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Equilibria in bottleneck games

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Abstract

This paper introduces a bottleneck game with finite sets of commuters and departing time slots as an extension of congestion games of Konishi et al. (J Econ Theory 72:225–237, 1997a). After characterizing Nash equilibrium of the game, we provide sufficient conditions for which the equivalence between Nash and strong equilibria holds. Somewhat surprisingly, unlike in congestion games, a Nash equilibrium in pure strategies may often fail to exist, even when players are homogeneous. In contrast, when there is a continuum of atomless players, the existence of a Nash equilibrium and the equivalence between the set of Nash and strong equilibria hold as in congestion games (Konishi et al. 1997a).

Keywords Bottleneck game · Nash equilibrium · Strong equilibrium

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1 Introduction

A bottleneck model is used in analyzing traffic congestion during rush hours, where commuters depart from their origins (e.g. their houses) to their destinations (e.g. their workplaces). The simplest model was independently analyzed by Vickrey (1969) and Hendrickson and Kocur (1981), where a continuum of commuters depart from a single origin to a single destination connected by a single road with continuous time horizon. Along the road, there is a bottleneck in which a queue forms if the number of commuters exceeds the capacity of the road at a given time, where the capacity is defined as the maximum number of commuters that can pass through it in each slot. In these papers, commuters decide on the departure time based on the trade-offs between congestion and their optimal arrival time. Players are assumed to have the same preferred time of arrival and a specific form of the trip cost function. Subsequent papers, such as Smith (1984) and Daganzo (1985), introduce heterogeneity in allowing differences in preferred time of the commuters, and Newell (1987), in addition, allow heterogeneity in the different cost functions of the commuters, albeit under the restriction of linear cost functions. Overall, the aforementioned papers have stuck to the simple model with a single bottleneck and preferences of commuters defined by a specific functional form. We refer the reader to the survey by Small (2015) that summarizes a broad line of research regarding the bottleneck model.

More recent papers have incorporated a general network within the model to expand on the single-bottleneck model of the papers. Most of these papers have retained the continuum assumption of the set of players and the specific functional form of the preferences in the literature but have expanded the network to incorporate many routes, examples of such are given in Koch and Skutella (2011) and Cominetti et al. (2015). The main reason for retaining the functional form is that these papers focus on the relationship between equilibrium and the social optimum through measures such as the price of anarchy and price of stability, and social optimum is easier to define in these models. Recently, Scarsini et al. (2018) consider a discretized model with atomic players but again retains a similar functional form for the utility function of the commuter. In another related and recent paper, Rivera et al. (2018) consider a model where the set of commuters and the set of departure times are discrete (finite) sets, but instead simplify the network so that they revert back to the original bottleneck model. However, unlike the previous papers, they show that nonexistence of a Nash equilibrium is a widespread phenomenon when the cost of being late to work is substantially high. Instead, they focus on mixed equilibria and correlated equilibria, which exist under these conditions, and use these equilibrium concepts to calculate the price of anarchy and the price of stability.

In this paper, we define a bottleneck game with a finite set of departure time slots to enter a single bottleneck, but we allow commuters to have more general preferences. Each commuter has preferences on two arguments: her departure time and the length of the queue in which she has to wait to pass through the bottleneck. Our game is an anonymous game with congestion generated by a queue structure without imposing a specific form of trip costs function. The model is a discrete version of the bottleneck model given in Konishi (2004) with one simple bottleneck. Also, the network structure we consider resembles very closely to that of a recent paper by Rivera et al. (2018), where they also consider a single bottleneck with a finite set of commuters and a finite set of departure times. However, unlike Rivera et al. (2018) and other papers mentioned before, we do not specify a functional form for the preferences of the commuter. In addition, because of the generality of preferences that we consider, we do not explicitly include players' preferences towards exit times out of the bottleneck. Instead, these factors could be implicitly assumed to be part of the players' attitudes towards congestion when they enter the bottleneck.

Despite the abstractness of preferences outlined in the previous paragraph, we can interpret our model in a different context other than traffic congestion. For example, consider a location choice problem along a river, in which residents pollute the river while the river has an ability to abate pollution up to some level at each location of the river. We can allow residents' arbitrary preferences over locations (such as scenic and/or convenient locations) on the river. However, if many residents prefer locations upstream *ceteris paribus* and do choose these locations, then residents either located at that spot or at more downstream locations are affected by the pollution caused by these residents. In this model, the natural cleansing ability of the river corresponds to the capacity of our bottleneck model. The residue of pollutants then corresponds to the congestion that results from the bottleneck. Moreover, the residents' preferences over locations corresponds to the commuters' preferences over time slots. While the assumption of preferences over just time slots may seem simplistic, it has the advantage of applicability to models outside of the transportation literature.

Mathematically, our model is also an extension of the congestion game given by Konishi et al. (1997a), which has the following three properties:[1] Anonymity (A), Congestion (C) and Independence of Irrelevant Choices (IIC). First, A requires that the payoff of each player depends on the number of players who choose each action and not on the players' names. Second, C states that the payoff of each player increases if another player who had chosen the same strategy chooses a different strategy. Finally, IIC states that the payoff of a player is not affected even if another player that chooses a different strategy from hers switches to another strategy that is also a different strategy from hers. In this game, Milchtaich (1996) shows that this class of games always admits a Nash equilibrium in pure strategies.[2] Konishi et al. (1997a) show that in the same model, any strictly improving coalitional deviation

[1] The name "congestion game" is sometimes attributed to a class of games introduced by Rosenthal (1973), who considers a situation in which players choose a combination of primary factors out of a certain number of alternatives. Each player's payoff is determined by the sum of the costs of each primary factor she chooses, while the cost of each primary factor depends on the number of players who choose it, and not on the players' names. Rosenthal (1973) proved that there always exists at least one pure-strategy Nash equilibrium by constructing a potential function, which is later formalized by Monderer and Shapley (1996). However, these congestion games do not require payoffs being negatively affected by the population while requiring that payoff functions have the same form among the players who take same factors.

[2] Although the formulation of the game of Milchtaich (1996) is different from Konishi et al. (1997a), Voorneveld et al. (1999) show that these two formulations are equivalent.

from a Nash equilibrium results in another Nash equilibrium, thus implying a congestion game also admits a strong equilibrium, introduced by Aumann (1959), that is immune to any strictly improving coalitional deviation. They also show that if there is a continuum of atomless players, then the sets of Nash and strong equilibria coincide with each other. Holzman and Law-Yone (1997) also provide several conditions under which the sets of strong equilibria and Nash equilibria coincide for congestion games of Rosenthal (1973), while Harks et al. (2013) also provide sufficient conditions for the existence of strong equilibrium for a class of games called bottleneck congestion games, which are different from the model considered in the present paper in that their model resembles more closely to communication networks.

Our bottleneck game does not satisfy IIC, whereas the other two conditions hold (though C applies in a strict sense only after a queue forms by exceeding the capacity). Specifically, IIC would be violated in the case where a player who had departed later then switched to an earlier departure time and thereby possibly creating a longer queue for some of those players which she leaps over. With this difference in mind, we analyze the Nash equilibria and strong equilibria of the bottleneck game. In contrast to Rivera et al. (2018) which consider mixed Nash equilibria and correlated equilibria that involve probabilistic choices by the players, we focus exclusively on pure strategies. One reason for this is to compare our results to Konishi et al. (1997a), which also consider only pure strategies, more directly. Another important reason is that we can compare the results in the game with a finite number players to a game with a continuum of atomless players, where it is customary to consider pure strategy equilibria due to their existence in such games [see Schmeidler (1973)]. Moreover, we can view strong equilibria to be an equilibrium concept that allows for correlation of strategies among a group of players, not confined to be the whole player set. When the possibility of such correlation by a group of players is not ruled out *a priori*, then the strong equilibrium is a suitable equilibrium concept to consider in such circumstances. This possibility may arise not only in the transportation context of the bottleneck game, but also in the location choice problem.

Specifically, we first show the equivalence between Nash and strong equilibria under some conditions (Propositions 1, 2, and 3), but we show that a Nash equilibrium may not exist even when players are Homogeneous (H) and other stringent conditions such as Single-Peakedness (SP) and Order-Preservation (OP) on the payoff function are satisfied (Examples 2 and 7). With an even more stringent condition, we show the existence of Nash equilibrium (Proposition 4), which yields the types of strategy profiles in the earlier sections that are also strong equilibria. These results are in stark contrast with the ones in congestion games: a Nash equilibrium always exists, but it is hard to ensure the equivalence between Nash and strong equilibrium due to coordination failures unless players are homogeneous. In contrast, when players are atomless, we can establish both the existence of Nash equilibrium and equivalence between Nash and strong equilibria exactly in the same way as in congestion games (Proposition 7).

The rest of the paper is organized as follows. In Sect. 2, we define our bottleneck game with a finite number of players. In Sect. 3, we provide three sufficient conditions under which Nash and strong equilibria are equivalent to each other. In

Sect. 4, we show that our bottleneck game may not have a Nash equilibrium in pure strategies even when players are homogeneous. We also provide a positive result for the existence although the conditions are very stringent. Section 5 introduces a bottleneck game with atomless players, and we show that the existence of Nash and the equivalence between Nash and strong equilibria all hold in this idealized environment. Sect. 6 concludes.

2 The model with a finite number of players

The following model we introduce is a discretized and simplified version of a model considered in Konishi (2004) with a single bottleneck and a finite set of departure time slots. Let $N = \{1, 2, \dots, n\}$ denote the finite set of players or commuters, and let $\mathcal{T} = \{1, 2, \dots, T\}$ be the set of available departing time slots where time period 1 represents the earliest time slot. For example, each discrete time unit can represent every minute or every five minutes.

For each $t \in \mathcal{T}$, denote by q_t the length of the resulting queue at departing time slot t. This length can be calculated inductively by

$$q_t = \max\left\{0, q_{t-1} + m_t - c\right\},$$

where m_t is the number of players who depart at time slot t, and $c \in \mathbb{Z}_+$ is the capacity of the bottleneck, and we let $q_0 = 0$. We also introduce the notation $\tilde{q}_t = q_{t-1} + m_t - c$ to describe possible slacks: $\tilde{q}_t < 0$ means that the road capacity is not binding at time slot t, and the queue at time slot t does not develop even if an additional car joins. For notational convenience, let $\tilde{q}_0 = 0$.

Each player i's choice (strategy) of departing time is denoted $\tau_i \in \mathcal{T}$. A strategy profile is $\tau = (\tau_1, \dots, \tau_n) \in \mathcal{T}^N$, and resulting queue lengths at all time slots are described by a vector $\tilde{q}(\tau) = (\tilde{q}_1(\tau), \dots, \tilde{q}_T(\tau))$ and $q(\tau) = (q_1(\tau), \dots, q_T(\tau))$, respectively. To denote the dependence of m_t on τ more explicitly, we use the notation $m_t(\tau)$ as well. Denote player i's payoff from choosing time slot t with queue length q_t $u^i(t, q_t)$, where each function u^i could be different for each player, thus allowing for heterogeneity. Note that by this specification, we are assuming the condition Anonymity (A) implicitly, the formal statement of which is as follows.

Anonymity (A) For each $i \in N$, the payoff function u^i depends on τ only through the strategy τ_i chosen by i and the overall distribution of strategies chosen by the players $(m_t(\tau))_{t\in\mathcal{T}}$ according to strategy profile τ.

In addition, we also assume that the game satisfies the following condition Congestion (C), which states that *ceteris paribus*, each player dislikes an additional unit of queuing. This condition can be encompass more than a player's dislike for queueing time and can include pure distaste for a congested environment caused by the queueing.

Congestion (C) For all $i \in N$, $u^i(t, k) > u^i(t, k + 1)$ holds for all $t \in \mathcal{T}$ and all $k \in \mathbb{Z}_+$.

Unless specified otherwise, we assume conditions A and C throughout, as they are integral to the setup of the bottleneck games that we want to consider.

As was explained in the introduction, while the motivation behind the above abstract model comes from bottleneck models, application of the abstract model need not be confined to just transportation settings. For example, consider a river and a set of firms who are choosing where to locate along the river. Each $t \in \mathcal{T}$ represents a section of the river, where a lower index represents a more upstream location. Each firm emits the same amount of pollutant onto the river, where now the capacity c can be interpreted as the amount of pollutant per section the river can sustain without being completely polluted and the queue at t, q_t, represents the residual amount of pollutants that have not been washed out by the natural ability of the water. This amount then flows down to the next section downstream. The equations regarding the queue q_t then represents the accumulation of pollutants at a particular section t with respect to its capacity t and the pollutants that are flowing from its immediate upstream neighbor. Moreover, each firm has preferences over combinations of location (t) and the amount of pollutant present in the river at that location (q_t), where in general, firms have a dislike for more pollutants, expressed by condition C. The problem is then for each firm to decide where to locate along the river. The abstract model is thus set up to seamlessly include these types of problems as well.

A strategy profile τ is a **Nash equilibrium** if and only if for all $i \in N$ and all $t \in \mathcal{T}$, $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t, q_t(t, \tau_{-i}))$ holds. In the following, we give a characterization for Nash equilibria in this model.

First, note that although $q_t = 0$ holds either when $\tilde{q}_t = 0$ or $\tilde{q}_t < 0$, these two cases are different when an additional car arrives at time slot t. In the former case, a queue develops with an additional car while in the latter case it does not develop. Based on this, we introduce the following concepts.

Definition 1

1. A single slot t is said to be a **basin** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) < 0$ and $\tilde{q}_{t-1}(\tau) \leq 0$.
2. A single slot t is a **single terrace** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) = 0$ and $\tilde{q}_{t-1}(\tau) \leq 0$.
3. A set of consecutive time slots $I = [t_1, t_2]$ with $1 \leq t_1 < t_2 \leq T$ is said to be a **connected terrace** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) > 0$ for all $t \in [t_1, t_2)$, $\tilde{q}_{t_1-1}(\tau) \leq 0$, and $\tilde{q}_{t_2}(\tau) \leq 0$ whenever $t_2 < T$.

Note that each time slot t in a particular strategy profile is a basin, a single terrace, or part of a connected terrace. Therefore, for each strategy profile, the set of time slots can be partitioned into basins, single terraces, and connected terraces.

To illustrate, consider the following example. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{T} = \{1, 2, 3, 4, 5\}$. Let $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$ be a strategy profile such that $\tau_1 = \tau_3 = \tau_4 = 3$ and $\tau_2 = 1$. Let the capacity $c = 1$. Figure 1 depicts the situation, where the upper line represents the queue length at each time period, and the lower line represents the capacity level, which is fixed at 1 in this example. The

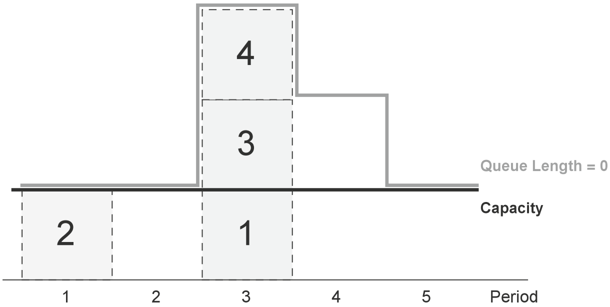

Fig. 1 Illustration of the concepts defined in Definition 1. The capacity level is set at 1, while the line representing the queue length is normalized to 0 when it hits the capacity line

queue lengths q_t and $\tilde{q}_t$ at each time period t can be calculated, resulting in the following.

$$\begin{aligned}\tilde{q}_1 &= 0+1-1=0, q_1 = \max\{0,0\}=0\\ \tilde{q}_2 &= 0+0-1=-1, q_2 = \max\{0,-1\}=0\\ \tilde{q}_3 &= 0+3-1=2, q_3 = \max\{0,2\}=2\\ \tilde{q}_4 &= 2+0-1=1, q_4 = \max\{0,1\}=1\\ \tilde{q}_5 &= 1+0-1=0, q_5 = \max\{0,0\}=0\end{aligned}$$

Using the above information, we can conclude that $t = 1$ is a single terrace, $t = 2$ is a basin, and the interval [3, 5] is a connected terrace.

The following is a characterization of Nash equilibria of the bottleneck game stated using the above terminology. Compared to the usual definition of Nash equilibrium, the following form is much easier to check as one needs to only compare the time-slot and queue-length combinations with some slight adjustments in the queue length (see the second statement of the lemma below).

Lemma 1 *Suppose that the bottleneck game satisfies condition A. A strategy profile τ is a Nash equilibrium if and only if for all $i \in N$,*

1. *for all $t' < \tau_i$, $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t', \max\{\tilde{q}_{t'}(\tau)+1, 0\})$*
2. *for all $t' > \tau_i$,*

(a) $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t', \max\{\tilde{q}_{t'}(\tau), 0\})$ *if $t' \in [t_1, t_2]$, where $[t_1, t_2]$ is a connected terrace at τ such that $\tau^i \in [t_1, t_2]$,*
(b) $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t', \max\{\tilde{q}_{t'}(\tau)+1, 0\})$, *otherwise.*

Proof Let τ be a Nash equilibrium and consider any $i \in N$. If player i chooses a time slot $t' < \tau_i$, by definition of Nash equilibrium, we must have the following inequality:

$$u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t', q_{t'}(t', \tau_{-i}))$$

Now, note that because all players other than i have not switched to a different time slot, $m_t(\tau) = m_t(t', \tau_{-i})$ for all $t < t'$ and $m_{t'}(t', \tau_{-i}) = m_{t'}(\tau) + 1$. This then implies that $q_{t'-1}(\tau) = q_{t'-1}(t', \tau_{-i})$, so that $\tilde{q}_{t'}(t', \tau_{-i}) = \tilde{q}_{t'}(\tau) + 1$. Using the definition of $q_{t'}$, part 1. of the proposition then follows.

Now, suppose $t' > \tau_i$ and that t' and τ_i are contained in the same connected terrace $[t_1, t_2]$ at strategy profile τ. Now, as before $m_t(\tau) = m_t(t', \tau_{-i})$ for all $t < \tau_i$, but now $m_{\tau_i}(t', \tau_{-i}) = m_{\tau_i}(\tau) - 1$, which implies that $\tilde{q}_{\tau_i}(t', \tau_{-i}) = \tilde{q}_{\tau_i}(\tau) - 1$. Now, because τ_i is in the interior of the connected terrace, $q_{\tau_i} > 0$ holds, which implies that $q_{\tau_i}(\tau) = \tilde{q}_{\tau_i}(\tau)$. Moreover, $m_t(\tau) = m_t(t', \tau_{-i})$ for all $t > \tau_i$ with $t \neq t'$. Putting all of these facts together, we have $q_{t'-1}(t', \tau_{-i}) = q_{t'-1}(\tau) - 1$. Thus, we have the following equality:

$$q_{t'}(t', \tau_{-i}) = \tilde{q}_{t'}(t', \tau_{-i}) = q_{t'-1}(\tau) - 1 + m_{t'}(\tau) + 1 = \tilde{q}_{t'}(\tau),$$

if the right-hand side is nonnegative, and is equal to 0 if it is negative. This proves part (a). For part (b), we must instead have $q_{t'-1}(t', \tau_{-i}) = q_{t'-1}(\tau)$ so that by the same logic as part 1, we have the inequality in (b).

It is relatively easy to check following the same argument as above that if the two conditions in the proposition are satisfied, then τ is a Nash equilibrium. □

With IIC, Konishi et al. (1997a) shows that with strict preferences, every Nash equilibrium has the same structure (the same distribution of strategies—the game satisfies anonymity) in their domain. However, in our domain, there may be Nash equilibria with multiple distinct queue structures even under strict preferences.

Example 1 Let $N = \{1, 2, 3, 4\}$ and $\mathcal{T} = \{1, 2, 3, 4\}$ with capacity $c = 1$. Players 1 and 2 are attached to time slots 1 and 2, respectively (that is, preferences are such that they will not choose to move to any other time slot under any circumstance). Players 3 and 4 have the following preferences:

$$u^3(2,0) > u^3(1,0) > u^3(2,1) > u^3(3,0) > u^3(4,0) > u^3(1,1) > \cdots$$
$$u^4(4,0) > u^4(1,1) > u^4(3,0) > u^4(2,1) > u^4(3,1) > \cdots$$

where, throughout this paper, the ellipses denote that slot-queue combinations that do not appear in the above list can be ordered in any way. Also, any other slot-queue combinations such as (1, 0) and (2, 0) not listed above for player 4 can be ordered in any way that does not contradict condition C.

There are two Nash equilibria: $\tau = (\tau_1, \tau_2, \tau_3, \tau_4) = (1, 2, 4, 1)$ and $\tau' = (1, 2, 2, 4)$. The queue vector for τ can be calculated in the following way: $q_1(\tau) = 2 - 1 = 1$, $q_2(\tau) = 1 - 1 + 1 = 1$, $q_3(\tau) = 0 - 1 + 1 = 0$, and $q_4(\tau) = 1 - 1 = 0$, which leads to $q(\tau) = (1, 1, 0, 0)$. To check that τ is a Nash equilibrium, note that we only need to consider players 3 and 4, as players 1 and 2 cannot be improved. Player 3 faces the slot-queue combination of (4, 0). If player 3 moves to slot 3, she faces the slot-queue combination of (3, 1) as the resulting queue at slot 3 still carries the residue from time slot 1. Similarly, player 3 moving to time slot 2 results in (2, 2) and moving to time slot 1 results in (1, 2). None of these are better than (4, 0) for player 3, so player 3 cannot benefit by moving to another time slot. Player 4 faces (1, 1) under τ. The only slot-queue combination that is better is (4, 0), but if player 4 moves to time slot 4, she faces (4, 1), since player 3 is also at time slot 4. Thus, player 4 cannot benefit by moving to another time slot, and thus, τ is a Nash equilibrium.

Now consider τ', whose queue vector is $q(\tau') = (0, 1, 0, 0)$. It is easy to check that under τ', all players face their most preferred slot-queue combination, and it can be shown that τ' is a strong equilibrium (see the next section for details). Note that the queue structure may be different across different Nash equilibria. □

3 Equivalence between Nash and strong equilibria

A **coalitional deviation** from τ is a pair of $(C, \hat{\tau}_C)$ such that (i) $C \neq \emptyset$, and (ii) for all $i \in C$, $u^i(\hat{\tau}) > u^i(\tau)$, where $\hat{\tau} = (\hat{\tau}_C, \tau_{-C})$. A **strong equilibrium** is a strategy profile such that there is no coalitional deviation from τ.[3] By definition, strong equilibria are Nash equilibria, but not all Nash equilibria need not be strong equilibria. Nonetheless, under some special cases, we can show that every Nash equilibria are strong Nash equilibria. This result in our domain is very distinct from the results in Konishi et al. (1997a), since in the latter model, inefficient Nash equilibria are prevalent due to coordination failure.

The first of the domains where we can establish the equivalence between the sets of Nash equilibria and strong equilibria is where each player has the same preferences. We state this property as homogeneity (H), which is formally given below.

Homogeneity (H). For all players $i, j \in N$, $u^i = u^j$.

We then have the following result.

Proposition 1 *Suppose that the bottleneck game satisfies conditions A, C, and H. Then, the set of Nash equilibria coincides with the set of strong equilibria.*

The proof of this result, along with the rest of the results in this section are given in the Appendix. Note that Proposition 1 says nothing about whether a Nash equilibrium exists or not. In fact, it is shown in Sect. 4 that even under the conditions of Proposition 1, a Nash equilibrium may not exist. Proposition 1 sheds light on the difficulties of the existence problem, as it is well-known that a strong equilibrium may not exist as well.

Proposition 1 is a strong statement when a pure strategy Nash equilibrium exists, but when it does not, the equivalence result holds trivially. Because the set of

[3] We use this version of a strong equilibrium, as it is the same as the one used in Konishi et al. (1997a). See the discussion at the end of this section regarding an alternative version.

strategies is a finite set, it is well known that a Nash equilibrium in mixed strategies exists. The next example shows that Proposition 1 does not hold for mixed strategy Nash equilibria, as there exists a mixed strategy Nash equilibrium from which a coalition can deviate.

Example 2 Consider the following example with three players and three time slots with $N = \{1, 2, 3\}$ and $\mathcal{T} = \{1, 2, 3\}$ and with capacity $c = 1$. Suppose that all the players have the same utility function defined by $u(t, q) = v(t) - q$ where $v(1) = 4.4$, $v(2) = 3.5$, and $v(3) = 3$.

First, note that there is no pure strategy Nash equilibrium. To see this, first note at least one player chooses 1 in a Nash equilibrium. Let player 1 be such a player. Without loss of generality, consider strategy profiles where player 2 chooses a time slot that is not later than what player 3 has chosen. There are five possible strategy profiles, each of which is not a Nash equilibrium: (i) at (1, 1, 1), one of the players would move to 3 since $u(3, 0) = 3 > u(1, 2) = 4.4 - 2 = 2.4$; (ii) at (1, 1, 2), player 3 would move to 3 since $u(3, 0) = 3 > u(2, 1) = 3.5 - 1 = 2.5$; (iii) at (1, 1, 3), player 1 or 2 moves to 2 since $u(2, 0) = 3.5 > u(1, 1) = 4.4 - 1 = 3.4$; (iv) at (1, 2, 2), player 2 or 3 moves to 3 since $u(3, 0) = 3 > u(2, 1) = 3.5 - 1 = 2.5$; and (v) at (1, 2, 3), player 3 moves to 1 since $u(1, 1) = 4.4 - 1 = 3.4 > u(3, 0) = 3$. Thus, there is no Nash equilibrium in pure strategies.

Next, we look at mixed strategies. In particular, by assumptions A and H, the bottleneck game is symmetric so that there exists a symmetric Nash equilibrium. By the observation that a pure strategy Nash equilibrium does not exist, the symmetric Nash equilibrium must be in mixed strategies. Let $p = (p_1, p_2, p_3)$ be the mixed strategy that is used by the players. Consider first the case where all time slots are played with positive probability. Then, each player must be indifferent between choosing any of the time slots 1,2, and 3. Because the utility functions are the same for all the players, it suffices to consider just one player. The expected utility when choosing time slot 1 when the other players play the mixed strategy p is given by

$$2.4p_1^2 + 4.4p_2^2 + 4.4p_3^2 + 2\big(3.4p_1p_2 + 3.4p_1p_3 + 4.4p_2p_3\big).$$

The expected utility when choosing time slot 2 is given by

$$2.5p_1^2 + 1.5p_2^2 + 3.5p_3^2 + 2\big(2.5p_1p_2 + 3.5p_1p_3 + 3.5p_2p_3\big),$$

and when choosing time slot 3, the expected utility is given by

$$3p_1^2 + 2p_2^2 + p_3^2 + 2\big(3p_1p_2 + 2p_1p_3 + 2p_2p_3\big).$$

Note that the first expression is bounded above 4.4, the second expression by 3.5, and the third by 3. Moreover, in equilibrium, these three expressions must be equal, so that the expected utilities must be bounded above 3. Moreover, because the probabilities are all positive, the expected utilities must be below 3. Consider then a deviation by N to the pure strategy profile (1, 2, 3). All the players are strictly improved, player 1 has a utility of $u(1, 0) = 4.4$, player 2 with $u(2, 0) = 3.5$, and player 3 with $u(3.0) = 3$, all of which are greater than the expected utility they receive when playing the mixed strategy equilibrium. Thus, the mixed strategy equilibrium is not a strong equilibrium.

Now, consider the case when one strategy is not used in the mixed strategy equilibrium. Slot 1 must be played with positive probability, since slot 1 is the unique best response to any combination of strategies of the other players that do not choose slot 1. We consider only the case $p_3 = 0$, as the logic for $p_2 = 0$ is similar. When $p_3 = 0$, we only need the expected utilities of choosing slots 1 and 2 to be equal. Moreover, the expected utility of choosing slot 2 is strictly below 2.5 since $p_3 = 0$ and $p_2 > 0$. At equilibrium, the expected utilities must be equal so that the expected utilities of the players must be below 2.5. As before, N has a deviation to (1, 2, 3). Thus, this mixed strategy equilibrium is not a strong equilibrium. □

Condition H is undoubtedly strong, as it forbids any form of heterogeneity as well. The next question then involves whether we can obtain a similar existence result while admitting some sort of heterogeneity. The next proposition analyzes this problem indirectly. It shows that if a Nash equilibrium has a certain shape, then it must also be a strong equilibrium.

Proposition 2 *Assume that the bottleneck game satisfies conditions A and C. Suppose that there is a Nash equilibrium τ with a unique connected terrace $[t_1, t_2]$, and $\tilde{q}_t(\tau) < 0$ for all $t \notin [t_1, t_2]$. Then, τ is a strong equilibrium.*

The condition in the previous proposition is satisfied, for example, when each commuter has a same preferred arrival time and would like to arrive as close to that time as possible without being late. This simple setup is used in Arnott et al. (1990), where the resulting queue structure resembles one large connected terrace. In the next section, we introduce a sufficient condition for the existence of a Nash equilibrium that has possible connections to these earlier literature on bottlenecks. The queue structure constructed in showing the existence results consists of one connected terrace, so that the previous result can be used to show that the Nash equilibrium must also be a strong equilibrium.

The above result relies both on the uniqueness of connected terrace and the absence of single terraces in equilibrium. The next example shows that the equivalence result may not hold if the conditions are not satisfied.

Example 3 Let $N = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{T} = \{1, 2, 3, 4, 5\}$ with capacity $c = 1$. Players 1, 2, 3 and 4 are attached to time slots 1, 2, 4, and 5, respectively. Players 5 and 6 have the following preferences, respectively:

$$u^5(1,0) > u^5(2,0) > u^5(4,0) > u^5(5,0) > u^5(1,1) > u^5(2,1) > u^5(4,1) > u^5(5,1) > \cdots$$
$$u^6(4,0) > u^6(5,0) > u^6(1,0) > u^6(2,0) > u^6(4,1) > u^6(5,1) > u^6(1,1) > u^6(2,1) > \cdots$$

In this example, player 5 prefers time slots 1, 2, 4, and 5 in that order, while player 6 prefers time slots 4, 5, 1, and 2 in that order. Both players want to avoid queues, but most importantly, both players want to avoid time slot 3. Therefore, time slot 3 is very unpopular among the players. There are two Nash equilibria: $\tau = (1, 2, 4, 5, 1, 4)$

and $\tau' = (1, 2, 4, 5, 4, 1)$. In these cases $\tilde{q}_3 = 0$. Only τ is a strong equilibrium, as $(\{5, 6\}, (1, 4))$ is a coalition deviation from τ' with player 5 preferring (1, 1) over (4, 1) and player 6 preferring (4, 1) over (1, 1). □

While we noted the importance of the single connected terrace condition, by imposing an additional condition, single-peakedness, we can still obtain an equivalence result between the set of Nash equilibria and the set of strong equilibria. To explain this concept, we first define the concept of an optimal time slot. We say that a time slot $t_i^* \in \mathcal{T}$ is an **optimal slot** for player $i \in N$ if $u^i(t_i^*, 0) > u^i(t, 0)$ for all $t \in \mathcal{T}, t \neq t_i^*$.

Single-Peakedness (SP). For each $i \in N$, there exists an optimal time slot $t_i^* \in \mathcal{T}$, and for all $t' < t < t_i^*$ or $t_i^* < t < t'$, $u^i(t, 0) > u^i(t', 0)$ holds.

Single-peakedness also appears quite frequently in models other than transportation models, such as voting, public goods, and division problems. Note that in the definition, each commuter may have a different optimal slot, but each commuter must have preferences such that absent any queue, a time slot closer to the optimal time slot must be strictly preferred over one that is further away. As long as commuters' preferences satisfy this structure, even if there are multiple connected terraces, as long as there are only basins between any two connected terraces, instead of single terraces, then the Nash equilibrium strategy profile behind this queue structure must be also be a strong equilibrium.

Proposition 3 *Suppose that conditions A, C, and SP are satisfied, and consider a Nash equilibrium τ in which (i) there is no single terrace, and (ii) all the time slots between any pair of connected terraces is contained in a basin. Then, τ is a strong equilibrium.*

If connected terraces are not separated by $\tilde{q}_t < 0$, the equivalence between Nash and strong equilibria need not hold.

Example 4 Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{T} = \{1, 2, 3, 4, 5\}$ with capacity $c = 2$. Players 1, 2, and 3 are attached to time slot 1, and players 4 and 5 are attached to time slot 3. Players 6 and 7 have the following preferences:

$$u^6(3,0) > u^6(3,1) > u^6(2,0) > u^6(1,0) > u^6(4,0) > u^6(3,2) > \cdots$$
$$u^7(2,0) > u^7(3,0) > u^7(3,1) > u^7(4,0) > u^7(2,1) > u^7(1,0) > \cdots$$

There is a Nash equilibrium $\tau = (1, 1, 1, 3, 3, 2, 3)$, but $(C, \hat{\tau}_C) = (\{6, 7\}, (3, 2))$ is a coalitional deviation from τ. The destination profile $\hat{\tau} = (1, 1, 1, 3, 3, 3, 2)$ is a strong equilibrium. In this example, SP is satisfied, but $\tilde{q}_2(\tau) = \tilde{q}_2(\hat{\tau}) = 0$, and the two connected terraces [1, 2] and [3, 4] are not separated by $\tilde{q}_2 < 0$. □

The following example also violates SP, but it shows that we cannot ensure the existence of strong equilibrium even if there exists a Nash equilibrium.

Example 5 Let $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{T} = \{1, 2, 3, 4, 5\}$ with capacity $c = 1$. Players 4 and 5 are attached to time slots 1 and 4, respectively. Players 1,2, and 3 have the following preferences, respectively:

$$u^1(1,1) > u^1(4,1) > u^1(1,2),$$
$$u^2(2,0) > u^2(1,1) > u^2(1,2) > u^2(2,1),$$
$$u^3(3,0) > u^3(1,2) > u^3(4,1) > u^3(3,1),$$

There is only one Nash equilibrium $\tau = (1, 1, 4, 1, 4)$. However, $(C, \hat{\tau}_C) = (\{1, 2, 3\}, (4, 2, 3))$ is a coalitional deviation. Hence, τ is not a strong equilibrium. □

To summarize, we have shown instances where every Nash equilibrium is a strong equilibrium, albeit under strong conditions. In general, this equivalence result need not hold, since it could be possible that even in a Nash equilibrium, a group of players may want to swap time slots. In the case of Konishi et al. (1997a), preventing these types of swaps was sufficient. However, for the bottleneck game analyzed in this paper, there could be a another type of deviation where one player leaves a time slot that decreases the congestion of nearby time slots and creating an opportunity of some other player to choose such time slot. The first deviator can then move to the vacated spot. The propositions presented in this section provide the conditions where this kind of "almost" swapping time slots cannot hold.

Alternatively, one can think of a stronger form of strong equilibrium where for a strategy profile τ, we cannot have a coalition $C \subset N$ such that $u^i(\tau_C, \tau_{-C}) \geq u^i(\tau)$ for all $i \in C$ with strict inequality holding for some $i \in C$. If preferences are such that for each $i \in N$ and each pair $(t, k) \neq (t', k')$, $u^i(t, k) \neq u^i(t', k')$, that is, each player $i \in N$ has strict preferences, then Proposition 1 still holds with this "stronger" version of strong equilibrium. Otherwise, the following example shows that the result may not hold if a player exhibits indifference between two distinct time-queue pairs.

Example 6 Let $N = \{1, 2, 3\}$, $\mathcal{T} = \{1, 2, 3\}$ and $c = 1$. Suppose that preferences of all players is given by the same function $u(t, q) = 3t - 6q$, which induces the following order:

$$u(1,1) < u(2,1) < u(3,1) = u(1,0) < u(2,0) < u(3,0).$$

It can be checked that the strategy profile $\tau = (2, 3, 3)$ is a Nash equilibrium. By Proposition 1, it must also be a strong equilibrium, since the preferences of the players satisfies condition H. However, it is not immune to the weakly improving coalitional deviation outlined in the previous paragraph. Let $C = N$ and $\tau' = (3, 2, 1)$. Note that the following inequalities hold:

$$u^1(\tau) = u(2,0) < u(3,0) = u^1(\tau')$$
$$u^2(\tau) = u(3,1) < u(2,0) = u^2(\tau')$$
$$u^3(\tau) = u(3,1) = u(1,0) = u^3(\tau')$$

Therefore, τ does not satisfy the stronger version of strong equilibrium. □

4 (Non)existence of Nash equilibrium

Unfortunately, even under homogeneity, the existence of (pure-strategy) Nash equilibrium is not guaranteed. In fact, the following simple example shows that there may not be a Nash equilibrium even under H together with SP and another stringent condition, Order Preservation (OP) introduced by Konishi et al. (1997b) that investigates positive externality games (see below).

Order Preservation (OP). For all $i \in N$, all $t, t' \in \mathcal{T}$ and all $k, k' \in \mathbb{Z}_+$,

$$u^i(t,k) \geq u^i(t',k') \iff u^i(t,k+1) \geq u^i(t',k'+1).$$

The following Boundedness (B) condition together with OP enables us a tractable representation of payoff functions.

Boundedness (B) For every $t, t' \in \mathcal{T}$, there exists a nonnegative integer $k_{tt'} \in \mathbb{Z}_+$ such that $u^i(t, k_{tt'}) < u^i(t', 0)$.

Furthermore, we can derive the following variation of the result in Konishi and Fishburn (1996).[4] The proof of this result follows the same steps as Konishi and Fishburn (1996) with a few modifications and can be found in the appendix.

Fact. *Under conditions A, C, B, and OP, utility function u^i has a quasi-linear representation. There is a vector $v^i = (v^i(1), \ldots, v^i(T)) \in \mathbb{R}^{\mathcal{T}}$ such that for all $t, t' \in \mathcal{T}$, and all $k, k' \in \mathbb{Z}_+$,*

$$u^i(t,k) \geq u^i(t',k') \iff v^i(t) - k \geq v^i(t') - k'.$$

While these conditions were sufficient for a Nash equilibrium to exist in Konishi et al. (1997b), the example below shows that these conditions are still not sufficient for a Nash equilibrium to exist. In Example 2, the utility function of the players satisfies conditions A, B, H, OP, and SP. However, a pure strategy Nash equilibrium failed to exist.

[4] Note that this Boundedness condition differs from the one in Konishi and Fishburn (1996). Their Boundedness goes: "For all $t, t' \in \mathcal{T}$, there exists $k_{tt'} \in \mathbb{Z}_+$ such that $u^i(t, k_{tt'}) > u^i(t', 0)$," and the resulting utility representation is $v^i_t + q_t$ (conformity instead of congestion).

Therefore we seek a stronger concept, which we call symmetric single-peakedness (SSP). Symmetric single-peakedness reflects a player who values the trade-off between departing at her optimal slot and the queue-length at a one-to-one ratio. That is, departing k slots later (earlier) than the optimal slot is equivalent to facing an added queue-length of k at her optimal slot. The formal definition is given in the following.

Symmetric single-peakedness (SSP). Suppose that for each $i \in N$, u^i satisfies SP, and let $t_i^* \in \mathcal{T}$ be an optimal slot. Then, player i's payoff function u^i satisfies

$$u^i(t_i^*, k) = u^i(t_i^* \pm k, 0) \quad \text{for all } k \in \mathbb{N} \text{ such that } t_i^* \pm k \in \mathcal{T}.$$

Finally, we show that additionally assuming SSP enables us to establish the existence of a Nash equilibrium. Thus, SSP is a stronger version of SP, which together with the conditions we have introduced so far, gives a sufficient condition for the existence of a Nash equilibrium. Moreover, because we assume condition H, by Proposition 1, this Nash equilibrium must also be a strong equilibrium. Condition SSP is quite strong, so Proposition 1 should also be viewed as a result highlighting the difficulty in deriving an existence result.

Proposition 4 *Under conditions A, C, B, H, SSP, and OP, there exists a Nash equilibrium with pure strategies, which is also a strong equilibrium.*

Before proving the result, we note a class of payoff functions that satisfy the hypotheses of Proposition 4. Let t^* be the optimal time, which is the same for all commuters by H. Then, the utility function u^i below satisfies all the conditions stated in Proposition 4.

$$u^i(t, k) = -|t - t^*| - k.$$

Intuitively, this utility function illustrates that quasi-linear preferences such that the commuter values a time difference from the optimal time the same as an additional unit of the queue.

Proposition 4 is proved by constructing a Nash equilibrium profile using the following procedure. Suppose that each player chooses her most preferred time slot one-by-one, given the choices of the players who have chosen before her. Based on the preferences, there may be ties in the preferences. The tie-breaking rule resolves ties by prioritizing first t^*, then $t^* + 1$ and then $t^* - 1$, then $t^* + 2$, etc. in that order. This process continues until all players have chosen their preferred time slots. The formal procedure given below explicitly describes the actual choices made by the players under this procedure.

Let n' denote the number of players yet to be allocated.

Procedure

Step 1 Set $n' = n$.
Step 2 At slot t^*, put $(c+1)$ players if $n' \geq c+1$, and proceed to Step 3. If $n' < c+1$, put all n' players at slot t^*, and stop.
Step 3 Update n' with $n' - (c+1)$, i.e., $n' \to n' - (c+1)$.
Step 4 Set $\kappa = 1$.
Step 5 While $t^* - \kappa > 0$ and $n' > 0$:

Step 5-1 If $t^* + \kappa \leq T$, then at slot $(t^* + \kappa)$, put $(c-1)$ players whenever possible, and proceed to Step 5-2. If $n' < c-1$, put all of the remaining n' players at slot $(t^* + \kappa)$, and stop. If $t^* + \kappa > T$, then proceed to Step 5-3.

Step 5-2 Update $n' \to n' - (c-1)$.
Step 5-3 At slot $t^* - \kappa$, put $(c+1)$ players whenever possible, and proceed to Step 5-4. Otherwise, put all n' players at slot $(t^* - \kappa)$, and stop.
Step 5-4 Update $n' \to n' - (c+1)$ and $\kappa \to \kappa + 1$.
Step 6 While $n' > 0$:

Step 6-1 If $t^* + \kappa \leq T$, then at slot $(t^* + \kappa)$, put $(c-1)$ players whenever $n' \geq c-1$, and proceed to Step 6-2. If $n' < c-1$, put all of the remaining n' players at slot $(t^* + \kappa)$, and stop. If $t^* + \kappa > T$, then skip to Step 6-3.
Step 6-2 Update $n' \to n' - (c-1)$.
Step 6-3 At slot 1, put one player in slot 1, and proceed to Step 6-4. Otherwise, stop.
Step 6-4 Update $n' \to n' - 1$ and $\kappa \to \kappa + 1$.

When $c = 1$, steps 5-1, 5-2, 6-1 and 6-2 can be skipped since $c - 1 = 0$. Also, note that if there are enough players so that Step 5 in the procedure is implemented, then the procedure allocates $(c+1)$ players to all slots $t \in [2, t^*]$, $(c-1)$ players to all slots $t \in [t^* + 1, t']$ for some $t' > t^*$, and the remaining players to slot 1.

Proof Let τ be a profile resulting from this procedure. If the total number of players $n \leq c+1$, then we have $\tau_i = t^*$, which is trivially a Nash equilibrium. Thus, we consider the case in which $n > c+2$.

(I) Suppose $t^* \geq 2$. There exists at most one connected terrace, which we label as $[t_1, t_2]$ at τ. We consider further two cases: (i) $t_1 > 1$ and (ii) $t_1 = 1$.

(i) When $t_1 > 1$, we have the following:

$$m_t(\tau) = c+1 \quad t \in \left[t_1, t^*\right],$$
$$m_t(\tau) = c-1 \quad t \in \left[t^*+1, t_2-1\right].$$

At this profile, the queue-length vector $q(\tau)$ becomes

$$\begin{aligned} q(\tau) &= \left(q_1, \dots, q_{t_1-1}, q_{t_1}, q_{t_1+1}, \dots, q_{t^*}, q_{t^*+1}, \dots, q_{t_2-1=2t^*-t_1}, q_{t_2}, \cdots\right) \\ &= \left(0, \dots, 0, 1, 2, \dots, t^* - t_1 + 1, t^* - t_1, \dots, 1, 0, \dots\right). \end{aligned} \tag{1}$$

SSP and OP imply

$$u(t_1-1,0)=u(t_1,1)=\cdots=u(t^*,t^*-t_1+1)$$
$$=u(t^*+1,t^*-t_1)=\cdots=u(t_2-1,1)=u(t_2,0).$$

First, note that player i with $\tau_i \in [t_1,t_2]$ cannot improve by departing later in $[t_1,t_2]$, since the queue-length at switched slot, τ_i' is the same as in (1), so player i is indifferent between τ_i and τ_i'.

In addition, these players cannot improve by departing earlier in $[t_1,t_2]$, since the queue-length at switched slot, τ_i', compared to (1), increases by one, so they are worse off by switching to τ_i'.

Next we consider the case when they depart later or earlier out of the connected terrace. At τ_i', they face a queue of length zero or one if $\tau_i'=t_1-1$ or of length zero otherwise. If $\tau_i'=t_1-1$ and $q_{t_1-1}(\tau_i',\tau_{-i})=0$, player i is indifferent between $\tau_i'=t_1-1$ and τ_i. If $\tau_i'=t_1-1$ and $q_{t_1-1}(\tau_i',\tau_{-i})=1$, $\tau_i'=t_1-1$ is worse than τ_i. If $\tau_i'\neq t_1-1, u(\tau_i',0)<u(t_1-1,0)=u(t_2,0)$, they making worse-off.

Player i in slot t_1-1, if any, does not depart earlier than slot t_1-1 or later than slot t_2 by the same logic in the above. Player i also does not switch to $\tau_i\in[t_1,t_2]$, since the queue-length at switched slot, τ_i', compared to (1), increases by one, so they are worse.

Thus, since no player has an incentive to switch their slots in the case (A)-(i), τ is a Nash equilibrium.

(ii) If $t_1=1$, we have the following:

$$\begin{aligned} &m_1(\tau)\geq c+1,\\ &m_t(\tau)=c+1 \quad t\in[2,t^*],\\ &m_t(\tau)=c-1 \quad t\in[t^*+1,t_2-1]. \end{aligned}$$

Depending on the value of n, there may be some player(s) at slot t_2, say

$$m_{t_2}(\tau)=k \quad \text{for some } k\in[0,c-1].$$

Let $m_1(\tau)\equiv q_1^*$. At this profile the queue-length vector $q(\tau)$ becomes

$$\begin{aligned} q(\tau)&=(q_1,q_2,\dots,q_{t^*},q_{t^*+1},\dots,q_{t_2-1},q_{t_2},\dots)\\ &=(q_1^*,q_1^*+1,\dots,q_1^*+(t^*-2+1),q_1^*+(t^*-2+1)-1,\dots,1,0,\dots). \end{aligned}$$

In this case, using a similar argument as in case (A)-(i), no player has an incentive to switch their slots.

(II) Suppose $t^*=1$. This is a variant of the case (A)-(ii), and it is shown that no player has an incentive to switch their slots. □

While we have shown that without SSP, a Nash equilibrium may not exist, we also note that removing OP also jeopardizes the existence of a Nash equilibrium, as the next example shows.

Example 7 Let $N=\{1,2,3,4\}$ and $\mathcal{T}=\{1,2,3,4\}$ with capacity $c=1$. Players have the following preferences.

$$u(2,0) > u(1,0) = u(2,1) = u(3,0) > u(1,1) > u(3,1) > u(2,2) = u(4,0) > \cdots.$$

In this example, H and SSP with optimal time slot $t^* = 2$ are satisfied, while OP is not, since $u(2,0) > u(1,0)$ but $u(2,1) = u(1,0) > u(1,1)$. This example does not admit a pure strategy Nash equilibrium. To see this, first consider four cases: (i) (1, 2, 2, 3) then player 4 moves to 1. (ii) (1, 2, 2, 1) then player 3 moves to 3. (iii) (1, 2, 3, 1) then player 4 moves to 2. (iv) (1, 2, 3, 2) then player 3 moves to 1. Since the queue-structure when (i) and (iv) are the same and H holds, the cycle is started by player 3, and never stops. Moreover, even when stating from an arbitrary profile, the deviation process is finally absorbed to this cycle. □

The assumptions used in Proposition 4 are quite strong, but the previous example shows how essential these conditions are. We can show that a Nash equilibrium exists under a slight relaxation of condition SSP in Proposition 4, which we will call (SSP*). While condition SSP required a one-to-one trade-off between an additional unit of congestion and a time unit away from the optimal time slot, the weakened condition SSP* requires a constant trade-off between one unit of time slot away from the optimal slot and α units of additional congestion, where α is a positive integer.

Symmetric single-peakedness* (SSP*). Suppose that for each $i \in N$, u^i satisfies SP, and let $t_i^* \in \mathcal{T}$ be an optimal slot. Then, for some positive integer α, player i's payoff function u^i satisfies

$$u^i(t_i^*, \alpha k) = u^i(t_i^* \pm k, 0) \quad \text{for all } k \in \mathbb{N} \text{ such that } t_i^* \pm k \in \mathcal{T}.$$

For an example of a utility function that satisfies the above condition SSP* and the other conditions in Proposition 4, consider a utility function given in the following form:

$$u^i(t,k) = -\alpha|t - t^*| - \beta k,$$

where α and β can be interpreted in the same way as the parameters used in the utility (or cost) function of the commuters in Hendrickson and Kocur (1981) or Arnott et al. (1990). Condition SSP requires that $\alpha = \beta = 1$ hold, while condition SSP* requires that $\beta = 1$ and α be a positive integer, or by considering an equivalent representation, that α/β be an integer. We can then show using a similar technique in Proposition 4 that if we replace SSP with SSP*, we still have an existence result for a Nash equilibrium and also a strong equilibrium by condition H and Proposition 1.

Proposition 5 *Under conditions A, C, B, H, SSP*, and OP, there exists a Nash equilibrium in pure strategies.*

Proof In the procedure used in the proof of Proposition 4, we replace $c + 1$ with $c + \alpha$ and with $c - 1$ with $\min\{c - \alpha, 0\}$. More specifically, in Step 2 and in Step 5-3, we place $c + \alpha$ players, whenever possible, in time slots $t \leq t^*$ where t^* is the peak time slot for the players. We place $c-\alpha$ players in time slots $t > t^*$ in Step 5 if $c-\alpha$ is positive and do not place any players there if it is negative.

Moreover, when Step 6-3 is implemented, we put α players at time slot 1 instead of just one player. The resulting strategy profile then induces a queue profile where within a connected terrace, the queue increases by α per time unit until the optimal time slot t^*, from which the queue decreases by α per unit time.

Suppose first that $t^* \geq 2$, and as before, let $[t_1, t_2]$ be the resulting connected terrace formed by running the procedure as outlined in the previous paragraphs. Again, we divide the proof into two cases: (i) $t_1 > 1$ and (ii) $t_1 = 1$.

(i) When $t_1 > 1$, we have the following:

$$m_t(\tau) = c + \alpha \quad t \in [t_1, t^*],$$
$$m_t(\tau) = \min\{c-\alpha, 0\} \quad t \in [t^*+1, t_2-1].$$

At this profile, the queue-length vector $q(\tau)$ becomes

$$\begin{aligned} q(\tau) &= \left(q_1, \dots, q_{t_1-1}, q_{t_1}, q_{t_1+1}, \dots, q_{t^*}, q_{t^*+1}, \dots, q_{t_2-1=2t^*-t_1}, q_{t_2}, \dots\right) \\ &= \left(0, \dots, 0, \alpha, 2\alpha, \dots, \left(t^*-t_1+1\right)\alpha, \left(t^*-t_1\right)\alpha - \min\{c-\alpha, 0\} - c, \dots, q_{t_2}, 0, \dots\right). \end{aligned} \tag{2}$$

Between time slots t_1 and t^*, the queue increases by α in each time slot towards t^*. By condition SSP*, each player i is indifferent between choosing a time slot in this interval and a later one within the same interval, and each player i would be worse off by moving to an earlier time slot within this interval. This logic extends easily to time slots beyond t^* for the case when $c-\alpha \geq 0$, as the queue then decreases by α for each time slot after t^*, and by SSP*, no player can improve by deviating to another time slot. For the case when $c-\alpha < 0$, the queue decreases by c instead of α with $c < \alpha$. Thus, no player from time slots t_1 to t^* has an incentive to deviate to a time slot beyond t^* by condition SSP* and the fact that the queue decreases by a rate c which is less than the level α that the player finds indifferent between choosing a unit time away from t^*. Also, there are no players located in time slots beyond t^*, so we have shown for the first case that no single player can deviate, and the resulting profile is a Nash equilibrium.

(ii) When $t_1 = 1$, the proof is similar as in the first case, except that $q_1 > 0$. We then have the following:

$$m_1(\tau) \geq c+1,$$
$$m_t(\tau) = c + \alpha, \quad t \in \left[2, t^*\right],$$
$$m_t(\tau) = \min\{c-\alpha, 0\}, \quad t \in \left[t^*+1, t_2-1\right].$$

The argument is the same as in (i) with the queue levels increased by q_1.

Finally, suppose $t^* = 1$. This is a variant of the case (ii) above, where it was shown that no player has an incentive to switch their slots. Therefore, the resulting profile is a Nash equilibrium in this case as well. □

The restriction that α/β is an integer is indispensable, as the following example shows that there may not be a Nash equilibrium when α/β is not an integer.

Example 8 Suppose that there are $n = 10$ players, and the capacity of the bottleneck is $c = 1$. For simplicity, assume that $\mathcal{T} = \{1, 2, 3, 4, 5\}$, although the exact upper bound does not affect the following analysis. Suppose that all players have the same utility function with parameters in the utility function given by $\alpha = 3$, $\beta = 2$, and $t^* = 4$. That is, the players' utility function is given by

$$u(t, k) = -3|t - 4| - 2k.$$

Note that conditions B, H, SP, and OP are still satisfied, but $\alpha/\beta = 1.5$, which is not an integer.

Suppose that there is a Nash equilibrium τ, and to keep the notation simple, we suppress the dependence of m_t and q_t on τ in the following when no confusion arises. First, we claim that $m_4 \geq 2$. If not, since $n > m_4$, we must have $q_4 \leq q_t + 1$ for some $t \neq 4$ with $m_t > 0$, but any player i with $\tau_i = t$ can benefit by deviating to time slot 4, which contradicts τ being a Nash equilibrium. Since $m_4 \geq 2$, we have $q_4 \geq 1$, which in turn implies $\tilde{q}_5 \geq 0$.

We must have $m_5 \leq 1$, since otherwise, we would have $q_5 \geq q_4 + 1$, and any player at time slot 5 can benefit by moving to time slot 4.

Observe that if $m_4 > 0$, then the inequalities $q_4 - q_3 \leq 2$, $q_4 - q_2 \leq 4$, and $q_4 - q_1 \leq 5$ must hold, or otherwise, any player i with $\tau_i = 4$ can benefit by switching to time slot 3, 2, or 1, depending on whether the first, second, or third inequality is violated. This immediately implies that $m_4 \leq 3$, and combined with $m_5 \leq 1$, we have $m_1 + m_2 + m_3 \geq 6$, which implies $q_3 > 0$ has to hold. Moreover, when considering the restricted game to time slots $\{1, 2, 3\}$, by the same argument applied to showing that $m_4 > 0$ must hold, we must have $m_3 > 0$ hold as well.

If in addition $m_3 > 0$ and $q_3 > 0$, then $q_4 - q_3 \geq 2$, or else players in time slot 3 can benefit by moving to time slot 4. Therefore, $q_4 - q_3 = 2$ has to hold, which implies $m_4 = 3$. Moreover, we must now have $m_5 = 0$, since with $m_5 = 1$, we have $q_5 = q_4 = q_3 + 2$, and the player at time slot 5 can deviate to time slot 3. Apply the same set of steps to time slot 3 as we did with time slot 4 to conclude that $q_3 - q_2 \leq 2$ holds, which implies $m_3 \leq 3$. Now, $m_1 + m_2 \geq 4$ must hold, which implies $q_2 > 0$ holds. Moreover, with τ being a Nash equilibrium, $m_2 > 0$ must also hold, thus implying $m_3 = 3$ as well.

Next, consider the restricted game with the four remaining players choosing between 1 and 2. The unique Nash equilibrium distribution of this restricted game involves $m_1 = 1$ and $m_2 = 3$. Now, putting the argument together, the only possible Nash equilibrium distribution must be $m_1 = 1, m_2 = m_3 = m_4 = 3$. However, simple calculation shows that $q_1 = 0$ and $q_4 = 6$, so that $q_4 - q_0 = 6 > 5$. Indeed, a player at time slot 4 can benefit by moving to time slot 1. Thus, there is no Nash equilibrium of this game. □

Another extension would be to consider the case when all players view choosing time slots after t^* to be arbitrarily unfavorable, while for retaining the same set of assumptions (H, SSP being applied only to slots $t^* - k \in \mathcal{T}$, and OP) for those time slots t with $t \leq t^*$. These types of preferences are similar in spirit to those made in Rivera et al. (2018) as well, although this condition is not directly comparable as we have not included an explicit departure time out of the bottleneck that is necessary to fully capture the notion of schedule delay. The procedure used in the proof of Proposition 4 can be modified to filling in only time slots at or before t^* and leaving time slots after t^* completely unfilled.

Another relaxation of condition H where we can still obtain an existence result is if there are two types of players with preferences that satisfy SSP and OP that differ in the optimal time slot. Let t_E^* and t_L^* be the two optimal times where $t_E^* < t_L^*$ ("E" for early and "L" for late), and for convenience, we call a player with optimal time t_E^* (resp. t_L^*) an early (resp. a late) player. Given that there are a sufficiently many players, we can show that a Nash equilibrium exists. Moreover, we can show that the strategy profile induces a single connected terrace, which by Proposition 3 implies that the strategy profile is also a strong equilibrium. The proof of the following proposition is rather lengthy and can be found in the Appendix.

Proposition 6 *Assume conditions A, B, SSP, and OP. Suppose that there are two types of players, early and late, whose peaks are t_E^* and t_L^* respectively with $t_E^* < t_L^*$. If $n \geq 2c(t_L^* - 1) + (c + 1)$, then there exists a Nash equilibrium in pure strategies, and this Nash equilibrium is also a strong equilibrium.*

We have found several combinations of conditions that ensure the existence of a Nash equilibrium in pure strategies, and the proof involved an explicit method in constructing a particular Nash equilibrium. This method differs from those such as in Rosenthal (1973) and Konishi et al. (1997b), where the game is shown to be a potential game, as defined in Monderer and Shapley (1996), and a Nash equilibrium can be obtained from maximizing a particular function called the potential function. We now show in the following that the bottleneck game is in almost all cases not a potential game. Moreover, the conditions in Proposition 1 may actually cause the bottleneck game to not be an ordinal potential game, which is a weaker version of a potential game also defined in Monderer and Shapley (1996).

For consistency in notation, we present the definitions of potential games in terms of the bottleneck game. First, the bottleneck game is an (exact) potential game if there exists a real-valued function P on $\mathcal{T}^N$ such that for all $\tau \in \mathcal{T}^N$, $i \in N$, and $t \in \mathcal{T}$,

$$u^i\big(\tau_i, q_{\tau_i}(\tau)\big) - u^i\big(t, q_t(t, \tau_{-i})\big) = P(\tau) - P\big(t, \tau_{-i}\big).$$

The following example shows that bottleneck games with 3 or more players and with 3 or more time slots can never be potential games.

Example 9 Let $N = \{1, 2, 3\}$ and $\mathcal{T} = \{1, 2, 3\}$ and $c = 1$. Consider the sequence of strategy profiles with $\tau^0 = (1, 3, 1)$, $\tau^1 = (2, 3, 1)$, $\tau^2 = (2, 2, 1)$, $\tau^3 = (1, 2, 1)$. If this game were a potential game, then the function P would need to satisfy the following:

$$
\begin{aligned}
P(\tau^1) - P(\tau^0) &= u^1(2,0) - u^1(1,1) \\
P(\tau^2) - P(\tau^1) &= u^2(2,1) - u^2(3,0) \\
P(\tau^3) - P(\tau^2) &= u^1(1,1) - u^1(2,1) \\
P(\tau^0) - P(\tau^3) &= u^2(3,0) - u^2(2,1)
\end{aligned}
$$

Adding both sides of the equations yields 0 on the left-hand side and $u^1(2,0) - u^1(2,1)$ which is positive by condition C, a contradiction. Thus, this game is not a potential game for any utility functions u^1, u^2, and u^3 that satisfy conditions A and C. The example can be extended to games with more than three players by considering strategy profiles where all the other players other than players 1 and 2 choose time slot 1, as the only change in the above analysis is that the second argument in each of the functions u^1 and u^2 increased by the same number that equals the number of players other players 1 and 2 in time slot 1. The extension to a game with more than three time slots is similarly trivial. □

The condition for a potential game may be strong, and thus, we consider whether the bottleneck game is an ordinal potential game, also defined in Monderer and Shapley (1996). A bottleneck game is an ordinal potential game if there exists a real-valued function P on $\mathcal{T}^N$ such that for all $\tau \in \mathcal{T}^N$, $i \in N$, and $t \in \mathcal{T}$,

$$
u^i(\tau_i, q_{\tau_i}(\tau)) - u^i(t, q_t(t, \tau_{-i})) > 0 \Leftrightarrow P(\tau) - P(t, \tau_{-i}) > 0.
$$

The condition states that the signs of the two expressions must be the same. We use Example 9 again to illustrate that condition SSP actually is the reason a game may not be an ordinal potential game.

Example 10 (Example 9 continued) Consider once again the setup as in Example 9 with 3 players and 3 time slots. For concreteness, suppose that the utility function for player 1 and player 2 is given by $u^1(t,q) = u^2(t,q) = -|t-1| - q$. While we do not have to impose any assumptions on the utility function of player 3, we assume that player 3 also has the same utility function, so that this game satisfies the sufficient conditions in Proposition 1. Assume that the game is an ordinal potential game with function P. Using the same sequence of strategy profiles $\tau^0 = (1,3,1), \tau^1 = (2,3,1), \tau^2 = (2,2,1), \tau^3 = (1,2,1)$, the following conditions hold:

$$
\begin{aligned}
P(\tau^1) &= P(\tau^0) \text{ since } && u^1(2,0) = u^1(1,1), \\
P(\tau^2) &= P(\tau^1) \text{ since } && u^2(2,1) = u^2(3,0), \\
P(\tau^3) &> P(\tau^2) \text{ since } && u^1(1,1) > u^1(2,1), \\
P(\tau^0) &= P(\tau^3) \text{ since } && u^2(3,0) = u^2(2,1).
\end{aligned}
$$

The conditions regarding P are contradictory, and thus, the above bottleneck game, which satisfies the conditions in Proposition 1 is not an ordinal potential game. □

5 Bottleneck games with atomless players

When there are a finite number of players, the existence of Nash equilibrium and in the equivalence between Nash and strong equilibria held under special circumstances. The primary reason for the number of negative results may be the asymmetry of the effect of a player deviating to an earlier slot versus deviating to a later slot within a connected terrace (Proposition 1). This is coming from the fact that players are atoms in that deviating players taken into account the change in the queue distribution caused by their deviations. In this section we will consider an idealized game in which players are atomless as in Vickrey (1969). The model is essentially a special case of the bottleneck model introduced in Konishi (2004) where there is only one origin-destination pair and one path. While (Konishi 2004) focused on the uniqueness of equilibria, we focus on the equivalence of Nash and strong equilibria in the atomless case.

The set of players is the interval $I = [0, 1]$ endowed with Lebesgue measure λ. There is a finite set of alternatives $\mathcal{T} = \{1, \dots, T\}$. A strategy profile is a measurable function $\tau : I \to \mathcal{T}$, and we use the notation τ_i in place of $\tau(i)$ to denote the strategy of player i in τ. We assume anonymity (A) in the sense of Schmeidler (1973), where each player's payoff function depends on her strategy and the overall distribution of the strategies. Formally, for each strategy profile τ, let $\mu(\tau)$ be a T-dimensional vector $\mu(\tau) = (\mu_1(\tau), \mu_2(\tau), \dots, \mu_T(\tau))$, where $\mu_t(\tau) = \lambda(\{i \in I : \tau(i) = t\})$ for each $t \in \mathcal{T}$. Given a strategy profile τ, define the queue at a slot t resulting from τ by $\tilde{q}_t(\tau) = q_{t-1}(\tau) + \mu_t(\tau) - c$, and $q_t(\tau) = \max\{\tilde{q}_t(\tau), 0\}$. To keep the same notation as the finite case, we assume that u^i depends on τ only through the time slot chosen by i, τ_i, and the queue that i faces at that slot, denoted by q_{τ_i}. As before, u^i is a function on $\mathcal{T} \times \mathbb{R}_+$. Notice u^i then depends on τ only through τ_i and $(\mu_t(\tau))_{t \in \mathcal{T}}$, which is a restatement of Anonymity (A).

Under the atomless player assumption, we will assume Schmeidler's technical assumption.

Regularity (R) (Schmeidler 1973). (i) For all $i \in I$, and all $t \in \mathcal{T}$, $u^i(t, \cdot)$ is continuous. Thus, all utility functions are uniformly bounded and there exists a positive constant K such that $|u^i(t, q)| < K$ for all $i \in I$, $t \in \mathcal{T}$, and $q \in \mathbb{R}_+$. (ii) For all $t, t' \in \mathcal{T}$ and $q \in \mathbb{R}_+$, the set $\{i \in I : u^i(t, q) > u^i(t', q)\}$ is measurable.

In this current context, we define the concept of Nash equilibrium as follows. A strategy profile τ is a **Nash equilibrium** if for almost all $i \in I$ and every $t \in \mathcal{T}$, $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t, q_t(\tau))$.

Proposition (Schmeidler 1973). *Under A and R, there exists a Nash equilibrium in pure strategies.*

A strategy profile is a **strong equilibrium** if there is no measurable subset $C \subset I$ with $\lambda(C) > 0$ and a strategy profile $\hat{\tau}$ of players in C such that $u^i(\hat{\tau}_i, q_{\hat{\tau}_i}(\hat{\tau})) > u^i(\tau_i, q_{\tau_i}(\tau))$ almost everywhere on C, where $\hat{\tau} = ((\hat{\tau}_i)_{i \in C}, (\tau_i)_{i \notin C})$.[5] We will impose the following congestion condition.

[5] The definition of strong equilibrium is from Konishi et al. (1997a), which is defined to be an analogue of the one by Aumann (1959) for games with atomless players. The same terminology also appears in Cominetti et al. (2015), but what they call strong equilibrium is equivalent to that of Nash equilibrium.

Congestion (C) $u^i(t, q_t)$ is strictly decreasing in q_t for all $t \in \mathcal{T}$ and all $q_t \in \mathbb{R}_+$.

Recall that q_t depends on q_{t-1} so that for bottleneck games, player i's payoffs not only depend on her choice and the number of players choosing the same strategy as her as is the case for congestion games, but also on the overall distribution of the players' strategy choices. Nonetheless, we can show that a similar equivalence result with Nash and strong equilibria as Konishi et al. (1997a) for atomless congestion games. Note that unlike Proposition 1, condition H is not needed here, and unlike Propositions 2, and 3, no condition on the structure of equilibria is needed as well.

Proposition 7 *Consider an atomless game. Under conditions A, C, and R, the set of Nash equilibria coincides with the set of strong equilibria.*

Proof Suppose that τ is a Nash equilibrium while it is not a strong equilibrium. Then, there exist a coalition C with $\lambda(C) > 0$ and a strategy profile $\hat{\tau}$ for C such that $u^i(\hat{\tau}_i, q_{\hat{\tau}_i}(\hat{\tau})) > u^i(\tau_i, q_{\tau_i}(\tau))$, where $\hat{\tau} = ((\hat{\tau}_i)_{i \in C}, (\tau_i)_{i \notin C})$. Consider the set $C' = \left\{ i \in C : \tilde{q}_{\hat{\tau}_i}(\tau) \leq 0 \right\}$. Note that $\lambda(C') = 0$ must hold. Indeed, for any $i \in C'$ except on some set of Lebesgue measure zero, we have the following inequality:

$$u^i\left(\hat{\tau}_i, q_{\hat{\tau}_i}(\tau)\right) \geq u^i\left(\hat{\tau}_i, q_{\hat{\tau}_i}(\hat{\tau})\right) > u^i\left(\tau_i, q_{\tau_i}(\tau)\right).$$

Thus, if $\lambda(C') > 0$, we have a contradiction to τ's being a Nash equilibrium.

Assume now that there is a time slot t with $q_t(\tau) > 0$ and $q_t(\hat{\tau}) > q_t(\tau)$. Take the earliest time slot of this kind t. Then, the set $C \cap \left\{ i' \in N : \hat{\tau}_{i'} = t \right\}$ must have positive measure, or otherwise the inequality $q_t(\hat{\tau}) > q_t(\tau)$ cannot hold. Let i be any player in this set. Since τ is a Nash equilibrium and by conditions A and C, for all such i, $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t, q_t(\tau)) > u^i(t, q_t(\hat{\tau})) = u^i(\hat{\tau}_i, q_t(\hat{\tau}))$ must hold. This contradicts that $\hat{\tau}$ is a deviation by C. This immediately implies $q_t(\hat{\tau}) \leq q_t(\tau)$ for t with $q_t(\tau) > 0$.

Suppose that for some t, $q_t(\hat{\tau}) < q_t(\tau)$. Because the set of players who deviate to a slot with $\tilde{q}_t(\tau) \leq 0$ has measure zero and the set of time slots is finite, there must exist a time slot t' with $q_{t'}(\hat{\tau}) \geq q_{t'}(\tau) > 0$ such that the set $\{i' \in C : \hat{\tau}_{i'} = t'\}$ has positive measure. For all such i in this set, $u^i(\hat{\tau}_i, q_{\hat{\tau}_i}(\tau)) \geq u^i(\hat{\tau}_i, q_{\hat{\tau}_i}(\hat{\tau}))$, which contradicts the condition for a deviation for coalition C. Therefore, we must have for all t with $q_t(\tau) > 0$, $q_t(\hat{\tau}) = q_t(\tau)$. Consequently, we must have the following hold for almost all $i \in C \setminus C'$:

$$u^i\left(\hat{\tau}_i, q_{\hat{\tau}_i}(\tau)\right) = u^i\left(\hat{\tau}_i, q_{\hat{\tau}_i}(\hat{\tau})\right) > u^i\left(\tau_i, q_{\hat{\tau}_i}(\tau)\right).$$

Recalling that $\lambda(C) > 0$ and $\lambda(C') = 0$, we must have $\lambda(C \setminus C') > 0$, which implies that the above inequality holds for a set of players with positive measure. This contradicts the fact that τ is a Nash equilibrium. □

6 Concluding remarks

We have investigated the bottleneck games with finite players and atomless players. Although the bottleneck game is a natural extension of congestion game by Milchtaich (1996) and Konishi et al. (1997a), the results of these two games differ from each other in the finite case. Somewhat surprisingly, the presence/absence of single-terraces (time slots that are chosen by the same number of players as the capacities) can alter the structure of the equilibria of the bottleneck game. This is because there is an asymmetry between an increase and a reduction in population at single-terraces: the former reduces payoffs while the latter has no effect on them. In contrast, in an atomless bottleneck game, we need essentially no condition for the result. There is no such asymmetry: players can simply choose the most preferable time slot given the queue structure without affecting the queues. This is why we can recover the nice equivalence result between Nash and strong equilibria as in Konishi et al. (1997a).

Thus, whether the traffic bottleneck model started by Vickrey (1969) would provide us useful insights or not depends on how we interpret the "atomless" assumption of the model. If we accept this assumption as an reasonable approximation of the real world, we can enjoy nice properties and rich results of the model. However, if we question the legitimacy of atomless players, then we need to suffer from the ill-behaved model coming from finite problems.

In this paper, we have focused on a very simple model with one bottleneck. Despite this simplicity, we have seen rather negative results in the discrete case, while we do have some positive results for the atomless case. In reality, however, more complex networks of roads with possibly many bottlenecks do exist. Although our results for the simple one-bottleneck model suggest the difficulty of extending the results to a more general network, one interesting direction would be whether we can derive similar results in such a more complex network, as is considered in Konishi (2004) and in other models of congestion networks. We conjecture that for the case in which there is one origin-destination pair with multiple routes that connect them, Proposition 1 holds.

Appendix: proofs omitted from the main text

Proofs of results from Sect. 3

Before proving Propositions 1, 2, and 3, we note some conditions that a coalitional deviation needs to satisfy when deviating from a Nash equilibrium. The following lemmas establish these properties that will be used to proving the equivalence results presented in this section. The first lemma establishes that in order a strategy profile τ' be part of a coalitional deviation from τ, the queue length at each time slot in τ' must weakly decrease relative to τ. The formal statement is given below.

Lemma 2 *Assume that the bottleneck game satisfies conditions A and C. Suppose that is a Nash equilibrium, and that* τ $(C, \hat{\tau}_C)$ *is a coalitional deviation from* τ*. Then,* $q_t(\hat{\tau}) \leq q_t(\tau)$ *for all* $t \in \mathcal{T}$*, where* $\hat{\tau} = (\hat{\tau}_C, \tau_{-C})$*.*

Proof Suppose not. Then, there exists at least one slot t such that

$$q_t(\hat{\tau}) > q_t(\tau). \tag{3}$$

If multiple slots are found, take the earliest such slot. Since the queue-length at slot t strictly increases, there must be at least one player who deviates to slot t at $\hat{\tau}$, i.e., $m_t(\hat{\tau}) > m_t(\tau)$. Then, we can find at least one member i of C who deviates to $\hat{\tau}_i = t$ from $\tau_i \neq t$ since only the members of C can change their strategies. Since the deviation is strictly improving, it must hold that

$$u^i\big(t, q_t(\hat{\tau})\big) > u^i\big(\tau_i, q_{\tau_i}(\tau)\big). \tag{4}$$

Note that (3) can be rewritten as

$$q_t(\hat{\tau}) \geq q_t(\tau) + 1 > q_t(\tau),$$

and by condition C we have

$$u^i\big(t, q_t(\hat{\tau})\big) \leq u^i\big(t, q_t(\tau) + 1\big) < u^i\big(t, q_t(\tau)\big).$$

Together with (4), we have

$$u^i(\tau_i, q_{\tau_i}(\tau)) < u^i(t, q_t(\tau) + 1).$$

This shows that under τ, player i could have switched to slot t and obtained a higher payoff. This contradicts that τ is a Nash equilibrium. □

The next lemma shows the limitations of coalitional deviations from a Nash equilibrium. In particular, if a coalitional deviation from a Nash equilibrium were to exist, there cannot be a member of that coalition that deviates to a slot that where the capacity is not binding.

Lemma 3 *Assume that the bottleneck game satisfies conditions A and C. Suppose that* τ *is a Nash equilibrium, and that* $(C, \hat{\tau}_C)$ *is a coalitional deviation from* τ*. Then, no member of C deviates to slots t such that* $\tilde{q}_t(\tau) < 0$*.*

Proof Suppose not. Then, there exists at least one member $i \in C$ such that $\tilde{q}_t(\tau) < 0$ with $t = \hat{\tau}_i$. Letting $\hat{\tau} = (\hat{\tau}_C, \tau_{-C})$, we consider two cases:

$$\text{(i)} \quad \tilde{q}_t(\tau) < \tilde{q}_t(\tau) + 1 = \tilde{q}_t\big(t, \tau_{-i}\big) \leq \tilde{q}_t(\hat{\tau}) \leq 0,$$
$$\text{(ii)} \quad \tilde{q}_t(\tau) < \tilde{q}_t(\tau) + 1 = \tilde{q}_t\big(t, \tau_{-i}\big) \leq 0 < \tilde{q}_t(\hat{\tau}).$$

Since the deviation is strictly improving, it must follow that

$$u^i\big(t, q_t(\hat{\tau})\big) > u^i\big(\tau_i, q_{\tau_i}(\tau)\big).$$

In case (i), noting $q_t(\tau) = q_t(t, \tau_{-i}) = q_t(\hat{\tau}) = 0$, we obtain

$$u^i\big(t, q_t\big(t, \tau_{-i}\big)\big) = u^i\big(t, q_t(\hat{\tau})\big) > u^i(\tau_i, q_{\tau_i}(\tau)).$$

This shows that under τ, player i could have switched to slot t and obtained higher payoff. This contradicts that τ is a Nash equilibrium.

In case (ii), we immediately obtain

$$0 = q_t(\tau) < q_t(\hat{\tau}),$$

contradicting Lemma 2. □

In proving Proposition 1, we prove that for a coalitional deviation $(C, \hat{\tau}_C)$, for each connected terrace $[t_1, t_2]$ at τ, the number of players choosing a slot in $[t_1, t_2]$ in τ does not change after the deviation at $\hat{\tau}$ as well (Lemma 6). These connected terraces play a similar role to the partition induced by looking at the set of players choosing an action, as in Konishi et al. (1997a). While looking at the time slots, the distribution of players with respect to each time slot may change in a coalitional deviation, when looking at a connected terraces at τ as one large unit, we can show that the distribution of players with respect to the connected terraces are unchanged. The proof is much more involved compared to that in Konishi et al. (1997a), since in congestion games, it is trivial to show the equivalence between Nash and strong equilibria if players are homogeneous since swapping strategies among members in a given coalition cannot improve all the members in that coalition.

Formally, for a time interval $[t_1, t_2]$ and strategy profile τ, with a slight abuse of notation, define $m([t_1, t_2], \tau)$ as the number of players in C choosing a time slot in $[t_1, t_2]$, which is given by

$$m\big(\big[t_1, t_2\big], \tau\big) = \Big|\big\{i \in C \,:\, \tau_i \in [t_1, t_2]\big\}\Big|.$$

Note that since in a coalitional deviation, players outside of C do not change their strategies, we only need to count the number of players in C. Our first goal is to prove that $m([t_1, t_2], \tau) = m([t_1, t_2], \hat{\tau})$ holds, and we do so by showing first that the left-hand side is at least as large as the right-hand side. Note that the following lemmas do not use condition H.

Lemma 4 *Assume conditions A and C. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ . Then, for any connected terrace $[t_1, t_2]$ at τ, we have*

$$m\big(\big[t_1, t_2\big], \tau\big) \geq m\big(\big[t_1, t_2\big], \hat{\tau}\big).$$

Proof Suppose, by way of contradiction, that there exists a connected terrace at τ, $[t_1, t_2]$ such that $m([t_1, t_2], \tau) < m([t_1, t_2], \hat{\tau})$. We consider two cases: (i) $\tilde{q}_{t_2}(\tau) = 0$, and (ii) $\tilde{q}_{t_2}(\tau) < 0$.

In case (i), we have

$$\tilde{q}_{t_2}(\hat{\tau}) \geq \sum_{t=t_1}^{t_2} m_t(\hat{\tau}) - c\big(t_2 - t_1 + 1\big) > \sum_{t=t_1}^{t_2} m_t(\tau) - c\big(t_2 - t_1 + 1\big) = \tilde{q}_{t_2}(\tau) = 0,$$

contradicting Lemma 2. In case (ii), by Lemma 3, $\left|\{i \in C : \hat{\tau}_i = t_2\}\right| = 0$ holds. Thus, we have $m([t_1, t_2 - 1], \tau) < m([t_1, t_2 - 1], \hat{\tau})$, implying

$$\tilde{q}_{t_2-1}(\hat{\tau}) \geq \sum_{t=t_1}^{t_2-1} m_t(\hat{\tau}) - c\big(t_2 - t_1 + 1\big) > \sum_{t=t_1}^{t_2-1} m_t(\tau) - c\big(t_2 - t_1 + 1\big) = \tilde{q}_{t_2-1}(\tau).$$

This is a contradiction with Lemma 2. Thus, $m([t_1, t_2], \tau) \geq m([t_1, t_2], \hat{\tau})$ must hold. □

While, Lemma 3 showed that no player in a deviating coalition can move to a time slot that was previously a basin, the following lemma shows that no player in a deviating coalition that was in a basin or the tail slot of a connected terrace where the capacity is not binding can move to a slot that was part of a connected terrace.

Lemma 5 *Assume conditions A and C. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ . Then, there does not exist $i \in C$ such that $\tilde{q}_{\tau_i}(\tau) < 0$ and $\hat{\tau}_j \in [t_1, t_2]$, where $[t_1, t_2]$ is a connected terrace at τ.*

Proof Suppose not. Then, there exists at least one $j \in C$ such that $\tilde{q}_{\tau_j}(\tau) < 0$ and $\hat{\tau}_j \in [t_1, t_2]$.

Note that no player $i \in C$ can deviate at $\hat{\tau}$ to the slots which is a basin at τ.

Due to the addition of this player j to $\hat{\tau}_j \in [t_1, t_2]$ and the finiteness of the number of connected terraces at any profile, we can find some connected terrace at τ, $[t'_1, t'_2]$ with $t'_1 \leq t'_2$ such that

$$m\big([t'_1, t'_2], \tau\big) < m\big([t'_1, t'_2], \hat{\tau}\big).$$

However, this contradicts Lemma 4. □

Now, we can show that the inequality in Lemma 4 must actually hold with equality.

Lemma 6 *Assume conditions A and C. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ . Then, for any connected terrace $[t_1, t_2]$ at τ with $t_1 \leq t_2$, we have*

$$m\big([t_1, t_2], \tau\big) = m\big([t_1, t_2], \hat{\tau}\big).$$

Proof Suppose not. By Lemma 4, for some connected terrace at τ, $[t_1, t_2]$ with $t_1 \leq t_2$, we have

$$m([t_1, t_2], \tau) > m([t_1, t_2], \hat{\tau}).$$

Note that from Lemma 3 no player involving an improving coalitional deviation takes the slots that is a basin at τ or at τ'.

Due to the finiteness of the number of players in C, $|C|$, we can find another connected terrace at τ, $[t'_1, t'_2]$ with $t'_1 \leq t'_2$ such that

$$m([t'_1, t'_2], \tau) < m([t'_1, t'_2], \hat{\tau}).$$

Again, this contradicts Lemma 4. □

We now prove Proposition 1. Suppose that τ is a Nash equilibrium, and that $(C, \hat{\tau}_C)$ is a coalitional deviation from τ. We will derive a contradiction through the following steps.

Step 1. Find $t \in \mathcal{T}$ such that $q_t(\hat{\tau}) < q_t(\tau)$. If there exist multiple such slots, take the earliest one. Denote by $[\underline{t}, \bar{t}]$ the connected terrace where t belongs. Note that some player $i \in C$ switches to $\hat{\tau}_i \notin [\underline{t}, \bar{t}]$ at $\hat{\tau}$.

Step 2. Find a player who deviates to a slot in $[\underline{t}, \bar{t}]$ at $\hat{\tau}$. By Lemma 6, there must be at least one such player. Among these players, let the player who chooses the latest slot at $\hat{\tau}$ be player $j \in C$. Note that player j chooses τ_j at τ which does not belong to $[\underline{t}, \bar{t}]$, say $[\underline{t}', \bar{t}']$. That is, player j chooses $\tau_j \in [\underline{t}', \bar{t}']$ at τ and $\hat{\tau}_j \in [\underline{t}, \bar{t}]$ at $\hat{\tau}$.

Step 3. Find a player who deviates to a slot in $[\underline{t}', \bar{t}']$ at $\hat{\tau}$, and name player k the one among such players who chooses the latest slot at $\hat{\tau}$. Likewise in Step 2, such player must be found due to player j's deviation from $[\underline{t}', \bar{t}']$. Let player k choose $\tau_k \in [\underline{t}'', \bar{t}''] \neq [\underline{t}', \bar{t}']$. That is, player k chooses $\tau_k \in [\underline{t}'', \bar{t}'']$ at τ and $\hat{\tau}_k \in [\underline{t}', \bar{t}']$ at $\hat{\tau}$.

Step 4. In this sequence of terraces, by finiteness in the number of connected terraces at τ, there must be a cycle; that is, there exists a player who deviated from a connected terrace we have identified earlier.

Let $[\underline{t}^{(1)}, \bar{t}^{(1)}], [\underline{t}^{(2)}, \bar{t}^{(2)}], \ldots, [\underline{t}^{(k)}, \bar{t}^{(k)}]$ be a cycle of connected terraces, where $[\underline{t}^{(k+1)}, \bar{t}^{(k+1)}] \equiv [\underline{t}^{(1)}, \bar{t}^{(1)}]$. Moreover, denote, by $i(1), i(2), \ldots, i(k) \in C$ with $k \in \mathbb{N}$ and $i(k+1) \equiv i(1)$, the player who takes $\tau_{i(l)} \in [\underline{t}^{(l)}, \bar{t}^{(l)}]$ at τ and $\hat{\tau}_{i(l)} \in [\underline{t}^{(l+1)}, \bar{t}^{(l+1)}]$ at $\hat{\tau}$ for $l = 1, \ldots, k$.

Since the payoffs of players $i(1), i(2), \ldots, i(k)$ must improve under the deviation,

$$u^{i(l)}\left(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})\right) > u^{i(l)}\left(\tau_{i(l)}, q_{\tau_{i(l)}}(\tau)\right) \tag{5}$$

for all $l = 1, \ldots, k$.

From Proposition 1, since τ is a Nash equilibrium, for all $l = 1, \ldots, k$,

$$u^{i(l)}\left(\tau_{i(l)}, q_{\tau_{i(l)}}(\tau)\right) \geq u^{i(l)}\left(t, q_t(t, \tau_{-i(l)})\right)$$

for all $t \in [\underline{t}^{(l)}, \bar{t}^{(l)}] \setminus \{\tau_{i(l)}\}$. Specifically, for $i(l+1)$-th player and $\hat{\tau}_{i(l)} \in [\underline{t}^{(l+1)}, \bar{t}^{(l+1)}]$,

$$u^{i(l+1)}\left(\tau_{i(l+1)}, q_{\tau_{i(l+1)}}(\tau)\right) \geq u^{i(l+1)}\left(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)})\right). \tag{6}$$

We would see that $q_{\hat{\tau}_{i(l)}}(\tau) = q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)}) \leq q_{\hat{\tau}_{i(l)}}(\hat{\tau})$.

Thus,

$$u^{i(l+1)}\Big(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)})\Big) \geq u^{i(l+1)}\Big(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})\Big). \tag{7}$$

Note that by H,

$$u^{i(l+1)}\Big(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})\Big) = u^{i(l)}\Big(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})\Big). \tag{8}$$

Hence, from (5), (6), (7) and (8), we obtain

$$\begin{aligned} u^{i(l+1)}\Big(\tau_{i(l+1)}, q_{\tau_{i(l+1)}}(\tau)\Big) &\geq u^{i(l+1)}\Big(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau}_{i(l)}, \tau_{-i(l+1)})\Big) \\ &\geq u^{i(l+1)}\Big(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})\Big) \\ &= u^{i(l)}\Big(\hat{\tau}_{i(l)}, q_{\hat{\tau}_{i(l)}}(\hat{\tau})\Big) \\ &> u^{i(l)}\Big(\tau_{i(l)}, q_{\tau_{i(l)}}(\tau)\Big). \end{aligned}$$

However, this yields a cycle on the preference:

$$\begin{aligned} u^{i(1)}\Big(\tau_{i(1)}, q_{\tau_{i(1)}}(\tau)\Big) &< u^{i(1)}\Big(\hat{\tau}_{i(1)}, q_{\hat{\tau}_{i(1)}}(\hat{\tau})\Big) \\ &< u^{i(2)}\Big(\tau_{i(2)}, q_{\tau_{i(2)}}(\tau)\Big) \\ &< u^{i(2)}\Big(\hat{\tau}_{i(2)}, q_{\hat{\tau}_{i(2)}}(\hat{\tau})\Big) \\ &\;\vdots \\ &< u^{i(k)}\Big(\tau_{i(k)}, q_{\tau_{i(k)}}(\tau)\Big) \\ &< u^{i(k)}\Big(\hat{\tau}_{i(k)}, q_{\hat{\tau}_{i(k)}}(\hat{\tau})\Big) \\ &< u^{i(k+1)}\Big(\tau_{i(k+1)}, q_{\tau_{i(k+1)}}(\tau)\Big) \\ &= u^{i(1)}\Big(\tau_{i(1)}, q_{\tau_{i(1)}}(\tau)\Big), \end{aligned}$$

which is a contradiction. □

Note that condition H was used only in the last step, where the cycle induced by the deviating players also induces, by H, a cycle of strict preference relations, which by transitivity of preferences yields a contradiction.

We now prove Proposition 2. Suppose that there is a coalitional deviation $(C, \hat{\tau}_C)$. By Lemma 3, all members of C choose time slots in $[t_1, t_2]$ under τ, and no member of C will not go out of $[t_1, t_2]$ under $\hat{\tau} = (\hat{\tau}_C, \tau_{-C})$.

Denote by $\bar{t}$, $\hat{t}$ the last slots which coalition members choose at τ, $\hat{\tau}$, respectively, i.e., $\bar{t} = \max\{\tau_i : i \in C\}$ and $\hat{t} = \max\{\hat{\tau}_j : j \in C\}$. We consider two cases: (i) $\hat{t} < \bar{t}$ and (ii) $\hat{t} \geq \bar{t}$.

In case (i), noting that $\left|\left\{i \in C : \tau_i \in [t_1, \hat{t}]\right\}\right| \leq |C| - 1$ and $\left|\left\{i \in C : \hat{\tau}_i \in [t_1, \hat{t}]\right\}\right| = |C|$, we have $\Delta q_{\hat{t}} := q_{\hat{t}}(\hat{\tau}) - q_{\hat{t}}(\tau) \geq 1$, since all slots in $[t_1, \bar{t}]$ belong to connected terrace $[t_1, t_2]$. However, this contradicts Lemma 2.

Then, we consider case (ii). First we have $\Delta q_{\hat{t}} = 0$, i.e., $\tilde{q}_{\hat{t}}(\tau) = \tilde{q}_{\hat{t}}(\hat{\tau})$, since $\left|\left\{i \in C : \tau_i \in [t_1, \hat{t}]\right\}\right| = \left|\left\{i \in C : \hat{\tau}_i \in [t_1, \hat{t}]\right\}\right| = |C|$ by the definitions of $\bar{t}$ and $\hat{t}$, and all slots in $[t_1, \hat{t}]$ belong to connected terrace $[t_1, t_2]$. Moreover, the deviation is strictly improving, and there must be member j of C such that $\hat{\tau}_j = \hat{t}$ and $\tau_j \neq \hat{t}$, which implies $\tau_j < \hat{\tau}_j = \hat{t}$. That is, player j delayed her departure time. These suggest that she could have done that under τ as well. This is a contradiction with τ's being a Nash equilibrium. □

Finally, we prove Proposition 3. Suppose that there is a Nash equilibrium τ consisting of K connected terraces $[t_1, t_2], \ldots, [t_{2k-1}, t_{2k}], \ldots, [t_{2K-1}, t_{2K}]$ each with $\tilde{q}_t < 0$ for all $t \in (t_{2k}, t_{2k+1})$ and $k = 1, 2, \ldots, K$. Focus on the k-th connected terrace $[t_{2k-1}, t_{2k}]$. Since τ is a Nash equilibrium, any player i with $\tau_i \in [t_{2k-1}, t_{2k}]$ satisfies $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t_{2k-1} - 1, 0)$ and $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t_{2k} + 1, 0)$. From the SP assumption, the optimal time slot of each player i with $\tau_i \in [t_{2k-1}, t_{2k}]$ must also be in that same set. If not, then the optimal time slot would be outside the connected terrace, and in the case where the optimal time slot is earlier than t_{2k-1}, player i would prefer to move to $t_{2k-1} - 1$, since by conditions SP and C,

$$u^i\left(t_{2k-1} - 1, 0\right) > u^i\left(\tau_i, 0\right) \geq u^i\left(\tau_i, q_{\tau_i}(\tau)\right),$$

which contradicts τ being a Nash equilibrium. A similar logic shows that the optimal time slot cannot be later than t_{2k}. By SP and the fact that τ is a Nash equilibrium, we have $u^i(\tau_i, q_{\tau_i}(\tau)) \geq u^i(t, 0)$ for all $t \leq t_{2k-1} - 1$ and $t \geq t_{2k} + 1$. Thus, for any coalitional deviation from τ, $(C, \hat{\tau}_C)$, if $i \in C$ with $\tau_i \in [t_{2k-1}, t_{2k}]$ then $\hat{\tau}_i \in [t_{2k-1}, t_{2k}]$ must hold. However, from the proof of Proposition 2, there cannot be such a coalitional deviation that involves all players in C moving within the same connected terrace at τ. Therefore, no coalitional deviation from τ can exist, and τ is a strong equilibrium. □

Proof of the quasi-linear representation

In this section, we show that conditions A, B, C, and OP imply that the utility function u^i has a quasi-linear representation. The proof follows the same lines as Konishi and Fishburn (1996) with very small changes. For simplicity, suppose that for $t \neq t'$, $u^i(t, 0) \neq u^i(t', 0)$. The case where such indifference between two distinct slots is allowed can be resolved in a similar manner as in Konishi and Fishburn (1996).

Without loss of generality, relabel the set of time slots so that $u^i(0, 0) > u^i(1, 0) > \cdots > u^i(T, 0)$.

For any two time slots t and t' with $t < t'$, by condition B and C, there exists a number $k_{tt'}$ such that the following inequalities hold:

$$u^i\left(t, k_{tt'}\right) < u^i(t', 0) < u^i\left(t, k_{tt'} - 1\right).$$

Then, we have the following lemma, which is the analogue of Lemma 2 in Konishi and Fishburn (1996).

Lemma 7 *Assume conditions A, C, B, and OP. If* $1 \leq t < t' \leq T$, *then*

$$k_{tt'} + k_{t't\varepsilon} \geq k_{tt''} \geq k_{tt'} + k_{t't''} - 1$$

Proof By definition of $k_{tt'}$, $u^i(t, k_{tt'}) < u^i(t', 0)$. By OP, we must have $u^i(t, k_{tt'} + k_{t't''}) < u^i(t', k_{t't''})$. Meanwhile, the definition of $k_{t't''}$ implies that $u^i(t', k_{t't''}) < u^i(t'', 0)$. Putting these inequalities together, we have

$$u^i\left(t, k_{tt'} + k_{t't''}\right) < u^i\left(t'', 0\right).$$

Then, by definition of $k_{tt''}$, we must have the following inequality:

$$k_{tt'} + k_{t't''} \geq k_{tt''}.$$

On the other hand, by definition of $k_{tt''}$, $u^i(t, k_{tt''}) < u^i(t'', 0) < u^i(t, k_{tt''} - 1)$ holds. The definition of $k_{t't''}$ implies $u^i(t'', 0) < u^i(t', k_{t't''} - 1)$, while the definition of $k_{tt'}$ implies $u^i(t', 0) < u^i(t, k_{tt'} - 1)$. Using OP, the second inequality is equivalent to $u^i(t', k_{t't''} - 1) < u^i(t, k_{tt'} + k_{t't''} - 2)$. Putting the inequalities together, we have

$$u^i\left(t'', 0\right) < u^i\left(t, k_{tt'} + k_{t't''} - 2\right),$$

which by definition of $k_{tt''}$ implies the following inequality:

$$k_{tt''} - 1 \geq k_{tt'} + k_{t't''} - 2,$$

or equivalently, $k_{tt''} \geq k_{tt'} + k_{t't''} - 1$. □

Using the above lemma, we prove our characterization result. As is noted in Konishi and Fishburn (1996), the quasi-linear representation is equivalent to finding numbers $v_t = v(t)$ for each $t = 1, 2, \dots, T$ such that if $1 \leq t < t' \leq T$, then the following inequality holds:

$$v_t - k_{tt'} < v_{t'} < v_t - \left(k_{tt'} - 1\right). \tag{9}$$

This can be shown by induction on T. This condition is vacuous when $T = 1$ and thus holds. Suppose that for some T, we can find $T - 1$ numbers v_t with $t = 1, 2, \dots, T - 1$ such that the above inequality holds. It is sufficient if we can find a number v_T so that along with numbers $v_1, v_2, \dots, v_{T-1}$, condition (9) is satisfied. A sufficient condition for (9) to be satisfied is the following inequality:

$$\max_{t<T} \left(v_t - k_{tT}\right) < \min_{t<T} \left(v_t - k_{tT} + 1\right) \tag{10}$$

Indeed, if (10) holds, then we can set v_T between the two sides of the above inequality. That is, for all $t < T$, the following inequalities hold:

$$v_t - k_{tT} < v_T < v_t - \left(k_{tT} - 1\right).$$

Because (9) already holds for $t < t' < T$, the above inequality implies that (9) holds for all $t < t' \leq T$.

Suppose (10) does not hold for some distinct pair t, t'. That is, for some t and t' with $t \neq t'$,

$$v_{t'} - k_{t'T} + 1 \leq v_t - k_{tT} \tag{11}$$

Consider first the case when $t < t'$. (9) implies that $-k_{tt'} < v_t' - v_t$. By (11), $v_{t'} - v_t \leq k_{t'T} - k_{tT} - 1$. Combining these two inequalities yields the following inequality:

$$-k_{tt'} < k_{t'T} - k_{tT} - 1$$

or equivalently,

$$k_{tT} < k_{tt'} + k_{t'T} - 1$$

which contradicts Lemma 7 where $t'' = T$.

Now, consider the case when $t' < t$. Because for these t' and t, (9) is satisfied so that we have

$$v_t' - k_{t't} < v_t < v_{t'} - k_{t't} + 1.$$

Using the above relationship and (11) we have

$$k_{tT} - k_{t'T} + 1 \leq v_{t'} - v_t < -k_{t't} + 1$$

which implies

$$k_{t't} + k_{tT} < k_{t'T}$$

The above inequality contradicts Lemma 7 again with $t'' = T$ and the roles of t' and t being switched in this case. Thus, (10) must hold, and a quasilinear representation of the form $v(t) - k_t$ must exist. □

Proof of Proposition 6

In this section, we prove Proposition 6, which gives a sufficient condition regarding the existence of a Nash equilibrium and strong equilibrium when we have two types of players: early and late, based on the property that early players' peak t_E^* is earlier than the late players' peak t_L^*.

First, run the procedure used to prove Proposition 4 as if the players' peaks are at t_L^*, but allocate the players such that no late player is allocated at a slot earlier than one occupied by an early player. Let τ^0 be the resulting strategy profile. By the condition $n \geq 2c(t_L^* - 1) + (c + 1)$, this implies that the connected terrace at τ^0 must start at time slot 1. To see this, τ^0 satisfies the following conditions, based on the procedure used in the proof of Proposition 4:

$$m_t(\tau^0) = c + 1, t \in [t_1, t_L^*],$$
$$m_t(\tau^0) = c - 1, t \in [t_L^* + 1, t_2],$$
$$\tau_i^0 \leq \tau_j^0 \text{ if } i \text{ is an early player and } j \text{ is a late player}$$

where the connected terrace at τ^0 is given by $[t_1, t_2]$. Recall that the procedure allocates players in $t_L^* + \kappa$ before $t_L^* - \kappa$, and suppose initially that $t_2 < T$. In this setup, the number of players allocated is equal to $(t_L^* - t_1 + 1)(c + 1) + (t_2 - t_L^*)(c - 1)$, as slots t_1 through t_L^* contain $c + 1$ players, while slots $t_L^* + 1$ through t_2 contain $c - 1$ players. This connected terrace starts at time slot 1 when $t_1 = 1$ and $t_2 = 2t_L^* - 1$ if this value is less than T, which gives the right-hand side of the inequality. When $2t_L^* - 1 > T$, then the procedure allocates players at earlier slots more quickly in the procedure and thus reaches time slot 1 with a fewer number of players. Thus, the inequality is sufficient for the connected terrace to start at time slot 1. In the following, we denote the connected terrace by $[1, t_2]$.

The queue vector q with respect to τ^0 is the same as equation (1) in the proof of Proposition 4, except with t^* replaced by t_L^*. Thus, the following conditions hold:

$$q_1(\tau^0) > 0,$$
$$q_t(\tau^0) = q_{t-1}(\tau^0) + 1, t \in (1, t_L^*],$$
$$q_t(\tau^0) = q_{t-1}(\tau^0) - 1, t \in [t_L^* + 1, t_2],$$

Also note that the same argument in the proof of Proposition 4 can be used to show that no late player cannot improve by choosing a different slot. Moreover, by the same argument, if $\tau_i^0 \leq t_E^*$ for all early players i, then no early player can improve by moving to a different slot. Thus, in this case τ^0 is a Nash equilibrium and also a strong equilibrium by Proposition 3, because there is only one connected terrace.

In the following, we use the following property of a connected terrace. Consider two strategy profiles τ and τ' such that for some player i, $\tau_i' < \tau_i$ and for all other players $j \neq i$, $\tau_j' = \tau_j$ and such that τ and τ' have the same connected terrace, but a different queue vector. That is, a player i moves from τ_i to τ_i'. Then, the following property holds:

$$q_t(\tau') = q_t(\tau) + 1, t \in [\tau_i', \tau_i - 1],$$
$$q_t(\tau') = q_t(\tau), t \in \mathcal{T} \setminus [\tau_i', \tau_i - 1],$$

The intuition behind this property is the following. Take any t with $t \geq \tau_i + 1$. The number of players choosing t is the same, and the total number of players choosing a time slot before t is the same. Thus, the queue at t should be unchanged. For $t = \tau_i$, the number of players choosing t decreases by 1, but the total number of players choosing a time slot before t increases by 1, so the queue at $t = \tau_i$ is also unchanged. For $\tau_i' < t \leq \tau_i$, the number of players choosing t is the same, but the total number of players choosing a time slot before t increases by 1, and the queue at t increases by

1. When $t = \tau'_i$, the number of players choosing τ'_i increases by 1, while the number of players choosing a time slot before τ'_i (if such exists) is unchanged, so the queue at t increases by 1. Finally, $t < \tau'_i$, the situation is unchanged under τ and τ', so the queue at t is unchanged.

Let t^{**} be the latest time slot chosen by an early player. The above argument shows that if $t^{**} \leq t^*_E$, then the strategy profile τ^0 is a Nash equilibrium. Now, suppose that $t^{**} > t^*_E$ so that there is an early player i with $\tau^0_i > t^*_E$, in which case τ^0 may not be a Nash equilibrium. In the following, we describe an adjustment process, which leads to a Nash equilibrium. First, begin by noting that no early player can deviate from τ^0 to a later time slot than the one she is currently choosing. The reasoning for this fact is the same as in the proof of Proposition 4. Thus, if an early player can improve by deviating, she must move to an earlier slot. Specifically, an early player i with $\tau^0_i = t^*_E + 1$ can improve by deviating to an earlier slot. To see this, first note that $u^i(1, q_1(\tau^0)) = u^i(2, q_2(\tau^0)) = \cdots = u^i(t^*_E, q_{t^*_E}(\tau^0))$. Recalling that player i must take into account the effect she has on the queue vector by this move, by the conditions of the queue vector of τ^0 and condition C, $u^i(t^*_E, q_{t^*_E}(\tau^0) + 1) = u^i(t^*_E + 1, q_{t^*_E}(\tau^0)) > u^i(t^*_E + 1, q_{t^*_E}(\tau^0) + 1) = u^i(t^*_E + 1, q_{t^*_E+1}(\tau^0))$. Thus, player i is better off by moving to any of the time slots from 1 to t^*_E, so move player i to time slot 1, and call the resulting strategy profile τ^1. That is, $\tau^1_j = \tau^0_j$ for all $j \neq i$ and $\tau^1_i = 1$. Then, we have the following:

$$q_t(\tau^1) = q_t(\tau^0) + 1, t \in [1, t^*_E],$$
$$q_t(\tau^1) = q_t(\tau^0), t \in [t^*_E + 1, t_2],$$

Consider the strategy profile τ^1. As was with τ^0, no late player benefits by moving to another time slot, since the queue length she faces is unchanged, and the queue length at all the other alternatives have not decreased. It is still the case that no early player at a time slot from 1 to t^*_E benefits by moving. Moreover, by construction, since $u^i(1, q_1(\tau^0)) = u^i(2, q_2(\tau^0)) = \cdots = u^i(t^*_E, q_{t^*_E}(\tau^0))$, and the queue length at these time slots all increase by 1, we have $u^i(1, q_1(\tau^1)) = u^i(2, q_2(\tau^1)) = \cdots = u^i(t^*_E, q_{t^*_E}(\tau^1))$. Thus, no player i in time slots 1 to t^*_E cannot benefit by moving to a time slot in that span. Also, a player in $t \in [1, t^*_E]$ cannot benefit by moving to $t^*_E + 1$, since that would imply $u^i(t, q_t(\tau^0) + 1) = u^i(t, q_t(\tau^1)) < u^i(t^*_E + 1, q_{t^*_E+1}(\tau^1)) = u^i(t^*_E + 1, q_{t^*_E+1}(\tau^0))$, which contradicts the inequality from the previous step. As will be seen throughout, this property holds throughout the process. For τ^1, $q_{t^*_E+1}(\tau^1) = q_{t^*_E}(\tau^1)$ holds, which implies that any player choosing $t^*_E + 1$ cannot benefit by moving to t^*_E since for such player i, $u^i(t^*_E + 1, q_{t^*_E+1}(\tau^1)) = u^i(t^*_E, q_{t^*_E+1}(\tau^1) + 1) > u^i(t^*_E, q_{t^*_E}(\tau^1) + 1)$. This also implies that this player cannot benefit by moving to any time slot from 1 to t^*_E. Moreover, since the queue length of time slots after $t^*_E + 1$ are unchanged, then any player choosing $t^*_E + 1$ at τ^1 cannot benefit by moving to any other time slot. If $t^{**} = t^*_E + 1$, then we are done. If not, we look at an early player i that is choosing $\tau^1_i = t^*_E + 2$. For such player i, note that $u^i(t^*_E + 2, q_{t^*_E+2}(\tau^1)) = u^i(t^*_E + 2, q_{t^*_E+2}(\tau^0))$ $= u^i(t^*_E, q_{t^*_E+2}(\tau^0) + 2) = u^i(t^*_E, q_{t^*_E}(\tau^0) + 4) = u^i(t^*_E, q_{t^*_E}(\tau^1) + 3) < u^i(t^*_E, q_{t^*_E}(\tau^1) + 1)$.

Therefore, player i benefits by moving to t_E^*, and hence to time slot 1 as well. Move this player i to time slot 1 and denote the resulting strategy profile by τ^2.

For strategy profile τ^2, the same argument can be used to show that another early player at $t_E^* + 2$ (again, if there is such a player) benefits by moving to time slot t_E^*, since $u^i(t_E^* + 2, q_{t_E^*+2}(\tau^2)) = u^i(t_E^* + 2, q_{t_E^*+2}(\tau^0)) = u^i(t_E^*, q_{t_E^*}(\tau^0) + 4) = u^i(t_E^*, q_{t_E^*}(\tau^2) + 2) < u^i(t_E^*, q_{t_E^*}(\tau^2) + 1)$. However, this also implies that this player i benefits by moving to time slot 1. Move this player to time slot 1 and denote the strategy profile by τ^3. Notice that at τ^3, no early player at $t_E^* + 2$ no longer benefits by moving to another time slot, and no early player at time slots 1 through $t_E^* + 1$ benefits by moving. Also, at this step, no late player benefits by moving as well.

Suppose first that $t^{**} \leq t_L^*$. Repeat the above process by moving two early players at each time slot past $t_E^* + 2$ and further until we reach t^{**}. Then, no player early or late can benefit by moving, and thus the strategy profile at this step is a Nash equilibrium. If $t^{**} > t_L^*$, we stop moving players, since once we remove two players from t_L^*, as at that point no player early or late can benefit by moving, and thus the strategy profile at this step is a Nash equilibrium.

Note that during this adjustment process, the queue length of time slot 1 never decreases, and no time slot has a nonpositive q by the assumption on n. Therefore, the connected terrace at the resulting strategy profile is still $[1, t_2]$. By Proposition 3, this strategy profile must also be a strong equilibrium. □

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References

Arnott R, De Palma A, Lindsey R (1990) Economics of a bottleneck. J Urban Econ 27(1):111–130

Aumann RJ (1959) Acceptable points in general cooperative n-person games. In: Tucker AW, Luce RD (eds) Contributions to the theory of games IV. Princeton University Press, Princeton, pp 287–324

Cominetti R, Correa J, Larré O (2015) Dynamic equilibria in fluid queueing networks. Oper Res 63(1):21–34

Daganzo CF (1985) The uniqueness of a time-dependent equilibrium distribution of arrivals at a single bottleneck. Transp Sci 19(1):29–37

Harks T, Klimm M, Möhring RH (2013) Strong equilibria in games with the lexicographical improvement property. Internat J Game Theory 42(2):461–482

Hendrickson C, Kocur G (1981) Schedule delay and departure time decisions in a deterministic model. Transp Sci 15(1):62–77
Holzman R, Law-Yone N (1997) Strong equilibrium in congestion games. Games Econ Behav 21(1–2):85–101
Koch R, Skutella M (2011) Nash equilibria and the price of anarchy for flows over time. Theory Comput Syst 49(1):71–97
Konishi H (2004) Uniqueness of user equilibrium in transportation networks with heterogeneous commuters. Transp Sci 38(3):315–330
Konishi H, Fishburn P (1996) Quasi-linear utility in a discrete choice model. Econ Lett 51(2):197–200
Konishi H, Le Breton M, Weber S (1997a) Equilibria in a model with partial rivalry. J Econ Theory 72(1):225–237
Konishi H, Le Breton M, Weber S (1997b) Pure strategy Nash equilibrium in a group formation game with positive externalities. Games Econom Behav 21(1/2):161–182
Milchtaich I (1996) Congestion games with player-specific payoff functions. Games Econ Behav 13(1):111–124
Monderer D, Shapley LS (1996) Potential games. Games Econ Behav 14(1):124–143
Newell GF (1987) The morning commute for nonidentical travelers. Transp Sci 21(2):74–88
Rivera TJ, Scarsini M, Tomala T (2018) Efficiency of correlation in a bottleneck game. HEC Paris Research Paper No ECO/SCD-2018-1289
Rosenthal RW (1973) A class of games possessing pure-strategy Nash equilibria. Internat J Game Theory 2(1):65–67
Scarsini M, Schröder M, Tomala T (2018) Dynamic atomic congestion games with seasonal flows. Oper Res 66(2):327–339
Schmeidler D (1973) Equilibrium points of nonatomic games. J Stat Phys 7(4):295–300
Small KA (2015) The bottleneck model: an assessment and interpretation. Econ Transp 4(1–2):110–117
Smith MJ (1984) The existence of a time-dependent equilibrium distribution of arrivals at a single bottleneck. Transp Sci 18(4):385–394
Vickrey WS (1969) Congestion theory and transport investment. Am Econ Rev 59(2):251–260
Voorneveld M, Borm P, Van Megen F, Tijs S, Facchini G (1999) Congestion games and potentials reconsidered. Int Game Theory Rev 1(3 & 4):283–299

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