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EQUILIBRIUM FLUCTUATION OF THE ATLAS MODEL

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We study the fluctuation of the Atlas model, where a unit drift is assigned to the lowest ranked particle among a semi-infinite ($\mathbb{Z}_+$-indexed) system of otherwise independent Brownian particles, initiated according to a Poisson point process on $\mathbb{R}_+$. In this context, we show that the joint law of ranked particles, after being centered and scaled by $t^{-\frac{1}{4}}$, converges as $t \to \infty$ to the Gaussian field corresponding to the solution of the Additive Stochastic Heat Equation (ASHE) on $\mathbb{R}_+$ with the Neumann boundary condition at zero. This allows us to express the asymptotic fluctuation of the lowest ranked particle in terms of a fractional Brownian Motion (fBM). In particular, we prove a conjecture of Pal and Pitman [*Ann. Appl. Probab.* **18** (2008) 2179–2207] about the asymptotic Gaussian fluctuation of the ranked particles.

1. Introduction. In this paper, we study the infinite particles Atlas model. That is, we consider the $\mathbb{R}^{\mathbb{Z}_+}$-valued process $\{X_i(t)\}_{i\in\mathbb{Z}_+}$, each coordinate performing an independent Brownian motion except for the lowest ranked particle receiving a drift of strength $\gamma > 0$. For suitable initial conditions, this process is given by the unique weak solution of

$$dX_i(t) = \gamma \mathbf{1}_{\{X_i(t)=X_{(0)}(t)\}}\,dt + dB_i(t), \qquad i \in \mathbb{Z}_+. \tag{1.1}$$

Hereafter, $B_i(t)$, $i \in \mathbb{Z}_+$, denote independent standard Brownian motions and $X_{(i)}(t)$, $i \in \mathbb{Z}_+$, denote the *ranked* particles, that is, $X_{(0)}(t) \le X_{(1)}(t) \le \cdots$. More precisely, recall that $(x_i) \in \mathbb{R}^{\mathbb{Z}_+}$ is rankable if there exists a bijection $\pi : \mathbb{Z}_+ \to \mathbb{Z}_+$ (i.e., permutation) such that $x_{\pi(i)} \le x_{\pi(j)}$ for all $i \le j \in \mathbb{Z}_+$. Such ranking permutation is unique up to ties, which we break in lexicographic order. The equation (1.1) is then well defined if $(X_i(t))_{i\in\mathbb{Z}_+}$ is rankable at all $t \ge 0$ with a measurable ranking permutation.

The Atlas model (1.1) is a special case of diffusions with *rank dependent drifts*. In finite dimensions, such systems are studied in [2], motivated by questions in filtering theory, and in [8, 9], in the context of stochastic portfolio theory. See also [4, 5, 11–13], for their ergodicity and sample path properties, and [6, 19]

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for their large deviations properties as the dimension tends to infinity. The Atlas model is a simple special case [where the drift vector is specialized to $(\gamma, 0, \dots, 0)$] that allows more detailed analysis. In particular, Pal and Pitman [18] consider the infinite dimensional Atlas model (1.1), establishing the well-posedness and the existence of an explicit invariant measure; see also [12, 22].

In this paper, we study the long-time behavior of the ranked particles, in particular the lowest ranked particle. This amounts to understanding competition between the drift γ and the push-back from the bulk of particles (due to ranking). These two effects act against each other, and balance exactly at the critical density 2γ. More precisely, recall from [18] that, starting from $\{X_{(i)}(0)\} \sim \mathrm{PPP}_+(2\gamma)$, the Poisson Point Process with density 2γ on $\mathbb{R}_+ := [0,\infty)$, (1.1) admits a unique weak solution (which is rankable) such that $\{X_{(i)}(t) - X_{(0)}(t)\}_{i\in\mathbb{Z}_+}$ retains the $\mathrm{PPP}_+(2\gamma)$ law for all $t \geq 0$. At this critical density, we show that, for large t and for all i, $X_{(i)}(t)$ fluctuates at $O(t^{\frac{1}{4}})$, and the joint law of the fluctuations of the particles scales to a Gaussian field characterized by ASHE.

Hereafter, we fix $\{X_i(t)\}_{i\in\mathbb{Z}_+}$ to be the unique weak solution of (1.1) starting from $\mathrm{PPP}_+(2\gamma)$. With $Y_i(t) := X_{(i+1)}(t) - X_{(i)}(t)$ denoting the ith gap, such initial condition are equivalent to $X_{(0)}(0) = 0$ and $\{Y_i(0)\}_{i\in\mathbb{Z}_+} \sim \bigotimes_{i\in\mathbb{Z}_+} \mathrm{Exp}(2\gamma)$. We consider the process

$$\mathcal{X}_t^\varepsilon(x) := \varepsilon^{\frac{1}{4}}\big(i_\varepsilon(x) - 2\gamma X_{(i_\varepsilon(x))}(\varepsilon^{-1}t)\big), \qquad i_\varepsilon(x) := \big(2\gamma\varepsilon^{\frac{1}{2}}\big)^{-1}x, \tag{1.2}$$

defined for all $x \in \frac{\varepsilon^{\frac{1}{2}}}{2\gamma}\mathbb{Z}_+$, and *linearly interpolated in* x so that $\mathcal{X}_\cdot^\varepsilon(\cdot) \in C(\mathbb{R}_+^2)$. Recall that the relevant solution of the ASHE, (1.5), is invariant under the scaling $\mathcal{X}_t(x) \mapsto a^{\frac{1}{4}}\mathcal{X}_{t/a}(x/a^{\frac{1}{2}})$, which suggests the scaling of (1.2). Alternatively, this scaling can be understood as choosing the diffusive scaling of (t,x) to respect $B_i(\cdot)$, and choosing the $\varepsilon^{\frac{1}{4}}$ factor to capture the Gaussian fluctuation of $\mathrm{PPP}_+(2\gamma\varepsilon^{-\frac{1}{2}})$.

Let $p_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2}$ denote the standard heat kernel, with the corresponding scaled error function $\Phi_t(x) := \int_\infty^x p_t(y)\,dy$. We use $p_t^{\mathrm{N}}(y,x) := p_t(y-x) + p_t(y+x)$ for the Neumann heat kernel and

$$\Psi_t(y,x) := 2 - \Phi_t(y-x) - \Phi_t(y+x) = \int_y^\infty p_t^{\mathrm{N}}(z,x)\,dz. \tag{1.3}$$

Hereafter, we endow the space $C(\mathbb{R}_+^2)$ the topology of uniform convergence on compact sets, and use $\Rightarrow$ to denote weak convergence of probability measures. Our main result is as follows.

THEOREM 1.1. *Let $\mathcal{X}_\cdot(\cdot)$ denote the $C(\mathbb{R}_+^2)$-valued centered Gaussian process with covariance*

$$\begin{aligned}
&\mathbf{E}(\mathcal{X}_t(x)\mathcal{X}_{t'}(x')) \\
&\quad = 2\gamma\biggl(\int_0^\infty \Psi_t(y,x)\Psi_{t'}(y,x')\,dy \\
&\qquad + \int_0^{t\wedge t'}\int_0^\infty p^N_{t-s}(y,x)p^N_{t'-s}(y,x')\,dy\,ds\biggr).
\end{aligned} \tag{1.4}$$

Then $\mathcal{X}^\varepsilon_\cdot(\cdot) \Rightarrow \mathcal{X}_\cdot(\cdot)$, as $\varepsilon \to 0$.

REMARK 1.2. The limiting process $\mathcal{X}_\cdot(\cdot)$ can be equivalently characterized by the solution of the ASHE (see, e.g., [23]) on $\mathbb{R}_+$,

$$\biggl(\partial_t - \frac{1}{2}\partial_{xx}\biggr)\mathcal{X}_t(x) = (2\gamma)^{\frac{1}{2}}\xi, \qquad t, x > 0, \tag{1.5}$$

with the initial condition $\mathcal{X}_0(x) = \sqrt{2\gamma}B(x)$ and a suitable boundary condition at $x = 0$. Here, $B(x)$ denotes a standard Brownian motion and ξ denotes a 2-dimensional white noise, independent of $B(\cdot)$. In the course of proving Theorem 1.1, extracting the boundary condition requires a *special choice* of the test function [see (1.13)]. From this, we end up with the *Neumann boundary condition*. That is, we declare the semigroup of (1.5) to be $p^{\mathrm{N}}_t(y,x)$, whereby obtaining

$$\mathcal{X}_t(x) = \mathcal{W}_t(x) + \mathcal{M}_t(x) \tag{1.6}$$

for

$$\mathcal{W}_t(x) := \int_0^\infty p^{\mathrm{N}}_t(y,x)\mathcal{X}_0(y)\,dy = \sqrt{2\gamma}\int_0^\infty \Psi_t(y,x)\,dB(y), \tag{1.7}$$

$$\mathcal{M}_t(x) := \sqrt{2\gamma}\int_0^t\int_0^\infty p^{\mathrm{N}}_{t-s}(y,x)\,d\mathscr{W}(s,y), \tag{1.8}$$

where $d\mathscr{W}(s,y) := \xi(s,y)\,ds\,dy$. The former and latter, measurable with respect to B and ξ, respectively, are independent. From (1.7) and (1.8), one then concludes the covariance as given in (1.4).

In retrospect, the Neumann boundary condition represents the conservation of particles at $x = 0$. It is shown in [3] that at the equilibrium density we consider here, $\sup_{s\in[0,t]}\{\varepsilon^{\frac{1}{2}}|X_{(0)}(\varepsilon^{-1}t)|\} \to 0$ almost surely. That is, at the scale $\varepsilon^{-\frac{1}{2}}$ of space, the lowest rank particle stays very close to $x = 0$. Consequently, the flux at $x = 0$ should be zero, which amounts to the Neumann boundary condition.

REMARK 1.3. The limiting process $\mathcal{X}(t,x)$ is the solution to (1.5) with $\mathcal{X}_0(x) = \sqrt{2\gamma}B(x)$, which is invariant in the sense that $\mathcal{X}(t,\cdot) - \mathcal{X}(t,0) \stackrel{\text{distr.}}{=} \sqrt{2\gamma}B(\cdot)$, $\forall t \in \mathbb{R}_+$. More generally, if one starts the Atlas model off equilibrium with $\{\varepsilon^{\frac{1}{2}} X_{(i)}(0)\}_{i\in\mathbb{Z}_+}$ converging in a suitable sense to a nonequilibrium limiting initial condition $\mathcal{X}'_0(\cdot)$, one should obtain the convergence of $\mathcal{X}^\varepsilon(t,x)$ to the solution $\mathcal{X}'(t,x)$ to (1.5) with the initial condition $\mathcal{X}'_0(x)$. A natural special case of such is the equally spaced initial condition $X_{(i)}(0) = i/(2\gamma)$, where $\mathcal{X}'_0(\cdot) = 0$, and hence $\mathcal{X}'(t,x) = \mathcal{M}(t,x)$. This, however, is not directly comparable with convergence of finite dimensional distributions of the gaps. Further, our proof of Theorem 1.1 requires the stationarity of $\{X_{(i)}(\cdot) - X_{(0)}(\cdot)\}_{i\in\mathbb{Z}_+}$ to obtain a priori estimates, and hence does not apply to off-equilibrium initial conditions.

An important consequence of Theorem 1.1 is the following.

COROLLARY 1.4.

(a) *Let $B^{(H)}(\cdot)$ denote the fractional Brownian motion with Hurst parameter H. As $\varepsilon \to 0$, $\varepsilon^{-\frac{1}{4}} X_{(0)}(\varepsilon^{-1}\cdot)$, the scaled fluctuation of the lowest ranked particle, weakly converges to $(2/\pi)^{\frac{1}{4}}\gamma^{-\frac{1}{2}} B^{(\frac{1}{4})}(\cdot)$.*

(b) *As $\varepsilon \to 0$, $\varepsilon^{\frac{1}{4}}(X_{(i_\varepsilon(x))}(\varepsilon^{-1}) - X_{(i_\varepsilon(x))}(0))$ weakly converges to a centered Gaussian with variance $\sigma^2(x)$, satisfying $\sigma(0) = (2/\pi)^{\frac{1}{4}}\gamma^{-\frac{1}{2}}$ and $\lim_{x\to\infty} \sigma(x) = (2\pi)^{-\frac{1}{4}}\gamma^{-\frac{1}{2}}$.*

Indeed, it is not difficult to deduce from (1.4) the covariance of the center Gaussian process $\mathcal{K}_\cdot(x) := (2\gamma)^{-1}(\mathcal{X}_\cdot(x) - \mathcal{X}_0(x))$ for the special case of $x = 0$ and $x \to \infty$, and to arrive at

$$\mathbf{E}\big(\mathcal{K}_t(0)\mathcal{K}_{t'}(0)\big) = \gamma^{-1}(2/\pi)^{\frac{1}{2}}\mathbf{E}\big(B^{(\frac{1}{4})}(t)B^{(\frac{1}{4})}(t')\big), \tag{1.9}$$

$$\lim_{x\to\infty}\mathbf{E}\big(\mathcal{K}_t(x)\mathcal{K}_{t'}(x)\big) = \gamma^{-1}(2\pi)^{-\frac{1}{2}}\mathbf{E}\big(B^{(\frac{1}{4})}(t)B^{(\frac{1}{4})}(t')\big). \tag{1.10}$$

From (1.9)–(1.10), Corollary 1.4(a) immediately follows, and part (b) follows by setting $t = t' = 1$ in (1.9)–(1.10).

Theorem 1.1 is the first result of asymptotic fluctuations of (1.1), with Corollary 1.4(b) resolving the conjecture of Pal and Pitman [18], Conjecture 3. Further, Theorem 1.1 establishes the previously undiscovered connection of (1.1) to ASHE.

REMARK 1.5. In [3], the hydrodynamic limits of the Atlas model (1.1) is studied. For out-of-equilibrium initial conditions, it is shown that $\varepsilon^{\frac{1}{2}} X_{(0)}(\varepsilon^{-1}\cdot)$ converges to a deterministic limit described by the one-sided Stefan's problem. For the symmetric simple exclusion process on $\mathbb{Z}$, [16] shows that the hydrodynamic limit of a driven tagged particle is described by the two-sided Stefan's problem. For the same model, [17] shows that the fluctuation scales to a generalized Ornstein–Uhlenbeck process related to ASHE.

REMARK 1.6. Harris [10] introduces a closely related model of i.i.d. $\mathbb{Z}$-indexed Brownian particles $B_i(t)$, which can be regarded as the bulk version of (1.1). Using an explicit formula for the law of $B_{(0)}(t)$, he shows that at equilibrium with density 2γ, $\lim_{t\to\infty} t^{-\frac{1}{4}}(B_{(0)}(t) - B_{(0)}(0)) \Rightarrow (2\pi)^{-\frac{1}{4}}\gamma^{-\frac{1}{2}}B(1)$. This result is further extended by [7] to the functional convergence $\varepsilon^{\frac{1}{4}}(B_{(0)}(\varepsilon^{-1}\cdot) - B_{(0)}(0)) \Rightarrow (2\pi)^{-\frac{1}{4}}\gamma^{-\frac{1}{2}}B^{(\frac{1}{4})}(\cdot)$.

Intuitively, we expect the Atlas model to behave similarly to the Harris model once we match the equilibrium density. This is indeed confirmed in (1.10). That is, at the bulk ($x \to \infty$) the asymptotic fluctuation of the two systems are approximately equal, to $(2\pi)^{-\frac{1}{4}}\gamma^{-\frac{1}{2}}B^{\frac{1}{4}}(\cdot)$. Somewhat unexpectedly, as shown in Corollary 1.4(a), the $\frac{1}{4}$-fBM fluctuation also appears at $x = 0$, but with a *different* prefactor.

REMARK 1.7. Applying our technique to the Harris model, one may rederive the results of [7, 10]. This provides an explanation of the scaling and the $\frac{1}{4}$-fBM limit as the fluctuation of ASHE at $x = 0$. Specifically, the scaling limit of the Harris model should be ASHE on $\mathbb{R}$ with no boundary condition. Since no drift presents in the Harris model, the latter scaling limit could be deduced directly from the time evolution equation.

REMARK 1.8. The Harris model is generalized in [21] by replacing the ordering with nearest neighbor repulsion through a potential. The authors show that the equilibrium fluctuation converges to an Ornstein–Uhlenbeck process. For the symmetric simple exclusion (without drift) on $\mathbb{Z}$, which is a discrete analog of Harris model, [1] proves a central limit theorem of the fluctuation of a tagged particle at equilibrium. This result is generalized in [14] to include off-equilibrium initial conditions, where the limiting fluctuation is characterized by an Ornstein–Uhlenbeck process.

Our strategy of proving Theorem 1.1 is to characterize, via the empirical measure, the asymptotic behaviors of ranked particles by the ASHE. While this strategy has been widely used for interacting particle systems, in the context of Atlas model (or more generally diffusions with rank-dependent drifts), this is a new approach of characterizing asymptotic behaviors of ranked particles, that has only been used here and in [3]. Further, by focusing on the empirical, we completely bypass the need of local times, which is a major a challenge when analyzing diffusions with rank-dependent drifts.

To define the empirical measure, we consider $w(y) := e^{-y} \wedge 1$, $|\phi|_{\mathscr{Q}} := \sup_{y\in\mathbb{R}} |\phi(y)|/w(y)$, and $\mathscr{Q} := \{\phi \in L^\infty(\mathbb{R}) : |\phi(y)|_{\mathscr{Q}} < \infty\}$. Let $X_i^\varepsilon(t) := \varepsilon^{\frac{1}{2}} X_i(\varepsilon^{-1}t)$, $X_{(i)}^\varepsilon(t) := \varepsilon^{\frac{1}{2}} X_{(i)}(\varepsilon^{-1}t)$ and, for any $\phi \in \mathscr{Q}$, we define the empirical measure Q_t^ε, together with its centered, scaled version $\widehat{Q}_t^\varepsilon$, by

$$\langle Q_t^\varepsilon, \phi\rangle := \sum_{i=0}^{\infty} \phi\big(X_i^\varepsilon(t)\big), \tag{1.11}$$

$$\langle \widehat{Q}_t^\varepsilon, \phi\rangle := \varepsilon^{\frac{1}{4}}\bigg(\langle Q_t^\varepsilon, \phi\rangle - 2\gamma\varepsilon^{-\frac{1}{2}}\int_0^\infty \phi(y)\,dy\bigg), \tag{1.12}$$

which are well defined (see Lemma 3.1). As we are at stationarity, Q_t^ε is a $\mathrm{PPP}_+(2\gamma\varepsilon^{-\frac{1}{2}})$ translated by $X_{(0)}^\varepsilon(t)$, so $\widehat{Q}_t^\varepsilon$ captures the Gaussian fluctuation of $\mathrm{PPP}_+(2\gamma\varepsilon^{-\frac{1}{2}})$ around $2\gamma\varepsilon^{-\frac{1}{2}}\mathbf{1}_{\mathbb{R}_+}(y)\,dy$.

Under this framework, the main challenge of proving Theorem 1.1 is to choose the test function $\mathcal{F}_t^{\varepsilon,a}(x)$ that (i) identifies the relevant boundary condition; and (ii) relates itself to the process $\mathcal{X}_t^\varepsilon(x)$. With

$$\mathcal{F}_t^{\varepsilon,a}(x) := \langle \widehat{Q}_t^\varepsilon, \Psi_{\varepsilon^a}(\cdot, x)\rangle, \tag{1.13}$$

establishing (ii) amounts to approximating the displacement of a ranked particle by the net flux of particles, which we achieve by using stationarity. In Sections 4 and 5, we prove Propositions 1.9 and 1.10, respectively, from which Theorem 1.1 follows immediately.

PROPOSITION 1.9. *Fix any $a \in (\frac{1}{2}, 1)$ and $b \in (0, \frac{1}{4})$. As $\varepsilon \to 0$, $\mathcal{F}_\cdot^{\varepsilon,a}(\cdot + \varepsilon^b) \Rightarrow \mathcal{X}_\cdot(\cdot)$, where $\mathcal{X}_t(x)$ given as in Theorem 1.1.*

PROPOSITION 1.10. *Fix any $a \in (\frac{1}{2}, 1)$ and $b \in (0, \frac{1}{4})$. As $\varepsilon \to 0$, $\mathcal{F}_\cdot^{\varepsilon,a}(\cdot + \varepsilon^b) - \mathcal{X}_\cdot^\varepsilon(\cdot) \Rightarrow 0$.*

2. Outline of the proof of Propositions 1.9 and 1.10. Without loss of generality, we scale the drift $\gamma > 0$ to unity by $X_i(t) \mapsto \gamma X_i(\gamma^{-2}t)$. Hereafter, we *fix* $\gamma := 1$ and use $C(a, b, \dots)$ to denote generic positive finite (deterministic) constant that depends only on the designated variables.

We proceed to describe the time evolution of $\widehat{Q}_t^\varepsilon$. To this end, let

$$\mathscr{Q}_T := \big\{\psi_t(x) \in C^2\big([0, T] \times \mathbb{R}\big) : |\psi|_{\mathscr{Q}_T} < \infty\big\},$$
$$|\psi|_{\mathscr{Q}_T} := \sup_{t\in[0,T]} \big(|\partial_t\psi_t|_{\mathscr{Q}} + |\partial_x\psi_t|_{\mathscr{Q}} + |\partial_{xx}\psi_t|_{\mathscr{Q}} + |\psi_t|_{\mathscr{Q}}\big).$$

We decompose $\widehat{Q}_t^\varepsilon = A_t^\varepsilon + W_t^\varepsilon$, where

$$\langle A_t^\varepsilon, \phi\rangle := -2\varepsilon^{-\frac{1}{4}}\int_0^{X_{(0)}^\varepsilon(t)} \phi(y)\,dy \tag{2.1}$$

records the fluctuation of the lowest ranked particle, and

$$\langle W_t^\varepsilon, \phi\rangle := \varepsilon^{\frac{1}{4}}\bigg(\langle Q_t^\varepsilon, \phi\rangle - 2\varepsilon^{-\frac{1}{2}}\int_{X_{(0)}^\varepsilon(t)}^\infty \phi(y)\,dy\bigg) \tag{2.2}$$

accounts for the fluctuations of the bulk of particles. For any $\psi \in \mathscr{Q}_T$ and $t_0 \in [0, T]$, let

$$M^{\varepsilon}_{t_0,t}(\psi, k) := \varepsilon^{\frac{1}{4}} \sum_{i=0}^{k} \int_{t_0}^{t} \partial_y \psi_s\big(X^{\varepsilon}_i(s)\big)\, dB^{\varepsilon}_i(s), \tag{2.3}$$

which is a $C([t_0, T], \mathbb{R})$-valued martingale in t, where $B^{\varepsilon}_i(\cdot) := \varepsilon^{\frac{1}{2}} B_i(\varepsilon^{-1}\cdot) \overset{\text{distr.}}{=} B_i(\cdot)$.

PROPOSITION 2.1. *For any $T \in \mathbb{R}_+$, $t_0 \in [0, T]$ and $\psi \in \mathscr{Q}_T$, there exists a $C([t_0, T], \mathbb{R})$-valued martingale $M^{\varepsilon}_{t_0,\cdot}(\psi, \infty)$ such that, for all $q \in [1, \infty)$,*

$$\Big\| \sup_{t \in [t_0, T]} \big| M^{\varepsilon}_{t_0,t}(\psi, k) - M^{\varepsilon}_{t_0,t}(\psi, \infty) \big| \Big\|_q \to 0. \tag{2.4}$$

Furthermore, almost surely

$$\begin{aligned} &\langle \widehat{Q}^{\varepsilon}_t, \psi_t \rangle - \langle \widehat{Q}^{\varepsilon}_0, \psi_0 \rangle \\ &\quad = \int_0^t \Big\langle W^{\varepsilon}_s, \Big(\partial_s + \frac{1}{2}\partial_{yy}\Big)\psi_s \Big\rangle ds + \int_0^t \langle A^{\varepsilon}_s, \partial_s \psi_s \rangle\, ds + M^{\varepsilon}_{0,t}(\psi, \infty), \end{aligned} \tag{2.5}$$

for all $t \in [0, T]$.

REMARK 2.2. Proposition 2.1 is established in Section 3, where we derive (2.5) via Itô calculus. In this derivation, the underlying Brownian motions $B_i(t)$, $i \in \mathbb{Z}_+$, collectively contribute

$$\Big(\varepsilon^{\frac{1}{4}} \langle Q^{\varepsilon}_t, (\partial_t + 2^{-1}\partial_{yy})\psi_t \rangle - 2\varepsilon^{-\frac{1}{4}} \int_0^{\infty} \partial_s \psi_s(y)\, dy \Big) dt + dM^{\varepsilon}_{0,t}(\psi, \infty)$$

whereas the drift $\gamma = 1$ at the lowest ranked particle contributes

$$\varepsilon^{-\frac{1}{4}} \partial_y \psi_s\big(X^{\varepsilon}_{(0)}(t)\big)\, dt = \Big(-\varepsilon^{-\frac{1}{4}} \int_{X^{\varepsilon}_{(0)}(t)}^{\infty} \partial_{yy} \psi_s(y)\, dy \Big) dt.$$

These, when combined together, give the expression (2.5).

Based on Proposition 2.1, in Section 3 we establish the following a priori estimate of $X^{\varepsilon}_{(0)}(\cdot)$.

PROPOSITION 2.3. *Fixing any $q \in (1, \infty)$, $b \in [0, \frac{1}{4})$ and $T \in \mathbb{R}_+$, we let $\tau^{\varepsilon}_b := \inf\{t \geq 0 : |X^{\varepsilon}_{(0)}(t)| \geq \varepsilon^b\}$. There exists $C = C(T, b, q) < \infty$ such that, for all $\varepsilon \in (0, (2q)^{-2}]$,*

$$\mathbf{P}\big(\tau^{\varepsilon}_b \leq T\big) \leq C \varepsilon^{(\frac{1}{4} - b)q - 1}. \tag{2.6}$$

REMARK 2.4. Proposition 2.3 implies, for any $T \in \mathbb{R}_+$ and $b \in (0, \frac{1}{4})$, we have $\mathbf{P}(\sup_{t \in [0,T]} |X^{\varepsilon}_{(0)}(t)| \le \varepsilon^b) \to 1$. This is almost optimal, since we know a posteriori from Theorem 1.1 that $\varepsilon^{-\frac{1}{4}} X^{\varepsilon}_{(0)}(t) = \mathcal{X}^{\varepsilon}_t(0)$ converges weakly.

The idea of proving Proposition 2.3 is to utilize the *stationarity*. This is done by inserting a suitable time-independent test function $\psi_t(x) = \psi(x)$ with $\psi(0) > 0$ into (2.5), and expressing the result as the sum of $\langle A^{\varepsilon}_t, \psi \rangle$ and other terms whose moments are bounded by using $\{X_{(i)}(t) - X_{(0)}(t)\}_{i \in \mathbb{Z}_+} \sim \mathrm{PPP}_+(2)$. This then yields $\mathbf{E}|\langle A^{\varepsilon}_t, \psi \rangle|^q \le C(q) < \infty$, $\forall q < \infty$, which, with A^{ε}_t defined as in (2.1) and with $\psi(0) > 0$, implies (2.6).

Turning to the proof of Proposition 1.9, for each $a \in (\frac{1}{2}, 1)$, $b \in (0, \frac{1}{4})$ and $t, x \in \mathbb{R}_+$, we apply Proposition 2.1 for $\psi_s(y) := \Psi_{t-s+\varepsilon^a}(y, x + \varepsilon^b) \in \mathcal{Q}_t$. With $\psi_s(y)$ solving the backward heat equation $(\partial_s + 2^{-1}\partial_{yy})\psi_s = 0$, one easily obtains that

$$\mathcal{F}^{\varepsilon,a}_t(x + \varepsilon^b) = \mathcal{W}^{\varepsilon}_t(x) + \mathcal{M}^{\varepsilon}_t(x) + \mathcal{A}^{\varepsilon}_t(x),$$

where

$$\Psi^{\varepsilon}_t(y, x) := \Psi_{t+\varepsilon^a}(y, x + \varepsilon^b), \qquad p^{\mathrm{N},\varepsilon}_t(y, x) := p^{\mathrm{N}}_{t+\varepsilon^a}(y, x + \varepsilon^b), \tag{2.7}$$

$$\mathcal{W}^{\varepsilon}_t(x) := \langle \widehat{Q}^{\varepsilon}_0, \Psi^{\varepsilon}_t(\cdot, x) \rangle, \tag{2.8}$$

$$\mathcal{M}^{\varepsilon}_t(x) := M^{\varepsilon}_{0,t}(\Psi^{\varepsilon}_{t-\cdot}(\cdot, x), \infty) = \varepsilon^{\frac{1}{4}} \sum_{i=0}^{\infty} \int_0^t p^{\mathrm{N},\varepsilon}_{t-s}(X^{\varepsilon}_i(s), x)\, dB^{\varepsilon}_i(s), \tag{2.9}$$

$$\mathcal{A}^{\varepsilon}_t(x) := \int_0^t \langle A^{\varepsilon}_s, \partial_s \Psi^{\varepsilon}_{t-s}(\cdot, x) \rangle\, ds. \tag{2.10}$$

Since $\mathcal{W}^{\varepsilon}_t(x)$ and $\mathcal{M}^{\varepsilon}_t(x)$, consisting respectively of the contribution of $\{X^{\varepsilon}_i(0)\}$ and $\{B^{\varepsilon}_i(\cdot)\}$, are independent, Proposition 1.9 is an immediate consequence of the following.

PROPOSITION 2.5. *Fix any $a \in (\frac{1}{2}, 1)$ and $b \in (0, \frac{1}{4})$:*

(a) *As $\varepsilon \to 0$, $\mathcal{A}^{\varepsilon}_{\cdot}(\cdot) \Rightarrow 0$.*
(b) *As $\varepsilon \to 0$, $\mathcal{W}^{\varepsilon}_{\cdot}(\cdot) \Rightarrow \mathcal{W}_{\cdot}(\cdot)$, where $\mathcal{W}_{\cdot}(\cdot)$ is defined as in (1.7) with $\gamma = 1$.*
(c) *As $\varepsilon \to 0$, $\mathcal{M}^{\varepsilon}_{\cdot}(\cdot) \Rightarrow \mathcal{M}_{\cdot}(\cdot)$, where $\mathcal{M}_{\cdot}(\cdot)$ is defined as in (1.8) with $\gamma = 1$.*

REMARK 2.6. Our special choice of $\psi_s(y)$ is what makes Proposition 2.5(a) valid. To see this, note that $X^{\varepsilon}_{(0)}(t) = O(\varepsilon^b)$ for all $b \in (0, \frac{1}{4})$ (by Proposition 2.3) and that $\mathcal{A}^{\varepsilon}_t(x) = \int_0^t \langle A^{\varepsilon}_s, \varphi_s \rangle\, ds$ for $\varphi_s(y) = \partial_s \Psi_{t+\varepsilon^a - s}(y, x)$. With $\varphi_s(0) = 0$, by (2.1) we can approximate $\langle A^{\varepsilon}_s, \varphi_s \rangle$ by $\varepsilon^{-\frac{1}{4}} O((X^{\varepsilon}_{(0)}(s))^2)$, which indeed tends to zero. Further, we expect Proposition 2.5(b) and (c) to hold by comparing (1.7) with (2.8), and (1.8) with (2.9), since $\widehat{Q}^{\varepsilon}_0$ approximates $\sqrt{2} dB_0(\cdot)$, and $\varepsilon^{\frac{1}{2}} Q^{\varepsilon}_t$ approximates $2\mathbf{1}_{\mathbb{R}_+}(x)\, dx$, respectively.

The proof of Proposition 1.10 requires the following notation:

$$\mathcal{G}^{\varepsilon}_t(x) := \langle \widehat{Q}^{\varepsilon}_t, \mathbf{1}_{(-\infty,x]}\rangle = \varepsilon^{\frac{1}{4}}\langle Q^{\varepsilon}_t, \mathbf{1}_{(-\infty,x]}\rangle - 2\varepsilon^{-\frac{1}{4}}x, \tag{2.11}$$

$$I^{\varepsilon}_t(x) := \inf\{i \in \mathbb{Z}_+ : X^{\varepsilon}_{(i)}(t) > x\} = \langle Q^{\varepsilon}_t, \mathbf{1}_{(-\infty,x]}\rangle, \tag{2.12}$$

$$\widetilde{\mathcal{X}}^{\varepsilon}_t(x) := \varepsilon^{\frac{1}{4}}\big(I^{\varepsilon}_0(x) - 2X_{(I^{\varepsilon}_0(x))}(\varepsilon^{-1}t)\big). \tag{2.13}$$

Up to a centering and scaling, $\mathcal{G}^{\varepsilon}_t(x)$ counts the total number of particles to the left of x, and $\widetilde{\mathcal{X}}^{\varepsilon}_t(x)$ records the trajectory of $X_{(I^{\varepsilon}_0(x))}(\cdot)$, where $X^{\varepsilon}_{(I^{\varepsilon}_0(x))}(0)$ the first particle to the right of x at time 0. Proposition 1.10 is then an immediate consequence of the following.

PROPOSITION 2.7. *Let $a \in (\frac{1}{2}, 1)$ and $b \in (0, \frac{1}{4})$:*

(a) *As $\varepsilon \to 0$, $\mathcal{F}^{\varepsilon,a}_{\cdot}(\cdot + \varepsilon^b) - \mathcal{G}^{\varepsilon}_{\cdot}(\cdot + \varepsilon^b) \Rightarrow 0$.*
(b) *As $\varepsilon \to 0$, $\mathcal{G}^{\varepsilon}_{\cdot}(\cdot + \varepsilon^b) - \widetilde{\mathcal{X}}^{\varepsilon}_{\cdot}(\cdot + \varepsilon^b) \Rightarrow 0$.*
(c) *As $\varepsilon \to 0$, $\widetilde{\mathcal{X}}^{\varepsilon}_{\cdot}(\cdot + \varepsilon^b) - \mathcal{X}^{\varepsilon}_{\cdot}(\cdot) \Rightarrow 0$.*

Recall that $Y_i(t) := X_{(i+1)}(t) - X_{(i)}(t)$ denotes the ith gap. Letting

$$\rho^{\varepsilon}_t(x) := X_{(I^{\varepsilon}_t(x))}(\varepsilon^{-1}t) - \varepsilon^{-\frac{1}{2}}x = \varepsilon^{-\frac{1}{2}}\big(X^{\varepsilon}_{(I^{\varepsilon}_t(x))}(t) - x\big), \tag{2.14}$$

$$\mathcal{D}^{\varepsilon}(j, j', t) := j - j' - 2\big(X_{(j)}(\varepsilon^{-1}t) - X_{(j')}(\varepsilon^{-1}t)\big) \tag{2.15}$$

$$= \operatorname{sign}(j - j') \sum_{i \in [j', j) \cup [j, j')} \big(1 - 2Y_i(\varepsilon^{-1}t)\big), \tag{2.16}$$

in Section 5, we establish Proposition 2.7 relying on the following exact relations:

$$\rho^{\varepsilon}_t(x) \in \big(0, Y_{I^{\varepsilon}_t(x)-1}(\varepsilon^{-1}t)\big) \qquad \forall x \text{ such that } x \geq X^{\varepsilon}_{(0)}(t), \tag{2.17}$$

$$\mathcal{G}^{\varepsilon}_t(x) - \widetilde{\mathcal{X}}^{\varepsilon}_t(x) = \varepsilon^{\frac{1}{4}}\mathcal{D}^{\varepsilon}\big(I^{\varepsilon}_t(x), I^{\varepsilon}_0(x), t\big) + 2\varepsilon^{\frac{1}{4}}\rho^{\varepsilon}_t(x), \tag{2.18}$$

$$\widetilde{\mathcal{X}}^{\varepsilon}_t(x + \varepsilon^b) - \mathcal{X}^{\varepsilon}_t(x) = \varepsilon^{\frac{1}{4}}\mathcal{D}^{\varepsilon}\big(I^{\varepsilon}_0(x + \varepsilon^b), i_{\varepsilon}(x), t\big) \qquad \forall x \in \frac{\varepsilon^{\frac{1}{2}}}{2}\mathbb{Z}_+. \tag{2.19}$$

Indeed, (2.17) holds since $\rho^{\varepsilon}_t(x)$ represents the gap between $\varepsilon^{-\frac{1}{2}}x$ and the first particle to its right, (2.18) follows by combining (2.11)–(2.12) and (2.14), and (2.19) follows by comparing the expressions (1.2) and (2.13).

The starting point of proving Proposition 2.7 is as follows. We establish part (a) based on using $\Psi_{\varepsilon^a}(y, x + \varepsilon^b) \approx \mathbf{1}_{(-\infty,-x-\varepsilon^b]}(y) + \mathbf{1}_{(-\infty,x+\varepsilon^b]}(y)$, for $b \in (0, \frac{1}{4})$ to ensure that $\langle \widehat{Q}^{\varepsilon}_t, \mathbf{1}_{(-\infty,-x-\varepsilon^b]}\rangle \approx 0$. As for parts (b) and (c), by shifting each x by ε^b, we use (2.17) to ensure that $\varepsilon^{\frac{1}{4}}\rho^{\varepsilon}_t(x + \varepsilon^b) \approx 0$, and by using stationarity, we have $\mathcal{D}^{\varepsilon}(j, j', t) = O(|j - j'|^{\frac{1}{2}})$. Consequently, we reduce showing parts (b) and (c) to showing

$$\varepsilon^{\frac{1}{4}}\big|I^{\varepsilon}_t(x) - I^{\varepsilon}_0(x)\big|^{\frac{1}{2}} \approx 0, \qquad \varepsilon^{\frac{1}{4}}\big|I^{\varepsilon}_0(x + \varepsilon^b) - i_{\varepsilon}(x)\big|^{\frac{1}{2}} \approx 0.$$

The former should hold since, by (2.11)–(2.12), we have $I_t^\varepsilon(x) - I_0^\varepsilon(x) = \varepsilon^{-\frac{1}{4}}(\mathcal{G}_t^\varepsilon(x) - \mathcal{G}_t^\varepsilon(x)) = O(\varepsilon^{-\frac{1}{4}})$, and we expect the latter to be true since $I_0^\varepsilon(x+\varepsilon^b) \sim \mathrm{Pois}(2\varepsilon^{-\frac{1}{2}}(x+\varepsilon^b))$ and $i_\varepsilon(x) = 2\varepsilon^{-\frac{1}{2}}x = 2\varepsilon^{-\frac{1}{2}}(x+\varepsilon^b) + O(\varepsilon^{-\frac{1}{2}+b})$.

The rest of this paper is organized as follows. Section 3 is primarily devoted to the proof of Propositions 2.1 and 2.3. In Sections 4 and 5, we prove Propositions 2.5 and 2.7, respectively.

3. A priori estimates: Proof of Propositions 2.1 and 2.3. Let $X_i^{\varepsilon,\ell}(t) := X_i(0) + B_i^\varepsilon(t)$, and $X_i^{\varepsilon,r}(t) := X_i^{\varepsilon,\ell}(t) + \varepsilon^{-\frac{1}{2}}t$, with $X_{(i)}^{\varepsilon,\ell}(t)$ and $X_{(i)}^{\varepsilon,r}(t)$ denoting the corresponding ranked processes. We have from (1.1) (for $\gamma = 1$) that, almost surely, for all $i \in \mathbb{Z}_+$ and $t \geq 0$,

$$X_i^{\varepsilon,\ell}(t) \leq X_i(t) \leq X_i^{\varepsilon,r}(t), \tag{3.1}$$

from which it easily follows that

$$X_{(i)}^{\varepsilon,\ell}(t) \leq X_{(i)}(t) \leq X_{(i)}^{\varepsilon,r}(t). \tag{3.2}$$

Based on (3.1)–(3.2), we next establish bounds on the mass of the empirical measure on intervals of the form $(-\infty, x]$.

LEMMA 3.1. *Fix any $a > 0$, $q \in [1,\infty)$, $t \in \mathbb{R}_+$ and $j \in \mathbb{Z}_+$. There exists $C = C(a,q,t) < \infty$ such that, for all $\varepsilon \in (0, (aq)^{-2}]$,*

$$\sum_{i=j}^{\infty} \Big\| \sup_{s\in[0,t]} \exp(-aX_i^\varepsilon(s)) \Big\|_q \leq C\varepsilon^{-\frac{1}{2}} e^{-j\varepsilon^{\frac{1}{2}}a/4}, \tag{3.3}$$

$$\Big\| \sum_{i=j}^{\infty} \sup_{s\in[0,t]} \exp(-aX_{(i)}^\varepsilon(s)) \Big\|_q \leq C\varepsilon^{-\frac{1}{2}} e^{-j\varepsilon^{\frac{1}{2}}a/4}. \tag{3.4}$$

PROOF. Fix $t \in \mathbb{R}_+$, $q \in [1,\infty)$, $a > 0$ and $j_* \in \mathbb{Z}_+$. Let $X_i^{\varepsilon,\ell,*}(s) := X_{i+j_*}^{\varepsilon,\ell}(s)$ be the ith (unranked) particle among $\{X_j^{\varepsilon,\ell}\}_{j\geq j_*}$. Let $F_i^\varepsilon := \sup_{s\in[0,t]} \exp(-a \times X_i^\varepsilon(s))$, $F_{(i)}^\varepsilon := \sup_{s\in[0,t]} \exp(-aX_{(i)}^\varepsilon(s))$, and similarly let $F_i^{\varepsilon,\ell}$, $F_{(i)}^{\varepsilon,\ell}$, $F_i^{\varepsilon,\ell,*}$ and $F_{(i)}^{\varepsilon,\ell,*}$ be the corresponding random variables for $X_i^{\varepsilon,\ell}$, $X_{(i)}^{\varepsilon,\ell}$, $X_i^{\varepsilon,\ell,*}$, $X_{(i)}^{\varepsilon,\ell,*}$, respectively.

By (3.1), $F_i^\varepsilon \leq F_i^{\varepsilon,\ell}$, hence $\sum_{i=j}^\infty \|F_i^\varepsilon\|_q \leq \sum_{i=j}^\infty \|F_i^{\varepsilon,\ell}\|_q$. Let $r := 2^{-1}aq\varepsilon^{\frac{1}{2}}$ and $\overline{B}_i^\varepsilon(t) := \sup_{s\in[0,t]} |B_i^\varepsilon(s)|$. With $X_i^{\varepsilon,\ell}(t)$ defined as in the preceding, we have

$$\mathbf{E}(F_i^{\varepsilon,\ell})^q \leq (\mathbf{E}e^{-2rY_0(0)})^i \mathbf{E}(e^{aq\overline{B}_i^\varepsilon(t)}) = (1+r)^{-i}\mathbf{E}(e^{aq\overline{B}_i^\varepsilon(t)}). \tag{3.5}$$

Further, by the reflection principle, $\mathbf{E}[\exp(-aq\overline{B}_i^{\varepsilon}(t))] \le 2\mathbf{E}[\exp(aqB_i^{\varepsilon}(t))] \le C(a,q,t)$. Consequently,

$$\sum_{i=j}^{\infty} \|F_i^{\varepsilon,\ell}\|_q \le \frac{(1+r)^{-(j-1)/q}}{(1+r)^{1/q}-1} C. \tag{3.6}$$

With $r \in (0,1]$, it is easy to show that $(1+r)^{1/q} \ge 1 + \frac{r}{2q}$ and $(1+r)^{-j/q} \le \exp(-jr/(2q))$. Using these in (3.6) yields (3.3).

We next show (3.4). Since $X_{(i)}^{\varepsilon,\ell,*}(s)$ is the ith rank particle among $\{X_j^{\varepsilon,\ell}(s)\}_{j\ge j_*}$ and $X_{(i+j_*)}^{\varepsilon,\ell}(s)$ is the $(i+j_*)$th rank particle among $\{X_j^{\varepsilon,\ell}(s)\}_{j\in\mathbb{Z}_+}$, we must have $X_{(i)}^{\varepsilon,\ell,*}(s) \le X_{(i+j_*)}^{\varepsilon,\ell}(s)$. Combining this with $X_{(i+j_*)}^{\varepsilon,\ell}(s) \le X_{(i+j_*)}^{\varepsilon}(s)$ yields $F_{(i+j_*)}^{\varepsilon} \le F_{(i)}^{\varepsilon,\ell,*}$. Summing both sides over i, we further obtain $\sum_{i=0}^{\infty} F_{(i+j_*)}^{\varepsilon} \le \sum_{i=0}^{\infty} F_{(i)}^{\varepsilon,\ell,*} = \sum_{i=0}^{\infty} F_i^{\varepsilon,\ell,*} = \sum_{i=j_*}^{\infty} F_i^{\varepsilon,\ell}$. From this and (3.3) we conclude (3.4). $\square$

Based on (3.1), we next establish the continuity of the process $X_{(i)}^{\varepsilon}(\cdot)$.

LEMMA 3.2. *There exists $C < \infty$ such that for any $[t_1,t_2] \subset [0,\infty)$, $j \in \mathbb{Z}_+$ and $\varepsilon \in (0,1]$,*

$$\mathbf{P}\Big(\sup_{t\in[t_1,t_2]} \big|X_{(j)}^{\varepsilon}(t) - X_{(j)}^{\varepsilon}(t_1)\big| \ge \alpha\Big) \le C\exp\big(-\alpha\varepsilon^{-\frac{1}{2}} + 2\varepsilon^{-1}(t_2-t_1)\big). \tag{3.7}$$

PROOF. It clearly suffices to show that

$$\mathbf{E}\Big[\exp\Big(\varepsilon^{-\frac{1}{2}} \sup_{t\in[t_1,t_2]} \big|X_{(j)}^{\varepsilon}(t) - X_{(j)}^{\varepsilon}(t_1)\big|\Big)\Big] \le C\exp\big(2\varepsilon^{-1}(t_2-t_1)\big). \tag{3.8}$$

Since $\{Y_i(\cdot)\}_{i\in\mathbb{Z}_+}$ is at stationarity, we have

$$\big(X_{(i)}^{\varepsilon}(\cdot+t_1) - X_{(i)}^{\varepsilon}(t_1)\big)_{i\in\mathbb{Z}_+} \stackrel{\text{distr.}}{=} \big(X_{(i)}^{\varepsilon}(\cdot) - X_{(i)}^{\varepsilon}(0)\big)_{i\in\mathbb{Z}_+},$$

so without loss of generality we assume that $t_1 = 0$. Let

$$U^{\varepsilon,r}(t,i,j) := \sup_{s\in[0,t]} \big\{\exp\big[\varepsilon^{-\frac{1}{2}}\big(X_i^{\varepsilon,r}(s) - X_{(j)}^{\varepsilon,r}(0)\big)\big]\big\}, \tag{3.9}$$

$$U^{\varepsilon,\ell}(t,i,j) := \sup_{s\in[0,t]} \big\{\exp\big[-\varepsilon^{-\frac{1}{2}}\big(X_i^{\varepsilon,\ell}(t) - X_{(j)}^{\varepsilon,\ell}(0)\big)\big]\big\}. \tag{3.10}$$

Similar to (3.5) we have

$$\begin{aligned} \mathbf{E}\big(U^{\varepsilon,r}(t,i,j)\big) &\le \big(\mathbf{E}(e^{-Y_0(0)})\big)^{j-i}\mathbf{E}\big(e^{\varepsilon^{-\frac{1}{2}}\overline{B}_i^{\varepsilon}(t)+\varepsilon^{-1}t}\big) \\ &\le (2/3)^{j-i}Ce^{2\varepsilon^{-1}t} \qquad \forall i \le j, \end{aligned} \tag{3.11}$$

$$\mathbf{E}\big(U^{\varepsilon,\ell}(t,i,j)\big) \le \big(\mathbf{E}\big(e^{-Y_0(0)}\big)\big)^{i-j}\mathbf{E}\big(e^{\varepsilon^{-\frac{1}{2}}\overline{B}_i^{\varepsilon}(t)}\big) \le (2/3)^{i-j} C e^{\varepsilon^{-1}t} \qquad \forall i \ge j. \tag{3.12}$$

By (3.1), $\exp[\varepsilon^{-\frac{1}{2}}|X^{\varepsilon}_{(j)}(t) - X^{\varepsilon}_{(j)}(0)|] \le \exp[\varepsilon^{-\frac{1}{2}}(X^{\varepsilon,r}_{(j)}(t) - X^{\varepsilon,r}_{(j)}(0))] + \exp[-\varepsilon^{-\frac{1}{2}}(X^{\varepsilon,\ell}_{(j)}(t) - X^{\varepsilon,\ell}_{(j)}(0))]$. For all $t \in [0,t_2]$, the last two terms are bounded by $\sum_{i\le j} U^{\varepsilon,r}(t_2,i,j)$ and $\sum_{i \ge j} U^{\varepsilon,\ell}(t_2,i,j)$, respectively. Combining this with (3.11)–(3.12), we conclude (3.8). □

Based on Lemma 3.1, we establish the following useful bounds on the empirical measure.

LEMMA 3.3. *Fix $T \in \mathbb{R}_+$, $q \in [1,\infty)$ and $a \in (0,\infty)$. Let $J^{\varepsilon}_j := [\varepsilon^{-\frac{1}{2}} j, \varepsilon^{-\frac{1}{2}}(j+1)) \cap \mathbb{Z}$ and f_i, $i \in \mathbb{Z}_+$, be $\mathbb{R}_+$-valued random variables. There exits $C = C(T,q,a) < \infty$ such that for all $t \in [0,T]$ and $\varepsilon \in (0,(aq)^{-2}]$,*

$$\Big\| \sum_{i=0}^{\infty} f_i e^{-aX^{\varepsilon}_{(i)}(t)} \Big\|_q \le C\varepsilon^{-\frac{1}{4}} \sum_{j=0}^{\infty} e^{-ja/4} \Big(\sum_{i \in J^{\varepsilon}_j} \|f_i\|^2_{2q} \Big)^{\frac{1}{2}}. \tag{3.13}$$

PROOF. For each $j \in \mathbb{Z}_+$, by the Cauchy–Schwarz inequality we have

$$\Big\| \sum_{i \in J^{\varepsilon}_j} f_i e^{-aX^{\varepsilon}_{(i)}(t)} \Big\|_q \le \Big\| \sum_{i \in J^{\varepsilon}_j} e^{-2aX^{\varepsilon}_{(i)}(t)} \Big\|_q^{\frac{1}{2}} \Big\| \sum_{i \in J^{\varepsilon}_j} (f_i)^2 \Big\|_q^{\frac{1}{2}}.$$

On the right-hand side, replacing $\|\sum_{i\in J^{\varepsilon}_j}(f_i)^2\|_q$ with $\sum_{i \in J^{\varepsilon}_j}\|(f_i)^2\|_q = \sum_{i\in J^{\varepsilon}_j}\|(f_i)\|^2_{2q}$, and replacing $\|\sum_{i\in J^{\varepsilon}_j} e^{-2aX^{\varepsilon}_{(i)}(t)}\|_q$ with $\|\sum_{i \ge \varepsilon^{-\frac{1}{2}} j} e^{-2aX^{\varepsilon}_{(i)}(t)}\|_q$, which, by (3.4), is bounded by $C\varepsilon^{-\frac{1}{2}}\exp(-ja/2)$, we conclude (3.13). □

Now we establish a decomposition of W^{ε}_t into $W^{\varepsilon,*}_t$ and R^{ε}_t as follows. As we show latter in (3.17), R^{ε}_t becomes negligible as $\varepsilon \to 0$, so $W^{\varepsilon}_t \approx W^{\varepsilon,*}_t$.

LEMMA 3.4. *Fix $t \in \mathbb{R}_+$, $\varepsilon \in (0,1]$ and $\phi \in \mathscr{Q}$ such that $\frac{d\phi}{dy} \in \mathscr{Q}$, and let*

$$\langle W^{\varepsilon,*}_t, \phi\rangle := \varepsilon^{\frac{1}{4}} \sum_{i=0}^{\infty} \phi\big(X^{\varepsilon}_{(i)}(t)\big)\big(1 - 2Y_i(\varepsilon^{-1}t)\big), \tag{3.14}$$

$$\langle R^{\varepsilon}_t, \phi\rangle := \varepsilon^{-\frac{1}{4}} \sum_{i=0}^{\infty} \int_{X^{\varepsilon}_{(i)}(t)}^{X^{\varepsilon}_{(i+1)}(t)} \big(X^{\varepsilon}_{(i+1)}(t) - y\big)\frac{d\phi}{dy}\,dy. \tag{3.15}$$

Then

$$\langle W^{\varepsilon}_t, \phi\rangle = \langle W^{\varepsilon,*}_t, \phi\rangle - 2\langle R^{\varepsilon}_t, \phi\rangle. \tag{3.16}$$

PROOF. Since the gaps are at stationarity, $X^\varepsilon_{(i)}(t) - X^\varepsilon_{(0)}(t)$ is the sum of the i.i.d. $\mathrm{Exp}(2\varepsilon^{-\frac{1}{2}})$ random variables, so by the law of large numbers we have $\lim_{k\to\infty} X^\varepsilon_{(k)}(t) = \infty$, hence

$$\langle W^\varepsilon_t, \phi\rangle = \varepsilon^{\frac{1}{4}} \sum_{i=0}^{\infty} \Big(\phi\big(X^\varepsilon_{(i)}(t)\big) - 2\varepsilon^{-\frac{1}{2}} \int_{X^\varepsilon_{(i)}(t)}^{X^\varepsilon_{(i+1)}(t)} \phi(y)\,dy\Big).$$

With $\int_{x_1}^{x_2} \phi(y)\,dy = (x_2 - x_1)\phi(x_1) + \int_{x_1}^{x_2}(x_2 - y)\frac{d\phi}{dy}\,dy$, we obtain the desired decomposition. □

Based on Lemma 3.3, we next establish bounds on $\langle R^\varepsilon_t, \phi\rangle$ and $\langle W^{\varepsilon,*}_t, \phi\rangle$. We note here that, while these bounds fall short of proving Proposition 2.5, they suffice for justifying the use of Itô calculus in Proposition 2.1.

Hereafter, when the context is clear, we sometimes use ϕ^ε_i, Y^ε_i and $X^\varepsilon_{(i)}$, respectively, to denote $\phi(X^\varepsilon_{(i)}(t))$, $Y_i(\varepsilon^{-1}t)$ and $X^\varepsilon_{(i)}(t)$.

LEMMA 3.5. *Fix $T < \infty$, $q \in [1, \infty)$ and $\phi \in \mathscr{Q}$ such that $\frac{d\phi}{dy} \in \mathscr{Q}$. There exists $C = C(T, q) < \infty$ such that, for all $t \in [0, T]$ and $\varepsilon \in (0, (2q)^{-2}]$,*

$$\big\|\langle R^\varepsilon_t, \phi\rangle\big\|_q \le C\varepsilon^{\frac{1}{4}} \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}}, \tag{3.17}$$

$$\big\|\langle W^{\varepsilon,*}_t, \phi\rangle\big\|_q \le C \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}}. \tag{3.18}$$

PROOF. Fixing $T \in \mathbb{R}_+$, $t \in [0, T]$, $q \in [1, \infty)$, $\varepsilon \in (0, (2q)^{-2}]$ and $\psi \in \mathscr{Q}$, we let $C = C(T, q) < \infty$. To show (3.17), in (3.15), we use $X^\varepsilon_{(i+1)} - y \le \varepsilon^{\frac{1}{2}} Y_i$ and

$$\sup_{y\in[X^\varepsilon_{(i)}, X^\varepsilon_{(i+1)}]} \left|\frac{d}{dy}\phi(y)\right| \le \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}} \exp\big(-X^\varepsilon_{(i)}\big)$$

to obtain $|\langle R^\varepsilon_t, \phi\rangle| \le \varepsilon^{3/4} |\frac{d\phi}{dy}|_{\mathscr{Q}} \sum_{i=0}^\infty (Y_i)^2 \exp(-X^\varepsilon_{(i)})$. Combining this with (3.13) for $f_i = (Y_i)^2$, we arrive at

$$\big\|\langle R^\varepsilon_t, \phi\rangle\big\|_q \le C\varepsilon^{\frac{1}{2}} \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}} \sum_{j=0}^{\infty} \exp(-j/4)\big(\|(Y_i)^2\|^2_{2q} |J^\varepsilon_j|\big)^{\frac{1}{2}}.$$

Further using $\|(Y_i)^2\|_{2q} \le C$ and $|J^\varepsilon_j| \le \varepsilon^{-\frac{1}{2}} + 1$, we conclude (3.17) upon summing j.

Turning to showing (3.18), we assume without loss of generality $q \in \mathbb{Z}_+ \cap [1, \infty)$. Letting $Z_k := \sum_{i=0}^k (1 - 2Y_i)$, with $\phi \in \mathscr{Q}$, using summation by parts in

(3.14), we obtain

$$\langle W_t^{\varepsilon,*}, \phi \rangle := \varepsilon^{\frac{1}{4}} \sum_{i=0}^{\infty} (\phi_i^\varepsilon - \phi_{i+1}^\varepsilon) Z_i. \tag{3.19}$$

To bound this expression, we combine

$$|\phi_{i+1}^\varepsilon - \phi_i^\varepsilon| \le \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}} \int_{X_{(i)}^\varepsilon}^{X_{(i+1)}^\varepsilon} e^{-y}\, dy \le \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}} \varepsilon^{\frac{1}{2}} Y_i^\varepsilon \exp(-X_{(i)}^\varepsilon)$$

(where the second inequality is obtained by using $e^{-y} \le e^{-X_{(i)}}$) and (3.13) for $f_i = Y_i Z_i$ to obtain

$$\|\langle W_t^{\varepsilon,*}, \phi \rangle\|_q \le C \varepsilon^{\frac{1}{2}} \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}} \sum_{j=0}^{\infty} e^{-j/4} \Big(\sum_{i \in J_j^\varepsilon} \|Z_i Y_i\|_{2q}^2 \Big)^{\frac{1}{2}}. \tag{3.20}$$

With $\|Y_i\|_{4q} \le C$ and $\|Z_i\|_{4q} \le (i+1)^{\frac{1}{2}} C$, we have $\|Y_i Z_i\|_{2q}^2 \le (i+1)C$. Plugging this into (3.20), we further obtain

$$\|\langle W_t^{\varepsilon,*}, \phi \rangle\|_q \le C \varepsilon^{\frac{1}{2}} \left|\frac{d\phi}{dy}\right|_{\mathscr{Q}} \sum_{j=0}^{\infty} \big[|J_j^\varepsilon| \varepsilon^{-\frac{1}{2}} (j+1)\big]^{\frac{1}{2}} e^{-j/4}.$$

With $|J_j^\varepsilon| \le \varepsilon^{-\frac{1}{2}} + 1$, upon summing over j we conclude (3.18). $\square$

Based on Lemma 3.3, we now establish a bound on $M_{t_0,t}^\varepsilon(\psi, j)$ [as in (2.3)]. Hereafter, we adopt the convention that $M_{t_0,t}^\varepsilon(\psi, -1) := 0$.

LEMMA 3.6. *Let $\sigma \in [0, \infty]$ be an arbitrary stopping time (with respect to the underlying filtration). Fix $T < \infty$ and $q \in (1, \infty)$. There exists $C = C(T, q) < \infty$ such that, for all $\psi \in \mathscr{Q}_T$, $t_0 \in [0, T]$, $j, j' \ge -1$ and $\varepsilon \in (0, 1]$,*

$$\begin{aligned} \Big\| \sup_{t \in [t_0, T]} |M_{t_0,t\wedge\sigma}^\varepsilon(\psi, j) - M_{t_0,t\wedge\sigma}^\varepsilon(\psi, j')| \Big\|_q^2 \\ \le C |\psi|_{\mathscr{Q}_T}^2 \exp\big(-(j \wedge j')\varepsilon^{\frac{1}{2}}/2\big). \end{aligned} \tag{3.21}$$

PROOF. Fixing such $T, q, t_0, j, j', \varepsilon, \psi$ and σ, we let $C = C(T, q) < \infty$. We assume without loss of generality $j > j'$. Applying Doob's L^q-inequality and the Burkholder–Davis–Gundy (BDG) inequality (e.g., [20], Theorem II.1.7 and Theorem IV.4.1) to the $C([t_0, T], \mathbb{R})$-valued martingale $t \mapsto M_t^{\varepsilon,*} := M_{t_0,t\wedge\sigma}^\varepsilon(\psi, j) - M_{t_0,t\wedge\sigma}^\varepsilon(\psi, j')$, we obtain

$$\begin{aligned} \Big\| \sup_{t \in [t_0, T]} |M_t^{\varepsilon,*}| \Big\|_q^2 &\le C \Big\| \varepsilon^{\frac{1}{2}} \int_{t_0}^{T\wedge\sigma} \sum_{i=j'+1}^{j} \big(\partial_y \psi_s(X_i^\varepsilon(s))\big)^2 ds \Big\|_{q/2} \\ &\le C \int_0^T \varepsilon^{\frac{1}{2}} \sum_{i=j'+1}^{j} \|(\partial_y \psi_s(X_i^\varepsilon(s)))^2\|_{q/2} ds. \end{aligned} \tag{3.22}$$

In the last expression, replacing $(\partial_y \psi_s(y))^2$ with $|\psi|^2_{\mathscr{Q}_T} e^{-2y}$ and replacing j with ∞, and then applying (3.3) for $a = 2$, we further obtain the bound $C|\psi|^2_{\mathscr{Q}_T} \exp(-j\varepsilon^{\frac{1}{2}}/2)$, thereby concluding (3.21). $\square$

PROOF OF PROPOSITION 2.1. Fix $\psi \in \mathscr{Q}_T$. The bound (3.21) implies that $\{M^\varepsilon_{t_0,\cdot}(\psi, j)\}_j$ is Cauchy in the complete space $L^q(C([t_0, T], \mathbb{R}), \mathscr{B}, \mathbf{P})$, so (2.4) follows. Further, for all $q > 1$,

$$\begin{aligned}\Big\| \sup_{t\in[t_0,T]} \big|M^\varepsilon_{t_0,t}(\psi,\infty)\big|\Big\|_q &\le \lim_{j\to\infty} \Big\| \sup_{t\in[t_0,T]} \big|M^\varepsilon_{t_0,t}(\psi,j)\big|\Big\|_q \\ &\le C(T,q)|\psi|_{\mathscr{Q}_T},\end{aligned} \tag{3.23}$$

where the last inequality follows by (3.21) for $j' = -1$.

To derive (2.5), we apply Itô's formula to

$$\langle \widehat{Q}^\varepsilon_{k,s}, \psi_s\rangle := \varepsilon^{\frac{1}{4}} \Bigg(\sum_{i=0}^k \psi_s\big(X^\varepsilon_i(s)\big) - 2\varepsilon^{-\frac{1}{2}} \int_0^\infty \psi_s(y)\,dy \Bigg)$$

to obtain

$$\begin{aligned}\langle \widehat{Q}^\varepsilon_{k,s}, \psi_s\rangle\big|_{s=0}^{s=t} &= \int_0^t \Big\langle \varepsilon^{\frac{1}{4}} Q^\varepsilon_{k,s}, \Big(\partial_s + \frac{1}{2}\partial_{yy}\Big)\psi_s \Big\rangle\,ds \\ &\quad - 2\varepsilon^{-\frac{1}{4}} \int_0^t \int_0^\infty \partial_s \psi_s(y)\,dy\,ds \\ &\quad + M^\varepsilon_{0,t}(\psi,k) + \varepsilon^{-\frac{1}{4}} \int_0^t (\partial_y \psi_s)\big(X^\varepsilon_{(0)}(s)\big) \sum_{i=0}^k \mathbf{1}_{\{X_{(i)}(s)=X_{(0)}(s)\}}\,ds.\end{aligned}$$

Clearly, almost surely for all $s \in [0, T]$, $\langle \widehat{Q}^\varepsilon_{k,s}, \phi\rangle \to \langle \widehat{Q}^\varepsilon_s, \phi\rangle$ and $\sum_{i=0}^k \mathbf{1}_{\{X_{(i)}(s)=X_{(0)}(s)\}} \to 1$ as $k \to \infty$. As for $M^\varepsilon_{0,t}(\psi, k)$, from (2.4) (for large enough q) we deduce that, almost surely for all $t \in [0, T]$, $M^\varepsilon_{0,t}(\psi, k) \to M^\varepsilon_{0,t}(\psi, \infty)$. Hence, letting $k \to \infty$ we arrive at

$$\begin{aligned}\langle \widehat{Q}^\varepsilon_s, \psi_s\rangle\big|_{s=0}^{s=t} &= \int_0^t \Big\langle \varepsilon^{\frac{1}{4}} Q^\varepsilon_s, \Big(\partial_s + \frac{1}{2}\partial_{yy}\Big)\psi_s \Big\rangle\,ds \\ &\quad - 2\varepsilon^{-\frac{1}{4}} \int_0^t \int_0^\infty \partial_s \psi_s(y)\,dy\,ds \end{aligned}\tag{3.24}$$

$$\quad + \varepsilon^{-\frac{1}{4}} \int_0^t (\partial_y \psi_s)\big(X^\varepsilon_{(0)}(s)\big)\,ds + M^\varepsilon_{0,t}(\psi,\infty). \tag{3.25}$$

With A^ε_t and W^ε_t defined as in (2.1)–(2.2), the right-hand side of (3.24) equals

$$\begin{aligned}&\int_0^t \langle W^\varepsilon_t, (\partial_s + 2^{-1}\partial_{yy})\psi_s\rangle\,ds + \int_0^t \langle A^\varepsilon_s, \partial_s \psi_s\rangle\,ds \\ &\quad + \varepsilon^{-\frac{1}{4}} \int_0^t \int_{X^\varepsilon_{(0)}(s)}^\infty \partial_{yy}\psi_s\,dy\,ds.\end{aligned}\tag{3.26}$$

The last term in (3.26) cancels the first term in (3.25), so (2.5) follows. □

COROLLARY 3.7. *For any $T \in \mathbb{R}_+$ and $q \in (1,\infty)$, there exists $C = C(T,q) < \infty$ such that for all $q > 1$, $\varepsilon \in (0,(2q)^{-2}]$ and $t \in [0,T]$,*

$$\left\| \int_0^{X^\varepsilon_{(0)}(t)} \operatorname{sech}(y)\,dy \right\|_q \le C\varepsilon^{\frac{1}{4}}. \tag{3.27}$$

PROOF. Applying Proposition 2.1 for the time-independent test function $\psi_s(y) = \psi(y) := \operatorname{sech}(y) \in \mathscr{Q}_T$, we obtain

$$\langle A^\varepsilon_s + W^\varepsilon_s, \operatorname{sech}\rangle\big|_{s=0}^{s=t} = 2^{-1}\int_0^t \left\langle W^\varepsilon_s, \frac{d^2}{dy^2}\operatorname{sech}\right\rangle ds + M^\varepsilon_{0,t}(\operatorname{sech},\infty),$$

or equivalently

$$\langle A^\varepsilon_t, \operatorname{sech}\rangle = \langle W^\varepsilon_0 - W^\varepsilon_t, \operatorname{sech}\rangle + 2^{-1}\int_0^t \left\langle W^\varepsilon_s, \frac{d^2}{dy^2}\operatorname{sech}\right\rangle ds + M^\varepsilon_{0,t}(\operatorname{sech},\infty).$$

Recall from (3.16) we have $\langle W^\varepsilon_s, \phi\rangle = \langle W^{\varepsilon,*}_s, \phi\rangle - 2\langle R^\varepsilon_s, \frac{d\phi}{dy}\rangle$. As $\psi \in C^\infty(\mathbb{R})$ and $\frac{d^k}{dy^k}\operatorname{sech} \in \mathscr{Q}$ for all $k \in \mathbb{Z}_+$, further applying (3.17)–(3.18) and (3.23), we conclude (3.27). □

PROOF OF PROPOSITION 2.3. Fix $T \in \mathbb{R}_+$, $b \in [0, \frac{1}{4})$ and $q > 1$. Applying Chebyshev's inequality in (3.27), we obtain that, for all $t \in [0,T]$, $q > 1$ and $\varepsilon \in (0,(2q)^{-2}]$,

$$\mathbf{P}\big(\big|X^\varepsilon_{(0)}(t)\big| \ge \lambda\big) \le \varepsilon^{q/4} C(T,q)\left(\int_0^\lambda \operatorname{sech}(y)\,dy\right)^{-q}. \tag{3.28}$$

Indeed, letting $t^\varepsilon_k := \varepsilon k$, we have

$$\begin{aligned}\{\tau^\varepsilon_b \le T\} \subset \bigcup_{k\le\varepsilon^{-1}T}&\left(\left\{\big|X^\varepsilon_{(0)}(t^\varepsilon_k)\big| \ge \frac{\varepsilon^b}{2}\right\}\right.\\ &\left.\cup\left\{\sup_{t\in[t^\varepsilon_k,t^\varepsilon_{k+1}]}\big|X^\varepsilon_{(0)}(t) - X^\varepsilon_{(0)}(t^\varepsilon_k)\big| \ge \frac{\varepsilon^b}{2}\right\}\right).\end{aligned} \tag{3.29}$$

From (3.28) and (3.7), we deduce

$$\mathbf{P}\big(\big|X^\varepsilon_{(0)}(t^\varepsilon_k)\big| \ge \varepsilon^b/2\big) \le C\varepsilon^{(\frac{1}{4}-b)q}, \tag{3.30}$$

$$\mathbf{P}\Big(\sup_{t\in[t^\varepsilon_k,t^\varepsilon_{k+1}]}\big|X^\varepsilon_{(0)}(t) - X^\varepsilon_{(0)}(t^\varepsilon_k)\big| \ge \varepsilon^b/2\Big) \le Ce^{-\varepsilon^{b-\frac{1}{2}}/2}. \tag{3.31}$$

In (3.29) applying the union bound using (3.30)–(3.31), we conclude (2.6). □

Recall Q^ε_t is defined as in (1.11). We next derive bounds on $\widetilde{Q}^\varepsilon_t := \varepsilon^{\frac{1}{2}} Q^\varepsilon_t$. To this end, we let

$$\langle Q^{\varepsilon,(0)}_t, \phi\rangle := \langle Q^\varepsilon_t, \phi(\cdot + X^\varepsilon_{(0)}(t))\rangle, \tag{3.32}$$

$$S^\varepsilon_b(t) := \mathbf{1}_{\{\sup_{s\in[0,t]}|X^\varepsilon_{(0)}(s)|\le \varepsilon^b\}}. \tag{3.33}$$

LEMMA 3.8. *Fix any $a \in (\frac{1}{2}, 1)$, $b \in (0, \frac{1}{4})$, $s \in (\varepsilon^a, \infty)$, $t \in [0, \infty)$, $x, y' \in \mathbb{R}$ and $q \in [1, \infty)$. There exists $C = C(a, q) < \infty$ such that, for all $\varepsilon \in (0, 1]$,*

$$\| S^\varepsilon_b(t)\langle \widetilde{Q}^\varepsilon_t, p^N_s(\cdot - y', x)\rangle\|_q \le (|\log s| + 1)C, \tag{3.34}$$

$$\|\langle \widetilde{Q}^\varepsilon_0, p^N_s(\cdot, x)\rangle\|_q \le C. \tag{3.35}$$

PROOF. With $p^{\mathrm{N}}_s(y, x) := p_s(y - x) + p_s(y + x)$ and $S^\varepsilon_b(t)$ decreasing in b, it clearly suffices to prove, for any fixed $x' \in \mathbb{R}$,

$$\| S^\varepsilon_0(t)\langle \widetilde{Q}^\varepsilon_t, p_s(\cdot - x')\rangle\|_q \le (|\log s| + 1)C, \tag{3.36}$$

$$\|\langle \widetilde{Q}^\varepsilon_0, p_s(\cdot - x')\rangle\|_q \le C. \tag{3.37}$$

Since $p(z)$ decreases in $|z|$, we have $p_s(z) \le s^{-\frac{1}{2}} \sum_{j=0}^\infty p(j)\mathbf{1}_{[j,j+1)}(|z|s^{-\frac{1}{2}})$. Using this, we obtain

$$S^\varepsilon_0(t)\langle \widetilde{Q}^\varepsilon_t, p_s(\cdot - x')\rangle = S^\varepsilon_0(t)\varepsilon^{\frac{1}{2}}\langle Q^\varepsilon_t, p_{s'}(\cdot - x')\rangle \le \sum_{j=0}^\infty S^\varepsilon_0(t)F^\varepsilon_j(t, s)p(j), \tag{3.38}$$

$$\langle \widetilde{Q}^\varepsilon_0, p_s(\cdot - x')\rangle = \varepsilon^{\frac{1}{2}}\langle Q^\varepsilon_0, p_s(\cdot - x')\rangle \le \sum_{j=0}^\infty G^\varepsilon_j(s)p(j), \tag{3.39}$$

where

$$F^\varepsilon_j(t, s) := s^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}\langle Q^\varepsilon_t, \mathbf{1}_{[j,j+1)}(|\cdot - x'|s^{-\frac{1}{2}})\rangle,$$

$$G^\varepsilon_j(s) := s^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}\langle Q^\varepsilon_0, \mathbf{1}_{[j,j+1)}(|\cdot - x'|s^{-\frac{1}{2}})\rangle.$$

As $Q^\varepsilon_0 \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$, we have $\langle Q^\varepsilon_0, \mathbf{1}_{[j,j+1)}(|\cdot - x'|s^{-\frac{1}{2}})\rangle \sim \mathrm{Pois}(2\varepsilon^{-\frac{1}{2}}s^{\frac{1}{2}})$. From this, with $\varepsilon^{-\frac{1}{2}}s^{\frac{1}{2}} \ge \varepsilon^{(a-1)/2} \to \infty$, we obtain $\|G^\varepsilon_j\|_q \le C(q)$. Combining this with (3.39), using $\sum_{j=0}^\infty p(j) < \infty$, we conclude (3.37). As for (3.34), letting

$$H^\varepsilon_j(t, s) := \sup_{|x'|\le 1} \{s^{-\frac{1}{2}}\langle Q^{\varepsilon,(0)}_t, \mathbf{1}_{[j,j+1)}(|\cdot - x'|s^{-\frac{1}{2}})\rangle\},$$

since Q^ε_t and $Q^{\varepsilon,(0)}_0$ differ only by the shift of $X^\varepsilon_{(0)}(s)$, with $S^\varepsilon_0(t)$ as in (3.33), we have $S^\varepsilon_0(t)F^\varepsilon_j(t, s) \le H^\varepsilon_j(t, s)$. With $Q^{\varepsilon,(0)}_t \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$, (3.36) now follows in a way similar to (3.37). The only difference is the maximum over $\{x' : |x'| \le 1\}$, which results in the extra $|\log s|$ factor. □

4. Proof of Proposition 2.5.

4.1. *Proof of part* (a). Fixing $b \in (0, \frac{1}{4})$, $b' \in (\frac{1}{8}, \frac{1}{4}) \cap [b, \infty)$ and $T \in \mathbb{R}_+$, we show

$$\lim_{\varepsilon \to 0} S^{\varepsilon}_{b'}(T)\Big(\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}_+} \big|\mathcal{A}^{\varepsilon}_t(x)\big|\Big) = 0. \tag{4.1}$$

The desired result $\mathcal{A}^{\varepsilon}_{\cdot}(\cdot) \Rightarrow 0$ then follows since $S^{\varepsilon}_{b'}(T) \underset{\mathrm{P}}{\to} 1$ (by Proposition 2.3).

Turning to proving (4.1), fixing $t \in [0, T]$, by (2.1) and (2.10) we have

$$S^{\varepsilon}_{b'}(T)\big|\mathcal{A}^{\varepsilon}_t(x)\big| \le 2\varepsilon^{-\frac{1}{4}} S^{\varepsilon}_{b'}(T) \int_0^t \int_0^{X^{\varepsilon}_{(0)}(s)} \big|\partial_s \Psi_{t+\sigma-s}(y, x+\eta)\big|\, dy\, ds,$$

where $\sigma := \varepsilon^a$ and $\eta := \varepsilon^b$. Since here $\sup_{s \in [0,T]}\{|X^{\varepsilon}_{(0)}(s)|\} \le \varepsilon^{b'}$, we may integrate over $\int_{-\sigma}^{T+1} \int_{-\varepsilon^{b'}}^{\varepsilon^{b'}}$ instead. After exchanging the order of integrations, we integrate over $s \in (-\sigma, T+1)$ using the readily verified identity $|\partial_s \Psi_s(y, x+\eta)| = -\operatorname{sign}(y)\partial_s \Psi(y, x+\eta)$ to obtain

$$S^{\varepsilon}_{b'}(T)\big|\mathcal{A}^{\varepsilon}_t(x)\big| \le 2\varepsilon^{-\frac{1}{4}} \int_{-\varepsilon^{b'}}^{\varepsilon^{b'}} \big|\Psi_{T+1+\sigma}(y, x+\eta) - \Psi_0(y, x+\eta)\big|\, dy. \tag{4.2}$$

Let $f(y) := \Psi_{T+1+\sigma}(y, x+\eta) - 1$. Since $\Psi_0(y, x+\eta) = 1$, for all $x \ge 0$ and $|y| \le \varepsilon^{b'} < \eta$, we have $|\Psi_{T+1+\sigma}(y, x+\eta) - \Psi_0(y, x+\eta)| = |f(y)|$. Further, since $f(0) = 0$ and $f'(y) = -p^{\mathrm{N}}_{T+\sigma+1}(y, x+\eta)$, we further deduce $|f(y)| \le C|y| \times (T+1+\sigma)^{-\frac{1}{2}} \le C|y|$. Plugging this into (4.2), we obtain $S^{\varepsilon}_{b'}(T)|\mathcal{A}^{\varepsilon}_t(x)| \le C\varepsilon^{-\frac{1}{4}+2b'}$, thereby, with $b' > 1/8$, concluding (4.1).

4.2. *Proof of part* (b). Recall the definitions of $\Psi^{\varepsilon}_t(y, x)$ and $p^{\mathrm{N},\varepsilon}_t(y, x)$ from (2.7). By Lemma 3.4, we have $\mathcal{W}^{\varepsilon}_t(x) = \mathcal{W}^{\varepsilon}_t(x) - 2\mathcal{R}^{\varepsilon}_t(x)$, for

$$\mathcal{W}^{\varepsilon}_t(x) := \varepsilon^{\frac{1}{4}} \sum_{i=0}^{\infty} \big(1 - 2Y_i(0)\big)\Psi^{\varepsilon}_t\big(X^{\varepsilon}_{(i)}(0), x\big), \tag{4.3}$$

$$\mathcal{R}^{\varepsilon}_t(x) := \varepsilon^{-\frac{1}{4}} \sum_{i=0}^{\infty} \int_{X^{\varepsilon}_{(i)}(0)}^{X^{\varepsilon}_{(i+1)}(0)} \big(X^{\varepsilon}_{(i+1)}(0) - y\big) p^{\mathrm{N},\varepsilon}_t(y, x)\, dy. \tag{4.4}$$

We first show that $\mathcal{R}^{\varepsilon}_{\cdot}(\cdot) \Rightarrow 0$, or more explicitly, for any fixed $T, L < \infty$,

$$\mathbf{E}\Big(\sup_{t \in [0,T]} \sup_{x \in [0,L]} \big|\mathcal{R}^{\varepsilon}_t(x)\big|\Big) \le C\varepsilon^{\frac{1}{4}} |\log \varepsilon|, \tag{4.5}$$

where $C = C(T, L) < \infty$ and $\varepsilon \in (0, \frac{1}{4}]$.

PROOF OF (4.5). Fixing $T, L < \infty$, we let $C = C(T, L) < \infty$ to simplify notation. To bound $\mathcal{R}^{\varepsilon}_t(x)$, in (4.4) we use $(X^{\varepsilon}_{(i+1)}(0) - y) \le \varepsilon^{\frac{1}{2}} Y_i(0)$, and then divide the sum into the sums over $i \le \varepsilon^{-1}$ and over $i > \varepsilon^{-1}$. For the former replacing each $Y_i(0)$ with $\overline{Y}^\varepsilon := \sup_{i\le\varepsilon^{-1}} Y_i(0)$, we obtain

$$\sup_{t\in[0,T]}\sup_{x\in[0,L]} \left|\mathcal{R}_t^\varepsilon(x)\right| \le R_1^\varepsilon + R_2^\varepsilon, \tag{4.6}$$

$$R_1^\varepsilon := \varepsilon^{\frac{1}{4}}\overline{Y}^\varepsilon \int_{X_{(0)}^\varepsilon(0)}^{X^\varepsilon_{(\lceil\varepsilon^{-1}\rceil)}(0)} p_t^{\mathrm{N},\varepsilon}(y,x)\,dy \le 2\varepsilon^{\frac{1}{4}}\overline{Y}^\varepsilon,$$

$$R_2^\varepsilon := \varepsilon^{\frac{1}{4}} \sum_{i>\varepsilon^{-1}} Y_i(0) \sup_{t\in[0,T]}\sup_{x\in[0,L]} \int_{X_{(i)}^\varepsilon(0)}^\infty p_t^{\mathrm{N},\varepsilon}(y,x)\,dy. \tag{4.7}$$

With $\{Y_i(0)\} \sim \bigotimes_{i\in\mathbb{Z}_+} \operatorname{Exp}(2)$, we have $\mathbf{E}(R_1^\varepsilon) \le C\varepsilon^{\frac{1}{4}}|\log\varepsilon|$. As for R_2^ε, from (1.3), we have

$$0 \le \Psi_t^\varepsilon(x,y) \le C(T,L)\big(e^{-y}\wedge 1\big) \qquad \forall t\in[0,T], x\in[0,L], y\in\mathbb{R}_+. \tag{4.8}$$

Plugging this into (4.7), we obtain $R_2^\varepsilon \le C\varepsilon^{\frac{1}{4}}\sum_{i>\varepsilon^{-1}} Y_i \exp(-X_{(i)}^\varepsilon(0))$. Further applying (3.13) for $f_i = Y_i \mathbf{1}_{\{i>\varepsilon^{-1}\}}$, we conclude

$$\mathbf{E}\big(R_2^\varepsilon\big) \le C\varepsilon^{-\frac{1}{4}} \sum_{j=0}^\infty e^{-j/4}\bigg(\sum_{i\in J_j^\varepsilon} \mathbf{1}_{\{i>\varepsilon^{-1}\}}\|Y_i\|_2^2\bigg)^{\frac{1}{2}} \le C\varepsilon^{-\frac{1}{4}}\exp\bigg(-\frac{1}{4}\varepsilon^{-\frac{1}{2}}\bigg).$$

Combining the preceding bounds on $\mathbf{E}(R_1^\varepsilon)$ and $\mathbf{E}(R_2^\varepsilon)$ with (4.6), we conclude (4.5). □

Recall that $\mathcal{W}_t(x)$ is defined as in (1.7). With (4.1), it then suffices to show the following.

LEMMA 4.1. *We have that* $\{\mathcal{W}^\varepsilon\}_\varepsilon \subset C(\mathbb{R}_+^2)$ *and the processes are tight in* $C(\mathbb{R}_+^2)$.

LEMMA 4.2. *As* $\varepsilon \to 0$, $\mathcal{W}_\cdot^\varepsilon(\cdot)$ *converges in finite dimensional distribution to* $\mathcal{W}_\cdot(\cdot)$.

For a convex compact $\mathcal{K} \subset \mathbb{R}^2$ and $\beta_1, \beta_2 > 0$, defining the $C^{\beta_1,\beta_2}(\mathcal{K})$-norm

$$|f|_{C^{\beta_1,\beta_2}(\mathcal{K})} := \sup_{(t,x)\in\mathcal{K}} \left|f(t,x)\right| + \sup_{(t,x)\neq(t',x')\in\mathcal{K}} \frac{|f(t,x)-f(t',x')|}{|t-t'|^{\beta_1}+|x-x'|^{\beta_2}},$$

we recall the the following mixed Kolmogorov-type estimate.

LEMMA 4.3 ([15], Theorem 1.4.1). *Let* $I := [0,T]\times[0,L]$ *be a bounded square in* $\mathbb{R}^2$, *and let* $K(t,x)$ *be a* $C([0,T]\times\mathbb{R})$*-valued process. If, for some*

$\gamma_1, \gamma_2, \gamma, C_1 < \infty$ *with* $\frac{1}{\gamma\gamma_1} + \frac{1}{\gamma\gamma_2} := \gamma_0 < 1$ *such that*

$$\|K(0,0)\|_\gamma \le C_1, \tag{4.9}$$

$$\|K(t,x) - K(t',x')\|_\gamma \le C_1(|t-t'|^{\gamma_1} + |x-x'|^{\gamma_2}), \tag{4.10}$$

$\forall t, t' \in [0,T]$, $x, x' \in [0,L]$, *then, for any* $(\beta_1, \beta_2) \in (0, \gamma_1(1-\gamma_0)) \times (0, \gamma_2(1-\gamma_0))$, *there exists* $C_2(C_1, T, L, \gamma_1, \gamma_2, \gamma, \beta_1, \beta_2) < \infty$ *such that*

$$\big\||K|_{C^{\beta_1,\beta_2}(I)}\big\|_\gamma \le C_2. \tag{4.11}$$

REMARK 4.4. Recall that $\|F\|_\gamma := (\mathbf{E}|F|^\gamma)^{1/\gamma}$, so for the special case $\gamma_1 = \gamma_2$, the condition (4.10) reduces to the conventional form

$$\mathbf{E}|K(t,x) - K(t',x')|^\gamma \le C_1(|t-t'| + |x-x'|)^{\gamma\gamma_1},$$

for some $\gamma\gamma_1 := \alpha > 2$, and, with $\gamma_0 = \frac{2}{\gamma\gamma_1}$, (4.11) holds for $\beta_1 = \beta_2 \in (0, \frac{\alpha-2}{\gamma})$. Here, we refer to the generalized form as in Lemma 4.3 as it suits our purpose.

REMARK 4.5. Although the dependence of C_2 is not explicitly designated in [15], Theorem 1.4.1, under the present setting, it is clear from the proof of [15], Lemma 1.4.2, Lemma 1.4.3, that $C_2 = C_2(C_1, T, L, \gamma_1, \gamma_2, \gamma, \beta_1, \beta_2)$.

PROOF OF LEMMA 4.1. For each $i \in \mathbb{Z}_+$, $(t,x) \mapsto (1-2Y_i(0))\Psi_t^\varepsilon(X_{(i)}^\varepsilon(0), x)$ is continuous. The series (4.3) defining $\mathcal{W}_\cdot^\varepsilon(\cdot)$ converges absolutely, so $\mathcal{W}_\cdot^\varepsilon(\cdot) \in C(\mathbb{R}_+^2)$.

Fixing $T, L < \infty$, $\gamma \in (1,\infty)$, $x, x' \in [0,L]$ and $t < t' \in [0,T]$, letting $C = C(T,L,\gamma) < \infty$, our goal is to show (4.9)–(4.10) for $K(t,x) = \mathcal{W}_t^\varepsilon(x)$. To this end, consider the discrete time martingale

$$k \longmapsto m_k^\varepsilon(t,x) := \varepsilon^{\frac{1}{4}} \sum_{i=0}^{k} (1-2Y_i(0))\Psi_t^\varepsilon(X_{(i)}^\varepsilon(0), x). \tag{4.12}$$

With $\mathcal{W}_t^\varepsilon(x) = m_\infty^\varepsilon(t,x)$, showing (4.9)–(4.10) amounts to bounding the quadratic variation of $m_\cdot^\varepsilon(t,x)$, which we do by using $Q_0^\varepsilon \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$.

Let $\langle \widetilde{Q}_0^{\varepsilon,k}, f\rangle := \varepsilon^{\frac{1}{2}} \sum_{i=0}^k f(X_{(i)}^\varepsilon(0))$ be the kth approximation of $\widetilde{Q}_t^\varepsilon$. The martingale $m_k^\varepsilon(t,x)$ has quadratic variation $\langle \widetilde{Q}_0^{\varepsilon,k}, \Psi_t^\varepsilon(\cdot,x)^2\rangle$. Consequently, by the BDG inequality and Fatou's lemma, letting $k \to \infty$ we have

$$\|\mathcal{W}_0^\varepsilon(0)\|_\gamma^2 \le C\|\langle \widetilde{Q}_0^\varepsilon, (\Psi_0^\varepsilon(\cdot,0))^2\rangle\|_{\gamma/2}, \tag{4.13}$$

$$\|\mathcal{W}_t^\varepsilon(x) - \mathcal{W}_t^\varepsilon(x')\|_\gamma^2 \le C\|\langle \widetilde{Q}_0^\varepsilon, (\Psi_t^\varepsilon(\cdot,x) - \Psi_t^\varepsilon(\cdot,x'))^2\rangle\|_{\gamma/2}, \tag{4.14}$$

$$\|\mathcal{W}_t^\varepsilon(x') - \mathcal{W}_{t'}^\varepsilon(x')\|_\gamma^2 \le C\|\langle \widetilde{Q}_0^\varepsilon, (\Psi_t^\varepsilon(\cdot,x') - \Psi_{t'}^\varepsilon(\cdot,x'))^2\rangle\|_{\gamma/2}. \tag{4.15}$$

Applying $\Psi_0^\varepsilon(y,0) \le Ce^{-y}$ [by (4.8)] to (4.13) and then using $\|\langle \widetilde{Q}_0^\varepsilon, \exp(-2\cdot)\rangle\|_{q/2} \le C$ [by (3.4) for $j=0$], we obtain (4.9). Turning to showing (4.10), since $0 \le \Psi_t^\varepsilon(y,x) \le 2$, we have

$$(4.16)\quad \big(\Psi_t^\varepsilon(y,x) - \Psi_t^\varepsilon(y,x')\big)^2 \le 2\int_x^{x'} \big|\partial_z \Psi_t^\varepsilon(z,x)\big|\,dz = 2\int_x^{x'} p_t^{\mathrm{N},\varepsilon}(y,z)\,dz.$$

Using this in (4.14), we bound the right-hand side of (4.14) by $C\int_x^{x'} \|\langle \widetilde{Q}_0^\varepsilon, p_t^{\mathrm{N},\varepsilon}(\cdot,z)\rangle\|_{q/2}\,dz$. This, by (3.35), is bounded by $C|x-x'|$, from which it follows

$$(4.17)\qquad \big\|\mathcal{W}_t^\varepsilon(x) - \mathcal{W}_t^\varepsilon(x')\big\|_\gamma^2 \le C|x-x'|.$$

Next, letting $\widetilde{\Psi}_{t,t'}^\varepsilon(y) := \Psi_t^\varepsilon(y,x') - \Psi_{t'}^\varepsilon(y,x')$, similar to (4.16) we have

$$\begin{aligned}\big(\widetilde{\Psi}_{t,t'}^\varepsilon(y)\big)^2 &\le 2\int_t^{t'} \big|\partial_s \Psi_s^\varepsilon(y,x)\big|\,ds \\ &= \int_t^{t'} (s+\sigma)^{-1}\big|(y+x+\eta)p_{s+\sigma}(y+x+\eta) \\ &\qquad + (y-x-\eta)p_{s+\sigma}(y-x-\eta)\big|\,ds,\end{aligned}$$

where $\sigma := \varepsilon^a$ and $\eta := \varepsilon^b$. Further using $s^{-1}|z|p_s(z) \le Cs^{-\frac12}p_{2s}(z)$ to bound the right-hand side, and combining the result with (4.15), we arrive at

$$\big\|\mathcal{W}_t^\varepsilon(x') - \mathcal{W}_{t'}^\varepsilon(x')\big\|_\gamma^2 \le C\int_t^{t'} s^{-\frac12}\big\|\langle \widetilde{Q}_0^\varepsilon, p_{2s+2\sigma}^{\mathrm{N}}(\cdot,x+\eta)\rangle\big\|_{\gamma/2}.$$

Using the bound (3.35) and integrating over s on the right-hand side, we conclude

$$\big\|\mathcal{W}_t^\varepsilon(x) - \mathcal{W}_{tt}^\varepsilon(x')\big\|_\gamma^2 \le C|t-t'|^{\frac12}.$$

Combining this with (4.17) using triangle inequality, we thus obtain (4.10) for $(\gamma_1,\gamma_2) = (\frac14,\frac12)$, that is,

$$\big\|\mathcal{W}_t^\varepsilon(x) - \mathcal{W}_{t'}^\varepsilon(x')\big\|_\gamma \le C\big(|t-t'|^{\frac14} + |x-x'|^{\frac12}\big).$$

With $(\gamma_1,\gamma_2) = (\frac14,\frac12)$, we now choose large enough $\gamma \in (1,\infty)$ to ensures that $\frac{1}{\gamma\gamma_1} + \frac{1}{\gamma\gamma_1} := \alpha_0 < 1$. With this, we apply Lemma 4.3 to obtain that $\||\mathcal{W}^\varepsilon|_{C^{\beta_1,\beta_2}(I)}\|_\gamma$ for some suitable $\beta_1,\beta_2 > 0$, where $I := [0,T]\times[0,L]$. It hence follows that $\{\mathcal{W}^\varepsilon\}_\varepsilon$ is tight in $C(\mathbb{R}_+^2)$. $\square$

PROOF OF LEMMA 4.2. Instead of showing $\mathcal{W}_\cdot^\varepsilon(\cdot)$ converges to $\mathcal{W}_\cdot(\cdot)$ in finite dimensional distribution, we consider the following approximation of $\mathcal{W}_t^\varepsilon(x)$:

$$(4.18)\qquad \mathcal{W}_t^{\varepsilon,\prime}(x) := \varepsilon^{\frac14}\sum_{i\le\varepsilon^{-1}} \big(1-2Y_i(0)\big)\Psi_t^\varepsilon\big(x_i^\varepsilon, x\big),$$

where each $X^{\varepsilon}_{(i)}(0)$ is replaced by its expected value $x^{\varepsilon}_i := \mathbf{E}(X^{\varepsilon}_{(i)}(0)) = \varepsilon^{\frac{1}{2}} 2^{-1} i$, and the infinite sum is truncated at $i = \varepsilon^{-1}$. As $k \mapsto \sum_{i=0}^{k} (1 - 2Y_i(0)) \Psi^{\varepsilon}_t(x^{\varepsilon}_i, x)$ is a discrete time L^2-martingale, following the argument in the proof of Lemma 4.1, we have that

$$
\begin{aligned}
\|\mathcal{W}^{\varepsilon}_t(x) - \mathcal{W}^{\varepsilon,\prime}_t(x)\|_2 \le C\mathbf{E}\Big|\varepsilon^{\frac{1}{2}} \sum_{i \le \varepsilon^{-1}} \big(\Psi^{\varepsilon}_t\big(X_{(i)}(0), x\big) - \Psi^{\varepsilon}_t\big(x^{\varepsilon}_i, x\big)\big)\Big| \\
+ C\mathbf{E}\Big|\varepsilon^{\frac{1}{2}} \sum_{i > \varepsilon^{-1}} \Psi^{\varepsilon}_t\big(X_{(i)}(0), x\big)\Big|.
\end{aligned}
\tag{4.19}
$$

With $\{X^{\varepsilon}_{(i)}(0)\}_{i \in \mathbb{Z}_+} \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$ and $\Psi_t(y, x) \le C\varepsilon^{-y}$ [by (4.8)] we clearly have that the right-hand side of (4.19) converges to zero. As this holds for each $(t, x) \in \mathbb{R}^2_+$, it suffices to show $\mathcal{W}^{\varepsilon,\prime}_{\cdot}(\cdot)$ converges to $\mathcal{W}_{\cdot}(\cdot)$ in finite dimensional distributions.

Fixing arbitrary $(t_k, x_k) \in \mathbb{R}^2_+$, $k = 1, \dots, \ell$, we let

$$
\mathbf{W}^{\varepsilon,\prime} := \big(\mathcal{W}^{\varepsilon,\prime}_{t_1}(x_1), \dots, \mathcal{W}^{\varepsilon,\prime}_{t_\ell}(x_\ell)\big), \qquad \mathbf{W} := \big(\mathcal{W}_{t_1}(x_1), \dots, \mathcal{W}_{t_\ell}(x_\ell)\big).
$$

Our goal is to show $\mathbf{W}^{\varepsilon,\prime} \Rightarrow \mathbf{W}$. With $\mathcal{W}_t(x)$ as in (1.7), we have that $\mathbf{W} \sim \mathcal{N}(0, \Sigma)$, where $\Sigma_{jk} := 2\int_0^\infty \Psi_{t_j}(y, x_j)\Psi_{t_k}(y, x_k)\,dy$. As $\{\mathbf{W}^{\varepsilon,\prime}\}_\varepsilon$ is tight (by Lemma 4.1), it suffices to show $\mathbf{v} \cdot \mathbf{W}^{\varepsilon,\prime} \Rightarrow \mathbf{v} \cdot \mathbf{W} \sim \mathcal{N}(0, \sigma_{\mathbf{v}})$ for any fixed unit vector $\mathbf{v} \in \mathbb{R}^\ell$, where $\sigma_{\mathbf{v}} := \mathbf{v} \cdot (\Sigma\mathbf{v})$. To this end, we express $\mathbf{v} \cdot \mathbf{W}^{\varepsilon,\prime}$ as

$$
\mathbf{v} \cdot \mathbf{W}^{\varepsilon,\prime} = \sum_{i \le \varepsilon^{-1}} U_\varepsilon(i), \qquad U_\varepsilon(i) := \varepsilon^{\frac{1}{4}} \sum_{j=1}^{\ell} v_j \big(1 - 2Y_i(0)\big)\Psi^{\varepsilon}_{t_j}\big(x^{\varepsilon}_i, x_j\big).
$$

The random variables $U_\varepsilon(i)$, $i < \varepsilon^{-\frac{1}{2}}$, are independent, with mean zero and variance:

$$
\sigma^2_\varepsilon(i) := \sum_{j,k=1}^{\ell} \varepsilon^{\frac{1}{2}} v_j v_k \Psi^{\varepsilon}_{t_j}\big(x^{\varepsilon}_i, x_j\big)\Psi^{\varepsilon}_{t_k}\big(x^{\varepsilon}_i, x_k\big).
$$

It is easy to show that $\sum_{i \le \varepsilon^{-1}} \sigma^2_\varepsilon(i) \to \sigma_{\mathbf{v}}$ and that $\|U^{\varepsilon}(i)\|_q \le \varepsilon^{\frac{1}{4}} C(q) < \infty$, for any $q \in [1, \infty)$. With this and $\sigma_{\mathbf{v}} = 2\int_0^\infty (\sum_{j=1}^{\ell} v_j \Psi_{t_j}(y, x_j)\,dy)^2 > 0$, Lyapunov's condition (for central limit theorem)

$$
\frac{1}{(\sum_{i \le \varepsilon^{-1}} \sigma^2_\varepsilon(i))^{2+\delta}} \sum_{i \le \varepsilon^{-1}} \mathbf{E}\big(\big|U_\varepsilon(i)\big|^{2+\delta}\big) \le C(\delta)\varepsilon^{-1+\frac{1}{2}+\frac{\delta}{2}} \to 0
$$

holds for any $\delta \in (2, \infty)$. From this, we conclude the desired result $\mathbf{v} \cdot \mathbf{W}^{\varepsilon,\prime} \Rightarrow \mathcal{N}(0, \sigma_{\mathbf{v}})$ using Lyapunov central limit theorem. $\square$

4.3. *Proof of part* (c). Let $B'_i(t) := \int_0^t \sum_{j=0}^\infty \mathbf{1}_{\{X_j(s)=X_{(i)}(s)\}}\, dB_j(s)$ denote the driving Brownian motion of the *ith ranked particle*. It is known (e.g., [18]) that $B'_i(t)$, $i \in \mathbb{Z}_+$, are independent standard Brownian motions. With this we express $\mathcal{M}^\varepsilon_t(x)$ in terms of ranked particles as

$$\mathcal{M}^\varepsilon_t(x) = \varepsilon^{\frac{1}{4}} \sum_{i=0}^\infty \int_0^t p^{\mathrm{N},\varepsilon}_{t-s}\big(X^\varepsilon_{(i)}(s), x\big)\, dB'^{,\varepsilon}_i(s). \tag{4.20}$$

Recall that $\mathcal{M}_t(x)$ is defined as in (1.8) and that $x^\varepsilon_i := \mathbf{E}(X^\varepsilon_{(i)}(0)) = i\varepsilon^{\frac{1}{2}}2^{-1}$. To the end of showing $\mathcal{M}^\varepsilon_\cdot(\cdot) \Rightarrow \mathcal{M}_\cdot(\cdot)$, for each $\varepsilon > 0$ we couple the processes $\mathcal{M}^\varepsilon_\cdot(\cdot)$ and $\mathcal{M}_\cdot(\cdot)$ by setting $B'^{,\varepsilon}_i(t) := \sqrt{2}\varepsilon^{-\frac{1}{4}} \int_0^t \int_{x^\varepsilon_i}^{x^\varepsilon_{i+1}} d\mathscr{W}(y,s)$, whereby

$$\mathcal{M}^\varepsilon_t(x) = \sqrt{2} \int_0^t \int_0^\infty p^{\mathrm{N},\varepsilon}_{t-s}\big(\overline{X}_\varepsilon(s,y), x\big)\, d\mathscr{W}(y,s), \tag{4.21}$$

where $\overline{X}_\varepsilon(s,y) := \sum_{i=0}^\infty \mathbf{1}_{[x^\varepsilon_i, x^\varepsilon_{i+1})}(y) X^\varepsilon_{(i)}(s)$. Further, recall from Proposition 2.3 that we have $\mathbf{P}(\tau_{1/8} \le T) \to 1$, for any fixed $T < \infty$, so without loss of generality we consider the localized processes $\mathcal{N}^\varepsilon_t(x) := \mathcal{M}^\varepsilon_{t\wedge\tau_{1/8}}(x)$ and $\mathcal{N}_t(x) := \mathcal{M}_{t\wedge\tau_{1/8}}(x)$, and show $\mathcal{N}^\varepsilon_\cdot(\cdot) - \mathcal{N}_\cdot(\cdot) \Rightarrow 0$ instead.

We begin by showing that $\{\mathcal{N}^\varepsilon\}_\varepsilon$ is tight in $C(\mathbb{R}^2_+)$. To this end, we fix arbitrary $T, L < \infty$, $\gamma \in (1,\infty)$, $\gamma_1 \in (0, \frac{1}{4})$ and $\gamma_2 \in (0, \frac{1}{2})$, and estimate $\|\mathcal{N}^\varepsilon_{t'}(x') - \mathcal{N}^\varepsilon_t(x)\|_\gamma$ for $t \le t' \in [0,T]$ and $x, x' \in [0,L]$. Recalling the definition of $S_b(s)$ from (3.33), we use the expression (4.21) of $\mathcal{M}^\varepsilon_t(x)$ to express $\mathcal{N}^\varepsilon_{t'}(x') - \mathcal{N}^\varepsilon_t(x) = \sqrt{2}(F^\varepsilon_1 + F^\varepsilon_2)$, where

$$F^\varepsilon_1 := \int_t^{t'} \int_0^\infty S^\varepsilon_{1/8}(s) p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(s,y), x'\big)\, d\mathscr{W}(y,s),$$
$$F^\varepsilon_2 := \int_0^t \int_0^\infty S^\varepsilon_{1/8}(s)\big(p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(s,y), x'\big) - p^{\mathrm{N},\varepsilon}_{t-s}\big(\overline{X}_\varepsilon(s,y), x\big)\big)\, d\mathscr{W}(y,s).$$

Let $C = C(T, L, \gamma, \gamma_1, \gamma_2, \delta) < \infty$ to simplify notation hereafter. Applying the BDG inequality [in the same way as we derive (3.22)] yields

$$\|F^\varepsilon_1\|^2_\gamma \le C \int_t^{t'} \left\| S^\varepsilon_{1/8}(s) \int_0^\infty p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(s,y), x'\big)^2\, dy \right\|_{\gamma/2} ds, \tag{4.22}$$

$$\|F^\varepsilon_2\|^2_\gamma \le C \int_0^t \left\| S^\varepsilon_{1/8}(s) \int_0^\infty \big(f^\varepsilon(s,y)\big)^2\, dy \right\|_{\gamma/2} ds, \tag{4.23}$$

where $f^\varepsilon(s,y) := p^{\mathrm{N},\varepsilon}_{t'-s}(\overline{X}_\varepsilon(s,y), x') - p^{\mathrm{N},\varepsilon}_{t-s}(\overline{X}_\varepsilon(s,y), x)$. The kernel functions $p^{\mathrm{N},\varepsilon}_{t'-s}(\overline{X}_\varepsilon(s,y), x')$ and $f^\varepsilon(s,y)$ appear in quadratic power in (4.22)–(4.23). We apply the elementary inequalities

$$p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(y,s),x'\big) \le \frac{C}{\sqrt{t'-s}},$$

$$\big|f^\varepsilon(s,y)\big| \le \frac{C}{\sqrt{t-s}}\bigg(\bigg(\frac{|t'-t|}{|t-s|}\bigg)^{2\gamma_1} + \bigg(\frac{|x'-x|}{\sqrt{t-s}}\bigg)^{2\gamma_2}\bigg)$$

to one copy of the kernel functions in (4.22)–(4.23), respectively, to obtain

$$\begin{aligned}\int_0^\infty p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(s,y),x'\big)^2\,dy \le (t-s)^{-\frac{1}{2}} S^\varepsilon_{1/8}(s) \\ \times \int_0^\infty p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(s,y),x'\big)\,dy,\end{aligned} \tag{4.24}$$

$$\begin{aligned}\int_0^\infty \big(f^\varepsilon(s,y)\big)^2\,dy \le (t-s)^{-\frac{1}{2}-\gamma_{12}} S^\varepsilon_{1/8}(s) \\ \times \int_0^\infty p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(s,y),x'\big)\,dy,\end{aligned} \tag{4.25}$$

where $\gamma_{12} := (2\gamma_1)\vee\gamma_2 \in (0,\frac{1}{2})$. Recall that $\widetilde{Q}^\varepsilon_s := \varepsilon^{\frac{1}{2}} Q^\varepsilon_s$. Using

$$\int_0^\infty p^{\mathrm{N},\varepsilon}_{t'-s}\big(\overline{X}_\varepsilon(s,y),x'\big)\,dy = \frac{1}{2}\big\langle \widetilde{Q}^\varepsilon_s, p^{\mathrm{N},\varepsilon}_{t'-\cdot}(\cdot,x')\big\rangle \tag{4.26}$$

and $\int_0^\infty |f^\varepsilon(s,y)|\,dy \le \frac{1}{2}\langle \widetilde{Q}^\varepsilon_s, p^{\mathrm{N},\varepsilon}_{t'-\cdot}(\cdot,x')\rangle + \frac{1}{2}\langle \widetilde{Q}^\varepsilon_s, p^{\mathrm{N},\varepsilon}_{t-\cdot}(\cdot,x)\rangle$ in (4.24)–(4.25), taking the $L^{\gamma/2}$-norm of the results, and integrating s over the relevant regions, we arrive at

$$\begin{aligned}\|F^\varepsilon_1\|^2_\gamma &\le C\int_t^{t'} (t'-s)^{-\frac{1}{2}} \big\|S^\varepsilon_{1/8}(s)\big\langle \widetilde{Q}^\varepsilon_s, p^{\mathrm{N},\varepsilon}_{t'-\cdot}(\cdot,x')\big\rangle\big\|_{\gamma/2}\,ds,\\ \|F^\varepsilon_2\|^2_\gamma &\le C\big(|t-t'|^{\gamma_1} + |x-x'|^{\gamma_2}\big)^2\\ &\quad\times\int_0^t (t-s)^{-\frac{1}{2}-\gamma_{12}}\big(\big\|S^\varepsilon_{1/8}(s)\big\langle \widetilde{Q}^\varepsilon_s, p^{\mathrm{N},\varepsilon}_{t'-\cdot}(\cdot,x')\big\rangle\big\|_{\gamma/2}\\ &\quad+ \big\|S^\varepsilon_{1/8}(s)\big\langle \widetilde{Q}^\varepsilon_s, p^{\mathrm{N},\varepsilon}_{t'-\cdot}(\cdot,x')\big\rangle\big\|_{\gamma/2}\big)\,ds.\end{aligned}$$

Further apply (3.35) to the terms involving $\widetilde{Q}^\varepsilon_s$. With $\gamma_{12} < \frac{1}{2}$, integrating over s yields $\|F_1\|^2_\gamma \le C|t'-t|^{\frac{1}{2}} \le C|t'-t|^{2\gamma_1}$, and $\|F_2\|^2_\gamma \le C(|t-t'|^{\gamma_1} + |x-x'|^{\gamma_2})^2$, so

$$\big\|\mathcal{N}^\varepsilon_{t'}(x') - \mathcal{N}^\varepsilon_t(x)\big\|_\gamma \le C\big(|t-t'|^{\gamma_1} + |x-x'|^{\gamma_2}\big).$$

With this and $\mathcal{N}^\varepsilon_0(0) = 0$, we apply Lemma 4.3 for $K(t,x) = \mathcal{N}^\varepsilon_t(x)$ [and for some large enough $\gamma \in (1,\infty)$ such that $\frac{1}{\gamma\gamma_1} + \frac{1}{\gamma\gamma_1} := \alpha_0 < 1$] to conclude

$\| |\mathcal{N}^\varepsilon|_{C^{\beta_1,\beta_2}(I)} \|_\gamma$, for some suitable $\beta_1, \beta_2 > 0$, where $I := [0, T] \times [0, L]$. It hence follows that $\{\mathcal{N}^\varepsilon\}_\varepsilon$ is tight in $C(\mathbb{R}_+^2)$.

With $\{\mathcal{N}^\varepsilon\}_\varepsilon$ being tight, it now suffices to prove, for each fixed $(t, x) \in \mathbb{R}_+^2$, $\mathcal{N}_t^\varepsilon(x) - \mathcal{N}_t(x) \Rightarrow 0$. To this end, we use the expressions (1.8) and (4.21) for $\mathcal{M}_t(x)$ and $\mathcal{M}_t^\varepsilon(x)$ to express

$$\begin{aligned} &\mathcal{N}^\varepsilon(t,x) - \mathcal{N}(t,x) \\ &\quad = \sqrt{2} \int_0^t \int_0^\infty S_{1/8}^\varepsilon(s) \big(p_{t-s}^{\mathrm{N},\varepsilon}\big(\overline{X}_\varepsilon(s,y), x\big) - p_{t-s}^{\mathrm{N}}(y,x)\big) \, d\mathscr{W}(y,s), \end{aligned} \tag{4.27}$$

and apply the BDG inequality to obtain

$$\|\mathcal{N}^\varepsilon(t,x) - \mathcal{N}(t,x)\|_2^2 \le C \int_0^t \int_0^\infty \mathbf{E}\big|H^\varepsilon(s,y)\big| \, dy \, ds, \tag{4.28}$$

where $H^\varepsilon(s,y) := S_{1/8}^\varepsilon(s)(p_{t-s}^{\mathrm{N},\varepsilon}(\overline{X}_\varepsilon(s,y),x) - p_{t-s}^{\mathrm{N}}(y,x))^2$. Our goal is to show the right-hand side of (4.28) converges to zero utilizing the fact that $\overline{X}_\varepsilon(y,s)$ approximates y. More precisely, with $\overline{X}_\varepsilon(y,s) = X_{(i_*)}^\varepsilon(s) - y$, where $i_* \in \mathbb{Z}_+$ is such that $y \in [x_{i_*}^\varepsilon, x_{i_*+1}^\varepsilon)$, we have

$$\big|\overline{X}_\varepsilon(y,s) - y\big| \le \big|X_{(0)}^\varepsilon(s)\big| + \big|X_{(i_*)}^\varepsilon(s) - X_{(0)}^\varepsilon(s) - x_{i_*}^\varepsilon\big| + \big|x_{i_*+1}^\varepsilon - x_{i_*}^\varepsilon\big|.$$

Further, using $S_{1/8}^\varepsilon(s)|X_{(0)}^\varepsilon(s)| \le \varepsilon^{\frac{1}{8}}$, $\{X_{(i)}^\varepsilon(s) - X_{(0)}^\varepsilon(s)\}_{i\in\mathbb{Z}_+} \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$ and $|x_{i_*+1}^\varepsilon - x_{i_*}^\varepsilon| = \varepsilon^{\frac{1}{2}} 2^{-1}$, it is easy to show that $S_{1/8}^\varepsilon(s)|\overline{X}_\varepsilon(y,s) - y| \underset{\mathrm{P}}{\to} 0$, $\forall (s,y) \in \mathbb{R}_+$. Consequently,

$$H^\varepsilon(s,y) \underset{\mathrm{P}}{\to} 0 \qquad \forall (s,y) \in \mathbb{R}_+^2. \tag{4.29}$$

Furthermore, $\{H^\varepsilon\}_\varepsilon$ is uniformly integrable with respect to $\int_0^t \int_0^\infty \mathbf{E}(\cdot) \, dy \, ds$. To see this, fixing arbitrary $\delta \in (0, \frac{1}{2})$, with $H^\varepsilon(s,y)$ defined as in the the preceding, we write $|H^\varepsilon(s,y)|^{1+\delta} \le C S_{1/8}^\varepsilon(s) p_{t-s}^{\mathrm{N},\varepsilon}(\overline{X}_\varepsilon(s,y),x)^{2+2\delta} + p_{t-s}^{\mathrm{N}}(y,x)^{2+2\delta}$. Applying $\int_0^t \int_0^\infty \mathbf{E}(\cdot) \, dy \, ds$ to both sides using $p_{t-s}^{\mathrm{N},\varepsilon}(y,x)^{1+2\delta} \le C(t-s)^{-\frac{1}{2}-\delta}$ and (4.26), we obtain

$$\begin{aligned} &\int_0^t \int_0^\infty \mathbf{E}\big(\big|H^\varepsilon(s,y)\big|^{1+\delta}\big) \, dy \, ds \\ &\qquad \le C \int_0^t (t-s)^{-\frac{1}{2}-\delta} \int_0^\infty \mathbf{E}\big|S_{1/8}^\varepsilon(s) \big\langle \widetilde{Q}_{t-s}^\varepsilon, p_{t-s}^{\mathrm{N},\varepsilon}(\cdot,x)\big\rangle\big| \, dy \, ds \\ &\qquad\quad + C \int_0^t (t-s)^{-\frac{1}{2}-\delta} \int_0^\infty p_{t-s}^{\mathrm{N}}(y,x) \, dy \, ds. \end{aligned}$$

With $\delta < \frac{1}{2}$, $\int_0^\infty p_{t-s}^{\mathrm{N}}(y,x) \, dy \le 2$ and the bound (3.34), we clearly have that the right-hand side is uniformly (in ε) bounded. Consequently, $\{H^\varepsilon\}_\varepsilon$ is uniformly integrable. Combining this with (4.28), we conclude that $\int_0^t \int_0^\infty \mathbf{E}|H^\varepsilon(s,y)|\,dy\,ds \to 0$. This together with (4.27) yields the desired result $\mathcal{N}_t^\varepsilon(x) - \mathcal{N}_t(x) \Rightarrow 0$.

5. Proof of Proposition 2.7. Throughout this section we fix $a \in (\frac{1}{2}, 1)$, $b \in (\frac{1}{4}, \frac{1}{2})$, and assume without loss of generality (by Proposition 2.3) $\sup_{t \le T} |X_{(0)}(t)| \le \varepsilon^b$, for any given $T < \infty$. A basic tool will be to take union bounds over polynomially many (in ε^{-1}) points $(t,x) \in \mathbb{R}_+^2$. When doing so, we say that events $\{\mathcal{A}^\varepsilon\}$ happen up to Super-Polynomially Decay (SPD) if, for each $n < \infty$, $\mathbf{P}((\mathcal{A}^\varepsilon)^c)\varepsilon^{-n}$ is uniformly bounded. Recall that $t_k^\varepsilon := \varepsilon^{-1}k$. We begin by establishing the following continuity estimates (in t) of $\mathcal{G}_t^\varepsilon(x)$, $\widetilde{\mathcal{X}}_t^\varepsilon(x)$ and $\mathcal{X}_t^\varepsilon(x)$.

LEMMA 5.1. *For any fixed $T, L < \infty$,*

$$F_{\mathcal{X}^\varepsilon}^\varepsilon(T,L) := \sup\{|\mathcal{X}_t^\varepsilon(x) - \mathcal{X}_{t_k^\varepsilon}^\varepsilon(x)| : k \le T\varepsilon^{-1}, t \in [t_k^\varepsilon, t_{k+1}^\varepsilon], x \in [0,L]\} \underset{\mathrm{P}}{\to} 0, \tag{5.1}$$

$$F_{\widetilde{\mathcal{X}}^\varepsilon}^\varepsilon(T,L) := \sup\{|\widetilde{\mathcal{X}}_t^\varepsilon(x) - \widetilde{\mathcal{X}}_{t_k^\varepsilon}^\varepsilon(x)| : k \le T\varepsilon^{-1}, t \in [t_k^\varepsilon, t_{k+1}^\varepsilon], x \in [0,L]\} \underset{\mathrm{P}}{\to} 0, \tag{5.2}$$

$$F_{\mathcal{G}^\varepsilon}^\varepsilon(T,L) := \sup\{|\mathcal{G}_t^\varepsilon(x) - \mathcal{G}_{t_k^\varepsilon}^\varepsilon(x)| : k \le T\varepsilon^{-1}, t \in [t_k^\varepsilon, t_{k+1}^\varepsilon], x \in [0,L]\} \underset{\mathrm{P}}{\to} 0. \tag{5.3}$$

REMARK 5.2. In the sequel, we use (5.3), (5.2)–(5.3) and (5.1)–(5.2) in proving parts (a), (b) and (c) of Proposition 2.7, respectively, and we omit the dependence of $F_{\mathcal{X}^\varepsilon}^\varepsilon(T,L)$, $F_{\widetilde{\mathcal{X}}^\varepsilon}^\varepsilon(T,L)$ and $F_{\mathcal{G}^\varepsilon}^\varepsilon(T,L)$ on ε to simplify notation.

PROOF OF LEMMA 5.1. With $\mathcal{X}_t^\varepsilon(x)$ defined as in (1.2), we have $\mathcal{X}_t^\varepsilon(x) - \mathcal{X}_{t_k^\varepsilon}^\varepsilon(x) = 2\varepsilon^{\frac{1}{4}}(X_{(i_\varepsilon(x))}(\varepsilon^{-1}t) - X_{(i_\varepsilon(x))}(\varepsilon^{-1}t_k^\varepsilon))$, for all $x \in \frac{1}{2}\varepsilon^{\frac{1}{2}}\mathbb{Z}_+$. Fixing arbitrary $\delta > 0$, from (3.7) we deduce that

$$\sup_{t \in [t_k^\varepsilon, t_{k+1}^\varepsilon]} \{\varepsilon^{\frac{1}{4}} |X_{(i)}(\varepsilon^{-1}t) - X_{(i)}(\varepsilon^{-1}t_k^\varepsilon)|\} \le \delta \qquad \text{up to SPD.} \tag{5.4}$$

By taking the union bound of this over $k \le T\varepsilon^{-1}$ and over $i \le L\varepsilon^{-\frac{1}{2}} + 1$, we conclude that

$$\left\{|\mathcal{X}_t^\varepsilon(x) - \mathcal{X}_{t_k^\varepsilon}^\varepsilon(x)| \le a, \forall k \le \varepsilon^{-1}T, t \in [t_k^\varepsilon, t_{k+1}^\varepsilon], x \in \left(\frac{\varepsilon^{\frac{1}{2}}}{2}\mathbb{Z}_+\right) \cap [0,L]\right\} \tag{5.5}$$

holds up to SPD. As $\mathcal{X}^{\varepsilon}_t(x)$ is defined on $x \in \mathbb{R}_+$ via linear interpolation, the desired result (5.1) follows.

Turning to showing (5.2), with $\widetilde{\mathcal{X}}^{\varepsilon}_t(x)$ defined as in (2.13), we have $\widetilde{\mathcal{X}}^{\varepsilon}_t(x) - \widetilde{\mathcal{X}}^{\varepsilon}_{t^{\varepsilon}_k}(x) = 2\varepsilon^{\frac{1}{4}}(X_{(I^{\varepsilon}_0(x))}(\varepsilon^{-1}t) - X_{(I^{\varepsilon}_0(x))}(\varepsilon^{-1}t^{\varepsilon}_k))$. Further, with $\{X^{\varepsilon}_{(i)}(0)\} \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$, we have

$$I^{\varepsilon}_0(x) \le (4L+1)\varepsilon^{-\frac{1}{2}} \qquad \text{up to SPD,} \tag{5.6}$$

so (5.2) follows by taking union bound of (5.4) as done in the preceding.

Proceeding to show (5.1), we note that, by stationarity, $|\mathcal{G}^{\varepsilon}_{t+t^{\varepsilon}_k}(x + X^{\varepsilon}_{(0)}(t^{\varepsilon}_k)) - \mathcal{G}^{\varepsilon}_{t^{\varepsilon}_k}(x + X^{\varepsilon}_{(0)}(t^{\varepsilon}_k))| \overset{\text{distr.}}{=} |\mathcal{G}^{\varepsilon}_t(x) - \mathcal{G}^{\varepsilon}_0(x)|$. With this and $|X^{\varepsilon}_{(0)}(t^{\varepsilon}_k)| \le \varepsilon^b \le 1$, it suffices to show, for some $\delta > 0$,

$$\Big\{\sup_{t\in[0,\varepsilon]}\sup_{x\in[-1,L+1]} \big|\mathcal{G}^{\varepsilon}_t(x) - \mathcal{G}^{\varepsilon}_0(x)\big| \le \varepsilon^{\delta}\Big\} \qquad \text{holds up to SPD.} \tag{5.7}$$

With $\mathcal{G}^{\varepsilon}_t(x)$ defined as in (2.11), we clearly have

$$\mathcal{G}^{\varepsilon}_t(x) - \mathcal{G}^{\varepsilon}_0(x) = -\varepsilon^{\frac{1}{4}} \qquad \big(\text{net flux of } X^{\varepsilon}_i\text{-particles across } x \text{ within } [0,t]\big). \tag{5.8}$$

To bound the right-hand side of (5.8), we note that, since $\{X^{\varepsilon}_i(0)\}_{i\in\mathbb{Z}_+} \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$, by Lemma 3.2 (with $[t_1,t_2] = [0,\varepsilon]$) we clearly have

$$\Big\{\inf_{t\in[0,\varepsilon]} X^{\varepsilon}_{(i)}(t) > L+1, \forall i > \varepsilon^{-1}\Big\} \qquad \text{holds up to SPD,} \tag{5.9}$$

so without loss of generality we ignore particles $X^{\varepsilon}_{(i)}$ with $i \ge \varepsilon^{-1}$. Next, we apply (3.7) for $[t_1,t_2] = [0,\varepsilon]$ and $\alpha = \varepsilon^{\frac{1}{4}}$, and take union of the result over $i \le \varepsilon^{-1}$ to conclude that

$$\Big\{\sup_{t\in[0,\varepsilon]} |X^{\varepsilon}_{(i)}(t) - X^{\varepsilon}_{(i)}(0)| \le \varepsilon^{\frac{1}{4}}, \forall i \le \varepsilon^{-1}\Big\} \qquad \text{holds up to SPD.} \tag{5.10}$$

Under the events of (5.9)–(5.10), we have

$$\sup_{t\in[0,\varepsilon]} \big|\mathcal{G}^{\varepsilon}_t(x) - \mathcal{G}^{\varepsilon}_0(x)\big| \le \varepsilon^{\frac{1}{4}}\langle Q^{\varepsilon}_0, \mathbf{1}_{J^{\varepsilon}(x)}\rangle \qquad \forall x \in [-1, L+1], \tag{5.11}$$

where $J^{\varepsilon}(x) := [x - \varepsilon^{\frac{1}{4}}, x + \varepsilon^{\frac{1}{4}}]$. With $Q^{\varepsilon}_0 \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$, the desired result (5.7) [with $\delta \in (0, \frac{1}{4})$] follows by taking the supremum over $x \in [-1, L+1]$ on both sides of (5.11). $\square$

PROOF OF PROPOSITION 2.7(a). Fixing $L, T < \infty$, our goal is to show

$$\sup_{x\in[\varepsilon^b,L]}\sup_{t\in[0,T]} \big|\mathcal{F}^{\varepsilon,a}_t(x) - \mathcal{G}^{\varepsilon}_t(x)\big| \underset{\mathrm{P}}{\to} 0.$$

To this end, we fix $x \in [\varepsilon^b, L]$, $t \in [0, T]$ and let $C = C(L, T, a, b) < \infty$ denote a generic finite constant.

With $\mathcal{F}_t^{\varepsilon,a}(x)$ and $\mathcal{G}_t^{\varepsilon}(x)$ defined as in (1.13) and (2.11), we have

$$\begin{aligned}\mathcal{F}_t^{\varepsilon,a}(x) - \mathcal{G}_t^{\varepsilon}(x) &= \langle \widehat{Q}_t^{\varepsilon}, f^{\varepsilon}(\cdot, x)\rangle \\ &= \varepsilon^{\frac{1}{4}}\langle Q_t^{\varepsilon}, f^{\varepsilon}(\cdot)\rangle - 2\varepsilon^{-\frac{1}{4}}\int_0^{\infty} f^{\varepsilon}(y, x)\, dx,\end{aligned} \tag{5.12}$$

where $f^{\varepsilon}(y) := \Psi_{\varepsilon^a}(y, x) - \mathbf{1}_{(-\infty,x]}(y)$. Recall the explanation at the end of Section 2. The idea is to bound the last two terms in (5.12) *separately*, using the fact that $f^{\varepsilon}(y)$ is approximately zero on $(-x, \infty)$. More precisely, writing

$$f^{\varepsilon}(x) = 1 - \Phi_{\varepsilon^a}(y + x) + \mathbf{1}_{(x,\infty)}(y) - \Phi_{\varepsilon^a}(y - x), \tag{5.13}$$

by the elementary inequality $|\mathbf{1}_{(0,\infty)}(z) - \Psi_t(z)| \le C \exp(-\frac{|z|}{\sqrt{t}})$ we have

$$\begin{aligned}\left|f^{\varepsilon}(y)\right| &\le C \exp(-\varepsilon^{-\frac{a}{2}}|y + x|) \\ &\quad + C \exp(-\varepsilon^{-\frac{a}{2}}|y - x|) \\ &\le C \exp(-\varepsilon^{b-\frac{a}{2}} - \varepsilon^{-\frac{a}{2}} y) \\ &\quad + C \exp(-\varepsilon^{-\frac{a}{2}}|y - x|) \qquad \forall y \ge -\varepsilon^b \ge -x,\end{aligned} \tag{5.14}$$

where we use $x \ge \varepsilon^b$ in the second inequality.

By (5.14), we clearly have $\varepsilon^{-\frac{1}{4}} \int_0^{\infty} |f^{\varepsilon}(y, x)|\, dy \le C\varepsilon^{-\frac{1}{4}} e^{-\varepsilon^{b-\frac{a}{2}}} + C\varepsilon^{-\frac{1}{4}+\frac{a}{2}} \to 0$. Turning to bounding the term $\varepsilon^{\frac{1}{4}}\langle Q_t^{\varepsilon}, f^{\varepsilon}(\cdot)\rangle$, we fix $a' \in (\frac{1}{2}, a)$ and let $J_{\varepsilon}(x) := (x - \varepsilon^{a'}, x + \varepsilon^{a'}]$. The expression $\exp(-\varepsilon^{-\frac{a}{2}}|y - x|)$ is small expect for $y \in J_{\varepsilon}(x)$. More precisely,

$$\begin{aligned}\exp(-\varepsilon^{-a/2}|y - x|) &\le \exp\left(-\frac{\varepsilon^{-a/2} - 1}{\varepsilon^{a'}}\right) \exp(-|y - x|) \\ &\le C \exp(-\varepsilon^{-\frac{a}{2} - a'}) \exp(-|y - x|) \qquad \forall y \notin J_{\varepsilon}(x).\end{aligned}$$

Using this and $\exp(-|y - x|) \le C \exp(-y)$ (since $x \le L$) in (5.14), with $b < \frac{1}{4} < a'$, we obtain

$$\left|f^{\varepsilon}(y)\right| \le C \exp(-\varepsilon^{a'-\frac{a}{2}}) \exp(-y) + C\mathbf{1}_{J_{\varepsilon}(x)}(y) \qquad \forall y \ge -\varepsilon^b.$$

Using this and $\sup_{t \le T} |X_{(0)}(t)| \le \varepsilon^b$ to bound $\langle Q_t^{\varepsilon}, f^{\varepsilon}(\cdot)\rangle$, we arrive at

$$\left|\varepsilon^{\frac{1}{4}}\langle Q_t^{\varepsilon}, f^{\varepsilon}(\cdot)\rangle\right| \le C F_1^{\varepsilon}(t) + C F_2^{\varepsilon}(t, x),$$

where $F_1^{\varepsilon}(t) := \exp(\frac{1}{4} - \varepsilon^{a'-\frac{a}{2}})\langle Q_t^{\varepsilon}, \exp(-\cdot)\rangle$, and $F_2^{\varepsilon}(t, x) := \varepsilon^{\frac{1}{4}}\langle Q_t^{\varepsilon}, \mathbf{1}_{J_{\varepsilon}(x)}\rangle$. For $F_1^{\varepsilon}(t)$, the bound (3.4) implies $\sup_{t \in [0,T]} F_1^{\varepsilon}(t) \to 0$ in L^1, and hence in probability. Turning to bounding $F_2^{\varepsilon}(t, x)$, we recall the definition of $Q_t^{\varepsilon,(0)}$ from (3.32).

Since Q^ε_t differs from $Q^{\varepsilon,(0)}_t$ only by the shift $X^\varepsilon_{(0)}(t)$, with $|X^\varepsilon_{(0)}(t)| \le \varepsilon^b \le 1$, we have

$$\sup_{|x|\le L} \left|F^\varepsilon_2(t,x)\right| \le \varepsilon^{\frac{1}{4}} \sup\{\langle Q^{\varepsilon,(0)}_t, \mathbf{1}_{J_\varepsilon(x')}\rangle : x' \in [-1, L+1]\}. \tag{5.15}$$

As $Q^{\varepsilon,(0)}_t \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$ and $|J_\varepsilon| = 2\varepsilon^{a'}$, fixing $a'' \in (0, a' - \frac{1}{2})$, we have that

$$\begin{aligned} &\{\langle Q^{\varepsilon,(0)}_t, \mathbf{1}_{J_\varepsilon(x')}\rangle \le \varepsilon^{a''}\} \\ &\qquad \text{holds up to SPD, for any fixed } (t, x') \in [0,T] \times \mathbb{R}. \end{aligned} \tag{5.16}$$

Taking union bound of (5.16) over $x' \in (\varepsilon^{a'}\mathbb{Z}_+) \cap [-1, L+1]$, and combining the result with (5.15), we arrive at

$$\Big\{\sup_{|x|\le L} \left|F^\varepsilon_2(t,x)\right| \le 2\varepsilon^{a''}\Big\} \qquad \text{holds up to SPD, for any fixed } t \le T. \tag{5.17}$$

The desired convergence $F^\varepsilon_2(\cdot,\cdot) \Rightarrow 0$ now follows by writing $F^\varepsilon_2(t,x) = \mathcal{G}^\varepsilon_t(x+\varepsilon^{a'}) - \mathcal{G}^\varepsilon_t(x-\varepsilon^{a'}) - 2\varepsilon^{-\frac{1}{4}+a'}$, and combining the continuity estimate (5.3) with (5.17). □

Recall the definition of $\mathcal{D}^\varepsilon(j, j', t)$ from (2.15). The following elementary bound on $\mathcal{D}^\varepsilon(j, j', t)$ is useful as we progress to proving Proposition 2.7(b)–(c).

LEMMA 5.3. *Letting*

$$\begin{aligned} \mathscr{I}_\mu(T, L) := \{&(j, j', k) \in \mathbb{Z}^3_+ : \\ &j, j' \le 4(L+1)\varepsilon^{-\frac{1}{2}}, |j - j'| \le \varepsilon^{-\mu}, k \le T\varepsilon^{-1}\}, \end{aligned} \tag{5.18}$$

for any fixed $T, L < \infty$ and $\mu \in (0, \frac{1}{2})$, we have

$$\sup_{(j,j',k)\in\mathscr{I}_\mu(T,L)} \varepsilon^{\frac{1}{4}}\left|\mathcal{D}^\varepsilon(j, j', t^\varepsilon_k)\right| \underset{\mathrm{P}}{\to} 0.$$

PROOF. By the exact relation (2.16), $\mathcal{D}^\varepsilon(j, j', t)$ is the sum of i.i.d. random variables $1 - 2Y_i(\varepsilon^{-1}t), i \in [j, j')$. With this and $|j - j'| \le \varepsilon^{-\mu}$, $\mu < \frac{1}{2}$, we clearly have

$$\{\varepsilon^{\frac{1}{4}}\mathcal{D}^\varepsilon(j, j', t) \le \varepsilon^\delta\} \qquad \text{holds up to SPD,} \tag{5.19}$$

for any fixed $t < \infty$, and $\delta \in (0, \frac{\mu}{2} - \frac{1}{4})$. The desired result now follows by taking union bound of (5.19) over $(j, j', k) \in \mathscr{I}_\mu(T, L)$. □

PROOF OF PROPOSITION 2.7(b). Fixing $L, T < \infty$, by the exact relation (2.18) and the continuity estimates (5.2)–(5.3), it suffices to show

$$\sup_{k\le T\varepsilon^{-1}} \sup_{x\in[\varepsilon^b, L]} \left|\varepsilon^{\frac{1}{4}}\mathcal{D}^\varepsilon(I^\varepsilon_{t^\varepsilon_k}(x), I^\varepsilon_0(x), t^\varepsilon_k) - 2\varepsilon^{\frac{1}{4}}\rho^\varepsilon_{t^\varepsilon_k}(x)\right| \underset{\mathrm{P}}{\to} 0,$$

as $\varepsilon \to 0$. We do this by bounding the terms G_1^ε and G_2^ε *separately*, where

$$G_1^\varepsilon := \sup_{k\le T\varepsilon^{-1}} \sup_{x\in[\varepsilon^b,L]} \{\varepsilon^{\frac{1}{4}}\rho^\varepsilon_{t_k^\varepsilon}(x)\},$$

$$G_2^\varepsilon := \sup_{k\le T\varepsilon^{-1}} \sup_{x\in[\varepsilon^b,L]} \{\varepsilon^{\frac{1}{4}}|\mathcal{D}^\varepsilon(I^\varepsilon_{t_k^\varepsilon}(x), I_0^\varepsilon(x), t_k^\varepsilon)|\}.$$

More precisely, we next show (i) $G_1^\varepsilon \underset{\mathrm{P}}{\to} 0$; and (ii) $G_2^\varepsilon \underset{\mathrm{P}}{\to} 0$, following the reasoning explained already at the end of Section 2.

(i) As $\sup_{t\le T}|X^\varepsilon_{(0)}(t)| \le \varepsilon^b$, by (2.17) we have $\rho^\varepsilon_{t_k^\varepsilon}(x) \le \varepsilon^{\frac{1}{4}} Y_{I^\varepsilon_{t_k^\varepsilon}(x)-1}(t)$. The desired result $G_1^\varepsilon \underset{\mathrm{P}}{\to} 0$ follows if $I_t^\varepsilon(x)$ were deterministic. With this in mind, we proceed to establish a bound on the range of $I^\varepsilon_{t_k^\varepsilon}(x)$ and bound $Y_{I^\varepsilon_{t_k^\varepsilon}(x)-1}(t)$ by taking the maximum of $Y_{i-1}(t)$ over such range. To this end, we use (2.12) and (3.32) to express $I^\varepsilon_{t_k^\varepsilon}(x)$ as $I^\varepsilon_{t_k^\varepsilon}(x) = \langle Q^\varepsilon_{t_k^\varepsilon}, \mathbf{1}_{(-\infty,x]}\rangle = \langle Q^{\varepsilon,(0)}_{t_k^\varepsilon}, \mathbf{1}_{(-\infty,x-X^\varepsilon_{(0)}(t_k^\varepsilon)]}\rangle$. With $|X^\varepsilon_{(0)}(t_k^\varepsilon)| \le \varepsilon^b \le 1$, we obtain $I^\varepsilon_{t_k^\varepsilon}(L) \le \langle Q^{\varepsilon,(0)}_{t_k^\varepsilon}, \mathbf{1}_{(-\infty,L+1]}\rangle$. Combining this with $Q^{\varepsilon,(0)}_{t_k^\varepsilon} \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$, we thus conclude

$$\{I^\varepsilon_{t_k^\varepsilon}(L) \le 4(L+1)\varepsilon^{-\frac{1}{2}}, \forall k \le T\varepsilon^{-1}\} \qquad \text{holds up to SPD.} \tag{5.20}$$

Consequently,

$$G_1^\varepsilon \le \sup_{k\le T\varepsilon^{-1}} \sup_{|j|\le 4(L+1)\varepsilon^{-\frac{1}{2}}} \{\varepsilon^{\frac{1}{4}} Y_i(t_k^\varepsilon)\} \qquad \text{holds up to SPD.} \tag{5.21}$$

As $Y_i(t_k^\varepsilon)$, $i \in \mathbb{Z}_+$, are i.i.d., the right-hand side of (5.21) clearly converges to zero in probability.

(ii) Fix $\mu \in (\frac{1}{4}, \frac{1}{2})$, and recall the definition of $\mathscr{I}_\mu(T,L)$ from (5.18). With Lemma 5.3, it suffices to show $(I^\varepsilon_{t_k^\varepsilon}(x), I_0^\varepsilon(x), k) \in \mathscr{I}_\mu(T,L)$, $\forall x \in [0,L]$, $k \le \varepsilon^{-1}T$ holds with high probability. By Proposition 1.9 and Proposition 2.7(a), the process $(t,x) \mapsto (\mathcal{G}_t^\varepsilon(x) - \mathcal{G}_0^\varepsilon(x))\mathbf{1}_{[\varepsilon^b,\infty)}(x)$ converges weakly. The latter, by (2.11)–(2.12), is equal to $\varepsilon^{\frac{1}{4}}(I_t^\varepsilon(x) - I_0^\varepsilon(x))\mathbf{1}_{[\varepsilon^b,\infty)}(x)$. From this we conclude that

$$\lim_{\varepsilon\to 0}\mathbf{P}\Big(\sup_{t\in[0,T]}\sup_{x\in[\varepsilon^b,L]}|I_t^\varepsilon(x) - I_0^\varepsilon(x)| \le \varepsilon^{-\mu}\Big) = 1.$$

Combining this with (5.20) yields the desired result:

$$\lim_{\varepsilon\to 0}\mathbf{P}\big((I^\varepsilon_{t_k^\varepsilon}(x), I_0^\varepsilon(x), k) \in \mathscr{I}_\mu(T,L), \forall x \in [0,L], k \le \varepsilon^{-1}T\big) = 1. \qquad \square$$

PROOF OF PROPOSITION 2.7(c). Fixing $L, T < \infty$, by the exact relation (2.19) and the continuity estimates (5.1)–(5.2), it suffices to show

$$\sup_{k\le T\varepsilon^{-1}}\sup_{x\in[0,L]}|\varepsilon^{\frac{1}{4}}\mathcal{D}^\varepsilon(I_0^\varepsilon(x+\varepsilon^b), i_\varepsilon(x), t_k^\varepsilon)| \underset{\mathrm{P}}{\to} 0, \qquad \text{as } \varepsilon \to 0.$$

Since $\mathcal{X}_t^\varepsilon(x)$ is defined for $x \in \mathbb{R}_+$ by linear interpolation from $x \in \frac{1}{2}\varepsilon^{\frac{1}{2}}\mathbb{Z}_+$, and since $\widetilde{\mathcal{X}}_\cdot^\varepsilon(\cdot) \Rightarrow \mathcal{X}_\cdot^\varepsilon(\cdot)$ [by Proposition 2.7(a)–(b) and Proposition 1.9], without loss of generality we consider only $x \in \frac{1}{2}\varepsilon^{\frac{1}{2}}\mathbb{Z}_+$, and prove

$$\sup\left\{\varepsilon^{\frac{1}{4}}\mathcal{D}^\varepsilon\big(I_0^\varepsilon(x+\varepsilon^b), i_\varepsilon(x), t_k^\varepsilon\big) : k \le T\varepsilon^{-1}, x \in [0, L] \cap \left(\frac{\varepsilon^{\frac{1}{2}}}{2}\mathbb{Z}\right)\right\} \underset{\mathrm{P}}{\to} 0,$$

as $\varepsilon \to 0$. This, as shown in the proof of Proposition 2.7(b), follows once we prove

$$\lim_{\varepsilon\to 0} \mathbf{P}\big(\big|I_0^\varepsilon(x+\varepsilon^b) - i_\varepsilon(x)\big| \le \varepsilon^{-\mu}, \forall x \in [0, L]\big) = 1, \tag{5.22}$$

for some $\mu \in (0, \frac{1}{2})$. With $I_0^\varepsilon(x')$ defined as in (2.12) and $Q_0^\varepsilon \sim \mathrm{PPP}_+(2\varepsilon^{-\frac{1}{2}})$, the process $x \mapsto (I_0^\varepsilon(x) - i_\varepsilon(x))$ is an L^2-martingale. Applying Doob's L^2 maximal inequality to this martingale yields

$$\mathbf{E}\Big(\sup_{x\le L}\big|I_0^\varepsilon(x+\varepsilon^b) - i_\varepsilon(x)\big|\Big)^2 \le C\,\mathrm{Var}\big(I_0^\varepsilon(L+\varepsilon^b)\big) + C\big(i_\varepsilon(x+\varepsilon^b) - i_\varepsilon(x)\big)^2.$$

The right-hand side is clearly bounded by $C\varepsilon^{-1+2b}$, so by Markov's inequality we conclude (5.22) for $\mu \in (0, b)$. □

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