Equilibrium Concepts for Social Interaction Models†

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Abstract

To Come

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1 Introduction

This paper describes the relationship between two different binary social interaction models found in the literature. The mean field model of Brock and Durlauf (1995) is in its essence a static Nash equilibrium model in which expected utility preferences have been replaced by a random utility model. The strategy adjustment model of Blume (1994) is a description of a stochastic population process similar to Blume (1993), Kandori, Mailath, and Rob (1993) and Young (1993).

The mean field model suggests a differential equation whose steady states are precisely the mean field equilibria. We show that for large player populations, the solution path of the differential equation from given initial conditions closely approximates the sample path of the population process from the same starting point. This result is well-known in the population biology literature and has also been demonstrated in some population game models.

More interesting is that the differential equation also carries information about the asymptotic behavior of the population process. As the population size becomes large, any (weak convergence) accumulation point of the sequence of invariant distributions has support contained in the set of stable steady states of the differential equation. We characterize (weak) accumulation points of the sequence of suitably scaled invariant distributions for the population process. In general, the limit distributions distribute their mass among the mean field equilibria. For two particular cases, the constant tremble probability model of Kandori, Mailath and Rob and Young and the logit choice model of Blume and Brock and Durlauf, we demonstrate that the sequence of invariant distributions converges and we compute the limit.

The typical population game analysis fixes a population size and investigates the limit behavior of the sequence of invariant distributions as the stochastic component of choice disappears. These so-called "stochastic stability results" have been used to justify a particular selection from the set of Nash equilibria of the static game which drives the population process. The noisy choice is just a means to an equilibrium selection technique. We take seriously both the dynamic models and noisy choice. Consequently for us the invariant distributions are interesting in their own right rather than as a means to an end, and we want to understand the behavior of these models when there is a significant random component to choice.

Density-dependent population processes arise frequently in economic analysis, and most often they are studied by examining a differential equation which describes the evolution of mean behavior. The rationale for this approach is an appeal to a law of large numbers. For a particular class of game-theoretic models we make the large numbers argument precise, and clarify what can be learned from it. We expect that our results can be extended to some of the literature on search and sorting which proceeds in this manner, and we believe this to be an important area for future research.

2 The Structure of Interactions-Based Models

The object of interactions-based models is is to understand the behavior of a population of economic actors rather than that of a single actor. The focus of the analysis is the externalities across actors. These externalities, the source of the social interactions, are taken to be direct. The decision problem of any one actor takes the decisions of other actors to be parametric. Hence the interactions approach treats aggregate social behavior as a statistical regularity of the individual interactions. A second feature of these models is that individual behavior is not as tightly modeled as it is in traditional economic equilibrium models. Individual choice is guided by payoffs, but has a random component. This randomness can be attributed to to some form of bounded rationality. In static equilibrium models it may also be interpreted as an unobserved agent characteristic.

In this paper we focus on a simple class of interaction models with strategic complementarities. Formally, consider a population of I individuals. Suppose that each individual chooses one of two actions, labeled -1 and $+1$. Suppose that each individual's utility is the sum of utilities from pairwise interactions with every other player. Actor i's expected utility is

$$V_i(\omega_i) = h_i\omega_i - E\big\{\sum_j J_{i,j}(\omega_i - \omega_j)^2\big\} + \epsilon(\omega_i)$$

This specification can be decomposed into a private component, $h_i\omega_i + \epsilon(\omega_i)$, and the interaction effect, $E\big\{\sum_j J_{i,j}(\omega_i - \omega_j)^2\big\}$. The private component can be further decomposed (without loss of generality) into its mean, $h_i\omega_i$, and the stochastic deviation $\epsilon(\omega_i)$. The terms $J_{i,j}$ is a a measure of the disutility of non-conformance. When the $J_{i,j}$ are all positive there is an incentive to conform. The presence of positive conformity effects gives rise to multiple equilibria and interesting dynamics. Our methods also encompass the case of negative conformity effects, but the results are less interesting both economically and technically.

For binary choice, this specification of preferences is quite general. Any model in which the utility of action ω_i to individual i is the sum of the utilities from pairwise interactions with other players can be modeled this way. This specification does not include some interesting models of strategic complementarities, such as the stag hunt game. Multiplying out the quadratic and renormalizing,

$$V_i(\omega_i) = h_i\omega_i + 2E\Big\{\sum_j J_{i,j}\omega_i\omega_j\Big\} + \epsilon(\omega_i) \tag{1}$$

The random terms are independent, and we assume that the random variable $\epsilon_i(-1) - \epsilon_i(1)$ has mean 0 and cdf $F(z)$. Then

$$\begin{aligned} \mathrm{Prob}(\omega_i = 1) &= \mathrm{Prob}\big(V_i(1) > V_i(-1)\big) \\ &= F(2h_i + 4E\sum_j J_{i,j}\omega_j) \end{aligned}$$

Different specifications of the h_i and J_{ij} coefficients give rise to models with very different kinds of behavior. In this paper we will study *uniform global interaction*. That is, $J_{ij} \equiv J/2(I-1)$ and $h_i \equiv h$. Interactions with all other players are weighted equally, and so mean utility is the sum of a private effect, the $h_i\omega_i$ term, and a social effect which places a weight of J on the covariance of i's play with mean play of all other players. Also, the private terms are identical across players.[1] Under these assumptions, the individual

[1]The other leading example of social interaction is *uniform local interaction*, studied by (Blume 1993) and (Ellison 1993). Here $h_i \equiv h$ and $J_{i,j} = J$ or 0 depending upon whether or not i and j are neighbors. (Ellison uses a different model of the stochastic component.)

choice probabilities are

$$\text{Prob}(\omega_i = 1) \;=\; F\Bigl(2h + 2\frac{J}{I-1}\sum_{j\neq i} E\omega_j\Bigr) \tag{2}$$

An important special case arises when the random terms are assumed to be distributed according to the extreme value distribution with parameter β_i. That is,

$$\text{Prob}\bigl(\epsilon(-1) - \epsilon(1) < z\bigr) = \frac{1}{1 + \exp(-\beta_i z)}, \qquad \beta_i > 0$$

This model reduces to an instance of the standard logit binary choice framework when there are no interaction effects; that is, when $J_{i,j} \equiv 0$.

From this distribution the individual choice probabilities can be computed.

$$\text{Prob}(\omega_i = 1) \;=\; \frac{1}{1 + \exp -2\beta\Bigl(h + \frac{J}{I-1} E\sum_j \omega_j)\Bigr)} \tag{3}$$

When β is very large, individual i will choose an action to maximize mean utility

$$EV_i(\omega_i) = \Bigl(h + \frac{J}{I-1}\sum_{j\neq i} E\omega_j\Bigr)\omega_i$$

with probability near 1. When β is 0 the player will choose by flipping a coin.

Another important special case is that where the random terms are assumed to be distributed such that with probability ϵ they take on the value $A > |EV(1) - EV(-1)|$ and with probability $1-\epsilon$ they take on the value 0. Then[2]

$$\text{Prob}(\omega_i = 1) \;=\; \begin{cases} \epsilon(1-\epsilon) & \text{if } EV_i(1) < EV_i(-1), \\ 1 - \epsilon(1-\epsilon) & \text{if } EV_i(1) > EV_i(-1), \\ 1/2 & \text{if } EV_i(1) = EV_i(-1). \end{cases} \tag{4}$$

[2]When $EV_i(1) = EV_i(-1)$, individual i draws until she gets a non-zero realization of $\epsilon(-1) - \epsilon(1)$.

This model implements the "tremble" or "mistakes" model of Kandori, Mailath, and Rob (1993) and Young (1993). As ϵ becomes small, the probability of best responding approaches 1.

Equation (2) describes the probabilities with which the actions available to player i will be taken. This choice model is not closed, however, because we have not specified how the expectation in (2) is to be taken. In fact it is a conditional probability, and different choices for on what it is conditioned give rise to the different models which we consider in this paper.

3 Static Equilibrium: The Brock-Durlauf Model

One approach to closing the model is that suggested by Nash equilibrium. That is, each individual i has beliefs about all the ω_j, and these beliefs are correct. This specification gives the Brock and Durlauf (1995) model. Formally, suppose that each player believes that the expectation of the action of each of his opponents is m. Equation (2) becomes

$$\mathrm{Prob}(\omega_i = 1) \;=\; F(2h + 2Jm) \tag{5}$$

If this guess is to be correct, it must be that

$$m = 1 \cdot \mathrm{Prob}(\omega_i = 1) + (-1) \cdot \mathrm{Prob}(\omega_i = -1) \tag{6}$$

which is the equilibrium condition that closes the model. It will be convenient to rewrite this condition in terms of the log-odds function $g(z) = \log F(z) - \log\bigl(1 - F(z)\bigr)$. In all that follows we need to make sure that the support of $F(\,\cdot\,)$ is large enough that $g(z)$ is everywhere defined and that the population externality is present.

Axiom 1. *$F(z) > 0$ for all z in the interval $[2h - 2J, 2h + 2J]$ and $F(2h - 2J) < F(2h + 2J)$.*

Substituting into equation (6),

$$\begin{aligned} m &= \frac{\exp g(2h+2Jm)}{1+\exp g(2h+2Jm)} - \frac{1}{1+\exp g(2h+2Jm)} \\ &= \frac{\exp \frac{1}{2} g(2h+2Jm)}{\exp \frac{1}{2} g(2h+2Jm) + \exp -\frac{1}{2} g(2h+2Jm)} \\ &= \tanh\left(\frac{1}{2} g(2h+2Jm)\right) \end{aligned} \tag{7}$$

For the logit model, equation (5) becomes

$$\mathrm{Prob}(\omega_i = 1) \; = \; \frac{1}{1+\exp -2\beta(h+Jm)}$$

The log-odds function is $g(z) = \beta z$, and so the equilibrium condition (7) is

$$m = \tanh \beta(h+Jm) \tag{8}$$

Equation (7) is well-known in the world of statistical physics, where it has an important physical interpretation, and is known as the Curie-Weiss model of magnetization. The following characterization of the solutions to (8) is well known:

Theorem 1 (Static Equilibrium).

1. *If $\beta J \leq 1$ and $h = 0$, then $m = 0$ is the unique solution to* (8)*.*
2. *If $\beta J > 1$ and $h = 0$, then there are three solutions: $m = 0$ and $m = \pm m^*(\beta J)$. Furthermore, $\lim_{\beta J \to \infty} m^*(\beta J) = 1$.*
3. *If $h \neq 0$ and $J > 0$, then there is a threshold $C(h) > 0$ (which equals $+\infty$ if $h \geq J$) such that (a) for $\beta h < C(h)$, there is a unique solution, which agrees with h in sign; and (b) for $\beta h > C(h)$ there are three solutions, only one of which agrees with h in sign. Furthermore, as β becomes large the extreme solutions converge to ± 1.*
4. *If $J < 0$, then there is a unique solution which agrees with h in sign.*

This theorem illustrates both the nonlinearities and the multiple steady states which are the hallmarks of interacting systems. The model is nonlinear with respect to a change in h, the private component of preference, on the mean behavior m of the population. Indeed, the effect of a change in h may be to increase the number of equilibria, which will exceed one when the strength of interactions is great enough.

The underlying strategic situation for $J > |h|$ corresponds to a coordination game played by a population of opponents, wherein player i's preferences are the mean preferences $h + J\sum_{j\neq i}\omega_j/(I-1)$. The strategy choice $+1$ (-1) is risk-dominant if $h \geq 0$ $(h \leq 0)$. As β becomes large, the two extreme solutions converge to the pure strategy Nash equilibria. When $h \neq 0$ the middle equilibrium will not converge to the mixed Nash equilibrium because the choice probabilities (3) impose a particular randomization when $V_i(1) = V_i(-1)$ which will be incompatible with that required to implement the mixed equilibrium.

For the mistakes model, equation (5) becomes

$$\mathrm{Prob}(\omega_i = 1) \;=\; \begin{cases} \epsilon(1-\epsilon) & \text{if } 2h + 2Jm < 0, \\ 1-\epsilon(1-\epsilon) & \text{if } 2h + 2Jm > 0, \\ 1/2 & \text{if } 2h + 2Jm = 0. \end{cases} \tag{9}$$

The equilibrium condition is that m is any solution to the following equations:

$$m = \begin{cases} 1 - 2\epsilon(1-\epsilon) & \text{if } h + J - 2J(\epsilon(1-\epsilon) > 0, \\ 2\epsilon(1-\epsilon) - 1 & \text{if } h - J + 2J(\epsilon(1-\epsilon) < 0, \\ 0 & \text{if } h = 0. \end{cases} \tag{10}$$

Again multiple solutions are possible for small enough ϵ. Due to the discontinuities in the choice probabilities (9), there will typically either be one or two solutions, but never three solutions unless $h = 0$.

The parameter m is of interest to the modeler as well as to the actors. Because this model preserves the factorization of the joint distribution of choices into the product of the distribution of individual choices, a strong law guarantees that m is approximately the (sample) average choice when I is large.

4 Dynamics

Following Blume (1993), Kandori, Mailath, and Rob (1993) and Young (1993), interest has developed in stochastic processes wherein individuals in a population of players adapt their strategic choice to the play of the population. At randomly chosen moments players observe the play of their opponents and respond by by choosing a new strategy according to a random utility model. The stochastic processes of individual response have implications for the emergent dynamics of population behavior.

4.1 The Population Process

We formalize this model by giving each individual a Poisson alarm clock. When it rings, she revises her choice. Formally, each actor i is endowed with a collection of random variables $\{\tau_n^i\}_{n=1}^{\infty}$ such that each $\tau_n^i - \tau_{n-1}^i$ is exponentially distributed with mean 1, and all such differences are independent of all others, hers and the other actors'. At each time τ_n^i individual i chooses a new action by applying the random utility model of equation (2). Here she takes the expectation given certain knowledge of the ω_j at time t. That is, she chooses according to the transition probability

$$\mathrm{Prob}(\omega_{i\,t+} = 1|\omega_t) \;=\; F\left(2h + 2\frac{J}{I-1}\sum_{j\neq i}\omega_{jt}\right) \tag{11}$$

Implicit in this equation is the fact that players are myopic in (stochastically) best-responding to the current play of the population rather than some forecast of future paths of play. This assumption is much discussed in the literature and will not be defended here.

The process of individual strategy revision is a continuous time Markov process that changes state in discrete jumps. We are interested in tracking only the aggregate $S_t = \sum_{i=1}^{I}\omega_{it}$ rather than the behavior of each individual. The process $\{S_t\}_{t=0}^{\infty}$ is also a Markov jump process, whose states are $S^I = \{-I, -I+2, \ldots, I-2, I\}$. This process changes state whenever an actor changes her choice. If an actor changes from -1 to $+1$ S_t increases by 2, and it decreases by 2 whenever an actor changes in the opposite direction.

These are the only possible transitions, and so the process $\{S_t\}_{t\geq 0}$ is a birth-death process. The transition rates can be computed from the conditional probability distribution (11). Suppose the system is in state S. It transits to state $S+2$ only when a revision opportunity comes to one of the $(S-I)/2$ actors currently choosing -1, and that actor chooses $+1$.[3] The probability of a -1 actor making this choice is

$$F\left(2h + 2\frac{J}{I-1}S_t\right)$$

It will be convenient to make use of the log-odds function $g(z) = \log F(z) - \log\big(1 - F(z)\big)$. In terms of $g(z)$, $F(z) = \exp g(z)/1 + \exp g(z)$.

Putting this together, the transition rate from S to $S+2$ in a population of size I is

$$\lambda_S^I = \frac{I-S}{2}\frac{\exp g\left(2h + \frac{2J}{I-1}(S+1)\right)}{1 + \exp g\left(2h + \frac{2J}{I-1}(S+1)\right)}.$$

A similar computation gives the transition rate in the other direction. To transit from $S+2$ back to S requires that one of the $(S+2+I)/2$ actors choosing $+1$ switches to -1. The transition rate is

$$\mu_{S+2}^I = \frac{I+S+2}{2}\frac{1}{1 + \exp g\left(2h + \frac{2J}{I-1}(S+1)\right)}.$$

Since we will study the behavior of processes with different population sizes, we scale them so they all sit in the same state space, $[-1,1]$, by defining $m_t = S_t/I$. The process with population size takes values in $\{-1, -1 + 2/I, \ldots, 1 - 2/I, 1\} = M_I \subset [-1,1]$. The process has birth rates and death

[3] There are other imaginable transitions, such as where two -1 actors switch to $+1$ and one $+1$ actor switches to -1, but these transitions all involve the simultaneous arrival of revision opportunities to more than one actor, and is thus a 0-probability transition.

rates

$$\begin{aligned}\lambda^I_m &= \frac{I}{2}(1-m)\frac{\exp g\big(\Delta(m)\big)}{1+\exp g\big(\Delta(m)\big)} \\ \mu^I_{m+2/I} &= \frac{I}{2}(1+m+\frac{2}{I})\,\frac{1}{1+\exp g\big(\Delta(m)\big)}\end{aligned} \tag{12}$$

respectively, where

$$\begin{aligned}\Delta(m) &= 2\left(h+\frac{J}{I-1}(S+1)\right) \\ &= 2\left(h+J\frac{I}{I-1}m+\frac{J}{I-1}\right) \\ &\approx 2(h+Jm)\end{aligned}$$

for large I.

4.2 Short Run Dynamics

The birth and death rates are the time derivatives of the transition probabilities. Thus they can be used to characterize the rates of change of expected values of functions of the state. Thus

$$\begin{aligned}\frac{d}{d\tau}E\{f(m_{t+\tau})|m_t=m\}\big|_{\tau=0} \;&=\; \lambda^I_m\left(f\big(m+\frac{2}{I}\big)-f(m)\right)+\mu^I_m\left(f\big(m-\frac{2}{I}\big)-f(m)\right) \\ &= \frac{I}{2}\frac{(1-m)\exp g\big(\Delta(m)\big)\big(f(m+\frac{2}{I})-f(m)\big)}{1+\exp g\big(\Delta(m)\big)} \\ &\quad +\frac{I}{2}\frac{(1+m)\big(f(m-\frac{2}{I})-f(m)\big)}{1+\exp g\big(\Delta(m)\big)} \\ &= \frac{(1-m)\exp g\big(\Delta(m)\big)f'(m)+(1+m)f'(m)}{1+\exp g\big(\Delta(m)\big)} \\ &= \left(\tanh\big(\frac{1}{2}g\big(\Delta(m)\big)-m\right)f'(m)+O(I^{-2})\end{aligned}$$

When $f(m) = m$, this differential equation gives

$$\frac{d}{d\tau} E\{m_{t+\tau} | m_t = m\}\Big|_{\tau=0} = \tanh\Big(\frac{1}{2} g \Big(\Delta(m)\Big)\Big) - m + O(I^{-2})$$

Taking the $I \to \infty$ limit suggests the following differential equation, called the *mean field equation*:

$$\dot{m} = \tanh\Big(\frac{1}{2} g(2h + 2Jm)\Big) - m \tag{13}$$

For large I solutions to the diffenrential equation (13) approximate the sample path behavior of the process $\{m^I_\tau\}$ on finite time intervals. The following theorem is an application of a standard strong law of large numbers for density dependent population processes. (An elementary proof is too long to be given here. A quick high-tech proof can be found in Chapter 11.2 of Ethier and Kurtz, 1986.)

Theorem 2 (Sample-Path Behavior). *Let $\{m^I_t\}_{t \geq 0}$ refer to the average process with population size I. Suppose $m^I_0 = m_0$ and let $m(\tau)$ be the solution to the mean field equation* (13) *with initial condition $m(0) = m_0$. Then for every $t \geq 0$,*

$$\lim_{I \to \infty} \sup_{\tau \leq t} |m^I_\tau - m_\tau| = 0 \qquad a.s.$$

The content of the Theorem is that when I is large, the stochastic perturbations from individuals' random choices more or less averages out, and so the mean field path is nearly followed for some time. But in the long run some large deviation will occur, and ultimately the sample path will diverge from its mean field approximation.

4.3 Asymptotic Behavior

It is apparent that the steady states of the mean field equations (13) are precisely those states which satisfy the equilibrium condition (7) of the Brock-Durlauf model. Furthermore the sample path theorem suggests that there is motion towards at least the stable steady states of (13). This suggests that the long run behavior of the process should tend to be concentrated around the stable steady states of (13), a subset of the mean field equilibria.

The birth-death process with transition rates given by (12) is irreducible, and so for each population size I the population process has a unique invariant distribution ρ_I, which describes the long-run behavior of the process. The next Theorem shows that the intuition of the previous paragraph is correct. For large I the invariant distribution tends to pile up mass near one or more of the stable steady states of (13).

Since the state space $[-1, 1]$ is compact, the sequence of invariant measures $\{\rho_I\}$ is relatively compact, and so has weakly convergent subsequences. The next Theorem demonstrates properties about the subsequential limits. In all the applications we have examined the sequence $\{\rho_I\}$ converges, and the proof of the Theorem suggests a sufficient condition for convergence. Define the function

$$r(m) = \left(\left(\frac{1+m}{2}\right)^{1+\frac{m}{2}}\left(\frac{1-m}{2}\right)^{1-\frac{m}{2}}\right)^{-1} \exp\frac{1}{2}\int_{-1}^{m} g\big(\Delta(x)\big)\,dx \tag{14}$$

Theorem 3 (Asymptotic Behavior). *Let ρ be a weak subsequential limit of the sequence $\{\rho_I\}_{I=2}^{\infty}$ of invariant distributions for population processes with population size I. Then* $\operatorname{supp}\rho$ *is the set of global maxima of $r(m)$, and is contained in the open interval $(-1, 1)$. If the set of stationary states of the mean field equations* (13) *is the finite union of points and intervals, then* $\operatorname{supp}\rho$ *is the finite union of points and intervals, all of which are locally stable.*

This Theorem implies that if the population I is large, mean behavior is most often near the stable states of the mean-field equation.

Proof. For a population of size I the invariant measure ρ^I on M_I satisfies the relationship

$$\rho^I(m)\lambda^I_m = \rho^I\left(m + \frac{2}{I}\right)\mu^I_{m+\frac{2}{I}}$$

Consequently

$$\begin{aligned}\rho^I(m) &= z_I \binom{I}{I\frac{1+m}{2}} \exp\bigl\{g\bigl(\Delta(-1)\bigr) + \cdots + g\bigl(\Delta(m)\bigr)\bigr\}\\ &\approx \tilde{z}_I I^{-\frac{1}{2}} \sqrt{\frac{2}{\pi(1-m^2)}} r(m)^I\\ &\equiv \tilde{\rho}^I(m)\end{aligned}$$

where z_I and $\tilde{z}_I$ are normalizing factors.

The approximation comes from Stirling's formula and the Riemann sum approximation to the integral. The approximation is such that $\rho^I - \tilde{\rho}^I$ converges uniformly to 0 on compact subsets of the interior of M. Let m^* denote a global maximum of $r(m)$. The function $r(m)$ is strictly increasing in a neighborhood of $m = -1$, strictly decreasing in a neighborhood of $m = 1$, and $r(-1) = 1$. Consequently all its critical points are interior, and the global maximum of $r(m)$ exceeds 1. Let O be an open neighborhood of $\operatorname{argmax} r(m)$ and let C be a compact set disjoint from the closure of O. Then $\tilde{\rho}^I(O)/\tilde{\rho}^I(C) \to +\infty$, so $\lim_{I\to\infty} \tilde{\rho}^I(O) = 1$. Consequently $\lim_{I\to\infty} \rho^I(O) = 1$ and so $\operatorname{supp} \rho \subset \operatorname{argmax} r(m)$. This proves the first part of the Theorem.

It remains only to show that $\operatorname{argmax} r(x)$ is contained in the set of stable equilibria of (13). The derivative of $\log r(m)$ is

$$\begin{aligned}\frac{d}{dm}\log r(m) &= -\frac{1}{2}\log\frac{1+m}{1-m} + \frac{1}{2}g\bigl(\Delta(m)\bigr)\\ &= -\operatorname{arctanh}(m) + \frac{1}{2}g\bigl(\Delta(m)\bigr),\end{aligned}$$

and so the critical points are those m which satisfy the equation

$$m = \tanh\Bigl(g\bigl(\Delta(m)\bigr)/2\Bigr).$$

By hypothesis, the solution set is the union of a finite collection of points $p_1, \dots, p_K$ and intervals, $[a_1, b_1], \dots, [a_L, b_L]$. This union is the set of all critical points of $\log r(m)$, and so the set of global maxima of $r(m)$ is the union of a sub-collection of these elements. Consider a point p_k or a left endpoint a_l in $\operatorname{supp}\rho$. For the point p in question there is an $\epsilon > 0$ such that on the interval $(p - \epsilon, p)$, $d/dm \log r(m) < 0$. Suppose at some point m in this interval, $\dot{m} \leq 0$. Then $m \geq \tanh\Big(g\big(\Delta(m)\big)\Big)/2$. Since arctanh is an increasing function, applying it to both sides of the inequality gives $d/dm \log r(m) \geq 0$, which is a contradiction. A similar argument works for the right side of all singletons and right endpoints to show that on some neighborhood to their right, $\dot{m} < 0$. Consequently they are all locally stable. □

The next result follows from the proof of the Asymptotic Behavior Theorem.

Corollary 1 (Convergence). *If the function $r(m)$ defined in equation (14) has a unique global maximum m^*, then the sequence $\{\rho^I\}$ converges weakly to δ_{m^*}, point mass at m^*.*

4.4 Examples

In the following examples we assume $J > 0$. There is no loss of generality in assuming $h > 0$ because our examples treat the different strategies symmetrically. For more on this, see Blume (1994).

For logit choice, $g(x) = \beta x$. The mean field equation (13) is

$$\dot{m} = \tanh \beta(h + Jm) - m$$

For generic values of the parameters there are either one or three equilibria, and if there are three, the center equilibrium is unstable. When β is small there is a uniqu stable equilibrium. When β is large and $h = 0$ the distributions ρ^I are symmetric, and so ρ^I converges to the distribution which places mass 1/2 on each stable steady state. If $h > 0$ then for all $m > 0$, $r(m) > r(-m)$. $r(m)$ has three critical points, two of which are negative and one which is positive. The positive critical point is the unique global maximizer of $r(m)$, and so ρ^I converges to point mass on the positive equilibrium. In the following picture, the bottom plot shows the mean field equilibrium and the top plot shows invariant distribution probability functions for $I = 50$ (flatter) and $I = 400$, for the logit choice model with $\beta = 1.5$, $h = 0.05$ and $J = 1$.

The Logit Model

For the tremble model, the mean field equation is

$$\dot{m} = \begin{cases} -1 + 2\delta - m & \text{if } h + Jm < 0, \\ 0 & \text{if } h + Jm = 0, \\ 1 - 2\delta - m & \text{if } h + Jm > 0. \end{cases}$$

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