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CORRIGENDUM

Equilibrium and stability of collisionless systems in the paraxial limit

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The boundary term in the energy integral (105) is superfluous and should be dropped. If we use the Jacobian relation (86) to transform the dummy variables of integration from ψ, θ to x, y (recalling that $B = B_{\text{max}} =$ const. at the ends), we see that this term reduces to the integral of a fixed function of x and y over a fixed region of the (x, y) plane. In other words, it reduces to a pure constant.

It was thought (erroneously) that the boundary term was needed in order that the minimum-energy configuration would satisfy the open-ended boundary condition $i = 0$. The conclusion (that $i = 0$ is a minimization condition) remains valid, but it requires a better proof. Let us introduce an auxiliary Lagrange multiplier, $\lambda(\psi, \theta, \pm L)$, to represent the Jacobian constraint at the ends. We can then replace (110) and (111) with

$$\begin{aligned}\Delta U &= \Delta U - \int \kappa \Delta(\mathbf{x}_\psi . \mathbf{x}_\theta^*)\, d\psi\, d\theta\, dz - \int \lambda \Delta(\mathbf{x}_\psi . \mathbf{x}_\theta^*)\, d\psi\, d\theta \big|_{-L}^{L} \\ &= \int \left[\frac{Q}{B}\mathbf{x}_z . \boldsymbol{\xi}_z + \kappa(\mathbf{x}_\psi^* . \boldsymbol{\xi}_\theta - \mathbf{x}_\theta^* . \boldsymbol{\xi}_\psi)\right] d\psi\, d\theta\, dz + \int \lambda(\mathbf{x}_\psi^* . \boldsymbol{\xi}_\theta - \mathbf{x}_\theta^* . \boldsymbol{\xi}_\psi)\, d\psi\, d\theta \big|_{-L}^{L} \\ &= \int \boldsymbol{\xi} . \left[-\left(\frac{Q\mathbf{x}_z}{B}\right)_z - \kappa_\theta \mathbf{x}_\psi^* + \kappa_\psi \mathbf{x}_\theta^*\right] d\psi\, d\theta\, dz \\ &\qquad + \int \boldsymbol{\xi} . (B_{\text{max}} \mathbf{x}_z - \lambda_\theta \mathbf{x}_\psi^* + \lambda_\psi \mathbf{x}_\theta^*)\, d\psi\, d\theta \big|_{-L}^{L}. \end{aligned} \tag{C 1}$$

The interior Euler–Lagrange condition is again (114), but the corresponding condition at the ends now takes the form

$$B_{\text{max}} \mathbf{x}_z = \lambda_\theta \mathbf{x}_\psi^* - \lambda_\psi \mathbf{x}_\theta^*, \tag{C 2}$$

or (using (96))

$$\lambda_\psi = -B_{\text{max}}^2 \mathbf{x}_\psi . \mathbf{x}_z, \quad \lambda_\theta = -B_{\text{max}}^2 \mathbf{x}_\theta . \mathbf{x}_z. \tag{C 3}$$

This, however, reduces immediately to $i = 0$, as was claimed. Indeed,

$$i = B_{\text{max}}(\mathbf{x}_{z\psi} . \mathbf{x}_\theta - \mathbf{x}_{z\theta} . \mathbf{x}_\psi) = B_{\text{max}}^{-1}(-\lambda_{\theta\psi} + \lambda_{\psi\theta}) = 0, \tag{C 4}$$

using (98). It also reduces to the form (118),

$$\mathbf{x}_z = \mathsf{C} . \mathbf{x}, \tag{C 5}$$

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by the same argument as in the original work. (This is the form in which it was used in all the subsequent calculations.)

Equations (C 3) may be integrated to yield

$$\lambda = -\tfrac{1}{2}B_{\max}^2 \mathbf{x}.\mathbf{x}_z = -\tfrac{1}{2}B_{\max}^2 \mathbf{C}:\mathbf{xx}, \tag{C 6}$$

plus an inessential constant of integration. Indeed, if α denotes either ψ or θ, then $(\mathbf{C}:\mathbf{xx})_\alpha = 2\mathbf{C}:\mathbf{xx}_\alpha = 2\mathbf{x}_z.\mathbf{x}_\alpha$, so that differentiation of (C 6) obviously leads back to (C 3).

The calculation of the second variation (in §5) is essentially unchanged, except for one detail. If (C 1) replaces (111) as the general formula for ΔU, then the expression

$$\begin{aligned}\Sigma_1 &= -\tfrac{1}{2}B_{\max}\int \boldsymbol{\xi}.\Delta(\mathbf{C}.\mathbf{x})\,d\psi\,d\theta|_{-L}^{L}\\ &= -\tfrac{1}{2}B_{\max}\int \mathbf{C}:\boldsymbol{\xi\xi}\,d\psi\,d\theta|_{-L}^{L},\end{aligned} \tag{C 7}$$

appearing on both sides of (161), should be amended to

$$\begin{aligned}\Sigma_2 &= -\frac{1}{2}\int \boldsymbol{\xi}.\Delta(\lambda_\theta \mathbf{x}_\psi^* - \lambda_\psi \mathbf{x}_\theta^*)\,d\psi\,d\theta|_{-L}^{L}\\ &= \tfrac{1}{2}B_{\max}^{-1}\int \boldsymbol{\xi}.\Delta(\lambda_\theta \nabla\theta + \lambda_\psi \nabla\psi)\,d\psi\,d\theta|_{-L}^{L}\\ &= \tfrac{1}{2}B_{\max}^{-1}\int \boldsymbol{\xi}.\Delta(\nabla\lambda)\,d\psi\,d\theta|_{-L}^{L}\end{aligned} \tag{C 8}$$

(see (94)). It will be shown, however, that Σ_1 and Σ_2 are actually identical, so that no *further* change is implied. (In particular, there is no change at all in the final result.)

Observing that $\nabla(\mathbf{C}:\mathbf{xx}) = 2\mathbf{C}.\mathbf{x}$, we put

$$\begin{aligned}\Sigma_2 - \Sigma_1 &= \tfrac{1}{2}B_{\max}^{-1}\int \boldsymbol{\xi}.\Delta(\nabla\zeta)\,d\psi\,d\theta|_{-L}^{L}\\ &= -\frac{1}{2}\int \boldsymbol{\xi}.\Delta(\zeta_\theta \mathbf{x}_\psi^* - \zeta_\psi \mathbf{x}_\theta^*)\,d\psi\,d\theta|_{-L}^{L},\end{aligned} \tag{C 9}$$

where $\zeta = \lambda + \tfrac{1}{2}B_{\max}^2\mathbf{C}:\mathbf{xx}$. Next, using the fact that $\zeta = 0$ in the unperturbed equilibrium, according to (C 6), we rewrite the last expression as

$$-\frac{1}{2}\int \boldsymbol{\xi}.(\mathbf{x}_\psi^*\Delta\zeta_\theta - \mathbf{x}_\theta^*\Delta\zeta_\psi)\,d\psi\,d\theta|_{-L}^{L}. \tag{C 10}$$

Finally, after an integration by parts using the boundary condition (109), we arrive at

$$\Sigma_2 - \Sigma_1 = \frac{1}{2}\int \Delta\zeta(\boldsymbol{\xi}_\theta.\mathbf{x}_\psi^* - \boldsymbol{\xi}_\psi.\mathbf{x}_\theta^*)\,d\psi\,d\theta|_{-L}^{L} = 0, \tag{C 11}$$

as claimed (see (139)).

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