PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 1, January 2014, Pages 205–212
S 0002-9939(2013)11746-6
Article electronically published on September 20, 2013

EQUILATERAL SETS IN INFINITE DIMENSIONAL BANACH SPACES

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(Communicated by Thomas Schlumprecht)

ABSTRACT. We show that every Banach space X containing an isomorphic copy of c_0 has an infinite equilateral set and also that if X has a bounded biorthogonal system of size α, then it can be renormed so as to admit an equilateral set of equal size.

INTRODUCTION

A subset S of a metric space (M, d) is said to be equilateral if there is a $\lambda > 0$ such that for $x, y \in S, x \neq y$ we have $d(x, y) = \lambda$; we also call S a λ-equilateral set. Equilateral sets have been studied mainly in finite dimensional spaces (see [15], [19], [20], [21]).

Our aim in this note is to study equilateral sets in infinite dimensions. We first prove (improving results of K.J. Swanepoel) that any infinite dimensional Banach space has an equivalent norm arbitrarily close to the original one admitting an infinite equilateral set (Theorem 1). Then we prove that every Banach space containing an isomorphic copy of c_0 admits an infinite equilateral set (Theorem 2). We also introduce a notion of antipodal sets (Definitions 1 and 2) which yields that a Banach space containing a bounded biorthogonal system of size α can be renormed so that in the new (equivalent) norm it has an equilateral set of equal size (Theorem 3). These results generalize results of Petty [15] and Swanepoel [19], [20].

If X is any (real) Banach space, then B_X denotes its closed unit ball. The Banach-Mazur distance between two isomorphic Banach spaces X and Y is $d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\|$, where $T : X \to Y$ is a linear isomorphism$\}$.

A sequence (e_n) in a Banach space X is said to be spreading if for any sequence $0 < p_1 < p_2 < \cdots < p_N$ of integers and any sequence $\alpha_1, \alpha_2, \ldots, \alpha_N$ of scalars we have $\|\sum_{k=1}^N \alpha_k e_k\| = \|\sum_{k=1}^N \alpha_k e_{p_k}\|$. It is clear that a non-constant spreading sequence is equilateral. Regarding the theory of spreading models we refer the reader to [3] and [1].

Received by the editors November 28, 2011 and, in revised form, December 23, 2011, February 24, 2012, and February 29, 2012.

2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 46B06, 46B04.

Key words and phrases. Equilateral set, antipodal set, biorthogonal system.

Equilateral and antipodal sets in infinite dimensions

The question of whether any infinite dimensional Banach space contains an infinite equilateral set has been answered in the negative by P. Terenzi in [17]. The question we are concerned with in this note is under which conditions an infinite dimensional Banach space contains an infinite equilateral set.

K.J. Swanepoel has proved that every infinite dimensional Banach space may be renormed so as to contain an infinite equilateral set and also that if the space is uniformly convexifiable, then we may choose the new norm to be arbitrarily close to the original norm (see [19], [20]).

P. Brass [4] and B.V. Dekster [7] have proved (both using Dvoretzky's Theorem) that each infinite dimensional Banach space contains arbitrarily large (finite) equilateral sets (see also [20]).

Our first result improves the results of Swanepoel mentioned above.

Theorem 1. *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. Then for every $\varepsilon > 0$ there exists an equivalent norm $|||\cdot|||$ on X such that:*

(1) $d((X, \|\cdot\|), (X, |||\cdot|||)) \leq 1 + \varepsilon$,
(2) $(X, |||\cdot|||)$ *admits an infinite equilateral set.*

Proof. It is enough to find a closed subspace Z of X and an equivalent norm $|||\cdot|||$ on Z, so that conditions (1) and (2) are satisfied for Z (see Remark 1(1) below).

Assume first that X contains an isomorphic copy, say Y of ℓ_1. By the classical non-distortion property of ℓ_1 ([13], Prop. 2.e.3), for every $\varepsilon > 0$ there is a normalized sequence (y_n) in $(Y, \|\cdot\|)$ such that for every sequence $(\alpha_n) \in c_{00}$ (the space of eventually zero real sequences) we have

$$\frac{1}{1+\varepsilon} \sum_{n=1}^{\infty} |\alpha_n| \leq \|\sum_{n=1}^{\infty} \alpha_n y_n\| \leq \sum_{n=1}^{\infty} |\alpha_n|.$$

Let $Z = \overline{\langle y_n \rangle}$ (i.e. the closed linear span of (y_n)) and for $y = \sum_{n=1}^{\infty} \alpha_n y_n \in Z$, set $|||y||| = \sum_{n=1}^{\infty} |\alpha_n|$. Clearly the space $(Z, |||\cdot|||)$ is isometric to ℓ_1 and $d((Z, \|\cdot\|), (Z, |||\cdot|||)) \leq 1 + \varepsilon$.

Now assume that $\ell_1 \not\hookrightarrow X$, so there exists a normalized weakly null (basic) sequence (x_n) in X with spreading model (e_n), which is a normalized unconditional basic sequence with suppression constant $K_s = 1$ (see [1], pp. 275-279). Given $\varepsilon > 0$ and $m \in \mathbb{N}$ ($m \geq 2$), it is enough to produce a subsequence of (x_n) (still denoted (x_n)) and a norm $|||\cdot|||$ on the span $\langle x_n \rangle$ of (x_n) satisfying the following:

(i) $\frac{1}{1+\varepsilon}\|x\| \leq |||x||| \leq \frac{1}{(1-\varepsilon)^2}\|x\|$ for $x = \sum \alpha_n x_n, (\alpha_n) \in c_{00}$ and

(ii) $|||\sum_{n\in F} \alpha_n x_n||| = \|\sum_{n\in F} \alpha_n e_n\|$ for all $F \subseteq \mathbb{N}$ with $|F| = m$.

Indeed, by passing to a subsequence we may assume that:

(a) $(1-\varepsilon)\|\sum_{n\in F} \alpha_n e_n\| \leq \|\sum_{n\in F} \alpha_n x_n\| \leq (1+\varepsilon)\|\sum_{n\in F} \alpha_n e_n\|$ for all $(\alpha_n) \in c_{00}$ and $F \subseteq \mathbb{N}$ with $|F| = m$ and

(b) $\|\sum_{n=1}^{\infty} \alpha_n x_n\| \geq (1-\varepsilon)\|\sum_{n\in F} \alpha_n x_n\|$ for all $(\alpha_n) \in c_{00}$ and $|F| = m$ by using Schreier unconditionality and keeping in mind that m is fixed (see [14]).

Let $x = \sum \alpha_n x_n, (\alpha_n) \in c_{00}$. We set

$$|||x||| = \max\{\frac{1}{1+\varepsilon}||x||, \sup_{|F|=m} \|\sum_{n\in F} \alpha_n e_n\|\}.$$

It is obvious that $\frac{1}{1+\varepsilon}\|x\| \leq |||x|||$, and since

$$\|\sum_{n\in F}\alpha_n e_n\| \leq \frac{1}{1-\varepsilon}\|\sum_{n\in F}\alpha_n x_n\| \leq \frac{1}{(1-\varepsilon)^2}\|x\|,$$

by (a) and (b) we get (i).

Now let $y = \sum_{n\in F_0}\alpha_n x_n, |F_0| = m$. Then (a) implies that

$$\|\sum_{n\in F_0}\alpha_n e_n\| \geq \frac{1}{1+\varepsilon}\|y\|,$$

so since the unconditional sequence (e_n) has suppression constant $K_s = 1$, it follows that $|||y||| = \|\sum_{n\in F_0}\alpha_n e_n\|$, and hence we get (ii).

Set $Z = \overline{\langle x_n\rangle}$. Since the sequence (e_n) is spreading, by (ii) we get the conclusion. $\square$

Remarks 1. (1) Let $(X, \|\cdot\|)$ be a normed space, Z a linear subspace of X and $|||\cdot|||$ an equivalent norm on Z such that

$$c_1|||x||| \leq \|x\| \leq c_2|||x|||\ \forall x \in Z. \tag{1}$$

Then the norm $|||\cdot|||$ of Z can be extended on X (using the Hahn-Banach Theorem) so that (1) is satisfied. Indeed, for $x \in X$ set $|||x|||_1 = \sup|\tilde{f}(x)|$, where the supremum is taken over all $f \in Z^*$ with $|||f||| \leq 1$ and $\tilde{f}$ is a Hahn-Banach extension of f on X with $||f|| = ||\tilde{f}||$. Then take $|||x||| = \max\{|||x|||_1, \frac{1}{c_2}\|x\|\}$, and it is easy to see that (1) is satisfied for all $x \in X$.

(2) By a result of Rosenthal [16] the sequence $(x_n) \subseteq X$ in the proof of Theorem 1 can be chosen so that the spreading sequence (e_n) is 1-unconditional (i.e. with unconditional constant $K_u = 1$). Therefore the proof gives more than equilateral; in particular, we get that $|||x_n \pm x_m||| = |||x_1 - x_2||| > 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$. More generally, $|||\sum_{i=1}^m a_i x_{n_i}||| = |||\sum_{i=1}^m b_i x_{k_i}|||$ whenever $n_1 < n_2 < \cdots < n_m$, $k_1 < k_2 < \cdots < k_m$ and $|a_i| = |b_i|$ for $i \leq m$.

Theorem 2. *Every Banach space X containing an isomorphic copy of c_0 admits an infinite equilateral set.*

Proof. We shall use the non-distortion property of c_0 and the following generalization of Theorem B of [21], with a similar proof (see also [4]).

Claim. Let $\|\cdot\|$ be an equivalent norm on c_0 with Banach-Mazur distance at most $\frac{3}{2}$ from the original norm $\|\cdot\|_\infty$ of c_0. Then $(c_0, \|\cdot\|)$ admits an infinite equilateral set.

Proof of the Claim. We may assume that $\|x\| \leq \|x\|_\infty \leq \frac{3}{2}\|x\|$ for $x \in c_0$. Let $I = \{(n,m) : n, m \in \mathbb{N}$ and $n < m\}$; denote by K the compact cube $[0, \frac{1}{2}]^I$.

For $\varepsilon = (\varepsilon_{(n,m)}) \in K$ we set $p_1(\varepsilon) = (-1, 0, \dots)$ and $p_n(\varepsilon) = (\varepsilon_{(1,n)}, \varepsilon_{(2,n)}, \dots, \varepsilon_{(n-1,n)}, -1, 0, \dots)$ for $n \geq 2$. Observe that for $n < m$ we have

$$\|p_n(\varepsilon) - p_m(\varepsilon)\|_\infty = 1 + \varepsilon_{(n,m)}.$$

We define a function $\varphi : K \to K$ by the rule $\varphi_{(n,m)}(\varepsilon) = 1 + \varepsilon_{(n,m)} - \|p_n(\varepsilon) - p_m(\varepsilon)\|$, $(n,m) \in I$, $\varepsilon \in K$. Note that $\varphi_{(n,m)}(\varepsilon) \geq 1+\varepsilon_{(n,m)} - \|p_n(\varepsilon) - p_m(\varepsilon)\|_\infty = 0$ and $\varphi_{(n,m)}(\varepsilon) \leq 1 + \varepsilon_{(n,m)} - \frac{2}{3}\|p_n(\varepsilon) - p_m(\varepsilon)\|_\infty = \frac{1}{3}(1 + \varepsilon_{(n,m)}) \leq \frac{1}{2}$, so φ is well defined. Since each coordinate function $\varphi_{(n,m)}$ is continuous, φ is also continuous. Hence by a classical result of Schauder φ has a fixed point $\varepsilon' = (\varepsilon'_{(n,m)}) \in K$; that is, $\varphi(\varepsilon') = \varepsilon'$, which implies that $\|p_n(\varepsilon') - p_m(\varepsilon')\| = 1$ for $n < m$. Therefore the set $\{p_n(\varepsilon') : n \in \mathbb{N}\}$ is equilateral in $(c_0, \|\cdot\|)$ and the Claim holds.

Denote by $\|\cdot\|$ the norm on X and let Y be a subspace of X isomorphic to c_0. By the non-distortion property of $(c_0, \|\cdot\|_\infty)$ there is a subspace Z of Y (isomorphic to c_0) such that $d((Z, \|\cdot\|), (c_0, \|\cdot\|_\infty)) \leq \frac{3}{2}$ (see also Theorem 1). It follows immediately from the Claim that the space $(Z, \|\cdot\|)$ admits an infinite equilateral set. □

Notes. (1) c_0 cannot be replaced by ℓ_1 in Theorem 2. Indeed Terenzi's example gives for every $\epsilon > 0$ a $(1+\epsilon)$-renorming of ℓ_1 so as not to contain an infinite equilateral set.

(2) Let K be an infinite compact Hausdorff space. Since (as it is well known) $(C(K), ||\cdot||_\infty)$ contains an isometric copy of c_0, its unit sphere contains a sequence (x_n) with $||x_n - x_m||_\infty = 2$ for $n \neq m$. So it seems natural to ask how large a 2-equilateral subset of the unit sphere of $C(K)$ can be, assuming further that K is (compact) non-metrizable. In "most" cases one can prove that such a set is uncountable, but the general case is open for us.

In the following definition we generalize a concept coming from finite dimensions to infinite dimensional spaces.

Definition 1. Let $(X, \|\cdot\|)$ be a normed space. A subset S of X is said to be antipodal if for every $x, y \in S$ with $x \neq y$ there exists $f \in X^*$ such that $f(x) < f(y)$ and $f(x) \leq f(z) \leq f(y)\, \forall z \in S$. That is, for every $x, y \in S$ with $x \neq y$ there exist closed distinct parallel support hyperplanes $P(= \{z \in X : f(z) = f(x)\})$ and $Q(= \{z \in X : f(z) = f(y)\})$ with $x \in P$ and $y \in Q$.

Remarks 2. (1) If X is a finite dimensional real vector space, then the concept of antipodality coincides with the classical one.

(2) It is well known by a result of Danzer and Grünbaum [5] that the maximum cardinality of an antipodal set in $\mathbb{R}^n$ is 2^n, and this is attained only if the points of the antipodal set are the vertices of an n-dimensional parallelotope. A typical example of such a set is the unit ball B of ℓ_∞^n; the vertices of B are (the extreme points of B) $\{(\varepsilon_1, \ldots, \varepsilon_n) : \varepsilon_i = \pm 1,\, i = 1, 2, \ldots, n\}$.

Let X be a Banach space. A family $\{(x_\gamma, x_\gamma^*), \gamma \in \Gamma\}$ of pairs in $X \times X^*$ is called a biorthogonal system if $x_\beta^*(x_\alpha) = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker δ, for all $\alpha, \beta \in \Gamma$. A family $\{x_\gamma : \gamma \in \Gamma\}$ in X is called a minimal system if there exists a family $\{x_\gamma^* : \gamma \in \Gamma\}$ in X^* such that $\{(x_\gamma, x_\gamma^*), \gamma \in \Gamma\}$ is a biorthogonal system.

Proposition 1. *Every minimal system in a Banach space is antipodal.*

Proof. Let $\{x_\gamma : \gamma \in \Gamma\}$ be a minimal system in X. Hence there exists $\{x_\gamma^* : \gamma \in \Gamma\} \subseteq X^*$ such that the family $\{(x_\gamma, x_\gamma^*), \gamma \in \Gamma\}$ is a biorthogonal system. Let $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \neq \gamma_2$. Then we have

$$0 = x_{\gamma_1}^*(x_{\gamma_2}) \leq x_{\gamma_1}^*(x_\gamma) \leq x_{\gamma_1}^*(x_{\gamma_1}) = 1 \ \ \forall \gamma \in \Gamma.$$

It follows immediately that $\{x_\gamma : \gamma \in \Gamma\}$ is an antipodal set in X. □

The following result generalizes a result of Petty with essentially the same proof ([15], Th. 1).

Proposition 2. *Let S be an equilateral set in a normed space $(X, \|\cdot\|)$; then S is antipodal.*

Proof. Let $x, y \in S, x \neq y$. Suppose that S is a λ-equilateral set. By the Hahn-Banach theorem there is an $f \in X^*, \|f\| = 1$ such that

$$f(y - x) = \|y - x\| = \lambda > 0.$$

Then $f(x) < f(y)$ and $f(y) = \sup\{f(z) : z \in B(x, \lambda)\}$. So f is a support functional of the ball $B(x, \lambda)$ through y and $f(z) \leq f(y) \,\forall z \in S$. Also if $g = -f$ we have

$$g(x - y) = f(y - x) = \|y - x\| > 0$$

and $\|g\| = 1$, so similarly g is a support functional of the ball $B(y, \lambda)$ through x and $g(z) \leq g(x) \,\forall z \in S$. Hence $f(x) \leq f(z) \leq f(y) \,\forall z \in S$, and the set S is antipodal. ☐

Petty has also proved that if S is an antipodal set in a finite dimensional real vector space X, then there exists a norm $\|\cdot\|$ on X such that S is equilateral in $(X, \|\cdot\|)$ ([15], Th. 2). In order to generalize this result in infinite dimensions we shall need a strengthening of the concept of an antipodal set introduced in Definition 1.

Definition 2. Let $(X, \|\cdot\|)$ be a normed space. We call an antipodal subset S of X (cf. Definition 1) <u>bounded</u> and <u>separated</u> if there are positive constants c_1, c_2 and d such that

(1) $\|x\| \leq c_1, \forall x \in S$ and
(2) for every $x, y \in S$ with $x \neq y$ there is an $f \in X^*$ with $\|f\| \leq c_2$ such that $0 < d \leq f(y) - f(x)$ and $f(x) \leq f(z) \leq f(y) \,\forall z \in S$.

Remarks 3. (1) Let S be a bounded and separated antipodal set in $(X, \|\cdot\|)$. It is easy to see that if $\lambda > 0$, then S is also bounded and separated with constants $c_1, \lambda c_2, \lambda d$ and the same is valid for the set $\lambda S = \{\lambda x : x \in S\}$ with constants $\lambda c_1, c_2, \lambda d$.

(2) It follows from the above remark that an antipodal bounded and separated set can be defined as a subset S of B_X satisfying the property that there is a constant $d > 0$ such that for every $x, y \in S$ with $x \neq y$ there exists $f \in B_{X^*}$ with $d \leq f(y) - f(x)$ and $f(x) \leq f(z) \leq f(y)$ for $z \in S$; that is, we may assume that $c_1 = c_2 = 1$. Given that formulation of Definition 2, it would be interesting to know if every infinite dimensional Banach space contains an infinite antipodal bounded and separated set with ($c_1 = c_2 = 1$ and) $d > 1$. (The answer is positive in the case when X contains some $\ell_p, 1 \leq p < \infty$ or c_0.) We note in this connection that by a result of Elton and Odell the unit sphere of every infinite dimensional Banach space contains an infinite $(1 + \varepsilon)$-separated set for some $\varepsilon > 0$ ([8]; see also [6] and [11]).

Examples. Let X be a Banach space.

(1) Each finite antipodal set in X is bounded and separated (obvious).

(2) Let $\{(x_\gamma, x^*_\gamma), \gamma \in \Gamma\}$ be a bounded biorthogonal system in X; that is, there is a constant $M > 0$ such that $\|x_\gamma\| \cdot \|x^*_\gamma\| \leq M$ for all $\gamma \in \Gamma$. We set $y_\gamma = \frac{x_\gamma}{\|x_\gamma\|}$ and $y^*_\gamma = \|x_\gamma\| \cdot x^*_\gamma$, for $\gamma \in \Gamma$. Clearly the system $\{(y_\gamma, y^*_\gamma), \gamma \in \Gamma\}$ is biorthogonal. Now it is easy to see that the minimal system $\{y_\gamma : \gamma \in \Gamma\}$ is (by Proposition 1) antipodal bounded and separated, with constants $c_1 = 1, c_2 = M$ and $d = 1$.

(3) Each equilateral set S in X is a bounded and separated antipodal set. Indeed, as it follows from the method of proof of Proposition 2, if S is λ-equilateral, then the desired constants are $c_1 = M, c_2 = 1$ and $d = \lambda$, where $M = \sup\{\|x\| : x \in S\}$. (Each equilateral set is clearly bounded.)

The following result simultaneously generalizes a result of Petty ([15], Th. 2) and a result of Swanepoel already mentioned in the introduction [19].

Theorem 3. *Let $(X, \|\cdot\|)$ be a Banach space and $S \subseteq X$ be a bounded and separated antipodal set. Then we have the following:*

(1) *There is an equivalent norm $|||\cdot|||$ on X, such that S is an equilateral set in $(X, |||\cdot|||)$.*
(2) *If the constants of S are $c_1 = 1, c_2 = c$ and d, then the Banach-Mazur distance between $(X, \|\cdot\|)$ and $(X, |||\cdot|||)$ satisfies the inequality*

$$d((X, \|\cdot\|), (X, |||\cdot|||)) \leq 2 \cdot \frac{c}{d}.$$

Proof. Assume (as we may) that the constants of S are $1, c$ and d (see Remarks 3). We set

$$K = \overline{\mathrm{conv}}(\frac{d}{c} \cdot B_X \cup \{x - y : x, y \in S\}).$$

Then K is a closed (bounded), convex symmetric set with $0 \in \mathrm{int}(K)$, so the corresponding Minkowski functional defines a norm on X

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$$

and the unit ball of the space $(X, \|\cdot\|_K)$ is exactly the set K. For $x, y \in S, x \neq y$ there is an $f \in cB_{X^*}$ such that

$$d \leq f(y) - f(x) \leq \|f\|\|x - y\| \leq 2c.$$

Hence $\frac{d}{c} \cdot B_X \subseteq K \subseteq 2 \cdot B_X$, so it follows that the Banach-Mazur distance of the two norms is $\leq 2 \cdot \frac{c}{d}$.

It suffices to show that if $x, y \in S$ with $x \neq y$, then $x - y \in \partial K$ (equivalently $\|x - y\|_K = 1$, where ∂K stands for the boundary of the set K), from which we have that S is a 1-equilateral set in $(X, \|\cdot\|_K)$. Let $x, y \in S$ with $x \neq y$. Then there is $f \in cB_{X^*}$ with $d \leq f(y) - f(x)$ and $f(x) \leq f(z) \leq f(y) \,\forall z \in S$. For every $z_1, z_2 \in S$ we have $f(z_1 - z_2) \leq f(y - x)$. Also, if $z \in \frac{d}{c} \cdot B_X$, then $f(z) \leq |f(z)| \leq \|f\|\|z\| \leq d \leq f(y - x)$. Hence f is a support functional of the set K through the point $y - x$, and so $y - x \in \partial K$. □

Corollary 1. *Let $\{(x_\gamma, x_\gamma{}^*) : \gamma \in \Gamma\}$ be a bounded biorthogonal system in the Banach space $(X, \|\cdot\|)$ such that $\|x_\gamma\| = 1$ and $\|x^*_\gamma\| \leq c$ for all $\gamma \in \Gamma$. Then there is an equivalent norm $|||\cdot|||$ on X of Banach-Mazur distance at most $2c$ from the original norm such that $\{x_\gamma : \gamma \in \Gamma\}$ is an equilateral set in $(X, |||\cdot|||)$.*

Proof. The set $\{x_\gamma : \gamma \in \Gamma\}$ is a bounded and separated antipodal set with constants $c_1 = 1, c_2 = c$ and $d = 1$. So Theorem 3 can be applied. □

Remarks 4. (1) For a Banach space $(X, \|\cdot\|)$ with $\dim X = \infty$, set $\mathrm{ant}(X) = \sup\{d > 0 :$ there is an infinite antipodal, bounded and separated set $S \subseteq X$ with constants $c_1 = c_2 = 1$ and $d\}$. We note that $\mathrm{ant}(X) \leq 2$. Since by a result of Day (Th. 1.20 in [9]) there is an infinite Auerbach system $\{(x_n, x^*_n) : n \geq 1\}$ in X, that is, a biorthogonal system with $\|x_n\| = \|x^*_n\| = 1$ for $n \in \mathbb{N}$, we get that

$\mathrm{ant}(X) \geq 1$. So Theorem 3 yields that for every $\varepsilon > 0$, X admits an equivalent norm with Banach-Mazur distance $\leq \frac{2}{\mathrm{ant}(X)-\varepsilon}$ from the original one, admitting an infinite equilateral set (cf. Theorem 1 and Remarks 3).

(2) A concept weaker than biorthogonality is that of semibiorthogonality. Let X be a Banach space. A family $\{(x_\alpha, x_\alpha^*) : \alpha < \omega_1\}$ is said to be ω_1-semibiorthogonal if it satisfies the following: (i) $x_\beta^*(x_\alpha) = 0$ for $\alpha < \beta < \omega_1$, (ii) $x_\beta^*(x_\beta) = 1$ for $\beta < \omega_1$ and (iii) $x_\beta^*(x_\alpha) \geq 0$ for $\beta < \alpha < \omega_1$.

If we replace condition (iii) by the stronger (iv) $0 \leq x_\beta^*(x_\alpha) \leq 1$ for all $\alpha, \beta < \omega_1$, and if the sets $\{x_\alpha : \alpha < \omega_1\}$,$\{x_\alpha^* : \alpha < \omega_1\}$ are bounded, then it is easy to see that $\{x_\alpha : \alpha < \omega_1\}$ is a bounded, separated and antipodal set. If for instance the compact space K contains a closed non-G_δ set, then the Banach space $C(K)$ admits an ω_1-semibiorthogonal system of the form $\{(f_\alpha, \delta_{t_\alpha}) : \alpha < \omega_1\}$, where $f_\alpha : K \to [0,1], \alpha < \omega_1$, are continuous functions and δ_t is the Dirac measure at $t \in K$ (see [9], Prop. 8.7). The compact scattered non-metrizable space constructed (under CH) by Kunen is such that the Banach space $C(K)$ admits an ω_1-semibiorthogonal system but no uncountable biorthogonal system ([9], Th. 8.8 and Th. 4.41). We also note that it is consistent with ZFC that there exist nonseparable Banach spaces (of the form $C(K)$, where K is compact) which admit no ω_1-semibiorthogonal system (see [12], [2] and [10]).

(3) It should be mentioned that there exist several interesting classes of nonseparable Banach spaces, such as weakly compactly generated (WCG) and their generalizations, that admit uncountable bounded biorthogonal systems (actually Markushevich bases); see [9]. Finally, notice that by a result of Todorcevic it is consistent with ZFC (under Martin's Maximum axiom) to assume that every nonseparable Banach space admits an uncountable bounded biorthogonal system (see [18] and [9], Ths. 4.48 and 8.12).

Added in proof. Very recently D. Freeman, E. Odell, B. Sari and T. Schlumprecht proved that every infinite dimensional uniformly smooth Banach space contains an infinite equilateral set [Equilateral sets in uniformly smooth Banach spaces, arXiv:1305.6750v1 [math.FA], 29 May 2013].

Acknowledgements

The authors wish to thank the referee for valuable suggestions and remarks. In particular, Theorem 1 in this general form is due to the referee. Initially the authors had proved a weaker result.

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