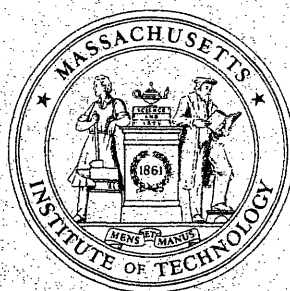


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EQUILIBRIA ON A CONGESTED
TRANSPORTATION NETWORK

by

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ABSTRACT

Network equilibrium models arise in applied contexts as varied as urban transportation, energy distribution, spatially separated economic markets, electrical networks, and water resource planning. In this paper, we propose and study an equilibrium model for one of these applications, namely for predicting traffic flow on a congested transportation network. The model is quite similar to those that arise in most contexts of network equilibria, though, and the methods that we use are applicable in these other settings as well.

Our transportation model includes such features as (i) multiple modes of transit, (ii) link interactions and their effect on congestion, (iii) limited choices (or perceptions) of paths for flow between any origin-destination pair, (iv) generalized cost or disutility for travel, and (v) demand relationships for travel between origin-destination pairs that depend upon the travel time (cost) between all other origin-destination pairs. Using Brouwer's fixed point theorem, we establish existence of an equilibrium solution to the model. By imposing monotonicity conditions on the delay and demand functions, we also show that travel times (costs) are unique and, in certain instances, that link flows are unique.

EQUILIBRIA ON A CONGESTED TRANSPORTATION NETWORK

1. Introduction

Network analysis draws its origins from several sources. Prominent among these is the study of passive electrical networks, particularly the prediction of a network's utilization when it is loaded with prescribed voltages and impedances. With given voltages applied to an electrical network, what is the resulting current flow? More recently, similar types of predictive questions are being posed in social and economic contexts. In transportation, travelers' demands for transportation services function, like voltages, as forces that generate network flow which, in this instance, are trips to be made between origin and destination points in the network. In this setting, travel time, travel cost, and other disutility measures replace electrical resistance as the impedance to flow. In economics, price differentials between spatially separated markets act like voltages as forces for generating commodity flow; transportation costs between the markets act as resistance to commodity movement. In each of these applications, the equilibration of forces and impedances has served as a model for predicting flow on the network. The nature of the specific equilibrium model depends upon the behavioral assumption, such as Ohm's Law, profit maximization, or cost minimization, that relate the forces, impedances, and network flow.

The advent of robust theories for constrained optimization has precipitated an attractive and common approach for studying network equilibrium problems, namely to view the equilibrium model as the Lagrange multiplier conditions or, more generally, the Karush-Kuhn-Tucker optimality conditions of well-conceived auxiliary optimization problems. For example, minimize

power loss instead of finding an equilibrium on an electrical network directly. Making this association permits the powerful and flexible solution techniques of constrained optimization to be used to compute an equilibrium and, moreover, permits optimization theory to serve as the methodology base to study questions such as existence and uniqueness of equilibrium solutions. On the other hand, the equivalent optimization approach limits the richness of equilibrium modeling by restricting the problem assumptions to those for which the equilibrium conditions can be interpreted as optimality conditions for an associated optimization problem.

In this paper, we study a class of network equilibrium problems with no known equivalent optimization problem. Although the approach that we take might apply to a variety of different network equilibrium applications, we restrict our discussion to transportation planning. In the next section, we propose a general model for network equilibrium of an urban transportation system. The model includes such features as (i) multiple (and interacting) modes of transit, (ii) link interactions and their effect on congestion, (iii) limited choices (or perceptions) of paths for flow between any origin-destination pair, (iv) generalized cost or disutility for travel on any path that depends upon the flow pattern on the entire transportation network, and (v) demand relationships for travel between origin-destination pairs that depend upon the travel time (cost) between *all* other origin-destination pairs. With the exception of (iii), any one of these modeling features invalidates the assumptions that are typically made when showing that the transportation equilibrium problem can be converted to an equivalent optimization model.

After stating this model and discussing some of its applications

and specializations, we show that only very mild restrictions need be imposed upon the problem data, restrictions that we would expect to be met almost always in practice, to insure that an equilibrium solution exists. We also establish conditions that will insure that an equilibrium solution is unique. To establish these results, we formulate the equilibrium model as an equivalent nonlinear complementarity problem. Then we use Brouwer's fixed-point theorem to establish existence and nonlinear complementarity results to establish uniqueness.

2. Background

The genesis of transportation equilibrium modeling was a behavioral assumption, known as Wardrop's user traffic equilibrium law, first proposed in 1952 by the traffic engineer J. G. Wardrop [52], namely

"At equilibrium, for each origin-destination pair the travel times on all the routes actually used are equal, and less than the travel times on all non-used routes."

This principle has spawned a great deal of research by transportation engineers, economists, and operations researchers aimed at enhancing the scope and realism of Wardrop's model, at developing algorithms to compute an equilibrium, and at applying the equilibrium model in practice to predict traffic flow patterns. Modeling efforts and methodological advancements have evolved to the point that one version of the equilibrium model now forms part of the Urban Mass Transit Authority's transportation planning system [51].

Since 1952, a large number of algorithms have been developed for the traffic assignment problem. Most of the earlier techniques were

heuristics and usually did not consider congestion effects or any formal concept of an equilibrium ([39], [40], [53], [19]). The goal of these approaches was to assign flow between different paths so that the paths have almost equal travel time. The next generation of heuristics, as embodied by the "capacity restrained" technique ([12], [28], [29], [48]), attempted to account for capacity of the system. Later techniques ([30], [38], [39]) loaded the system incrementally, attempting to approximate an equilibrium solution.

The mathematical programming approach to traffic equilibrium originated in 1956 when Beckman, McGuire and Winsten [7] formulated a version of the equilibrium problem as the optimality conditions of an equivalent optimization problem.¹ They assumed:

- 1) a single mode of transit (private vehicle traffic has been the primary application since then);
- 2) that the demand function $D_i(u_i)$ between every origin-destination pair i depends only upon the impedance or shortest travel time u_i between that origin-destination pair;
- 3) that the delay functions for the links are separable; that is, the delay $t_a(v_a)$ for each link "a" depends only upon the total volume of traffic flow v_a on that link.

Since then several researchers have proposed algorithms for solving the equivalent optimization problem (Bruynooghe, Gibert and Sakarovitch [11], Bertsekas [8], Bertsekas and Gafni [9], Dafermos [13-16],

¹ Samuelson had earlier proposed a similar transformation in the context of spatially separated economics markets.

Dembo and Klincewicz [18], Leventhal, Nemhauser and Trotter [36], Leblanc [34,35], Nguyen [41-45], Golden [26], and Florian and Nguyen [23-25]).

There are a number of ways to enrich the modeling assumptions (1)-(3). Modeling multi-modal (for example, private vehicle and a public transit mode) and multi-class (for example, high vs. low income) traffic equilibrium would be extensions with great practical relevance. Incorporating demand functions for an O-D pair that depend upon impedance between other O-D pairs would permit destination choice to be modeled more realistically than in models based upon (1) - (3). For example, the distribution of trips from a residential district to two shopping centers depends, in part, upon the travel time to both centers. Residential home selection might be modeled as an origin choice version of this extension. Another extension would be to let delay on a link depend on volume flow on other links. This latter extension permits modeling of traffic equilibrium with two-way traffic in one link, traffic equilibrium with right and left turn penalties, and the like.

Some attempts have been made to generalize the equivalent optimization approach to traffic equilibrium to incorporate these modeling extensions. Dafermos [13, 15] has considered multiple classes of users and Florian [22] and Abdulaal and Leblanc [4] have considered the multi-modal problem. In addition, the equivalent optimization problem has been used to prove existence and uniqueness of an equilibrium for certain specializations of the general model (Dafermos [13, 15], Florian and Nguyen [23] and Steenbrink [49]). Nevertheless, the optimization based approach is limited since the assumptions required to insure an equivalent convex optimization problem are generally too severe to be applicable

in practice for modeling the type of extensions to assumptions (1)-(3) suggested above. The approach adopted in this paper originates with Aashtiani [1] who formulated an extended equilibrium model and studied existence of a solution by viewing the model as a nonlinear complementarity problem. In [2] he elaborates on this approach and proposes a computational scheme for solving for an extended equilibrium. Independently, Kuhn [27] devised a fixed point method, equipped with a special pivoting scheme, to solve equilibrium problems with fixed demands and with separable link delay functions. Asmuth [6] has proposed an additive model similar to the one discussed in this paper that includes point-to-set delay functions and demand functions. He has also studied existence and uniqueness, existence being a consequence of a constructive fixed point algorithm. The proof of existence given in this paper, which is adopted from Aashtiani and Magnanti [3], is shorter than these earlier proofs and relies on the classical fixed point theorem of Brouwer.

In related developments, Dafermos [14, 15], by assuming differentiability and strong monotonicity of the link delay function, has recently used the theory of variational equalities to establish the existence of a traffic equilibrium and to devise an algorithm for computing an equilibrium. Ahn [5] has used similar methods to study equilibrium for spatially separated markets arising in energy planning. Recently, Braess and Koch [10] and Smith [47] have used a proof different than that given in this paper, but also based upon Brouwer's fixed point theorem, to establish existence of an equilibrium for a special version of the model that we study here; they assume that the demand is fixed independent of the network congestion and that the cost on any path is the sum of costs on arcs in that path. Braess and Koch also impose a monotonicity assumption on the arc costs.

3. Traffic Equilibrium Model

The equilibrium model is defined on a transportation network $[N,A]$ with nodes N , directed arcs A , and with a given set I of origin-destination (O-D) node pairs. Nodes represent centroids of population, business districts, street intersections and the like, and arcs model streets and arteries or might be introduced to model connections (and wait time) between legs of a trip, between modes, or between streets at an intersection. The model is formulated as:

$$(3.1) \left\{ \begin{array}{ll} (T_p(h) - u_i)h_p = 0 & \text{for all } p \in P_i \text{ and } i \in I \quad (3.1a) \\ T_p(h) - u_i \geq 0 & \text{for all } p \in P_i \text{ and } i \in I \quad (3.1b) \\ \sum_{p \in P_i} h_p - D_i(u) = 0 & \text{for all } i \in I \quad (3.1c) \\ h \geq 0 & (3.1d) \\ u \geq 0 & (3.1e) \end{array} \right.$$

In this formulation

- I is the set of O-D pairs,
- P_i is the set of "available" paths for flow for O-D pair i (which might, but need not, be all paths joining the O-D pair),
- h_p is the flow on path p ,
- h is the vector of $\{h_p\}$ with dimension $n_1 = \sum_{i \in I} |P_i|$ equal to the total number of O-D pairs and path combinations,
- u_i is an accessibility variable, shortest travel time (or generalized cost) for O-D pair i ,
- u is the vector of $\{u_i\}$ with dimension $n_2 = |I|$,

$D_i(u)$ is the demand function for O-D pair i , $D_i : R_+^{n_2} \rightarrow R_+^1$

$T_p(h)$ is the delay time, or general disutility, function for path p ,

$$T(h) : R_+^{n_1} \rightarrow R_+^1 .$$

We also let $P = \cup\{P_i : i \in I\}$ denote the set of all "available" paths in the network and assume that the network is strongly connected, i.e., for any O-D pair $i \in I$ there is at least one path joining the origin to the destination; i.e., $|P_i| \geq 1$.

The first two equations in (3.1) model Wardrop's traffic equilibrium law requiring that for any O-D pair i , the travel time (generalized travel time) for all paths, $p \in P_i$, with positive flow $h_p > 0$, is the same and equal to u_i , which is less than or equal to the travel time for any path with zero flow. Equation (3.1c) requires that the total flow among different paths between any O-D pair i equal the total demand, $D_i(u)$, which in turn depends upon the congestion in the network through the shortest path variable u . Conditions (3.1d) and (3.1e) state that both flow on paths and minimum travel times should be nonnegative.

An important special case of the equilibrium problem (3.1) is an *additive model* in which

$$(3.2) \quad T_p(h) = \sum_{a \in A} \delta_{ap} t_a^i(h) \quad \text{for all } p \in P_i \text{ and } i \in I$$

where

$$\delta_{ap} = \begin{cases} 1 & \text{if link } a \text{ is in path } p \\ 0 & \text{otherwise} \end{cases}$$

and

$t_a^i(h)$ is the delay function for arc a and O-D pair i ,

$$t_a^i : R_+^{n_1} \rightarrow R_+^1 .$$

That is, the delay time on path p is the sum of the delays of the arcs in that path. More compactly, $T_p(h) = \Delta^T t^i(h)$ where $\Delta = (\delta_{ap})$ is the arc-path incidence matrix for the network and $t^i(h) = (t_a^i(h))$ is the vector of arc delay functions for O-D pair i .

Several features of the equilibrium model are worth noting. In a large transportation network, users generally will not perceive, or choose from, all possible paths joining every origin-destination pair. If we identify the paths P_i available for flow between O-D pair i as the available set of routes from which the user chooses, the equilibrium conditions model this type of limited route choice.² In addition, since the path disutility functions $T_p(h)$ are arbitrary and depend upon the full vector h of path flows, the model can account for path interactions, as at intersections, and the generalized costs $T_p(h)$ can, in principle, incorporate a variety of attributes that are relevant to route selection such as travel time, travel costs, and route attractiveness. To the best of our knowledge, no previous existence proof of traffic equilibrium incorporates both of these modeling features.

The equilibrium model (2.1) is more general than first appearance might indicate. A judicious choice of network structure permits the formulation to model a wide range of equilibrium applications including multi-modal transit, multiple classes of users, and destination or origin choice. To model multi-modal situations, we might conceptualize

²Several authors (e.g. Asmuth [6], Dafermos [14, 16], and Smith [47]) formulate the traffic equilibrium problem in terms of arc flows. The path flow formulation with limited path choice appears to be more general. If A_i is the union of the arcs contained on the paths in

an extended network with a distinct component for each mode of transit. (Dafermos [13] and Sheffi [46] adopt this approach as well.) The component networks might be identical copies of the underlying physical transportation network, as when autos and buses share a common street network. Since the delay $T_p(h)$ for paths on the automobile component network depend upon the full vector h of path flows, the delay function can account for congestion added by buses sharing common links. Note, though, that the networks for each mode need not be the same. Consequently, bus routes might be fixed and subway links might be distinct from those of other modes.

The model also provides flexibility in modeling demand. Suppose, for instance, that O-D pairs i and j in the extended network introduced above correspond to the same physical origin and destination points, but different modes of transit. If we introduce a source node s and terminal node t connected, respectively, to the origin and destination points of O-D pairs i and j , then a demand function $D_{st}(u)$ would model total trips between the origin and destination points as a function of network congestion. The equilibrium model would distribute these trips between the two modes to equalize the disutility $T_p(h)$ on all flow carrying paths by both modes. As an alternative, the modeler could prescribe the nature of modal split by introducing demand functions such as the well-known logit model:

$$D_i(u) = d \frac{e^{\theta u_i + A_i}}{e^{\theta u_i + A_i} + e^{\theta u_j + A_j}}, \quad D_j(u) = d - D_i(u)$$

which would distribute the total number of trips d between the two

in P_i , then the arc formulation implies that any path with arcs in A_i and joining O-D pair i belongs to P_i . In formulation (3.1), P_i is an arbitrary collection of paths joining O-D pair i , thereby permitting more flexibility in modeling user's perception of "available" paths.

modes i and j depending upon delay times u_i and u_j by the two modes and the given negative constant θ and nonnegative constants A_i and A_j .

As Dial [20] has noted, a generalized version of the logit model permits destination choice and modal split to be made simultaneously. If $i = pqm$ denotes an origin destination pair p - q distinguished by transit mode m , the model is of the form

$$D_{pqm}(u) = d_p \frac{r_q e^{\theta u_{pqm}}}{\sum_{q', m'} r_{q'} e^{\theta u_{pq'm'}}$$

where d_p is the total number of trips generated at origin p to be sent to the destinations q' , and $r_{q'}$ is an index of attraction for destination q' .

4. Equivalent Non-linear Complementarity Problem

Let $F(x) = (F_1(x), \dots, F_n(x))$ be a vector valued function from an n -dimensional space R^n into itself. The well-known nonlinear complementarity problem of mathematical programming is to find a vector x that satisfies the following system:

$$(4.1) \quad \begin{cases} x \cdot F(x) = 0 \\ F(x) \geq 0 \\ x \geq 0 \end{cases}$$

This problem has wide ranging applications. Karamardian [31, 32] illustrates several examples. For instance, the primal-dual optimality conditions of linear and quadratic programming and the Kuhn-Tucker conditions for certain other nonlinear programming problems can be cast in this form.

In this section we show that the traffic equilibrium problem (3.1) can be formulated as a complementarity problem. By definition, equations (3.1a), (3.1b), and (3.1d) are complementary in nature. To show that the remaining equations can be expressed in a complementarity form requires some mild assumptions that we would expect to be met always in practice.

First some simplification in the formulation helps to clarify our discussion. Let $x = (h, u) \in R^n$ where $n = n_1 + n_2$ and furthermore, let

$$f_p(x) = T_p(h) - u_i \quad \text{for all } p \in P_i \text{ and } i \in I$$

and

$$g_i(x) = \sum_{p \in P_i} h_p - D_i(u) \quad \text{for all } i \in I .$$

Also, let

$$F(x) = (f_p(x) \text{ for all } p \in P_i \text{ and } i \in I, g_i(x) \text{ for all } i \in I) \in R^n .$$

Then F is a vector-valued function from an n -dimensional space R^n into itself. Now consider the following nonlinear complementarity system:

$$(4.2) \quad \left\{ \begin{array}{ll} f_p(x) h_p = 0 & \text{for all } p \in P_i \text{ and } i \in I \\ f_p(x) \geq 0 & \text{for all } p \in P_i \text{ and } i \in I \\ g_i(x) u_i = 0 & \text{for all } i \in I \\ g_i(x) \geq 0 & \text{for all } i \in I \\ x \geq 0 & \end{array} \right.$$

which is a specialization of (4.1).

Since any solution $\bar{x} = (\bar{h}, \bar{u})$ to the traffic equilibrium problem satisfies $g_i(\bar{x}) = 0$ for all $i \in I$, the solution \bar{x} solves the nonlinear

complementarity problem (4.2) as well, independent of the nature of the delay functions $T_p(h)$ and the demand functions $D_i(u)$. The following result establishes a partial converse.

PROPOSITION 4.1: Suppose, for all $p \in P$, that $T_p : R_+^{n_1} \rightarrow R_+^{n_1}$ is a positive function. Also, suppose, for all $i \in I$, that $D_i : R_+^{n_2} \rightarrow R_+^{n_2}$ is a non-negative function. Then the traffic equilibrium system (3.1) is equivalent to the nonlinear complementarity system (4.2).

PROOF: In light of our comment preceding the proposition, it is sufficient to show that any solution to (4.2) is a solution to (3.1). Suppose to the contrary that there is an $x = (h, u)$ satisfying (4.2), but that $g_i(x) = \sum_{p \in P_i} h_p - D_i(u) > 0$ for some $i \in I$. Then $g_i(x)u_i = 0$ implies that $u_i = 0$. Also, since D_i is non-negative $\sum_{p \in P_i} h_p > D_i(u) \geq 0$ which implies that $h_p > 0$ for some $p \in P_i$. But, for this particular p , equation $f_p(x)h_p = 0$ implies that:

$$f_p(x) = T_p(h) - u_i = 0$$

or

$$T_p(h) = u_i .$$

But since $u_i = 0$, $T_p(h) = 0$ which contradicts the assumption $T_p(h) > 0$. \square

When the traffic equilibrium problem is additive $T_p(h) = \sum_{a \in A} \delta_{ap} t_a^i(h)$, $T_p(h)$ is positive whenever the arc delay functions are positive, or more generally, whenever the arc delay functions are nonnegative and at least one is positive on an arc a in path p .³

³Notice that we have suppressed explicit dependence of the arc delay functions $t_a^i(h)$ on the origin-destination pair i since the generality of the equilibrium problem (3.1) permits us, at least conceptually, to duplicate the network, as indicated in the previous section, so that each arc carries the flow for a single O-D pair.

PROPOSITION 4.2: Suppose, for all $a \in A$, that $t_a : R_+^{n_1} \rightarrow R_+^1$ is a positive function. Also, suppose, for all $i \in I$, that $D_i : R_+^{n_2} \rightarrow R_+^1$ is a nonnegative function. Then the additive traffic equilibrium system (3.1) and (3.2) is equivalent to the nonlinear complementarity system (4.2).

Neither of the previous two propositions is valid if either the assumption that each demand function $D_i(u)$ is nonnegative or the assumption that each delay function $T_p(h)$ is positive is eliminated. See Aashtiani [2] for examples.

5. Existence

Rather extensive theory (see, for example, Karamardian [31] and Kojima [33]) provides necessary conditions that assure the existence of a solution to the nonlinear complementarity problem. Unfortunately, most of the conditions are too strong to be applied directly to the traffic equilibrium problem. To illustrate this situation and at the same time introduce concepts that will be useful in section 6 when we discuss uniqueness of solutions, we introduce a prototype of this theory by considering results due to Karamardian. First, we require some definitions.

DEFINITION 5.1: Let $F : D \rightarrow R^n$, $D \subset R^n$. The function F is *monotone* on D if, for every pair $x \in D$ and $y \in D$,

$$(x - y)(F(x) - F(y)) \geq 0 .$$

F is *strictly monotone* on D if, for every pair $x \in D$, $y \in D$ with $x \neq y$,

$$(x - y)(F(x) - F(y)) > 0 .$$

F is said to be *strongly monotone* on D if there is a scalar $k > 0$ such that, for every pair $x \in D$, $y \in D$,

$$(x - y)(F(x) - F(y)) \geq k|x - y|^2$$

where $|\cdot|$ denotes the usual Euclidean norm.

THEOREM 5.1: (Karamardian [31]) *If $F : R_+^n \rightarrow R^n$ is continuous and strongly monotone on R_+^n , then the nonlinear complementarity system (4.1) has a unique solution.*

THEOREM 5.2: (Karamardian [31]) *If $F : R_+^n \rightarrow R^n$ is strictly monotone on R_+^n , then the nonlinear complementarity system (4.1) has at most one solution.*

Notice that for traffic equilibrium problems, these theorems require that $F(x) = (\sum_{a \in A} T_p(h) - u_i \text{ for all } p \in P_i \text{ and } i \in I, \sum_{p \in P_i} h_p - D_i(u) \text{ for all } i \in I)$ and necessarily $T_p(h)$ be strictly or strongly monotone in terms of *path flows*. In most instances, this condition is not applicable; usually, the delay functions T_p depend upon arc flows each of which depends upon the sum of the flows on different paths. In these situations, whenever $x = (h, u)$ and $y = (h', u)$ correspond to two path flows h and h' that give rise to identical arc flows, $T_p(h) = T_p(h')$ and $\sum_{p \in P_i} h_p = \sum_{p \in P_i} h'_p$ for all $i \in I$. Consequently, $F(x) = F(y)$ and $(x - y)[F(x) - F(y)] = 0$ so that neither strict nor strong monotonicity applies.

Generally, however, for transportation applications the delay functions $T_p(h)$ are monotone, and frequently even strictly monotone, in terms of *link volumes*. Later we use this property to show the uniqueness of the solution in terms of link flows. In Theorem 5.3 to follow, though, we show that no monotonicity assumption is required for the existence of the solution.

To establish this result we use a well-known [50] transformation that permits us to convert the nonlinear complementarity problem and, in particular, the nonlinear complementarity version (4.2) of the traffic equilibrium problem into a Brouwer fixed point problem. Let us define $\phi : R^n \rightarrow R^n$ by defining its component functions ϕ_i for $i = 1, 2, \dots, n$ as:

$$\phi_i(x) = [x_i - F_i(x)]^+$$

where $[y]^+$ denotes $\max\{0, y\}$. Then \bar{x} is a fixed point to ϕ , i.e., $\bar{x} = \phi(\bar{x})$ if, and only if, \bar{x} solves the nonlinear complementarity problem $x \geq 0$, $F(x) \geq 0$, and $x F(x) = 0$.

This equivalence shows that we can, in principle, study any nonlinear complementarity problem by invoking fixed point theory. Note that we cannot use Brouwer's fixed point theorem directly, though, because the mapping $\phi(x)$ defined on R_+^n need not map any compact set into itself. Consequently, we will restrict the domain of ϕ to some large cube C . To apply the theorem, we must be assured that ϕ maps C into itself, which we accomplish by redefining $\phi(x)$ for any $x \in C$ if it lies outside of C by projecting $\phi(x)$ onto C . By Brouwer's fixed point theorem the modified map ϕ' has a fixed point. We must show that it has no

false fixed points, though; that is, no point \bar{x} contained on the boundary of C with the property that $\phi(\bar{x}) \notin C$ but the projection $\phi'(\bar{x})$ of $\phi(x)$ on C satisfies $\phi'(\bar{x}) = \bar{x}$. The essence of the following equilibrium proof is that ϕ' as derived from the complementarity version (4.2) of the traffic equilibrium problem admits no false fixed points.

THEOREM 5.3: Suppose (N,A) is a strongly connected network. Suppose that $T_p : R_+^{n_1} \rightarrow R^1$ is a non-negative continuous function for all $p \in P$. Also suppose that for all $i \in I$, $D_i : R_+^{n_2} \rightarrow R^1$ is a continuous function that is bounded from above. Then the nonlinear complementarity system (4.2) has a solution.

Proof: Let $F_i(h) = \sum_{p \in P_i} h_p$ denote the flow between O-D pair i and let e and \hat{e} denote vectors of ones with $|P|$ and $|I|$ components. We must show that the following complementarity problem has a solution:

$$\left. \begin{aligned} h_p [T_p(h) - u_i] &= 0 \\ u_i [F_i(h) - D_i(u)] &= 0 \\ T_p(h) - u_i &\geq 0 \\ F_i(h) - D_i(u) &\geq 0 \\ u_i \geq 0, h_p &\geq 0 \end{aligned} \right\} \text{ for all } i \in I \text{ and all } p \in P_i$$

Let $K_1 > 0$ satisfy $K_1 > \max_i \max_{u \geq 0} D_i(u)$ and

let $K_2 \geq K_1$ satisfy $K_2 > \max_{p \in P} \max_{0 \leq h \leq K_1 e} T_p(h)$.

K_1 exists because of the hypothesis that each $D_i(u)$ is bounded and K_2 exists because each $T_p(h)$ is continuous.

Define the continuous mapping ϕ of the cube $\{0 \leq h \leq K_1 e, 0 \leq u \leq K_2 \hat{e}\}$ into

itself by

$$\phi_p(h, u) = \min\{K_1, [h_p + u_i - T_p(h)]^+\} \text{ for all } p \in P_i \text{ and all } i \in I,$$

and

$$\phi_i(h, u) = \min\{K_2, [u_i + D_i(u) - F_i(h)]^+\} \text{ for all } i \in I .$$

By Brouwer's fixed point theorem this mapping has a fixed point (\hat{h}, \hat{u}) ; that is, $\hat{h}_p = \phi_p(\hat{h}, \hat{u})$ and $\hat{u}_i = \phi_i(\hat{h}, \hat{u})$ for all $i \in I$ and all $p \in P$. We show that this fixed point solves the complementarity problem by showing that for all $p \in P_i$ and $i \in I$

$$\hat{h}_p = [\hat{h}_p + \hat{u}_i - T_p(\hat{h})]^+ \quad (*)$$

$$\hat{u}_i = [\hat{u}_i + D_i(\hat{u}) - F_i(\hat{h})]^+.$$

First note that $\hat{u}_i < K_2$ for all $i \in I$, for if some $\hat{u}_i = K_2$ then for any $p \in P_i$, $\hat{h}_p + \hat{u}_i - T_p(\hat{h}) > \hat{h}_p$ by definition of K_2 which implies from $\hat{h}_p = \phi_p(\hat{h}, \hat{u})$ that $\hat{h}_p = K_1$. But then the definition of K_1 implies that $D_i(\hat{u}) < F_i(\hat{h})$ so that $[\hat{u}_i + D_i(\hat{u}) - F_i(\hat{h})] < \hat{u}_i$; therefore \hat{u}_i must equal 0 in order that $\hat{u}_i = \phi_i(\hat{h}, \hat{u})$, contradicting $\hat{u}_i = K_2$.

Next note that if $\hat{h}_p = K_1$ for some $i \in I$ and $p \in P_i$, then $D_i(\hat{u}) < F_i(\hat{h})$ by definition of K_1 which implies as above that $\hat{u}_i = 0$. By nonnegativity of T_p , $[\hat{h}_p + \hat{u}_i - T_p(\hat{h})] \leq \hat{h}_p$ with a strict inequality if $T_p(\hat{h}) > 0$. Consequently, in order that $\hat{h}_p = K_1 > 0$ equal $\phi_p(\hat{h}, \hat{u})$, $T_p(\hat{h})$ must equal 0 and thus $\hat{h}_p = [\hat{h}_p + \hat{u}_i - T_p(\hat{h})]^+$.

We have now established the expressions (*) which imply by considering the cases $\hat{h}_p > 0$ or $\hat{h}_p = 0$ and $\hat{u}_i > 0$ or $\hat{u}_i = 0$ that (\hat{h}, \hat{u}) solves the complementarity problem (4.2). □

As a consequence of Theorem 5.3 and Proposition 4.1 we have the following result.

THEOREM 5.4: (Existence) Suppose (N,A) is a strongly connected network. Suppose that $T_p : R_+^{n_1} \rightarrow R_+^1$ is a positive continuous function for all $p \in P$. Also suppose that for all $i \in I$, $D_i : R_+^{n_2} \rightarrow R_+^1$ is a nonnegative continuous function that is bounded from above. Then the traffic-equilibrium system (3.1) has a solution.

An important version of this theorem is its specialization for additive traffic equilibrium.

THEOREM 5.5: (Existence) Suppose (N,A) is a strongly connected network. Suppose that $t_a : R_+^{n_1} \rightarrow R_+^1$ is a positive continuous function for all $a \in A$. Also suppose that for all $i \in I$, $D_i : R_+^{n_2} \rightarrow R_+^1$ is a nonnegative continuous function that is bounded from above. Then the additive traffic equilibrium system (3.1) and (3.2) has a solution.

PROOF: Since every t_a is positive and continuous, so is $T_p(h) = \sum_{a \in A} \delta_{ap} t_a(h)$ and, consequently, Theorem 5.4 applies. \square

Asmuth [6] has suggested what appears to be a stronger version of Theorem 5.5 by not requiring that the demand functions $D_i(u)$ be bounded. To see the relevance of this result, suppose that $D_i(u)$ denotes the number of trips to be made between a particular origin-destination pair by automobiles. One possibility for modeling this situation is a Cobb-Douglas product form demand model given by:

$$D_i(u) = A \frac{(u_j)^\beta}{(u_i)^\alpha}$$

where A is a given constant and $\alpha \geq 0$ is a "direct elasticity" and

$\beta \geq 0$ is a "cross elasticity". In this model, u_i denotes the travel time between the O-D pair by auto and u_j denotes the travel time by an alternate mode such as bus. Note that $D_i(u)$ is not bounded unless we require $u_i \geq \epsilon$ for some, possibly small, number $\epsilon > 0$.

The next result shows that Theorem 5.4 can be modified easily to include settings of this nature.

THEOREM 5.6: (Existence) Suppose (N,A) is a strongly connected network. Suppose that for all $p \in P$, $T_p : R_+^{n_1} \rightarrow R_+^1$ is a continuous function and for all $h \in R_+^{n_1}$ that $T_p(h) > \epsilon$ for some $\epsilon \geq 0$. Also suppose that for all $i \in I$, $D_i : R_+^{n_2} \rightarrow R_+^1$ is a nonnegative continuous function that is bounded from above on the set $\{u \in R_+^{n_2} : u_i \geq \epsilon \text{ for all } i\}$. Then the traffic equilibrium system (3.1) has a solution.

PROOF: Let \hat{e} be a vector of ones with $|I|$ components and define

$$T'_p(h) = T_p(h) - \epsilon > 0$$

and

$$D'_i(u) = D_i(u + \epsilon \hat{e}) .$$

These functions satisfy the hypothesis of Theorem 5.4 and so they are guaranteed to have a complementarity (or equilibrium) solution (h', u') . But then $(\hat{h}, \hat{u}) = (h', u' + \epsilon \hat{e})$ is a complementarity (equilibrium) solution for T_p and D_i . □

6. Uniqueness

In situations in which the traffic equilibrium problem can be formulated as an equivalent convex optimization problem, the Kuhn-Tucker vector associated with the flow constraints $\sum_{p \in P_i} h_p = D_i(u)$ can be identified with the vector u of shortest travel times (generalized costs).

Since the gradients of these constraints as i varies are linearly independent, the theory of convex optimization implies that in equilibrium the shortest travel times are unique even if the flow vector h is not unique. This situation reflects practice as well. Generally, flow patterns in urban transportation networks vary, sometimes considerably, from day to day though travel times remain essentially constant.

In this section, we show that these observations apply to the additive version (3.1) and (3.2) of the general traffic equilibrium model as well. We first recall conditions due to Asmuth [6] that insure that link flows and shortest travel times are both unique. We then show that imposing weaker conditions will still imply that shortest travel times are unique.

To facilitate our discussion in this section, we represent the traffic equilibrium problem in a matrix form. Let v_a denote the total flow on arc a , that is, $v_a = \sum_{i \in I} \sum_{p \in P_i} \delta_{ap} \cdot h_p$, and let v with dimension $|A|$ denote the vector of arc flows. Since we are assuming an additive model, $T_p(h) = \Delta^T t(h) = \sum_{a \in A} \delta_{ap} t_a(h)$ for every path $p \in P$. In fact, we will assume that the arc delay term $t_a(h)$ can be expressed as a function of link flows v and write $t_a(v)$.

Also, let $t(v)$ be the vector of arc delay functions and $D(u)$ be the vector of demand functions. Recall that $\Delta = (\delta_{ap})$ is the arc-path incidence matrix with dimension $|A| \times n_1$. Let $\Gamma = (\gamma_{pi})$ be the path O-D pair incidence matrix with dimension $n_1 \times n_2$, i.e., $\gamma_{pi} = 1$ when path p joins O-D pair i and $\gamma_{pi} = 0$ otherwise.

Then the traffic-equilibrium problem can be written as:

$$(6.1) \quad \begin{cases} (\Delta^T \cdot t(\Delta h) - \Gamma \cdot u) \cdot h = 0 \\ \Delta^T \cdot t(\Delta h) - \Gamma \cdot u \geq 0 \\ \Gamma^T \cdot h - D(u) = 0 \\ h \geq 0, \quad u \geq 0. \end{cases}$$

Now let $x = (h, u)^T$ and let $F(x): R_+^n \rightarrow R^m$ be defined as in section 4 as $F(x) = (\Delta^T t(\Delta h) - \Gamma u, \Gamma^T h - D(u))$. Then (4.1) is the nonlinear complementarity version (4.2) of (6.1).

Whenever $F(x)$ is strictly monotone, the solution to the general nonlinear complementarity problem (4.1) is unique (see Theorem 5.2) Asmuth [6] has extended this result to establish the following uniqueness result, which we state without proof.

THEOREM 6.1: (Uniqueness) *For a strongly connected network (N, A) supposed that t , the vector of the volume delay functions, and $-D$, the vector of the negative of the demand functions, are strictly monotone. Then the arc volumes v and the accessibility vector u for the additive traffic equilibrium problem (3.1) and (3.2) are unique, and the set of equilibrium path flows is convex.*

Observe the distinction between the hypothesis of this theorem and the assumption that $F(x)$ is strictly monotone. The theorem requires that the vector t of volume delay functions be strictly monotone in terms of arc volumes v whereas the latter assumption requires strict monotonicity in terms of path flows h . As we have noted earlier, the path flows need not be unique since two collections of path flows might correspond to the same arc flows.

Note that to insure the uniqueness of (v,u) , Theorem 6.1 requires that both of the functions t and $-D$ are strictly monotone. Our next result shows that the strict monotonicity of $-D$ can be relaxed and, moreover, that uniqueness of u is maintained if either t or $-D$ is strictly monotone.

THEOREM 6.2: *For a complete network (N,A) suppose that t and $-D$ in the additive traffic equilibrium problems (3.1) and (3.2) are both monotone functions. If either of t or $-D$ is strictly monotone, then u is unique. Also, if t is strictly monotone, then (v,u) is unique.*

PROOF: Suppose that $x^1 = (h^1, u^1)$ and $x^2 = (h^2, u^2)$, $x^1 \neq x^2$, are two solutions to the equilibrium problem. Nonnegativity of x^1 , x^2 , $F(x^1)$, and $F(x^2)$ and the complementarity conditions $x^1 F(x^1) = 0$ and $x^2 F(x^2) = 0$ imply that

$$(x^1 - x^2)^T [F(x^1) - F(x^2)] \leq 0$$

or, substituting for F and $(h,u)^T$ for x

$$\begin{aligned} & (h^1 - h^2)^T (\Delta^T t(\Delta h^1) - \Gamma u^1 - \Delta^T t(\Delta h^2) + \Gamma u^2) \\ & + (u^1 - u^2)^T (\Gamma^T h^1 - D(u^1) - \Gamma^T h^2 - D(u^2)) \leq 0 \end{aligned}$$

or,

$$(6.2) \quad (\Delta h^1 - \Delta h^2)^T (t(\Delta h^1) - t(\Delta h^2)) + (u^1 - u^2)^T (-D(u^1) + D(u^2)) \leq 0 .$$

But both t and $-D$ are monotone functions, thus each term in (6.2) is zero; that is,

$$(6.3) \quad (\Delta h^1 - \Delta h^2)^T (t(\Delta h^1) - t(\Delta h^2)) = 0 ,$$

and

$$(6.4) \quad - (u^1 - u^2)^T (D(u^1) - D(u^2)) = 0 .$$

If $-D$ is strictly monotone, then equation (6.4) implies that $u^1 = u^2$, or u is unique.

Now, suppose that t is strictly monotone. Then (6.3) implies that $v^1 = \Delta h^1 = \Delta h^2 = v^2$, or that the arc volume vector v is unique. But uniqueness of the arc volume vector implies that the travel time, $t_a(v)$, on each arc is unique, which obviously implies that u is unique. \square

Whenever t_a is a function only of the total volume in the arc, as when all the traffic from different origins have the same effect on the travel time of each arc, and there is no interaction between opposing lanes of two-way traffic or right or left turn penalties, then the strictly monotone condition on t can be relaxed for the uniqueness results.

COROLLARY 6.1: (Special case) *For a strongly connected network (N,A) suppose that each t_a of the additive traffic equilibrium problems (3.1) and (3.2) is a function only of v_a , and that it is monotone. Also, suppose that $-D$ is monotone. Then u is unique.*

PROOF: By definition t , the vector of the volume delay function, is monotone because each of its components is monotone. Thus equation (6.3) in the proof of Theorem 6.2 is valid. But since each component of t is monotone, (6.3) can be separated into a single term for each arc:

$$(v_a^1 - v_a^2)(t_a(v_a^1) - t_a(v_a^2)) = 0$$

This equation implies that $t_a(v_a^1) = t_a(v_a^2)$, or that the travel time on each arc is unique and, consequently, that u , the minimum path travel time, is unique. □

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