\sum_{m=1}^{M}\phi(m\alpha+iy)\right|<5\varepsilon\,.$$

This concludes the proof. □

Proposition 4.4. *Let $\nu>0$, $\alpha\in\mathbb{R}$, $A\in\mathbb{Z}^+$, $s_0\in(1/2,1)$, and assume that* (4.5) *in Lemma 4.2 holds; also let $v\in\mathbb{Z}$, $k\in\{1,\dots,\kappa\}$, and assume that*

$$\lim_{M\to\infty}\frac{1}{M^A}\sum_{m=1}^{M^A}E_k\left(m\alpha+icM^{-A\nu},\frac{1}{2}+it,2v\right)=0 \tag{4.16}$$

holds for all $c,t\in\mathbf{D}$. We then have, for any fixed constants $0<C_1<C_2$ and $T>0$,

$$\frac{1}{M}\sum_{m=1}^{M}E_k\left(m\alpha+iy,\frac{1}{2}+it,2v\right)\to 0\,, \tag{4.17}$$

uniformly as $M\to\infty$, $C_1M^{-\nu}\leqq y\leqq C_2M^{-\nu}$ and $0\leqq t\leqq T$.

Proof. We first note that by [H2, pp. 280 (Prop. 3.7), 297(F), 368 (5.21), 374 (item 15)], there is a constant $C_5=C_5(\Gamma,T,v)>0$ such that

$$\left|E_k\left(z,\tfrac{1}{2}+it,2v\right)\right|\leqq C_5\mathcal{Y}_\Gamma(z)^{1/2},\qquad\forall t\in[0,T]\,,\ z\in\mathcal{H}\,. \tag{4.18}$$

Hence, since $\frac{1}{2} - s_0 < 0$, we may imitate the proof of Proposition 4.3, using (4.16), to prove that for any *fixed* $t \in \mathbf{D}$,

$$\frac{1}{M}\sum_{m=1}^{M} E_k\left(m\alpha + iy, \frac{1}{2} + it, 2v\right) \to 0\,, \tag{4.19}$$

uniformly as $M \to \infty$ and $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$.

Now let $\varepsilon > 0$ be given. We let K_2 be as in Lemma 4.2. We claim that there is a number $t_0 > 0$ such that for all $z \in \mathcal{H}$ and all $t_1, t_2 \in [0, T]$,

$$|t_1 - t_2| < t_0 \Longrightarrow \left|E_k\left(z, \tfrac{1}{2} + it_1, 2v\right) - E_k\left(z, \tfrac{1}{2} + it_2, 2v\right)\right| \leqq \tfrac{\varepsilon}{K_2}\mathcal{Y}_\Gamma(z)^{s_0} \tag{4.20}$$

To prove this, first note that we may assume $z \in \mathcal{F}$, by Γ-invariance. We take $Y > 0$ so large that $2C_5 Y^{\frac{1}{2} - s_0} \leqq \varepsilon/K_2$; then (4.20) holds automatically for all $z \in \mathcal{F}$ with $\mathcal{Y}_\Gamma(z) \geqq Y$, by (4.18). But the region $\mathcal{F} \cap \{\mathcal{Y}_\Gamma(z) \leqq Y\}$ is *compact* (as is the interval $[0, T]$), and $E_k\big(z, \frac{1}{2} + it, 2v\big)$ is a continuous function of $\langle z, t\rangle$. Hence we can indeed choose $t_0 > 0$ so small that (4.20) holds.

Next, we fix a finite subset $S \subset [0, T] \cap \mathbf{D}$ such that for each $t \in [0, T]$ there is at least one $t_1 \in S$ with $|t - t_1| < t_0$. It follows from (4.19) that there is a number M_0 such that, for all integers $M \geqq M_0$ and all $y \in [C_1 M^{-\nu}, C_2 M^{-\nu}]$, $t_1 \in S$,

$$\left|\frac{1}{M}\sum_{m=1}^{M} E_k\left(m\alpha + iy, \frac{1}{2} + it_1, 2v\right)\right| \leqq \varepsilon\,. \tag{4.21}$$

Now, for $t \in [0, T]$ arbitrary, we take $t_1 \in S$ such that $|t - t_1| < t_0$. It then follows from (4.21), (4.20) and (4.6) that, for all $M \geqq M_0$ and all $y \in [C_1 M^{-\nu}, C_2 M^{-\nu}]$,

$$\left|\frac{1}{M}\sum_{m=1}^{M} E_k\left(m\alpha + iy, \frac{1}{2} + it, 2v\right)\right| \leqq 2\varepsilon\,.$$

This concludes the proof. □

5 Equidistribution for Unbounded Test Functions

In this section we will prove Theorem 2, using the results from the preceding section together with the spectral expansion (2.6), (2.9). In fact, we will prove the slightly stronger Theorem 2′ below, which extends Theorem 2 to cases of unbounded test functions.

Definition. Given $\gamma \geqq 0$, we let $B_\gamma(\Gamma \setminus T_1\mathcal{H})$ be the family of continuous Γ-invariant functions $f : T_1\mathcal{H} \to \mathbb{C}$ such that $|f(z, \theta)| \leqq C\mathcal{Y}_\Gamma(z)^\gamma$ holds for all $(z,\theta) \in T_1\mathcal{H}$ and some constant $C > 0$. Given $\nu > 0$ and $\alpha \in \mathbb{R}$, we say that $\langle \Gamma, \nu, \alpha \rangle$-PSE$_\gamma$ holds ($\langle \Gamma, \nu, \alpha \rangle$-Point Set Equidistribution) if, for any fixed $f \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$ and any fixed numbers $0 < C_1 < C_2$, we have

$$\frac{1}{M} \sum_{m=1}^{M} f(m\alpha + iy, 0) \to \langle f \rangle \,, \tag{5.1}$$

uniformly as $M \to \infty$ and $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$.

Hence, Theorem 2 says that for any given $\nu > 0$, $\langle \Gamma, \nu, \alpha \rangle$-PSE$_0$ holds for almost all $\alpha \in \mathbb{R}$.

Theorem 2$'$. *Let $\nu > 0$. Then there is a set $P = P(\Gamma, \nu) \subset \mathbb{R}$ of full Lebesgue measure such that for each $\alpha \in P$ and each*

$$0 \leqq \gamma < \min\left(1, \tfrac{1}{2}(1 + 1/\nu)\right) \,, \tag{5.2}$$

$\langle \Gamma, \nu, \alpha \rangle$-PSE$_\gamma$ holds.

Remark 5.1. It is essential that the summation in (5.1) does not include $m = 0$. Indeed, if Γ has a cusp at the point 0, then for $f \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$, we may have $f(iy, 0)$ increasing like $y^{-\gamma}$ as $y \to 0^+$, thus causing $\frac{1}{M} f(iy, 0) \to \infty$ as $M \to \infty$, if $\nu\gamma > 1$. An example where $\nu\gamma = 1$ and where the contribution for $m = 0$ has to be treated separately is found in the proof of Proposition 8.1 below.

On the other hand, if $\nu\gamma < 1$ then for any Γ and any fixed $x \in \mathbb{R}$ we have $\frac{1}{M} f(x + iy, 0) \to 0$ as $M \to \infty$ and $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$. (To see this, write $z = x + iy$ and $\mathcal{Y}_\Gamma(z) =$ Im $N_k W(z)$ with $W \in \Gamma$; for y small one then knows that $|c| \geqq 1$ in $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = N_k W$, and this forces $|f(z, 0)| \leqq C \mathcal{Y}_\Gamma(z)^\gamma \leqq C(c^{-2} y^{-1})^\gamma \leqq C y^{-\gamma}$.) Hence when $\nu\gamma < 1$, Theorem 2$'$ remains true if $\sum_{m=1}^{M}$ in (5.1) is replaced by (say) $\sum_{m=0}^{M}$ or $\sum_{m=0}^{M-1}$.

Proof of Theorem 2$'$. Since any countable intersection of sets of full Lebesgue measure is again of full Lebesgue measure, it suffices to prove that for any *fixed* γ satisfying (5.2), there is a subset $P = P(\Gamma, \nu, \gamma) \subset \mathbb{R}$ such that $\langle \Gamma, \nu, \alpha \rangle$-PSE$_\gamma$ holds for all $\alpha \in P$.

From now on, we keep ν, γ fixed as above. We choose a number s_0 with $\max(1/2, \gamma) < s_0 < \min(1, 1/2(1 + 1/\nu))$ such that $s = s_0$ is not a pole of any of the Eisenstein series $E_k(z, s, 0)$, $k = 1, \ldots, \kappa$. For each $v \in \mathbb{Z}$, we let $\mathbf{D}_{2v}$ be a complete set of discrete eigenfunctions of $-\Delta_{2v}$, taken to be orthonormal. Using Corollaries 3.2 and 3.4, and the fact that any countable intersection of sets of full Lebesgue measure is again of full Lebesgue measure, it follows that there exists a positive integer $A = A(\Gamma, \nu, s_0)$ and a set $P \subset \mathbb{R}$ of full Lebesgue measure such that for all $\alpha \in P$, the assumption in Lemma 4.2 is fulfilled, as well as the assumption in Proposition 4.3 for each nonconstant $\phi \in \cup_v \mathbf{D}_{2v}$, and the assumption in Proposition 4.4 for all $v \in \mathbb{Z}$, $k \in \{1, \ldots, \kappa\}$.

Now take any $\alpha \in P$, and fix some numbers $0 < C_1 < C_2$. We are going to prove that for each fixed $f \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$, we have

$$\frac{1}{M} \sum_{m=1}^{M} f(m\alpha + iy, 0) \to \langle f \rangle \,, \tag{5.3}$$

uniformly as $M \to \infty$ and $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$. This will complete the proof of Theorem 2′.

Let us define a norm on $B_\gamma(\Gamma \setminus T_1\mathcal{H})$ through

$$\|f\|_{(s_0)} := \sup_{(z,\theta) \in T_1\mathcal{H}} \big(|f(z,\theta)| \mathcal{Y}_\Gamma(z)^{-s_0} \big) \,.$$

It follows from the definition of $B_\gamma(\Gamma \setminus T_1\mathcal{H})$ that $\|f\|_{(s_0)}$ is finite for each $f \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$, since $\gamma < s_0$ and $\mathcal{Y}_\Gamma(z)$ is bounded from below by a positive constant.

Now by Lemma 4.2, we have

$$\left| \frac{1}{M} \sum_{m=1}^{M} f_1(m\alpha + iy, 0) - \frac{1}{M} \sum_{m=1}^{M} f_2(m\alpha + iy, 0) \right| \leqq K_2 \|f_1 - f_2\|_{(s_0)} \,,$$

for all $f_1, f_2 \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$ and all $M \in \mathbb{Z}^+$, $y \in [C_1 M^{-\nu}, C_2 M^{-\nu}]$. Furthermore, there is a constant $C_6 > 0$ which only depends on Γ and s_0 such that

$$\big| \langle f_1 \rangle - \langle f_2 \rangle \big| \leqq C_6 \|f_1 - f_2\|_{(s_0)} \,, \qquad \forall f_1, f_2 \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$$

(for recall (2.5) and $s_0 < 1$). Because of these last two inequalities, it now suffices to prove (5.3) for all $f \in S$, where S is any $\|\cdot\|_{(s_0)}$-*dense subset* of $B_\gamma(\Gamma \setminus T_1\mathcal{H})$.

We claim that $C_c(\Gamma \setminus T_1\mathcal{H})$ is $\|\cdot\|_{(s_0)}$-dense in $B_\gamma(\Gamma \setminus T_1\mathcal{H})$. For take $f \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$ arbitrary. Given $Y \geqq 100$ we define $H : \mathbb{R}^+ \to [0,1]$ as in the proof of Proposition 4.3, and let $f_1(z,\theta) = H(\mathcal{Y}_\Gamma(z)) \cdot f(z,\theta)$. Then $f_1 \in C_c(\Gamma \setminus T_1\mathcal{H})$. Arguing as in the proof of Proposition 4.3, we find that $\|f - f_1\|_{(s_0)}$ can be made arbitrarily small by taking Y sufficiently large. This proves our claim.

Next, by a standard convolution argument, the space $C_c^\infty(\Gamma \setminus T_1\mathcal{H})$ is dense in $C_c(\Gamma \setminus T_1\mathcal{H})$ with respect to the usual supremum norm, and hence, a fortiori, with respect to $\|\cdot\|_{(s_0)}$.

Now let $f \in C_c^\infty(\Gamma \setminus T_1\mathcal{H})$ be given. We expand f in a Fourier series as in (2.6). Integrating by parts twice in (2.7) we find that $|\widehat{f}_v(z)| \leqq (2\pi)^{-1}|v|^{-2} \int_0^{2\pi} \big|(\partial^2/\partial\theta^2) f(z,\theta)\big| \, d\theta$ for all $|v| \geqq 1$. Hence, since f is smooth and of compact support modulo Γ, we have uniform convergence in (2.6), over all of $T_1\mathcal{H}$. In particular, the space of *finite* sums $\sum_v f_v(z)e^{iv\theta}$, with $f_v \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H}, 2v)$, is $\| \cdot \|_{(s_0)}$-dense in $B_\gamma(\Gamma \setminus T_1\mathcal{H})$.

Hence, by linearity, it will now be sufficient to prove (5.3) for functions f of the form $f(z,\theta) = f_v(z)e^{iv\theta}$, where $v \in \mathbb{Z}$ and $f_v \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H}, 2v)$. For such f, we clearly have $\langle f \rangle = 0$ if $v \neq 0$ (since $\int_0^{2\pi} f(z,\theta)\, d\theta = 0$ for all z), and $\langle f \rangle = \mu(\mathcal{F})^{-1} \int_{\mathcal{F}} f_v(z)\, d\mu(z)$ if $v = 0$. Hence, our goal is now to prove, for any fixed $v \in \mathbb{Z}$, and any fixed $f_v \in C^\infty(\mathcal{H}) \cap C_c(\Gamma \setminus \mathcal{H}, 2v)$,

$$\frac{1}{M} \sum_{m=1}^{M} f_v(m\alpha + iy) \to \delta_{v0}\mu(\mathcal{F})^{-1} \int_{\mathcal{F}} f_v(z) d\mu(z)\,, \tag{5.4}$$

uniformly as $M \to \infty$ and $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$.

We will now apply the spectral expansion (2.9). By an approximation argument of the same type as above, using the analog of the $\| \cdot \|_{(s_0)}$-norm for functions $\mathcal{H} \to \mathbb{C}$ of weight $2v$, and using Lemma 2.2, the inequality $1 + \sqrt{\mathcal{Y}_\Gamma(z)} \leqq O(\mathcal{Y}_\Gamma(z)^{s_0})$ and Lemma 4.2, we find that it suffices to prove (5.4) for f_v in the family of *finite* spectral expansions;

$$f_v(z) = \sum_{\lambda_n \leqq 1/4 + T^2} d_n \phi_n(z) + \sum_{k=1}^{\kappa} \int_0^T g_k(t) E_k\left(z, \frac{1}{2} + it, 2v\right) dt\,,$$

where $\phi_n \in \mathbf{D}_{2v}$, $T \geqq 1$, $d_0, d_1, \ldots$ are arbitrary complex numbers, and $g_k(t)$ are arbitrary continuous functions on $[0,T]$. But for *such* a function f_v, Proposition 4.4 and Proposition 4.3 can be applied directly. Concerning Proposition 4.3, we notice that if $v \neq 0$ then all ϕ_n's are nonconstant; if $v = 0$ then ϕ_n is nonconstant for all $n \neq 0$, while $\phi_0 \equiv \mu(\mathcal{F})^{-1/2}$. Hence we obtain

$$\frac{1}{M} \sum_{m=1}^{M} f_v(m\alpha + iy) \to \delta_{v0} d_0 \mu(\mathcal{F})^{-1/2},$$

uniformly as $M \to \infty$ and $C_1 M^{-\nu} \leqq y \leqq C_2 M^{-\nu}$. But here, if $v = 0$, we have $d_0 = \langle f_0, \phi_0 \rangle$, and hence $d_0 \mu(\mathcal{F})^{-1/2} = \mu(\mathcal{F})^{-1} \int_{\mathcal{F}} f_0(z) d\mu(z)$. Hence (5.4) holds. This concludes the proof of Theorem 2′. □□□

6 Proof of Theorem 3

For $x \in \mathbb{R}$ we use $\|x\|$ to denote the distance from x to the closest integer, i.e. $\|x\| = \inf_{n \in \mathbb{Z}} |x - n|$.

Lemma 6.1. *Let $\alpha \in \mathbb{R}$ be of type $K \geqq 2$. We then have, for all integers*

$M \geqq 1$, $N_2 \geqq N_1 \geqq 1$,

$$\sum_{n=N_1}^{N_2} \min\left(M, \frac{1}{\|n\alpha\|}\right) \leqq O\big(N_2^{K-1} \log(2N_2)\big)\,. \tag{6.1}$$

Writing $N = N_2 - N_1$, the same sum is also

$$\leqq O\big(M + ((NM)^{\frac{K-1}{K}} + N) \log(NM+1)\big)\,. \tag{6.2}$$

The implied constants depend only on α, K, not on M, N_1, N_2.

We remark that whenever $N_2 \geqq M^{\frac{1}{K-1}}$ holds, the bound in (6.2) is better than (or at least as good as) the bound in (6.1).

Proof. By assumption, there exists a constant $C \in (0,1)$ such that $|\alpha - a/q| > Cq^{-K}$ for all $a \in \mathbb{Z}$, $q \in \mathbb{Z}^+$.

We first prove (6.2). The bound is trivial when $N = 0$, so we may now assume $N \geqq 1$. By Dirichlet's Theorem, given any real number $Q \geqq 1$ there exist integers a, q with $1 \leqq q \leqq Q$, $\gcd(a,q) = 1$ and $|\alpha - a/q| < (qQ)^{-1} \leqq q^{-2}$. Using $|\alpha - a/q| > Cq^{-K}$ we then have $(CQ)^{\frac{1}{K-1}} < q \leqq Q$.

Splitting the summation range $N_1 \leqq n \leqq N_2$ into consecutive blocks of the form $\{hq + r \mid 1 \leqq r \leqq q\}$ (possibly overshooting on both ends), and using [N, Lemma 4.9] for each block, we now have

$$\sum_{n=N_1}^{N_2} \min\left(M, \frac{1}{\|n\alpha\|}\right) \leqq O\left(\left(\frac{N_2 - N_1}{q} + 2\right)(M + q\log q)\right)$$

$$= O\left(\frac{NM}{q} + M + (N+q)\log q\right) = O\big(NMQ^{-\frac{1}{K-1}} + M + (N+Q)\log Q\big)\,.$$

Taking $Q = (NM)^{\frac{K-1}{K}}$ (remembering $N \geqq 1$), we obtain (6.2).

To prove (6.1), we instead let $Q = C^{-1}(2N_2)^{K-1} > 1$, and find integers a, q as above. Now $Q \geqq q > (CQ)^{\frac{1}{K-1}} = 2N_2$, so by [N, Lemma 4.8],

$$\sum_{n=N_1}^{N_2} \min\left(M, \frac{1}{\|n\alpha\|}\right) \leqq \sum_{1 \leqq n \leqq q/2} \frac{1}{\|n\alpha\|} = O(q \log q) = O\big(N_2^{K-1} \log(2N_2)\big)\,.$$

□

PROPOSITION 6.2. *Let β, α, K, ν be as in Theorem 3. Let ϕ be a non-constant Maass waveform of even integer weight. We then have*

$$\frac{1}{M} \sum_{m=1}^{M} \phi(m\alpha + iy) \to 0\,, \tag{6.3}$$

uniformly as $M \to \infty$, $y \to 0^+$ so long as $y \geqq M^{-\nu}$.

Proof. We write $\phi \in L_2(\lambda, 2v)$ and $\lambda = s(1-s)$ as in Lemma 2.3, and $s = \sigma + it$. As in the proof of Proposition 3.1, the sum in (6.3) is equal to

$$c_0 y^{1-s} + \frac{1}{M} \sum_{n \neq 0} \frac{c_n}{\sqrt{|n|}} W_{v \cdot \mathrm{sgn}(n), s-\frac{1}{2}}\big(4\pi |n| y\big) \sum_{m=1}^{M} e(nm\alpha) \,. \tag{6.4}$$

We will always keep $y < 1$. Recall that we have either $\sigma < 1$, or else $s = \sigma \geqq 1$ and $c_0 = 0$ (cf. Lemma 2.5). Hence we certainly have $c_0 y^{1-s} \to 0$ as $y \to 0^+$. We now turn to the sum over $n \neq 0$ in (6.4). We will first work under the assumption that either $\sigma \leqq \frac{1}{2} + \beta$ or $s = \sigma \geqq 1$. By reviewing the proof of (3.3) on p. 1256, we then find that we have the following convenient bound for all $Y > 0$, in all non-vanishing terms in (6.4):

$$W_{v \cdot \mathrm{sgn}(n), s-\frac{1}{2}}(Y) = O(Y^{\frac{1}{2}-\beta} e^{-Y/4}) \,. \tag{6.5}$$

Furthermore, it follows from the definition of β and Lemma 2.4 or Lemma 2.5 that

$$|c_n| = O\big(|n|^{\beta+\varepsilon}\big) \,, \qquad \text{for all } n \neq 0 \,.$$

(Here and in all later "big-O" estimates in this proof, the implied constant depends on ϕ and ε.) We also note that

$$\left| \sum_{m=1}^{M} e(nm\alpha) \right| = \left| \frac{e(Mn\alpha) - 1}{e(n\alpha) - 1} \right| \leqq \min\big(M, \|n\alpha\|^{-1}\big) \,,$$

since $|e(x) - 1| = 2|\sin(\pi x)| \geqq 2\|x\|$ for all $x \in \mathbb{R}$. Hence (6.4) minus the trivial term $c_0 y^{1-s}$ is

$$\begin{aligned} &= O\big(M^{-1} y^{\frac{1}{2}-\beta}\big) \sum_{n=1}^{\infty} e^{-\pi n y} n^{\varepsilon} \min\big(M, \|n\alpha\|^{-1}\big) \\ &= O\big(M^{-1} y^{\frac{1}{2}-\beta}\big) \sum_{k=0}^{\infty} e^{-\pi k} \left(\frac{k+1}{y}\right)^{\varepsilon} \sum_{k/y < n \leqq (k+1)/y} \min\big(M, \|n\alpha\|^{-1}\big) \,. \end{aligned} \tag{6.6}$$

Applying the bound (6.2) in Lemma 6.1, and using $y \geqq M^{-\nu}$, we get

$$\begin{aligned} &= O\big(M^{-1} y^{\frac{1}{2}-\beta-\varepsilon}\big) \sum_{k=0}^{\infty} e^{-3k} \big(M + (M/y)^{\frac{K-1}{K}+\varepsilon} + M^{\varepsilon} y^{-1-\varepsilon}\big) \\ &= O\big(y^{\frac{1}{2}-\beta-\varepsilon} + M^{-\frac{1}{K}+\varepsilon+\nu(\frac{1}{2}+\beta-\frac{1}{K}+2\varepsilon)} + M^{\varepsilon-1+\nu(\frac{1}{2}+\beta+2\varepsilon)}\big) \,. \end{aligned} \tag{6.7}$$

Clearly, if $\beta < 1/2$, it is possible to keep ε so small that $\frac{1}{2} - \beta - \varepsilon > 0$; then the first term above tends to 0 as $M \to \infty$, $y \to 0^+$. One also checks that when $\beta < 1/2$, (1.8) implies that *both* $\nu < \frac{2}{1+2\beta}$ and $\nu < \frac{2}{2K\beta+K-2}$ hold; hence the last two terms above tend to 0 as well, provided that ε is sufficiently small.

On the other hand, if $\beta \geqq 1/2$, we instead apply the bound (6.1) (in the crude form "$O(N_2^{K-1+\varepsilon})$"); this gives that (6.6) is

$$= O\big(M^{-1}y^{\frac{1}{2}-\beta-\varepsilon}\big)\sum_{k=0}^{\infty} e^{-3k}\left(\frac{k+1}{y}\right)^{K-1+\varepsilon} = O\big(M^{-1+\nu(K+\beta-\frac{3}{2}+2\varepsilon)}\big)\,.$$

By (1.8), this tends to 0 as $M \to \infty$, provided that ε is sufficiently small.

We now turn to the remaining case; $\frac{1}{2}+\beta < \sigma < 1$. (Notice that $0 < \beta < 1/2$ must hold in this case.) Let us write

$$B(X) = \sum_{1\leqq n\leqq X} \min\big(M, \|n\alpha\|^{-1}\big)\,. \tag{6.8}$$

The bound (6.5) still holds as $Y \to \infty$, and thus all terms with $|n| > 1/y$ in (6.4) can be treated exactly as in (6.6), (6.7). When $Y \to 0^+$, we have $W_{v\cdot\mathrm{sgn}(n),s-\frac{1}{2}}(Y) = O(Y^{1-\sigma})$ (cf. (3.4)). Hence we obtain the following bound on the remaining part ("$0 < |n| \leqq 1/y$") of (6.4):

$$O(M^{-1}y^{1-\sigma})\int_{1/2}^{1/y} X^{\beta+\frac{1}{2}-\sigma+\varepsilon}\,dB(X)$$

$$= O(M^{-1}y^{1-\sigma})\Big\{\big[X^{\beta+\frac{1}{2}-\sigma+\varepsilon}B(X)\big]_{X=1/2}^{X=1/y} + \int_{1/2}^{1/y} X^{\beta-\frac{1}{2}-\sigma+\varepsilon}B(X)\,dX\Big\}.$$

Here $B(X) = 0$ for $0 < X < 1$, and it follows from Lemma 6.1 that $B(X) = O(X^{K-1+\varepsilon})$ for all $X \geqq 1$, and also, by (6.2), $B(X)=O((XM)^{\frac{K-1}{K}+\varepsilon}+X^{1+\varepsilon})$ whenever $X > M^{\frac{1}{K-1}}$; hence we get

$$\leqq O(M^{-1}y^{1-\sigma})\Big\{\big(M^{\frac{1}{K-1}}\big)^{\beta+\frac{1}{2}-\sigma+K-1+2\varepsilon}$$

$$+\, y^{-\beta-\frac{1}{2}+\sigma-\varepsilon}\big((M/y)^{\frac{K-1}{K}+\varepsilon} + y^{-1-\varepsilon}\big) + \int_1^{M^{\frac{1}{K-1}}} X^{\beta-\frac{3}{2}-\sigma+K+2\varepsilon}dX$$

$$+ \int_1^{1/y} X^{\beta-\frac{1}{2}-\sigma+\varepsilon}\big((XM)^{\frac{K-1}{K}+\varepsilon} + X^{1+\varepsilon}\big)dX\Big\}.$$

Here all the total X-exponents in the integrals are > -1, and hence we obtain

$$= O\big(M^{\frac{\beta+1/2-\sigma+2\varepsilon}{K-1}}y^{1-\sigma} + M^{-\frac{1}{K}+\varepsilon}y^{-\beta-\frac{1}{2}+\frac{1}{K}-2\varepsilon} + M^{-1}y^{-\beta-\frac{1}{2}-2\varepsilon}\big)\,.$$

But $\frac{1}{2}+\beta < \sigma < 1$, so it is possible to keep ε so small that $\beta+1/2-\sigma+2\varepsilon < 0$; then first term above certainly tends to 0 as $M \to \infty$, $y \to 0^+$. The second and third terms also tend to 0 so long as we keep ε sufficiently small, and $y \geqq M^{-\nu}$; cf. the discussion concerning (6.7) above. This concludes the proof of Proposition 6.2. □

Proposition 6.3. *Let $s_0, \beta, \alpha, K, \nu$ be as in Theorem 3. Let $v \in \mathbb{Z}$, $k \in \{1, \dots, \kappa\}$, and $s \in \frac{1}{2} + i[0, \infty)$ or $s = s_0$. We then have*

$$\frac{1}{M} \sum_{m=1}^{M} E_k(m\alpha + iy, s, 2v) \to 0\,,$$

uniformly as $M \to \infty$, $y \to 0^+$ so long as $y \geqq M^{-\nu}$.

Proof. We take the Fourier expansion of $E_k(z, s, 2v)$ to be as in (2.23), and it then follows from the definition of β and Lemma 2.6 that $|c_n| = O(|n|^{\beta+\varepsilon})$ holds for all $n \neq 0$. The proof of Proposition 6.3 is now almost identical to the proof of Proposition 6.2. □

We are now ready to prove Theorem 3. Just as with Theorem 2, we will actually prove a stronger result, wherein unbounded test functions are allowed:

Theorem 3$'$. *Let $s_0, \beta, \alpha, K, \nu$ be as in Theorem 3 (on p. 1243), and let $\gamma \in [0, s_0)$. Then for any function $f \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$ and any constant $C_1 > 0$, we have*

$$\frac{1}{M} \sum_{m=1}^{M} f(m\alpha + iy, 0) \to \langle f \rangle\,,$$

uniformly as $M \to \infty$, $y \to 0^+$ so long as $y \geqq C_1 M^{-\nu}$.

Proof of Theorem 3$'$. This proof is similar to the arguments given on pp. 1258–1266, except that the present case is easier. We will therefore only give an outline of the argument.

It is sufficient to prove Theorem 3$'$ in the case $C_1 = 1$; for we may always increase ν slightly, keeping (1.8) true.

We first prove that there exists a constant $K_2 = K_2(\Gamma, s_0, \alpha, \nu) > 0$ such that

$$\frac{1}{M} \sum_{m=1}^{M} \mathcal{Y}_\Gamma(m\alpha + iy)^{s_0} \leqq K_2, \qquad \forall M \in \mathbb{Z}^+,\ y \in [M^{-\nu}, 1]\,. \tag{6.9}$$

To this end, we define $G(z)$, B and $G_1(z)$ as in the proof of Lemma 4.2. It then follows from Proposition 6.3 (with $s = s_0$ and $v = 0$) that there are numbers $M_0 \in \mathbb{Z}^+$ and $y_0 > 0$ such that $\frac{1}{M}\sum_{m=1}^{M} G_1(m\alpha + iy) < |B| + 2$ holds whenever $M \geqq M_0$ and $M^{-\nu} \leqq y \leqq y_0$. Now (6.9) follows from (4.7), and the fact that $\mathcal{Y}_\Gamma(z)$ is bounded in the region $\min(y_0, M_0^{-\nu}) \leqq y \leqq 1$.

Secondly, we claim that Proposition 6.3 can be sharpened as follows:

Given $v \in \mathbb{Z}$, $k \in \{1, \dots, \kappa\}$ and $T > 0$, we have

$$\frac{1}{M} \sum_{m=1}^{M} E_k\left(m\alpha + iy, \frac{1}{2} + it, 2v\right) \to 0\,, \tag{6.10}$$

uniformly over all $t \in [0, T]$ as $M \to \infty$, $y \to 0^+$, so long as $y \geqq M^{-\nu}$. This is proved by imitating part of the proof of Proposition 4.4, using (6.9) and Proposition 6.3 in the place of Lemma 4.2 and (4.16).

The proof is now completed by mimicking the approximation argument on pp. 1265–1266, using (6.9), (6.10) and Proposition 6.2 in the place of Lemma 4.2, Proposition 4.4 and Proposition 4.3. □□□

REMARK 6.4. The restriction on ν in (1.8) is optimal for our method of proof in the following precise sense:

Given any $\beta \geqq 0$, $K \geqq 2$, $v \in \mathbb{Z}$ and $s \in \frac{1}{2} + i[0, \infty)$ or $s \in \left(\frac{1}{2}, \infty\right)$, let ν_0 be the number given on the right-hand side of (1.8). *Then there exists a number α of type K, and complex numbers $\{c_n\}_{n \neq 0}$ satisfying $c_n = O(|n|^\beta)$, such that the following sum of absolute values*

$$\frac{1}{M} \sum_{n \neq 0} \frac{|c_n|}{\sqrt{|n|}} \cdot \left|W_{v \cdot \mathrm{sgn}(n), s-\frac{1}{2}}(4\pi|n|y)\right| \cdot \left|\sum_{m=1}^{M} e(nm\alpha)\right| \tag{6.11}$$

does not *tend to* 0 *as $y = M^{-\nu_0}$, $M \to \infty$. (For $\beta \leqq 1/2$ we may even take the c_n's to satisfy a "Rankin–Selberg type formula" $\sum_{1 \leqq |n| \leqq N} |c_n|^2 =$ [const] $\cdot N + O(N^{2\beta})$ as $N \to \infty$.)*

We omit the proof.

Notice that (6.11) is the sum which we have to treat if we want to bound the sum (6.4) in Proposition 6.2 term by term. (The same type of sum arises also for the Eisenstein series, cf. Proposition 6.3.) Hence, our remark shows that to obtain any improvement upon (1.8) in Theorem 3 using (6.4), one would have to prove cancellation between the terms in (6.4).

Proof of the last statement in Remark 1.6. Take $\alpha \in \mathbb{R}$ of type $K \geqq 2$, and take $0 < \nu < (K-1)^{-1}$. We will show how to use Sarnak's theorem on the asymptotic equidistribution of closed horocycles to prove that, for any bounded continuous Γ-invariant function $f : \mathcal{H} \to \mathbb{C}$, and any fixed number $C_1 > 0$, we have

$$\frac{1}{M} \sum_{m=1}^{M} f\left(m\alpha + iy, 0\right) \to \langle f \rangle\,, \tag{6.12}$$

uniformly as $M \to \infty$, $y \to 0^+$ so long as $y \geqq C_1 M^{-\nu}$.

By standard approximation arguments, if suffices to prove (6.12) for $f \in C_c^\infty(\Gamma \setminus T_1\mathcal{H})$. For each $y > 0$ the function $f(x + iy, 0)$ is invariant under $x \mapsto x+1$, and hence we have

$$f(x+iy,0)=\sum_{n\in\mathbb{Z}} a(y,n)e(nx)\,,$$

where

$$a(y,n)=\int_0^1 f(x+iy,0)e(-nx)dx\,. \tag{6.13}$$

It follows from Sarnak's theorem (cf. Theorem 1 above) that

$$a(y,0)\to\langle f\rangle\,,\qquad \text{as }\ y\to 0^+. \tag{6.14}$$

Next, we will prove a bound on $a(y,n)$ for $n\neq 0$. Given a fixed integer $A\geqq 0$, we may apply integration by parts A times in (6.13) to obtain

$$\big|a(y,n)\big|\leqq O(|n|^{-A})\sup_{x\in[0,1]}\left|\frac{\partial^A}{\partial x^A}f(x+iy,0)\right|\,. \tag{6.15}$$

But we have, for arbitrary $z=x+iy\in\mathcal{H}$:

$$\left|\frac{\partial^A}{\partial x^A}f(x+iy,0)\right|=O(y^{-A}) \tag{6.16}$$

(the implied constant depends on f, A and Γ). To prove (6.16), it is convenient to use the standard identification of $T_1\mathcal{H}$ with the Lie group $G=\mathrm{PSL}(2,\mathbb{R})$, given by $G\ni U\mapsto U(i,0)\in T_1\mathcal{H}$. Under this identification, f is a function in $C^\infty(G)$ which is Γ-left invariant and which has compact support modulo Γ. We let $X: C^\infty(G)\to C^\infty(G)$ be the left invariant differential operator given by

$$(XF)(g)=\frac{d^A}{dt^A}F\left(g\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)\bigg|_{t=0}.$$

One then verifies that under our identification $G\leftrightarrow T_1\mathcal{H}$, we have

$$(Xf)\Big(\Big(\begin{smallmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{smallmatrix}\Big)\Big)=y^A\frac{\partial^A}{\partial x^A}f(x+iy,0)\,.$$

However, since X is a left invariant operator and f is a Γ-left invariant function, Xf is a Γ-left invariant function too. Furthermore, Xf has compact support modulo Γ, since this is true for f. Hence Xf is uniformly bounded over all of G. Clearly, (6.16) follows from this.

We now obtain, from (6.15) and (6.16)

$$\big|a(y,n)\big|\leqq O\big((|n|y)^{-A}\big)\,. \tag{6.17}$$

Let us keep $M\in\mathbb{Z}^+$ and $0<y<1$. We have

$$\left|\frac{1}{M}\sum_{m=1}^{M} f(m\alpha+iy,0)-a(y,0)\right|$$
$$=\left|\frac{1}{M}\sum_{n\neq 0}a(y,n)\sum_{m=1}^{M}e(mn\alpha)\right|\leqq\frac{1}{M}\sum_{n\neq 0}\big|a(y,n)\big|\min\big(M,\|n\alpha\|^{-1}\big)\,.$$

We use the inequality (6.17) with $A=0$ for $|n| \leqq 1/y$, and with some A yet to be fixed for $|n| > 1/y$. We then get

$$\leqq O(M^{-1}) \sum_{1 \leqq n \leqq 1/y} \min\big(M, \|n\alpha\|^{-1}\big) + O(M^{-1}) \sum_{k=1}^{\infty} k^{-A} \sum_{k/y < n \leqq (k+1)/y} \min\big(M, \|n\alpha\|^{-1}\big)$$

We now fix $\varepsilon > 0$ so small that $\nu < \frac{1}{K-1+\varepsilon}$. By (6.1) in Lemma 6.1, the above sum is

$$\leqq O(M^{-1} y^{1-K-\varepsilon}) + O(M^{-1}) \sum_{k=1}^{\infty} k^{-A} \left(\frac{k+1}{y}\right)^{K-1+\varepsilon}.$$

If we fix A so large that $K-1+\varepsilon-A < -1$, then we conclude

$$\left| \frac{1}{M} \sum_{m=1}^{M} f(m\alpha + iy, 0) - a(y,0) \right| = O(M^{-1} y^{1-K-\varepsilon}). \tag{6.18}$$

Now (6.12) follows from (6.14) and (6.18), since $\nu < (K-1+\varepsilon)^{-1}$. □ □ □

7 Negative Results

PROPOSITION 7.1. *Let Γ be a subgroup of $PSL(2,\mathbb{Z})$ of finite index, and let $\nu > 0$. Let $\alpha \in \mathbb{R}$ be any irrational number such that there are sequences of integers $p_1, p_2, \ldots$ and $0 < q_1 < q_2 < \ldots$ satisfying*

$$\left|\alpha - \tfrac{p_j}{q_j}\right| \leqq (3q_j)^{-2-2/\nu}, \qquad \textit{for } j = 1, 2, \ldots$$

Then there exist a non-negative function $f \in C_c^\infty(\Gamma \setminus T_1\mathcal{H})$ with $\langle f \rangle = 1$, and a sequence of integers $0 < M_1 < M_2 < \ldots$, such that

$$\frac{1}{M_j} \sum_{m=1}^{M_j} f(m\alpha + iM_j^{-\nu}, 0) = 0, \qquad \textit{for all } j = 1, 2, \ldots \tag{7.1}$$

Proof. Clearly, it suffices to prove the result for $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$.

Let us consider any fixed $M \geqq 1$, $y > 0$ and $j \geqq 1$, and write $p = p_j$, $q = q_j$. We will use the fact that $\alpha \approx p/q$ to show that, under certain conditions on the sizes of M and y, *all* points $\alpha + iy, 2\alpha + iy, \ldots, M\alpha + iy$ lie far out in the cusp on $\Gamma \setminus \mathcal{H}$.

Given $m \in \{1, \ldots, M\}$ we write $d = \gcd(q, mp)$; then $\gcd(-q/d, mp/d) = 1$, and hence there is a $\mathrm{PSL}(2,\mathbb{Z})$-transformation of the form

$$T = \begin{pmatrix} * & * \\ -q/d & mp/d \end{pmatrix} \in \Gamma = \mathrm{PSL}(2,\mathbb{Z}).$$

We now have

$$\operatorname{Im} T(m\alpha+iy) = y\left|-\frac{q}{d}(m\alpha+iy)+\frac{mp}{d}\right|^{-2} = \frac{d^2y}{q^2y^2+m^2(p-q\alpha)^2}$$
$$\geqq \frac{y}{q^2y^2+M^2q^2(3q)^{-4-4/\nu}}\,.$$

Clearly, this is $\geqq 2$ whenever

$$q^2y^2 \leqq y/4 \quad \text{and} \quad M^2q^2(3q)^{-4-4/\nu} \leqq y/4\,. \tag{7.2}$$

Hence, for M, y satisfying (7.2), we have $\mathcal{Y}_\Gamma(m\alpha+iy) \geqq 2$ for all $m \in \{1,\dots,M\}$.

Taking $y = M^{-\nu}$, one finds by a quick computation that (7.2) holds if and only if

$$4^{1/\nu}q^{2/\nu} \leqq M \leqq 4^{-\frac{1}{2+\nu}}3^{\frac{4(1+\nu)}{\nu(2+\nu)}}q^{2/\nu}\,. \tag{7.3}$$

Notice that $4^{1/\nu} < 4^{-\frac{1}{2+\nu}}3^{\frac{4(1+\nu)}{\nu(2+\nu)}}$, since $4 < 3^2$. Hence, whenever q is sufficiently large, there exists at least one integer M satisfying (7.3). Because of $q = q_j \to \infty$ as $j \to \infty$, we can now certainly find a sequence of integers $0 < M_1 < M_2 < M_3 < \dots$ such that for each $j = 1,2,3,\dots$, and each $m \in \{1,\dots,M_j\}$, we have $\mathcal{Y}_\Gamma(m\alpha + iM_j^{-\nu}) \geqq 2$.

But $R = \{z \in \mathcal{H} \mid \mathcal{Y}_\Gamma(z) < 2\}$ is an open, non-empty region on $\Gamma \setminus \mathcal{H}$. ($R$ is open because of the continuity of $\mathcal{Y}_\Gamma(z)$, and to see that R is non-empty we need only check that, e.g. $\mathcal{Y}_\Gamma(i) = 1$.) Hence there exists a smooth, Γ-invariant function $f_0 : \mathcal{H} \to [0,\infty)$ which has compact support contained in R, and which is not identically 0. We define $f(z,\theta) = f_0(z)$. Then $f \in C_c^\infty(\Gamma \setminus T_1\mathcal{H})$, and (7.1) holds. Rescaling f, we can also make $\langle f \rangle = 1$ hold. □

8 The Pair Correlation Density of $n^2\alpha$ mod 1

The objective of this final section is to establish that the equidistribution of the Kronecker sequence $m\alpha$ along closed horocycles implies Poisson statistics for the pair correlation density of the sequence $n^2\alpha$ mod 1. Rudnick and Sarnak's recent result [RuS, Theorem 1 for $d = 2$], which says that the pair correlation density of $n^2\alpha$ mod 1 is Poissonian for generic α (in Lebesgue measure sense), is therefore implied by our equidistribution theorem (Theorem 2).

It would be interesting to see to what extent the convergence properties of the spacing distributions studied in [RuSZ], where α is taken to be well approximable by rationals, are related to the equidistribution of Kronecker sequences.

Statistical properties of $n^2\alpha$ mod 1 were also considered in connection with the Berry–Tabor conjecture [BeT] on the energy level statistics of integrable Hamiltonian quantum systems [the relevant system is here the "boxed oscillator" with energy levels $n^2\alpha + m$, where $m, n \in \mathbb{Z}_+$], integrable quantum maps [Z], the "quantum kicked-rotator" [CGI], [Si], [P], and scattering problems on certain surfaces of revolution [ZZ].

For any interval $[a, b]$ the *pair correlation function* is defined as

$$R_2\big([a,b],\alpha,N\big) = \tfrac{1}{N}\left|\left\{1 \leqq j \neq k \leqq N \;\middle|\; j^2\alpha - k^2\alpha \in \left[\tfrac{a}{N}, \tfrac{b}{N}\right] + \mathbb{Z}\right\}\right|. \quad (8.1)$$

In the following, we will consider the special Fuchsian group

$$\overline{\Gamma}_1(4) = \left\{(\pm T) \in \mathrm{PSL}(2,\mathbb{R}) \;\middle|\; T \in \Gamma_1(4)\right\}, \quad (8.2)$$

where $\Gamma_1(4)$ is the congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$,

$$\Gamma_1(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; c \equiv 0\,,\ a \equiv d \equiv 1 \pmod 4 \right\}.$$

(Notice that $\overline{\Gamma}_1(4)$ has a normalized cusp at ∞.)

Proposition 8.1. *If $\langle\overline{\Gamma}_1(4), 2, \alpha\rangle$-$PSE_{1/2}$ (cf. the definition on p. 1263) holds for some $\alpha \in \mathbb{R}$, then, for any interval $[a,b] \subset \mathbb{R}$,*

$$R_2\big([a,b],\alpha,N\big) \to b - a \quad \text{as } N \to \infty\,. \quad (8.3)$$

That is, the pair correlation function of $n^2\alpha$ mod 1 converges to the one of independent random variables from a Poisson process.

We remark that by Theorem 2′, the assumption in Proposition 8.1 is satisfied for almost all $\alpha \in \mathbb{R}$, in Lebesgue measure sense.

To prepare for the proof of Proposition 8.1, let us note that (for arbitrary Γ as in earlier sections) if both $\langle\Gamma,\nu,\alpha\rangle$-$\mathrm{PSE}_\gamma$ and $\langle\Gamma,\nu,-\alpha\rangle$-$\mathrm{PSE}_\gamma$ hold, then we also have asymptotic equidistribution of point sets with an arbitrary weight function h, as follows.

Lemma 8.2. *Let Γ,ν,α,γ be such that both $\langle\Gamma,\nu,\alpha\rangle$-$PSE_\gamma$ and $\langle\Gamma,\nu,-\alpha\rangle$-$PSE_\gamma$ hold. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded, piecewise continuous function of compact support. We then have, for any given $f \in B_\gamma(\Gamma \setminus T_1\mathcal{H})$,*

$$\frac{1}{M}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} h\left(\tfrac{m}{M}\right) f(m\alpha + iM^{-\nu}, 0) \to \int_{\mathbb{R}} h(u)du \cdot \langle f\rangle\,, \quad (8.4)$$

as $M \to \infty$.

Proof. It suffices to prove (8.4) for functions h which satisfy $h(u) = 0$ for all $u < 0$, for then functions h which satisfy $h(u) = 0$ for all $u > 0$ can be treated by replacing α by $-\alpha$, and the general case follows by adding the two cases.

If $h(u) = \chi_{(0,b]}$ for some $b > 0$, i.e. if $h(u)$ is the characteristic function of the interval $(0, b]$, then (8.4) follows immediately from $\langle\Gamma, \nu, \alpha\rangle$-PSE$_\gamma$ and the fact that $\lim_{M\to\infty}[bM]/M = b$. Using the relation $\chi_{(b_1,b_2]} = \chi_{(0,b_2]} - \chi_{(0,b_1]}$, we now find that (8.4) holds for any function h in the family $\mathcal{R}$ of finite linear combinations of characteristic functions of intervals $(b_1, b_2] \subset [0, \infty)$.

Next, if $h(u)$ is a piecewise continuous function on $[0, \infty)$ of compact support, then $h(u)$ is Riemann integrable, and thus there are sequences of functions $h_i^+, h_i^- \in \mathcal{R}$ such that $h_i^-(u) \leqq h(u) \leqq h_i^+(u)$ for all $u > 0$, and $\int_0^\infty (h_i^+ - h_i^-)du \to 0$ as $i \to \infty$. Now if $f \geqq 0$ then $\sum_m h_i^-(m/M)f(\dots) \leqq \sum_m h(m/M)f(\dots) \leqq \sum_m h_i^+(m/M)f(\dots)$, and applying (8.4) for h_i^+ and h_i^-, and letting $i \to \infty$, we find that (8.4) holds for $h(u)$. The case of arbitrary f then follows by applying the preceding result separately to the positive and negative parts of $\mathrm{Re}(f)$ and of $\mathrm{Im}(f)$ (each of these four functions belongs to $B_\gamma(\Gamma \setminus T_1\mathcal{H})$). □

REMARK 8.3. Let $T \mapsto \widetilde{T}$ be the automorphism of $\mathrm{PSL}(2, \mathbb{R})$ given by $\widetilde{T} = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) T \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$ (i.e. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{\sim} = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$). Concerning the assumption in Lemma 8.2, we then have

If the lattice Γ is invariant under $T \mapsto \widetilde{T}$, then $\langle\Gamma, \nu, -\alpha\rangle$-PSE$_\gamma$ is equivalent to $\langle\Gamma, \nu, \alpha\rangle$-PSE$_\gamma$, for any given ν, α, γ.

In particular, this applies when $\Gamma = \overline{\Gamma}_1(4)$.

To prove our claim, we assume that Γ is invariant under $T \mapsto \widetilde{T}$. We define $V(z, \theta) := (-\overline{z}, -\theta)$; one then checks that $V \circ T \equiv \widetilde{T} \circ V : T_1\mathcal{H} \to T_1\mathcal{H}$, for all $T \in \mathrm{PSL}(2, \mathbb{R})$. Hence, a function $f : T_1\mathcal{H} \to \mathbb{C}$ is Γ-invariant if and only if $f \circ V$ is Γ-invariant. Furthermore, if η is a cusp, then so is $-\eta$, and $\Gamma_{-\eta} = \{\widetilde{T} \mid T \in \Gamma_\eta\}$. It follows that the set $\mathcal{S}_\Gamma$ in (2.4) is invariant under $N \mapsto \widetilde{N}$, and using this one easily checks that $\mathcal{Y}_\Gamma(-\overline{z}) = \mathcal{Y}_\Gamma(z)$ for all $z \in \mathcal{H}$.

Now our claim follows directly from the definition of $\langle\Gamma, \nu, \alpha\rangle$-PSE$_\gamma$ on p. 1263, since replacing f with $f \circ V$ transforms the sum $\frac{1}{M}\sum_{m=1}^M f(m\alpha + iy, 0)$ into $\frac{1}{M}\sum_{m=1}^M f(m(-\alpha) + iy, 0)$.

Proof of Proposition 8.1. Fix α as in the proposition. We will consider the following smoothed version of the pair correlation function (8.1):

$$R_2(g, \psi, \alpha, N) = \frac{1}{N} \sum_{\substack{j,k\in\mathbb{Z} \\ |j|\neq|k|}} \psi\big(\tfrac{j}{N}\big)\psi\big(\tfrac{k}{N}\big) \sum_{m\in\mathbb{Z}} g\big(N(j^2\alpha - k^2\alpha + m)\big). \tag{8.5}$$

Here we assume ψ to be an even real-valued function in $C_c^\infty(\mathbb{R})$, and g to be a smooth function such that $g(x) = O(|x|^{-2})$ as $|x| \to \infty$, and such that the Fourier transform $\widehat{g}(u) = \int_{\mathbb{R}} g(x)e(-ux)\,dx$ has compact support. We will show that for any such functions ψ and g, we have

$$R_2(g,\psi,\alpha,N) \to \int_{\mathbb{R}} g(x)dx \cdot \left(\int_{\mathbb{R}} \psi(x)dx\right)^2, \tag{8.6}$$

as $N \to \infty$.

Let us first prove that (8.6) implies (8.3). Define $\eta(x)$ to be 1 for $0 < |x| \leqq 1$, and 0 for $x = 0$ and for $|x| > 1$. We can now pick two sequences of even non-negative functions $\psi_i^+, \psi_i^- \in C_c^\infty(\mathbb{R})$ with $\psi_i^- \leqq \eta \leqq \psi_i^+$ (hence in particular $\psi_i^-(0) = 0$, $\psi_i^+(0) \geqq 1$), such that $\int_{\mathbb{R}}(\psi_i^+ - \psi_i^-)dx \to 0$ as $i \to \infty$. Given $[a,b] \subset \mathbb{R}$, we can also pick two sequences of functions $g_i^+, g_i^- \in C^\infty(\mathbb{R})$, each satisfying $g_i^\pm(x) = O_i(|x|^{-2})$ as $|x| \to \infty$ and $\widehat{g_i^\pm} \in C_c(\mathbb{R})$, such that $g_i^- \leqq \chi_{[a,b]} \leqq g_i^+$ and $\int_{\mathbb{R}}(g_i^+ - g_i^-)dx \to 0$ as $i \to \infty$. (Such sequences g_i^+, g_i^- can be constructed following [M2, 8.6.1-2]. Cf. also [V].) Applying (8.6) for $g = g_i^-$, $\psi = \psi_i^-$ and for $g = g_i^+$, $\psi = \psi_i^+$, and then letting $i \to \infty$, we find that (8.6) also holds for $g = \chi_{[a,b]}$, $\psi = \eta$, i.e.

$$\tfrac{1}{N} \sum_{0<|j|\neq|k|\leqq N} \sum_{m\in\mathbb{Z}} \chi_{[a,b]}\big(N(j^2\alpha - k^2\alpha + m)\big) \to 4(b-a)\,, \tag{8.7}$$

as $N \to \infty$. Notice that if $N > b - a$, then the innermost sum in (8.7) is 1 if $j^2\alpha - k^2\alpha \in [a/N, b/N] + \mathbb{Z}$, otherwise 0. Notice also that $j^2\alpha - k^2\alpha$ is invariant under $j \leftrightarrow -j$ and $k \leftrightarrow -k$. It follows that for $N > b - a$, the left-hand side in (8.7) equals $4R_2([a,b],\alpha,N)$. Hence (8.6) implies (8.3).

It now remains to prove (8.6). Let $R_2^{(a)}(g,\psi,\alpha,N)$ be the same as in (8.5) but without the restriction $|j| \neq |k|$, i.e. with the outer sum taken over *all* $j,k \in \mathbb{Z}$. Since ψ is even, we then have

$$R_2(g,\psi,\alpha,N) = R_2^{(a)}(g,\psi,\alpha,N) - \tfrac{2}{N}\Big(\sum_{j\in\mathbb{Z}} \psi\left(\tfrac{j}{N}\right)^2 - \tfrac{1}{2}\psi(0)^2\Big) \cdot \sum_{m\in\mathbb{Z}} g(Nm)\,. \tag{8.8}$$

Notice here that

$$\lim_{N\to\infty} \tfrac{2}{N}\Big(\sum_{j\in\mathbb{Z}} \psi\left(\tfrac{j}{N}\right)^2 - \tfrac{1}{2}\psi(0)^2\Big) \cdot \sum_{m\in\mathbb{Z}} g(Nm) = 2g(0)\int_{\mathbb{R}} \psi(x)^2 dx\,, \tag{8.9}$$

since $\psi \in C_c^\infty(\mathbb{R})$ and $g(x) = O(|x|^{-2})$. Also, by the Poisson summation formula, we have

$$\begin{aligned} R_2^{(a)}(g,\psi,\alpha,N) &= \tfrac{1}{N^2} \sum_{j,k\in\mathbb{Z}} \psi\big(\tfrac{j}{N}\big)\psi\left(\tfrac{k}{N}\right) \sum_{m\in\mathbb{Z}} \widehat{g}\left(\tfrac{m}{N}\right) e\big(m(j^2-k^2)\alpha\big) \\ &= \tfrac{1}{N} \sum_{m\in\mathbb{Z}} \widehat{g}\left(\tfrac{m}{N}\right) \big|\theta_\psi(m\alpha + iN^{-2})\big|^2, \end{aligned}$$

where we have defined

$$\theta_\psi(x+iy) = y^{1/4} \sum_{j\in\mathbb{Z}} \psi(jy^{1/2}) e(j^2 x) \,.$$

This is a theta sum with a smooth cutoff function; such sums were studied in [M1] (both for smooth and sharp cutoff functions). To apply [M1] to our situation, we take $\Gamma = \overline{\Gamma}_1(4)$ (cf. (8.2)) and $f = |\Theta_\psi|^2$, where Θ_ψ is as in [M1, Prop. 3.1]. Notice that Θ_ψ is a function on a 4-*fold cover* of $\Gamma \setminus \mathrm{PSL}(2,\mathbb{R})$. However, we claim that $|\Theta_\psi|^2$ is a function on $\Gamma \setminus \mathrm{PSL}(2,\mathbb{R})$ itself. To see this, we first use [M1, (10), (12)] to check that the function θ_k in [M1, (24)] satisfies $\theta_k\big(\big[\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right), \beta_{-1}\big](z,\phi)\big) = e^{-i(k+\frac12)\pi}\theta_k(z,\phi)$. We also notice that for $k\in\mathbb{Z}^+$ odd, $\theta_k \equiv 0$. Hence $\Theta_\psi\big(\big[\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right), \beta_{-1}\big](z,\phi)\big) = e^{-i\pi/2}\Theta_\psi(z,\phi)$, by [M1, (26)], and thus $|\Theta_\psi(z,\phi)|^2$ is invariant under $\big[\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right), \beta_{-1}\big]$. This gives the desired result, since we already know that $|\Theta_\psi(z,\phi)|^2$ is invariant under $\Delta_1(4)$, cf. [M1, (15)]. In conclusion, we see from [M1, Prop. 3.1] that the function $f = |\Theta_\psi|^2$ is smooth and Γ-invariant, and

$$f(z,0) = \big|\theta_\psi(z)\big|^2, \qquad \forall z \in \mathcal{H}\,.$$

It follows from [M1, Prop. 3.2] that $f \in B_{1/2}(\Gamma \setminus T_1\mathcal{H})$. Hence, since $\langle \overline{\Gamma}_1(4), 2, \alpha\rangle$-$\mathrm{PSE}_{1/2}$ holds by assumption, Lemma 8.2 and Remark 8.3 apply, and we obtain

$$\lim_{N\to\infty} \Big(R_2^{(a)}(g,\psi,\alpha,N) - \tfrac{1}{N}\widehat{g}(0)\big|\theta_\psi(iN^{-2})\big|^2 \Big) = \int_{\mathbb{R}} \widehat{g}(u)\,du \cdot \langle f\rangle\,. \tag{8.10}$$

But here $\int_{\mathbb{R}} \widehat{g}(u)\,du = g(0)$, and it follows from [M1, (48)] and $\mu(\Gamma\backslash\mathcal{H}) = 2\pi$ that $\langle f\rangle = 2\int_{\mathbb{R}} \psi(x)^2\,dx$. Finally, the definition of $\theta_\psi(x+iy)$ implies that

$$\tfrac{1}{\sqrt{N}}\theta_\psi(iN^{-2}) = \tfrac{1}{N}\sum_{j\in\mathbb{Z}} \psi(j/N) \to \int_{\mathbb{R}} \psi(x)dx\,, \qquad \text{as } N\to\infty\,. \tag{8.11}$$

Now (8.6) follows from (8.8), (8.9), (8.10) and (8.11), and the proof is complete. □

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