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EQUILIBRIUM FLUCTUATIONS FOR ONE-DIMENSIONAL GINZBURG-LANDAU LATTICE MODEL

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Dedicated to Professor Takeyuki Hida on his 60th birthday

§ 1. Introduction

We shall investigate a system of spin configurations $S = \{S(t, x); t \geq 0, x \in \mathbb{Z}\}$ on a one-dimensional lattice $\mathbb{Z}$ changing randomly in time. The evolution law is described by an infinite-dimensional stochastic differential equation (SDE):

$$\begin{aligned} dS(t, x) = \{&U'(S(t, x + 1)) - 2U'(S(t, x)) + U'(S(t, x - 1))\}dt \\ &+ \sqrt{2}\,(d\beta(t, x + 1) - d\beta(t, x)), \quad x \in \boldsymbol{Z} \end{aligned} \tag{1.1}$$

where $\{\beta(t, x); t \geq 0, x \in \boldsymbol{Z}\}$ is a family of independent standard Wiener processes and U' is the derivative of a self-potential $U: \boldsymbol{R} \to \boldsymbol{R}$. Throughout this paper we are assuming that U has two times continuous derivatives and

$$a - A \leq U''(x) \leq a + A \tag{1.2}$$

with some constants $a > 0$ and $A > 0$. The system (1.1) is called one-dimensional Ginzburg-Landau lattice model (cf. [1], [2]), which has a unique strong solution in a certain class of configuration spaces (see Section 2, Theorem 2.1).

The purpose of the present paper is to investigate the hydrodynamical behavior, especially the equilibrium fluctuation problem, for (1.1). We introduce the space-time scaling:

$$x \to [x/\varepsilon]\,, \quad t \to t/\varepsilon^2\,, \quad \varepsilon > 0 \tag{1.3}$$

for the equation (1.1). Here $[u]$ denotes the integral part of $u \in \boldsymbol{R}$. After this scaling the process $S_\varepsilon(t, x) = S(t/\varepsilon^2, [x/\varepsilon])$ solves the following scaled

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equation:

$$dS_\varepsilon(t, x) = \Delta_\varepsilon U'(S_\varepsilon(t, x))dt + \sqrt{2\varepsilon}\nabla_\varepsilon dw_\varepsilon(t, x)\,, \quad t > 0\,, \quad x \in \boldsymbol{R} \tag{1.4}$$

where $w_\varepsilon(t, x) = \sqrt{\varepsilon}\,\beta(t/\varepsilon^2, [x/\varepsilon])$ and

$$\begin{aligned} \nabla_\varepsilon \varphi(x) &= \varepsilon^{-1}[\varphi(x+\varepsilon) - \varphi(x)] \\ \Delta_\varepsilon \varphi(x) &= \varepsilon^{-2}[\varphi(x+\varepsilon) - 2\varphi(x) + \varphi(x-\varepsilon)] \end{aligned} \tag{1.5}$$

for functions φ of x. The operations ∇_ε and Δ_ε are the lattice approximations of step size ε to the differential operators $\partial/\partial x$ and $\partial^2/\partial x^2$, respectively. We are interested in the asymptotic behavior of $S_\varepsilon(t, x)$ as ε tends to 0.

Two kinds of problems are formulated concerning the hydrodynamical limit: the law of large numbers and the central limit theorem. For the lattice model (1.1), Fritz [2] proved the law of large numbers in the non-stationary case (in fact, he investigated more general lattice system) and Guo, Papanicolaou and Varadhan [3] gave a quite different approach to the same problem but in a finite volume case. It is known that $S_\varepsilon(t, x)$ converges as $\varepsilon \to 0$ to a deterministic limit $\gamma(t, x)$ which satisfies a diffusion equation

$$\frac{\partial \gamma}{\partial t} = \frac{\partial}{\partial x}\left[D(\gamma)\frac{\partial \gamma}{\partial x}\right] \tag{1.6}$$

with a certain diffusion coefficient $D(\gamma)$.

On the other hand, the equilibrium fluctuation problem which is the main problem of this paper is to investigate the asymptotic behavior of $V_\varepsilon(t, x) = (S_\varepsilon(t, x) - \gamma)/\sqrt{\varepsilon}$ for lattice model (1.1) in the stationary case, where $\gamma = \boldsymbol{E}[S_\varepsilon(t, x)]$ is independent of (t, x). The result will be formulated as the central limit theorem for the SDE (1.1). We shall prove that $V_\varepsilon = (V_\varepsilon(t, x); t \geq 0, x \in \boldsymbol{R})$ converges as $\varepsilon \to 0$ to a generalized Ornstein-Uhlenbeck process $V(t)$ characterized by an SDE

$$dV(t) = D(\gamma)\Delta V(t)dt + \sqrt{2}\,\nabla dw(t) \tag{1.7}$$

where the constant $D(\gamma)$ is the same one as in (1.6) (see Section 2, Theorem 2.2 and Remark in detail), $\Delta = \partial^2/\partial x^2$, $\nabla = \partial/\partial x$, and $w(t)$ is a cylindrical Brownian motion on $L^2(\boldsymbol{R})$. Spohn [10] investigated the equilibrium fluctuation problem for an interacting Brownian particles' model. In this paper we shall follow the method due to Rost [7] and Spohn [10].

§ 2. Main result

Let $\boldsymbol{R}^{\boldsymbol{Z}} = \{\sigma = (\cdots, \sigma_{-1}, \sigma_0, \sigma_1, \cdots); \sigma_k \in \boldsymbol{R}, k \in \boldsymbol{Z}\}$ the space with usual product topology and denote its Borel field by $\mathscr{B}(\boldsymbol{R}^{\boldsymbol{Z}})$.

Define product measures μ_λ, $\lambda \in \boldsymbol{R}$, on $(\boldsymbol{R}^{\boldsymbol{Z}}, \mathscr{B}(\boldsymbol{R}^{\boldsymbol{Z}}))$ by

$$\mu_\lambda(d\sigma) = \prod_{k=-\infty}^{\infty} q_\lambda(\sigma_k) d\sigma_k , \tag{2.1}$$

where

$$q_\lambda(x) = M(\lambda)^{-1} \exp[\lambda x - U(x)] \tag{2.2}$$

and

$$M(\lambda) = \int_{\boldsymbol{R}} \exp[\lambda x - U(x)] dx . \tag{2.3}$$

The probability measure μ_λ can be regarded as a Gibbs state associated with the (formal) Hamiltonian:

$$H_\lambda(\sigma) = \sum_{k \in \boldsymbol{Z}} U(\sigma_k) - \lambda \sum_{k \in \boldsymbol{Z}} \sigma_k . \tag{2.4}$$

We develop some more notation

$$\rho(\lambda) = \log M(\lambda) , \tag{2.5}$$

$$h(\gamma) = \sup_\lambda [(\lambda\gamma - \rho(\lambda)] , \quad \gamma \in \boldsymbol{R} . \tag{2.6}$$

Then $h(.)$ and $\rho(.)$ are a pair of conjugate convex functions and

$$\lambda = h'(\gamma) \text{ if and only if } \gamma = \rho'(\lambda) . \tag{2.7}$$

Elementary calculation shows

$$\int x q_\lambda(x) dx = \rho'(\lambda) . \tag{2.8}$$

Moreover, $\rho''(\lambda)$ is the variance of $q_\lambda(x)dx$ *i.e.*

$$\int (x - \rho'(\lambda))^2 q_\lambda(x) dx = \rho''(\lambda) . \tag{2.9}$$

One knows also that ρ' and h' are smooth strictly increasing functions.

Let $r > 0$ be fixed throughout this paper. Let $\xi(x) \in C^\infty(\boldsymbol{R})$ be a positive function such that $\xi(x) = |x|$ if $|x| \geq 1$. We define a Hilbert space as

$$(2.10)\qquad \boldsymbol{L}_r^2 = \left\{\sigma \in \boldsymbol{R}^{\boldsymbol{Z}};\ |\sigma|_r^2 = \sum_{k\in\boldsymbol{Z}} |\sigma_k|^2 \int_k^{k+1} \exp[-r\xi(x)]\,dx < \infty\right\}.$$

One can check that $\mu_\lambda(\boldsymbol{L}_r^2) = 1$.

Now we turn to the study of the SDE (1.1). In view of (1.2), the drift term of (1.1) is linearly bounded and uniformly Lipschitz continuous in the space $\boldsymbol{L}_r^2$. Therefore, a standard argument yields the existence and uniqueness of strong solutions to (1.1) in $\boldsymbol{L}_r^2$ (cf. [9]):

THEOREM 2.1. *For each $\sigma \in \boldsymbol{L}_r^2$, the* SDE (1.1) *has a unique $\boldsymbol{L}_r^2$-valued continuous strong solution S_t starting from σ (i.e. $S_0 = \sigma$).*

Let T_t, $t \geq 0$ be defined by

$$(T_t F)(\sigma) = \boldsymbol{E}_\sigma[F(S_t)]\,, \quad F \in C(\boldsymbol{L}_r^2)$$

where $\boldsymbol{E}_\sigma[\ ]$ means the expectation under the probability law of (1.1)'s solution S_t starting from $\sigma \in \boldsymbol{L}_r^2$. Then we can easily extend $\{T_t\}_{t\geq 0}$ to a self-adjoint strongly continuous contraction semigroup on $\boldsymbol{L}^2(\boldsymbol{R}^{\boldsymbol{Z}}, \mu_\lambda)$ and check that the Gibbs states μ_λ, $\lambda \in \boldsymbol{R}$, are reversible measures of T_t.

Let $\mathscr{E}_r = \mathscr{S} \exp[-r\xi(x)]$ the nuclear space with a topology introduced from $\mathscr{S}$, where $\mathscr{S} = \mathscr{S}(\boldsymbol{R})$ is Schwartz space. Let $\mathscr{E}_r'$ be the dual space of $\mathscr{E}_r$ with the strong topology and $\mathscr{C} = C([0, \infty); \mathscr{E}_r')$. Let $\{S(t, x); t \geq 0, x \in \boldsymbol{Z}\}$ be the solution of (1.1) with initial distribution μ_λ. Then by Theorem 2.1, we know $S_\varepsilon(t, x) = S(t/\varepsilon^2, [x/\varepsilon])$ is in $\mathscr{C}$ (a.s.). Now we can state our main result:

THEOREM 2.2. *Let $V_\varepsilon(t, x) = \varepsilon^{-1/2}(S_\varepsilon(t, x) - \rho'(\lambda))$ and P_ε be the probability distribution of V_ε on $\mathscr{C}$. Then P_ε converges as $\varepsilon \to 0$ to a distribution of a generalized Ornstein-Uhlenbeck process $V = \{V_t\}_{t>0}$ weakly on $\mathscr{C}$. The process $\{V_t\}$ satisfies the following equation*

$$(2.11)\qquad dV_t = \rho''(\lambda)^{-1} \Delta V_t dt + \sqrt{2}\,\nabla dw_t$$

where $\Delta = \partial^2/\partial x^2$, $\nabla = \partial/\partial x$ and w_t is a cylindrical Brownian motion on $\boldsymbol{L}^2(\boldsymbol{R})$.

Remark. From the relationship (2.7), we have $\rho''(\lambda)^{-1} = h''(\rho'(\lambda))$. However, it is known that $h''(\gamma) = D(\gamma)$; the diffusion coefficient appearing in (1.6) (cf. [1] [2] [3]).

§ 3. Sketch of the proof and Boltzmann-Gibbs principle

Let V_ε be the stationary process defined as in Section 2. From (1.4), we get an equation for $V_\varepsilon(t, x)$

$$dV_\varepsilon(t, x) = \frac{1}{\sqrt{\varepsilon}} \Delta_\varepsilon U'(\sqrt{\varepsilon}\, V_\varepsilon(t, x) + \rho'(\lambda))dt + \sqrt{2}\, \nabla_\varepsilon dw_\varepsilon(t, x)\,, \quad x \in \boldsymbol{R}\,,\quad t > 0\,. \tag{3.1}$$

Tending ε to 0 in (3.1), the second term converges to $\sqrt{2}\, \nabla dw(t)$ (at least formally). The difficulty in the proof of Theorem 2.2 lies in the computation of the first term. Although it is nonlinear, Rost [7] and [8] suggest that it should converge to a linear term $\rho''(\lambda)^{-1}\Delta V(t)$; precisely saying, our goal will be the following:

PROPOSITION 3.1 (Boltzmann-Gibbs principle). *For each $t > 0$ and $f \in \mathscr{S}_r$,*

$$\boldsymbol{E}\left[\left(\int_0^t ds \int_{\boldsymbol{R}} dx \frac{1}{\sqrt{\varepsilon}} \{U'(S_\varepsilon(s, x)) - \rho''(\lambda)^{-1} S_\varepsilon(s, x)\} \Delta_\varepsilon f(x)\right)^2\right] \to 0\,, \quad \text{as } \varepsilon \to 0\,. \tag{3.2}$$

In the rest of this section, we give an outline of the proof of this proposition. For convenience, we intrdoce some notation:

$$\Phi(x) = U'(x) - \rho''(\lambda)^{-1}x\,, \quad x \in \boldsymbol{R}\,,$$
$$f_\Delta^{(\varepsilon)}(x) = \varepsilon^{1/2}(\Delta_\varepsilon f)(\varepsilon x)\,, \quad \text{for } f \in \mathscr{S}_r\,,$$
$$\Phi(f)(\sigma) = \int_{\boldsymbol{R}} \Phi(\sigma_{[x]}) f(x)dx\,, \quad \text{for } f \in \mathscr{S}_r\,, \quad \sigma \in \boldsymbol{L}_r^2\,,$$
$$\Phi(f, t) = \Phi(f)(S_t)\,, \quad S_t = \{S(t, x); x \in \boldsymbol{Z}\} \in \boldsymbol{L}_r^2 \text{ (a.s.)}\,, \quad t \geq 0\,,$$
$$R(\varepsilon) = \boldsymbol{E}\left[\left(\int_0^t ds \Phi(f_\Delta^{(\varepsilon)}, s/\varepsilon^2)\right)^2\right]\,.$$

It is easy to check that $R(\varepsilon)$ = the l.h.s. of (3.2). Hence our goal is to show that $\lim_{\varepsilon\to 0} R(\varepsilon) = 0$. We define a class of shift operators $\{\tau_q\}_{q\in\boldsymbol{R}}$ as follows: For $q \in \boldsymbol{R}$, $\sigma \in \boldsymbol{L}_r^2$, and any functional F of σ,

$$(\tau_q \sigma)_x = \sigma_{[x+q]}\,,$$
$$(\tau_q F)(\sigma) = F(\tau_q \sigma)\,.$$

Now take $g \in C_0^\infty(\boldsymbol{R})$ satisfying $\int g(x)dx = 1$ and fix $t > 0$, $f \in \mathscr{S}_r$. For every T, $\varepsilon > 0$, choose $N = [T^{-1}\varepsilon^{-2}t]$, then we have from the stationarity of $S(t, x)$:

$$\begin{aligned}
R(\varepsilon) &= \boldsymbol{E}\left[\left(\varepsilon^2 \int_0^{\varepsilon-2t} ds\Phi(f_\Delta^{(\varepsilon)}, s)\right)^2\right] \\
&\leq 2\varepsilon^4\boldsymbol{E}\left[\left(\sum_{n=0}^{N-1}\int_{nT}^{(n+1)T} ds\Phi(f_\Delta^{(\varepsilon)}, s)\right)^2\right] + 2\varepsilon^4\boldsymbol{E}\left[\left(\int_{NT}^{\varepsilon-2t} ds\Phi(f_\Delta^{(\varepsilon)}, s)\right)^2\right] \\
&\leq 2\varepsilon^4\boldsymbol{E}\left[N\cdot\sum_{n=0}^{N-1}\left(\int_{nT}^{(n+1)T} ds\Phi(f_\Delta^{(\varepsilon)}, s)\right)^2\right] + R_3(\varepsilon) \\
&= 2\varepsilon^4 N^2\boldsymbol{E}\left[\left(\int_0^T ds\Phi(f_\Delta^{(\varepsilon)}, s)\right)^2\right] + R_3(\varepsilon) \\
&\leq 2t^2T^{-2}\boldsymbol{E}\left[\int_0^T ds\int_0^T du\Phi(f_\Delta^{(\varepsilon)}, s)\Phi(f_\Delta^{(\varepsilon)}, s)\right] + R_3(\varepsilon) \\
&\leq R_1(\varepsilon) + R_2(\varepsilon) + R_3(\varepsilon)\,,
\end{aligned}$$

where

$$\begin{aligned}
R_1(\varepsilon) &= 4t^2T^{-2}\int_0^T ds\int_0^T du\langle(T_{|s-u|/2}\Phi(g*f_\Delta^{(\varepsilon)}))^2\rangle, \\
R_2(\varepsilon) &= 4t^2T^{-2}\int_0^T ds\int_0^T du\langle(T_{|s-u|/2}\Phi(f_\Delta^{(\varepsilon)}) - T_{|s-u|/2}\Phi(g*f_\Delta^{(\varepsilon)}))^2\rangle, \\
R_3(\varepsilon) &= 2\varepsilon^4\boldsymbol{E}\left[\left(\int_0^{\varepsilon-2t-NT} ds\Phi(f_\Delta^{(\varepsilon)}, s)\right)^2\right],
\end{aligned}$$

and $\langle\cdot\rangle$ stands for the expectation with respect to μ_λ; it will be sometimes denoted by $\langle\cdot\rangle_\lambda$ to be made its dependence on λ clear (Section 6). These three terms can be estimated as follows.

LEMMA 3.2.

(1) *If* $\int_{\boldsymbol{R}} dq|\langle\Phi(g)\tau_q T_{2t}\Phi(g)\rangle - \langle\Phi(g)\rangle^2| < \infty$, *then*

$$\lim_{\varepsilon\to 0}\langle(T_t\Phi(g*f_\Delta^{(\varepsilon)}))^2\rangle = \|\Delta f\|^2\int_{\boldsymbol{R}} dq\{\langle\Phi(g)\tau_q T_{2t}\Phi(g)\rangle - \langle\Phi(g)\rangle^2\}\,. \tag{3.3}$$

(2) $\lim_{\varepsilon\to 0}\langle(T_t\Phi(g*f_\Delta^{(\varepsilon)}) - T_t\Phi(f_\Delta^{(\varepsilon)}))^2\rangle = 0$, *for all* $t > 0$.

(3) $\lim_{\varepsilon\to 0} R_3(\varepsilon) = 0$.

Proof. (1) By the uniqueness of solutions of eq. (1.1), it is easy to see that $T_t(\tau_q\Phi(g)) = \tau_{-q}(T_t\Phi(g))$. Thus

$$T_t\Phi(g*f_\Delta^{(\varepsilon)})(\sigma) = \int_{\boldsymbol{R}} dq f_\Delta^{(\varepsilon)}(q)\tau_{-q}T_t\Phi(g)\,.$$

Noting that $\langle T_t\Phi(g*f_\Delta^{(\varepsilon)})\rangle = 0$, we have

$$\begin{aligned}
\langle(T_t\Phi(g*f_\Delta^{(\varepsilon)}))^2\rangle &= \langle(T_t\Phi(g*f_\Delta^{(\varepsilon)}))^2\rangle - \langle T_t\Phi(g*f_\Delta^{(\varepsilon)})\rangle^2 \\
&= \int_{\boldsymbol{R}} dp(\Delta_\varepsilon f)(p)\int_{\boldsymbol{R}} dq(\Delta_\varepsilon f)(\varepsilon q + p)\{\langle\Phi(g)\tau_q T_{2t}\Phi(g)\rangle - \langle\Phi(g)\rangle^2\}\,.
\end{aligned}$$

Therefore (3.3) is established by letting $\varepsilon \to 0$.

(2) We compute

$$\langle (T_t\Phi(g*f_\Delta^{(\varepsilon)}) - T_t\Phi(f_\Delta^{(\varepsilon)}))^2\rangle \le \langle (\Phi(g*f_\Delta^{(\varepsilon)}) - \Phi(f_\Delta^{(\varepsilon)}))^2\rangle$$
$$= \langle \Phi(\sigma_0)^2\rangle \int dxdy\, \mathbf{1}_{\{[x]=[y]\}}(g*f_\Delta^{(\varepsilon)} - f_\Delta^{(\varepsilon)})(x)(g*f_\Delta^{(\varepsilon)} - f_\Delta^{(\varepsilon)})(y)\,.$$

Besauce the r.h.s. tends to 0 as $\varepsilon \to 0$, the assertion is proved.

(3) $$R_3(\varepsilon) \le 2\varepsilon^4 T\boldsymbol{E}\left[\int_0^{\varepsilon^{-2}t-NT} ds\,\Phi^2(f_\Delta^{(\varepsilon)}, s)\right] = 2\varepsilon^4 T\int_0^{\varepsilon^{-2}t-NT} ds\langle \Phi^2(f_\Delta^{(\varepsilon)})\rangle$$
$$\le 2\varepsilon^4 T^2\langle \Phi^2(\sigma_0)\rangle \int dxdy\, \mathbf{1}_{\{[x]=[y]\}} f_\Delta^{(\varepsilon)}(x) f_\Delta^{(\varepsilon)}(y)\,.$$

Taking the limit $\varepsilon \to 0$ proves the conclusion. □

This lemma shows

$$\overline{\lim_{\varepsilon\to 0}}\, R(\varepsilon) \le 4t^2T^{-2}\|\Delta f\|^2\cdot\int_0^T ds\int_0^T du\int dq\{\langle \Phi(g)\tau_q T_{|s-u|}\Phi(g)\rangle - \langle \Phi(g)\rangle^2\}\,.$$

Hence, it is sufficient to show that

$$\lim_{T\to\infty} T^{-2}\cdot\int_0^T ds\int_0^T du\int dq\{\langle \Phi(g)\tau_q T_{|s-u|}\Phi(g)\rangle - \langle \Phi(g)\rangle^2\} = 0\,. \tag{3.4}$$

Clearly, this is equivalent to the following statement:

$$\lim_{t\to\infty}\int_{\boldsymbol{R}} dq\{\langle \Phi(g)\tau_q T_t\Phi(g)\rangle - \langle \Phi(g)\rangle^2\} = 0\,. \tag{3.5}$$

However a simple calculation proves

$$\int dq\{\langle \Phi(g)\tau_q T_t\Phi(g)\rangle - \langle \Phi(g)\rangle^2\} = \sum_{n=-\infty}^{\infty}\{\langle \Phi(g)\tau_n T_t\Phi(g)\rangle - \langle \Phi(g)\rangle^2\},\quad t\ge 0\,.$$

Therefore (3.5) is equivalent to its lattice form:

$$\lim_{t\to\infty}\sum_{n=-\infty}^{\infty}\{\langle \Phi(g)\tau_n T_t\Phi(g)\rangle - \langle \Phi(g)\rangle^2\} = 0. \tag{3.6}$$

Now, we introduce a Hilbert space $\mathscr{H}$ with inner product $\langle F|G\rangle = \sum_{n=-\infty}^{\infty}\{\langle F\tau_n G\rangle - \langle F\rangle\langle G\rangle\}$, $F, G\in\mathscr{H}$. This space will be discussed in detail in Section 4. By Proposition 6.2, T_t is ergodic in $\mathscr{H}$, and

$$\lim_{t\to\infty} T_tF = \rho''(\lambda)^{-1}\langle F|F_0(g)\rangle F_0(g) \qquad \text{in } \mathscr{H},\ \text{for } F\in\mathscr{H} \tag{3.7}$$

where $F_0(g) = \int (\sigma(x) - \rho'(\lambda))g(x)dx$. Therefore

$$\lim_{t\to\infty}\sum_{n=-\infty}^{\infty}\{\langle\Phi(g)\tau_n T_t\Phi(g)\rangle-\langle\Phi(g)\rangle^2\}=\rho''(\lambda)^{-1}\langle\Phi(g)\,|\,F_0(g)\rangle^2\,.$$

A simple calculation shows that $\langle\Phi(g)\,|\,F_0(g)\rangle=0$. Consequently, we establish (3.6). Thus Boltzmann-Gibbs principle is shown.

The definition of the Hilbert space $\mathscr{H}$ and the ergodicity of T_t in $\mathscr{H}$ will be dealt with in Sections 4, 5 and 6. The martingale approach will be applied for showing the main theorem in Sections 7 and 8.

§ 4. Construction of the Hilbert space $\mathscr{H}$

As explained in Section 3, we want to introduce a Hilbert space $\mathscr{H}$ with the inner product $\langle\cdot\,|\,\cdot\rangle$. In this section, we shall define the space $\mathscr{H}$ by completing a class of local functions and investigate the relation between the L^2-norm approximation and the $\mathscr{H}$-norm approximation.

First we define the classes of local functions:

$$\mathscr{F}_{2,[-k,k]}=\left\{F(\sigma_{-k},\cdots,\sigma_k)\colon F\in L^2\left(R^{2k+1},\prod_{i=-k}^{k}q_\lambda(\sigma_i)d\sigma_i\right)\right\}$$

$$\mathscr{F}_{2,\mathrm{loc}}=\bigcup_{k\in Z^+}\mathscr{F}_{2,[-k,k]}$$

LEMMA 4.1. *Assume* $F_i\in\mathscr{F}_{2,\mathrm{loc}}$ *satisfy* $\langle F_i\rangle=0$, $i=1,2$. *Then,*

(4.1) (1) $\sum_{n=-\infty}^{\infty}|\langle F_1\tau_nF_1\rangle|\le(4\alpha+1)\langle F_1^2\rangle<\infty,$ *if* $F_1\in\mathscr{F}_{2,[-\alpha,\alpha]}$, $\alpha\in Z^+$

(4.2) (2) $\sum_{n=-\infty}^{\infty}\langle F_1\tau_nF_2\rangle=\lim_{n\to\infty}\frac{1}{2n+1}\left\langle\left(\sum_{k=-n}^{n}\tau_kF_1\right)\left(\sum_{k=-n}^{n}\tau_kF_2\right)\right\rangle$

(4.3) (3) $\sum_{n=-\infty}^{\infty}\langle F_1\tau_nF_1\rangle\ge0$

Proof. (1) Since $F_1(\sigma)=F_1(\sigma_{-\alpha},\cdots,\sigma_\alpha)\in\mathscr{F}_{2,[-\alpha,\alpha]}$, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|\langle F_1\tau_nF_1\rangle|&=\sum_{n=-2\alpha}^{2\alpha}|\langle F_1(\sigma_{-\alpha},\cdots,\sigma_\alpha)F_1(\sigma_{-\alpha+n},\cdots,\sigma_{\alpha+n})\rangle|\\&\le\sum_{n=-2\alpha}^{2\alpha}\langle F_1(\sigma_{-\alpha},\cdots,\sigma_\alpha)^2\rangle^{1/2}\langle F_1(\sigma_{-\alpha+n},\cdots,\sigma_{\alpha+n})^2\rangle^{1/2}\\&=(4\alpha+1)\langle F_1^2\rangle\end{aligned}$$

(2) First we note

$$\sum_{n=-2n}^{2n}\langle F_1\tau_kF_2\rangle=\frac{1}{2n+1}\left\langle\left(\sum_{k=-n}^{n}\tau_kF_1\right)\left(\sum_{l=-n}^{n}\tau_lF_2\right)\right\rangle+R(n)\,,\tag{4.4}$$

where

$$R(n) = (2n+1)^{-1} \sum_{k=1}^{2n} k(\langle \tau_k F_1 F_2 \rangle + \langle F_1 \tau_k F_2 \rangle) . \tag{4.5}$$

However, since $F_1, F_2 \in \mathscr{F}_{2,[-\beta,\beta]}$ with some $\beta \in \boldsymbol{Z}^+$, $\langle \tau_k F_1 F_2 \rangle = \langle F_1 \tau_k F_2 \rangle = \langle F_1 \rangle \langle F_2 \rangle = 0$ for $k > 2\beta$. Therefore

$$|R(n)| \leq \frac{1}{2n+1} \sum_{k=1}^{2\beta} k |\langle \tau_k F_1 F_2 \rangle + \langle F_1 \tau_k F_2 \rangle| \to 0 , \qquad \text{as } n \to \infty .$$

Taking the limit $n \to \infty$ in (4.4), we prove (4.2).

(3) is consequence of (4.2). □

Lemma 4.1 enables us to define the Hilbert space $\mathscr{H}$:

DEFINITION. For $F_1, F_2 \in \mathscr{F}_{2,\mathrm{loc}}$, set

$$\langle F_1 | F_2 \rangle = \sum_{n=-\infty}^{\infty} (\langle F_1 \tau_n F_2 \rangle - \langle F_1 \rangle \langle F_2 \rangle) . \tag{4.6}$$

We define the Hilbert space $\mathscr{H}$ as the completion of $\mathscr{F}_{2,\mathrm{loc}}$ with inner product $\langle \cdot | \cdot \rangle$ modulo $\{F: \langle F|F \rangle = 0\}$. We shall denote the norm corresponding to $\langle \cdot | \cdot \rangle$ by $\|\cdot\|_{\mathscr{H}}$.

Finally, we discuss the relationship between the convergences in two spaces $\boldsymbol{L}^2(\boldsymbol{R}^{\boldsymbol{Z}}, \mu_\lambda)$ and $\mathscr{H}$.

LEMMA 4.2. *Suppose* $F_n \in \mathscr{F}_{2,[-n,n]}$ *satisfies*

$$\lim_{n\to\infty} n \langle F_n^2 \rangle = 0 . \tag{4.7}$$

Then

$$\lim_{n\to\infty} \langle F_n | F_n \rangle = 0 . \tag{4.8}$$

Proof. The conclusion follows since Lemma 4.1 (1) implies

$$\begin{aligned} 0 \leq \langle F_n | F_n \rangle &= \sum_{k=-\infty}^{\infty} \langle (F_n - \langle F_n \rangle) \tau_k (F_n - \langle F_n \rangle) \rangle \\ &\leq (4n+1) \langle (F_n - \langle F_n \rangle)^2 \rangle \leq (4n+1) \langle F_n^2 \rangle . \end{aligned}$$ □

LEMMA 4.3. *Suppose* $F_1, F_2 \in \mathscr{H}$ *satisfy* $\langle F_1 \rangle = \langle F_2 \rangle = 0$ *and* $\sum_{n=-\infty}^{\infty} n |\langle F_1 \tau_n F_2 \rangle| < \infty$. *Then*

$$\langle F_1 | F_2 \rangle = \lim_{n\to\infty} \frac{1}{2n+1} \left\langle \left(\sum_{k=-n}^{n} \tau_k F_1 \right) \left(\sum_{k=-n}^{n} \tau_k F_2 \right) \right\rangle . \tag{4.9}$$

Proof. This is a consequence of (4.4) and (4.5). □

LEMMA 4.4. *Let* $F \in L^2(R^Z, \mu_\lambda)$ *and assume there exists* $F_n \in \mathscr{F}_{2,[-n,n]}$ *satisfying* $\langle F_n \rangle = 0$, $n = 1, 2, \cdots$, *and* $\delta > 2$ *such that*

$$\langle (F_n - F)^2 \rangle \leq Cn^{-\delta} \tag{4.10}$$

with C independent of n. Then

$$|\langle F\tau_k F\rangle| \leq C'(1 + |k|)^{1-\delta/2}, \quad k \in Z \tag{4.11}$$

where C' *is independent of k. Moreover if* $\delta > 4$, $F = \lim_{n\to\infty} F_n$ *in* $\mathscr{H}$ *and therefore* $F \in \mathscr{H}$.

Proof. Let $G_1 = F_1$ and $G_n = F_n - F_{n-1}$, $n = 2, 3, \cdots$. Then by (4.10) $F = \sum_{n=1}^{\infty} G_n$ in $L^2(R^Z, \mu_\lambda)$ and there exists a constant $C_1 > 0$ such that

$$\langle G_n^2 \rangle^{1/2} \leq C_1 n^{-\delta/2}. \tag{4.12}$$

Note that $m + n < |k|$ implies $\langle G_n \tau_k G_m \rangle = \langle G_n \rangle \langle \tau_k G_m \rangle = 0$. We can therefore compute by Schwarz inequality and (4.12)

$$\begin{aligned} |\langle F\tau_k F\rangle| &\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle G_n \tau_k G_m \rangle| \leq C_1^2 \sum_{n+m\geq|k|} m^{-\delta/2} n^{-\delta/2} \\ &\leq C'(1 + |k|)^{1-\delta/2} \end{aligned}$$

where C' is independent of k. Thus (4.11) is established and we also have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle G_n \tau_k G_m \rangle| \leq C'(1 + |k|)^{1-\delta/2}. \tag{4.13}$$

Finally, by (4.12) and (4.13) we have

$$\begin{aligned} \|F_N - F\|_{\mathscr{H}}^2 &\leq \sum_{|k|\leq N} \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} |\langle G_n \tau_k G_m \rangle| + \sum_{|k|\geq N+1} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle G_n \tau_k G_m \rangle| \\ &\leq C_1^2 2N \Big(\sum_{n=N+1}^{\infty} n^{-\delta/2} \Big)^2 + C' \sum_{|k|\geq N+1} (1 + |k|)^{1-\delta/2}. \end{aligned} \tag{4.14}$$

If $\delta > 4$, then $\lim_{N\to\infty} N \sum_{n=N+1}^{\infty} n^{-\delta/2} = 0$ and $\sum_{k\in Z} (1 + |k|)^{1-\delta/2} < \infty$. Consequently, the r.h.s. of (4.14) tends to 0 as $N \to \infty$. Therefore $\lim_{N\to\infty} F_N = F$ in $\mathscr{H}$ and $F \in \mathscr{H}$. □

§ 5. Semigroup and its generator in $\mathscr{H}$

In this section we shall discuss properties of the semigroup T_t and its generator L, which will be defined in $\mathscr{H}$. We define a class of nice functions, which will be the core for L:

$$\mathscr{D}_0 \equiv \{F(\sigma) = F(\sigma_{-m}, \cdots, \sigma_m) \in \mathscr{F}_{2,[-m,m]}\colon F \in C_0^\infty(R^{2m+1}), m \in N\}. \tag{5.1}$$

LEMMA 5.1. *$\mathscr{D}_0$ is dense in $\mathscr{H}$.*

Proof. Since $\mathscr{D}_0$ is dense in $L^2 = L^2(R^Z, \mu_\lambda)$, we have

$$\overline{\mathscr{D}_0 \cap \mathscr{F}_{2,[-m,m]}}^{L^2} = \mathscr{F}_{2,[-m,m]}, \qquad \text{for each } m \in N. \tag{5.2}$$

By Lemma 4.1(1)

$$\langle F|F\rangle \leq (4m+1)\langle F^2\rangle, \qquad \text{for } F \in \mathscr{F}_{2,[-m,m]}.$$

Hence, (5.2) implies that

$$\overline{\mathscr{D}_0 \cap \mathscr{F}_{2,[-m,m]}}^{\mathscr{H}} = \mathscr{F}_{2,[-m,m]}, \qquad \text{for each } m \in N.$$

Thus

$$\begin{aligned}\overline{\mathscr{D}_0} &= \overline{\bigcup_{m\in N} (\mathscr{D}_0 \cap \mathscr{F}_{2,[-m,m]})} \supset \bigcup_{m\in N} \overline{\mathscr{D}_0 \cap \mathscr{F}_{2,[-m,m]}} = \bigcup_{m\in N} \mathscr{F}_{2,[-m,m]} \\ &= \mathscr{F}_{2,\mathrm{loc}} \qquad \text{in } \mathscr{H}.\end{aligned}$$

Therefore, $\mathscr{H} = \overline{\mathscr{F}_{2,\mathrm{loc}}} \subset \overline{\overline{\mathscr{D}_0}} = \overline{\mathscr{D}_0}$. □

Now we discuss the properties of T_t and L. First, we show that the function T_tF with $F \in \mathscr{D}_0$ is in $\mathscr{H}$. To this end, consider the following local SDE's on $[-n, n]$: For each $n \in N$,

$$\begin{cases} dS(t,-n) = \{U'(S(t,-n+1)) - 2U'(S(t,-n)) + U'(S(t,n))\}dt \\ \qquad\qquad + \sqrt{2}\,(d\beta(t,-n+1) - d\beta(t,-n)), \\ dS(t,k) = \Delta_1 U'(S(t,k))dt + \sqrt{2}\,\nabla_1 d\beta(t,k), \ k = -n+1, \cdots, n-1, \\ dS(t,n) = \{U'(S(t,-n)) - 2U'(S(t,n)) + U'(S(t,n-1)\}dt \\ \qquad\qquad + \sqrt{2}\,(d\beta(t,-n) - d\beta(t,n)), \end{cases} \tag{5.3}$$

where $\Delta_1 a_k = a_{k+1} - 2a_k + a_{k-1}$ and $\nabla_1 a_k = a_{k+1} - a_k$, for sequence $\{a_k\}$. The generator of the process determined by the SDE (5.3) is denoted by L_n with domain $\mathscr{D}(L_n)$ and the corresponding semigroup by $T_{t,n} = e^{L_n t}$. Then

$$L_n = \sum_{i=-n}^{n} \left(\nabla_1 \frac{\partial}{\partial\sigma_i}\right)^2 - \sum_{i=-n}^{n} \nabla_1 U'(\sigma_i) \nabla_1 \frac{\partial}{\partial\sigma_i} \tag{5.4}$$

where $\partial/\partial\sigma_{n+1} \equiv \partial/\partial\sigma_{-n}$, and $\sigma_{n+1} = \sigma_{-n}$. Note that $\mu_\lambda^{(n)}(d\sigma_{-n}\cdots d\sigma_n) = \prod_{k=-n}^{n} q_\lambda(\sigma_k)d\sigma_k$, $\lambda \in R$, are the reversible measures of the SDE (5.3).

LEMMA 5.2. *Let $F \in \mathscr{D}_0$ satisfy $\langle F\rangle = 0$. Then, for every $t_0 > 0$ and $\delta > 0$, there exists a constant C such that*

$$|\langle F\tau_k T_t F\rangle| \leq C(1+|k|)^{-\delta}, \qquad \textit{for } k \in Z, \quad t \in [0, t_0]. \tag{5.5}$$

Moreover, $T_{t,n}F \to T_tF$ *is* $\mathscr{H}$ *as* $n \to \infty$ *and especially* $T_tF \in \mathscr{H}$.

Proof. Let $S(t, \sigma) = \{S(t, k, \sigma)\}_{k\in Z}$ be the solution of (1.1) with initial value $\sigma = \{\sigma_k\}_{k\in Z}$ and $S^{(n)}(t, \sigma) = \{S^{(n)}(t, k, \sigma)\}_{k=-n}^{n}$ the solution of (5.3) with initial value $\{\sigma_k\}_{k=-n}^{n}$. Since $F \in \mathscr{D}_0$ has a form $F(\sigma) = F(\sigma_{-\alpha}, \cdots, \sigma_\alpha)$, with some $\alpha \in N$, we see for $n \geq \alpha$

$$\begin{aligned}|T_{t,n}F - T_tF| &= |E[F(S^{(n)}(t, \sigma))] - E[F(S(t, \sigma))]|\\ &\leq C_F \sup_{k\in[-\alpha,\alpha]} E[|S^{(n)}(t, k, s) - S(t, k, \sigma)|]\end{aligned}$$

where $C_F = \sum_{i=-\alpha}^{\alpha} \left\|\frac{\partial F}{\partial \sigma_i}\right\|_\infty$. Now we set

$$I_m(t) = \sup_{k\in[-\alpha-m,\alpha+m]} E[|S^{(n)}(t, k, \sigma) - S(t, k, \sigma)|^2]\,.$$

Then for every m: $0 \leq m \leq n - \alpha - 2$ and t_m: $0 < t_m \leq t$,

$$\begin{aligned}I_m(t_m) &= \sup_{k\in[-\alpha-m,\alpha+m]} E\left[\left|\int_0^{t_m} \{\Delta_1 U'(S^{(n)}(t_{m+1}, k, \sigma)) \right.\right.\\ &\qquad\left.\left. - \Delta_1 U'(S(t_{m+1}, k, \sigma))\} dt_{m+1}\right|^2\right]\\ &\leq 16(a + A)^2 t \int_0^{t_m} dt_{m+1} I_{m+1}(t_{m+1})\,.\end{aligned}$$

Consequenly,

$$I_0(t) \leq (16(a + A)^2 t)^{n-\alpha-1} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-\alpha-2}} dt_{n-\alpha-1} I_{n-\alpha-1}(t_{n-\alpha-1})\,.$$

Noting that

$$\begin{aligned}\langle I_{n-\alpha-1}(s)\rangle &\leq \int d\mu_\lambda \sum_{k=-n+1}^{n-1} 2E[S^{(n)}(s, k, \sigma)^2 + S(s, k, \sigma)^2]\\ &= 4M_2(\lambda)(2n - 1)\,,\end{aligned}$$

where $M_2(\lambda) = \int_R x^2 q_\lambda(x)\,dx$, we have

$$\begin{aligned}\langle (T_{t,n}F - T_tF)^2\rangle &\leq C_F^2 \langle I_0(t)\rangle\\ &\leq 4C_F^2 M_2(\lambda)(2n - 1)(16(a + A)^2 t)^{n-\alpha-1} t^{n-\alpha-1}/(n - \alpha - 1)!\end{aligned}$$

This implies that there exists a constant C_1 such that

$$\langle (T_{t,n}F - T_tF)^2\rangle \leq C_1 n^{-\delta}\,, \qquad \text{for each } \delta \in Z^+.$$

Lemma 4.4 gives an estimate on $\langle T_tF\tau_k T_tF\rangle$ and therefore on $\langle F\tau_k T_tF\rangle$ by replacing t by $t/2$. This completes the proof. □

Since (5.5) verifies that $\sum_{k=-\infty}^{\infty} k|\langle T_tF\tau_kT_tF\rangle| < \infty$ for $F \in \mathscr{D}_0$, by Lemma 4.3, we obtain the following form of $\langle T_tF | T_tF\rangle$:

$$\langle T_tF | T_tF\rangle = \lim_{n\to\infty} \frac{1}{2n+1}\left\langle\left(\sum_{k=-n}^{n} \tau_kT_tF\right)^2\right\rangle \tag{5.6}$$

PROPOSITION 5.3. *T_t can be extended uniquely to a strongly continuous self-adjoint contraction semigroup on $\mathscr{H}$.*

Proof. By (5.6), for each $t > 0$ and $F \in \mathscr{D}_0$ satisfying $\langle F\rangle = 0$,

$$\|T_tF\|_{\mathscr{H}}^2 = \lim_{n\to\infty} \frac{1}{2n+1}\left\| T_t\left(\sum_{k=-n}^{n} \tau_{-k}F\right)\right\|_{L^2}^2 \le \lim_{n\to\infty} \frac{1}{2n+1}\left\| \sum_{k=-n}^{n} \tau_{-k}F\right\|_{L^2}^2 = \|F\|_{\mathscr{H}}^2 .$$

Thus $\|T_tF\|_{\mathscr{H}} \le \|F\|_{\mathscr{H}}$ for all $F \in \mathscr{D}_0$. We can therefore extend T_t from $\mathscr{D}_0$ to $\mathscr{H}$ in such a manner that

$$\|T_tF\|_{\mathscr{H}} \le \|F\|_{\mathscr{H}} \qquad \text{for all } F \in \mathscr{H} . \tag{5.7}$$

It is easy to check that for $F, G \in \mathscr{D}_0$,

$$\langle F | T_tG\rangle = \langle T_tF | G\rangle . \tag{5.8}$$

This implies the symmetry of T_t with the help of (5.7).

Finally we show the strong continuity of T_t, *i.e.*

$$\|T_tF - F\|_{\mathscr{H}} \to 0 , \quad \text{as } t \to 0 , \quad \text{for all } F \in \mathscr{H} . \tag{5.9}$$

In fact, it is enough to show that (5.9) holds for $F \in \mathscr{D}_2$; use (5.7) noting that $\mathscr{D}_0$ is dense in $\mathscr{H}$. We see from (5.5) that for each $F \in \mathscr{D}_0$,

$$\begin{aligned} |\langle (T_tF - F)\tau_kF\rangle| &\le |\langle T_tF\tau_kF| + |\langle F\tau_kF\rangle| \\ &\le C(1+|k|)^{-2} + |\langle F\tau_kF\rangle| , \quad k \in \boldsymbol{Z} , \end{aligned}$$

and the r.h.s. is summable in k. Moreover, we know that $\langle (T_tF - F)\tau_kF\rangle \to 0$ as $t \to 0$ by the fact T_t is L^2-strongly continuous. Thus Lebesgue's dominated convergence theorem proves

$$\langle (T_tF - F) | F\rangle = \sum_{k=-\infty}^{\infty} \langle (T_tF - F)\tau_kF\rangle \to 0 , \quad \text{as } t \to 0 .$$

Consequently, we obtain (5.9) for $F \in \mathscr{D}_0$ by noting

$$\|T_tF - F\|_{\mathscr{H}}^2 = \langle T_{2t}F | F\rangle - 2\langle T_tF | F\rangle + \langle F | F\rangle . \qquad \square$$

Let L be the generator of T_t in $\mathscr{H}$. Its domain is denoted by $\mathscr{D}(L)$. We shall see that L has the same form on $\mathscr{D}_0$ as the generator of T_t in L^2.

LEMMA 5.4. *We have* $\mathscr{D}_0 \subset \mathscr{D}(L)$ *and, for every* $F(\sigma) = F(\sigma_{-\alpha}, \cdots, \sigma_\alpha) \in \mathscr{D}_0$,

$$(5.10) \qquad (LF)(\sigma) = -\sum_{k \in Z} e^{U'(\sigma_k)} \frac{\partial}{\partial \sigma_k} \left\{ e^{-U'(\sigma_k)} \left(\frac{\partial F}{\partial \sigma_{k+1}} - 2 \frac{\partial F}{\partial \sigma_k} - \frac{\partial F}{\partial \sigma_{k-1}} \right) \right\}.$$

Proof. Let L' be the generator of T_t in L^2. We know $\mathscr{D}_0$ is in the domain of L' and on $\mathscr{D}_0$, L' is given by (5.10). Thus $L'F \in \mathscr{F}_{2,\mathrm{loc}} \subset \mathscr{H}$ for $F \in \mathscr{D}_0$, and $\|T_t L'F - L'F\|_{\mathscr{H}} \to 0$ as $t \to 0$. Moreover,

$$T_t F - F = \int_0^t ds T_s L'F, \qquad \mu_\lambda - \text{a.e.}$$

Therefore

$$\left\| \frac{1}{t}(T_t F - F) - L'F \right\|_{\mathscr{H}} = \left\| \frac{1}{t} \int_0^t ds (T_s L'F - L'F) \right\|_{\mathscr{H}}$$
$$\leq \frac{1}{t} \int_0^t ds \|T_s L'F - L'F\|_{\mathscr{H}} \to 0 \quad \text{as } t \to 0.$$

This means that $LF = L'F$. □

We shall see that $\mathscr{D}_0$ is a domain of essential self-adjointness for L in the following weak sense:

PROPOSITION 5.5. *Let* $F \in \mathscr{D}(L)$. *Then there exist* $F_n \in \mathscr{D}_0$ *such that*

$$(5.11) \qquad \lim_{n \to \infty} F_n = F \qquad \text{in } \mathscr{H}$$

and

$$(5.12) \qquad \lim_{n \to \infty} \langle F_n | LF_n \rangle = \langle F | LF \rangle .$$

The first task for the proof of this proposition is to derive the following estimates.

LEMMA 5.6. *Let* $F = F(\sigma_{-\alpha}, \cdots, \sigma_\alpha) \in \mathscr{D}_0$. *Then for* $n \geq \alpha$

$$(5.13) \qquad \left| \frac{\partial T_{t,n} F}{\partial \sigma_l} \right| \leq \begin{cases} C_F e^{8(a+A)t} & \text{if } |l| \leq \alpha, \\ C_F e^{8(a+A)t} (4(a+A)t)^{|l|-\alpha} / (|l| - \alpha)! & \text{if } \alpha < |l| \leq n \end{cases}$$

where $C_F = \sum_{k=-\alpha}^{\alpha} \left\| \frac{\partial F}{\partial \sigma_k} \right\|_\infty$.

Proof. For every $\varepsilon > 0$ and $\sigma = \{\sigma_k\} \in L_r^2$, set $\sigma(l, \varepsilon) = \{\sigma_k + \delta_{kl}\varepsilon\}$ and $\sigma_{[-n,n]} = \{\sigma_{-n}, \cdots, \sigma_n\}$. Then

$$
\begin{aligned}
&|T_{t,n}F(\sigma(l,\varepsilon)_{[-n,n]}) - T_{t,n}F(\sigma_{[-n,n]})| \\
&\quad = |E[F(S^{(n)}(t,\sigma(l,\varepsilon))) - F(S^{(n)}(t,\sigma))]| \\
&\quad \leq C_F \sup_{k\in[-\alpha,\alpha]} E[|S^{(n)}(t,k,\sigma(l,\varepsilon)) - S^{(n)}(t,k,\sigma))|]
\end{aligned} \tag{5.14}
$$

where $S^{(n)}$ is defined as in the proof of Lemma 5.2. To get further estimates on the r.h.s. of (5.14), set

$$
J^{\varepsilon}_m(t) = \sup_{k\in[-\alpha-m,\alpha+m]} E[|S^{(n)}(t,k,\sigma(l,\varepsilon)) - S^{(n)}(t,k,\sigma))|] \quad \text{for } m = 0, 1, \cdots, n-\alpha\,. \tag{5.15}
$$

We have, from the SDE (5.3), for $m = 0, 1, \cdots, n - \alpha - 1$

$$
\begin{aligned}
J^{\varepsilon}_m(t) &= \sup_{k\in[-\alpha-m,\alpha+m]} E\left[\left|\delta_{kl}\varepsilon + \int_0^t \{\Delta_1 U'(S^{(n)}(s,k,\sigma(l,\varepsilon))) - \Delta_1 U'(S^{(n)}(t,k,\sigma))\}ds\right|\right] \\
&\leq \begin{cases} 4(a+A)\displaystyle\int_0^t ds J^{\varepsilon}_{m+1}(s)\,, & \text{if } |l| > \alpha + m\,, \\ \varepsilon + 4(a+A)\displaystyle\int_0^t ds J^{\varepsilon}_{m+1}(s)\,, & \text{if } |l| \leq \alpha + m\,. \end{cases}
\end{aligned} \tag{5.16}
$$

For $m = n - \alpha$, similarly, we have

$$
J^{\varepsilon}_{n-\alpha}(t) \leq \varepsilon + 4(a+A)\int_0^t ds J^{\varepsilon}_{n-\alpha}(s)\,. \tag{5.17}
$$

This implies with the help of Gronwall's lemma

$$
J^{\varepsilon}_{n-\alpha}(t) \leq \varepsilon e^{4(a+A)t}\,. \tag{5.18}
$$

Therefore, combining (5.14), (5.16) and (5.18), we can easily show that the l.h.s. of (5.14) divided by ε is bounded by the r.h.s. of (5.13) for every l; $|l| \leq n$. □

Proof of Proposition 5.5. Since the space $\bigcup_{t\geq 0} T_t \mathscr{D}_0$ is a core for L (see Reed and Simon [6], II. Th. X. 49), the proof is completed if for every $F = F(\sigma_{-\alpha}, \cdots, \sigma_\alpha) \in \mathscr{D}_0$ and $t \geq 0$, we can find functions $F_n \in \mathscr{D}_0$ such that

$$
\lim_{n\to\infty} F_n = T_t F \qquad \text{in } \mathscr{H}
$$

and

$$
\lim_{n\to\infty} \langle F_n | LF_n \rangle = \langle T_t F | L T_t F \rangle\,.
$$

Take $F_n = T_{t,n}F$. Then, although $T_{t,n}F$ may not be in $\mathscr{D}_0$, there exist functions $G_m \in \mathscr{D}_0$ such that $G_m \to T_{t,n}F$ and $LG_m \to LT_{t,n}F$ as $m \to \infty$ in $L^2(\boldsymbol{R}^{2\alpha+1}, \mu^{(\alpha)})$ and therefore in $\mathscr{H}$; remind Lemma 4.1(1). Thus, it is sufficient to show that

$$\lim_{n\to\infty} \langle T_{t,n}F | LT_{t,n}F \rangle = \langle T_tF | LT_tF \rangle ,$$

since Lemma 5.3 proves $T_{t,n}F \to T_tF$ in $\mathscr{H}$. Noting that $L_nF = LF$ for n large enough, we have

$$\begin{aligned}|\langle T_{t,n}F | LT_{t,n}F\rangle - \langle T_tF | LT_tF\rangle| &\le |\langle T_{t,n}F | LT_{t,n}F - L_nT_{t,n}F\rangle| \\ &\quad + \|T_{t,n}L_nF\|_{\mathscr{H}} \|T_{t,n}F - T_tF\|_{\mathscr{H}} + \|T_tF\|_{\mathscr{H}} \|T_{t,n}LF - T_tLF\|_{\mathscr{H}} .\end{aligned}$$

Here, the second and third terms tend to 0 as $n \to \infty$ by Lemma 5.2. For the first term, noting the facts:

$$\langle F | LG\rangle = - \sum_{i\in \boldsymbol{Z}} \sum_{k \in \boldsymbol{Z}} \left\langle \nabla_{1,i} \frac{\partial \tau_k F}{\partial \sigma_i} \nabla_{1,i} \frac{\partial G}{\partial \sigma_i} \right\rangle , \tag{5.19}$$

$$\begin{aligned}\langle F | L_nG\rangle = - \sum_{k\in \boldsymbol{Z}} \sum_{i=-n}^{n-1} \Bigg\{ &\left\langle \nabla_{1,i} \frac{\partial \tau_k F_1}{\partial \sigma_i} \nabla_{1,i} \frac{\partial F_2}{\partial \sigma_i} \right\rangle \\ &+ \left\langle \left(\frac{\partial \tau_k F}{\partial \sigma_{-n}} - \frac{\partial \tau_k F}{\partial \sigma_n} \right) \left(\frac{\partial G}{\partial \sigma_{-n}} - \frac{\partial G}{\partial \sigma_n} \right) \right\rangle \Bigg\} ,\end{aligned} \tag{5.20}$$

we can use Lemma 5.6 to obtain

$$\begin{aligned}
&|\langle T_{t,n}F | LT_{t,n}F - L_nT_{t,n}F\rangle| \\
&= \Bigg| - \sum_{k=1}^{2n+1} \left\langle \tau_k \frac{\partial T_{t,n}F}{\partial \sigma_{n+1-k}} \frac{\partial T_{t,n}F}{\partial \sigma_n} \right\rangle + \sum_{k=-2n}^{0} \left\langle \tau_k \frac{\partial T_{t,n}F}{\partial \sigma_{-n-k}} \frac{\partial T_{t,n}F}{\partial \sigma_n} \right\rangle \\
&\quad + \sum_{k=0}^{2n} \left\langle \tau_k \frac{\partial T_{t,n}F}{\partial \sigma_{n-k}} \frac{\partial T_{t,n}F}{\partial \sigma_{-n}} \right\rangle - \sum_{k=-2n-1}^{-1} \left\langle \tau_k \frac{\partial T_{t,n}F}{\partial \sigma_{-n-1-k}} \frac{\partial T_{t,n}F}{\partial \sigma_{-n}} \right\rangle \Bigg| \\
&\le \Bigg(\sum_{k=1}^{2n+1} \left\langle \tau_k \left| \frac{\partial T_{t,n}F}{\partial \sigma_{n+1-k}} \right| \right\rangle + \sum_{k=-2n}^{0} \left\langle \tau_k \left| \frac{\partial T_{t,n}F}{\partial \sigma_{-n-k}} \right| \right\rangle + \sum_{k=0}^{2n} \left\langle \tau_k \left| \frac{\partial T_{t,n}F}{\partial \sigma_{n-k}} \right| \right\rangle \\
&\quad + \sum_{k=-2n-1}^{-1} \left\langle \tau_k \left| \frac{\partial T_{t,n}F}{\partial \sigma_{-n-1-k}} \right| \right\rangle \Bigg) C_F e^{8(a+A)t} \frac{1}{(n-\alpha)!} (4(a+A)t)^{n-\alpha} \\
&\le 8 e^{16(a+A)t} \left(\alpha - \frac{1}{2} + e^{4(a+A)t} \right) \frac{1}{(n-\alpha)!} (4(a+A)t)^{n-\alpha} .
\end{aligned}$$

This tends to 0 as $n \to \infty$. □

We conclude this paragraph by showing the following lemma which will be used in Section 6.

LEMMA 5.7. *Let* $F_1 \in \mathscr{D}(L)$ *and* $F_2 \in \mathscr{D}_0 \cap \mathscr{D}(L_n)$. *Then*

$$\langle F_1 | L_n F_2\rangle^2 \leq C(n)\langle F_1 | LF_1\rangle\langle F_2 L_n F_2\rangle .$$

where $C(n) = 2(2n+1)(4n^2+2n+1)$.

Proof. First, assume that $F_1 \in \mathscr{D}_0$ with $\langle F_1\rangle = 0$. Notice (5.19) and use the fact that for a sequence $\{a_l\}_{l\in \boldsymbol{Z}}$,

$$\sum_{l\in \boldsymbol{Z}} a_l = \sum_{l=-n}^{n} \sum_{m\in \boldsymbol{Z}} a_{l+(2n+1)m} ,$$

if $a_l = 0$ for all $l \in \boldsymbol{Z}$ but finite l's. Then some tedious but straightforward calculations prove that

$$\text{(5.21)} \qquad \frac{1}{2n+1} \sum_{k\in \boldsymbol{Z}} \sum_{l\in \boldsymbol{Z}} \Big\langle \sum_{i=-n}^{n} \nabla_{1,i} \frac{\partial \tau_k F_1}{\partial \sigma_i} \nabla_{1,i} \frac{\partial \tau_l F_1}{\partial \sigma_i} \Big\rangle = -\langle F_1 | LF_1\rangle .$$

By (5.20) and (5.21) we have for

$$\begin{aligned}
\langle F_1 | L_n F_2\rangle^2 &= \Big(\sum_{k\in \boldsymbol{Z}} \sum_{i=-n}^{n} \Big\langle \nabla_{1,i} \frac{\partial \tau_k F_1}{\partial \sigma_i} \nabla_{1,i}^{+} \frac{\partial F_2}{\partial \sigma_i} \Big\rangle \\
&\quad + \sum_{k\in \boldsymbol{Z}} \Big\langle \Big(\frac{\partial \tau_k F_1}{\partial \sigma_{-n}} - \frac{\partial \tau_k F_1}{\partial \sigma_{n+1}}\Big)\Big(\frac{\partial F_2}{\partial \sigma_{-n}} - \frac{\partial F_2}{\partial \sigma_n}\Big)\Big\rangle\Big)^2 \\
&\leq 2\Big\langle \sum_{i=-n}^{n} \Big(\sum_{k\in \boldsymbol{Z}} \nabla_{1,i} \frac{\partial \tau_k F_1}{\partial \sigma_i}\Big) \nabla_{1,i}^{+} \frac{\partial F_2}{\partial \sigma_i} \Big\rangle^2 \\
&\quad + 2\Big\langle \Big(\sum_{i=-n}^{n} \sum_{k\in \boldsymbol{Z}} \nabla_{1,i} \frac{\partial \tau_k F_1}{\partial \sigma_i}\Big)\Big(\sum_{i=-n}^{n-1} \nabla_1 \frac{\partial F_2}{\partial \sigma_i}\Big)\Big\rangle^2 \\
&\leq 2 \sum_{k\in \boldsymbol{Z}} \sum_{l\in \boldsymbol{Z}} \sum_{i=-n} \Big\langle \nabla_{1,i} \frac{\partial \tau_k F_1}{\partial \sigma_i} \nabla_{1,i} \frac{\partial \tau_l F_1}{\partial \sigma_i} \Big\rangle \sum_{i=-n}^{n} \Big\langle \Big(\nabla_1^{+} \frac{\partial F_2}{\partial \sigma_i}\Big)^2 \Big\rangle \\
&\quad + 4n(2n+1) \Big\langle \sum_{i=-n}^{n} \Big(\sum_{k\in \boldsymbol{Z}} \nabla_{1,i} \frac{\partial \tau_k F_1}{\partial \sigma_i}\Big)^2 \Big\rangle \Big\langle \sum_{i=-n}^{n-1} \Big(\nabla_1 \frac{\partial F_2}{\partial \sigma_i}\Big)^2 \Big\rangle \\
&\leq 2(2n+1)(4n^2+2n+1)\langle F_1 | LF_1\rangle\langle F_2 L_n F_2\rangle ,
\end{aligned}$$

where ∇_1^{+} is defined by $(\nabla_1^{+} G)(i) = \nabla_1 G(i)$, $-n \leq i \leq n-1$ and $(\nabla_1^{+} G)(n) = G(-n) - G(n)$. Consequently, the desired inequality is verified for $F_1 \in \mathscr{D}_0$ and $F_2 \in \mathscr{D}_0 \cap \mathscr{D}(L_n)$. However, this concludes the proof with the help of Proposition 5.5. □

§ 6. Invariant subspace

In this section, we show the ergodicity of T_t in $\mathscr{H}$. Denote by $P\mathscr{H}$ the subspace of $\mathscr{H}$ invariant under $\{T_t\}$. Then the spectral theorem implies that

$$\text{(6.1)} \qquad \lim_{t\to\infty} T_t F = G \in P\mathscr{H}$$

exists for every $F\in\mathscr{H}$. What we prove is that $P\mathscr{H}$ is one-dimensional subspace of $\mathscr{H}$. Let us denote the conditional expectation under $\mu_\lambda^{(n)}$ of $F\in\mathscr{F}_{2,[-n,n]}$, $n\in\boldsymbol{Z}^+$ on the hyperplane $\{\sigma\,|\,1/(2n+1)\sum_{k=-n}^{n}\sigma_k=y\}$ by

$$\nu_y^{(n)}(F)=\mu_\lambda^{(n)}\Big(F\,\Big|\,\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k=y\Big)\,,\qquad y\in\boldsymbol{R}\,,\tag{6.2}$$

and

$$(\Gamma_nF)(\sigma)=\nu^{(n)}_{1/(2n+1)\sum_{k=-n}^{n}\sigma_k}(F)\,.\tag{6.3}$$

Note that $\nu_y^{(n)}$ is determined independently of λ.

First, we show the following property of $P\mathscr{H}$:

PROPOSITION 6.1. *Let $G\in P\mathscr{H}$. Then for every $F\in\mathscr{D}_0\cap\mathscr{F}_{2,[-n,n]}$*

$$\langle G\,|\,\Gamma_nF\rangle=\langle G\,|\,F\rangle\,.\tag{6.4}$$

Proof. Proposition 5.3 verifies $G\in\mathscr{D}(L)$ and $LG=s\text{-}\lim_{t\to0}\frac{1}{t}(T_tG-G)=0$ in $\mathscr{H}$. Moreover by Lemma 5.7

$$\langle G\,|\,L_nF\rangle^2\leq C(n)\langle G\,|\,LG\rangle\langle FL_nF\rangle\qquad\text{for } F\in\mathscr{D}_0\cap\mathscr{D}(L_n)\,,\tag{6.5}$$

and therefore

$$\langle G\,|\,L_nF\rangle=0\qquad\text{for each } F\in\mathscr{D}_0\cap\mathscr{D}(L_n)\,.\tag{6.6}$$

For every $F\in\mathscr{D}_0\cap\mathscr{F}_{2,[-n,n]}$, noting $T_{t,n}F\in\mathscr{D}(L_n)$ for $t\geq0$, we choose $F_m\in\mathscr{D}_0\cap\mathscr{D}(L_n)$ such that $F_m\to T_{t,n}F$ and $L_nF_m\to L_nT_{t,n}F$ as $m\to\infty$ in $L^2(\boldsymbol{R}^{2n+1},\mu_\lambda^{(n)})$. This is actually possible because $\mathscr{D}_0\cap\mathscr{D}(L_n)$ is a core for L_n in $L^2(\boldsymbol{R}^{2n+1},\mu_\lambda^{(n)})$. However, Lemma 4.1(1) proves that $F_m\to T_{t,n}F$ and $L_nF_m\to L_nT_{t,n}F$ as $m\to\infty$ also in $\mathscr{H}$. Hence, from (6.6), we have

$$\langle G\,|\,L_nT_{t,n}F\rangle=0\qquad\text{for each } t\geq0 \text{ and } F\in\mathscr{D}_0\,.\tag{6.7}$$

Lemma 4.1(1) verifies also that $T_{t,n}F$ is strongly differentiable in $\mathscr{H}$ as well as in $L^2(\boldsymbol{R}^{2n+1},\mu_\lambda^{(n)})$ and $(d/dt)T_{t,n}F=L_nT_{t,n}F$. We therefore have

$$\frac{d}{dt}\langle G\,|\,T_{t,n}F\rangle=0\qquad\text{for each } t\geq0 \text{ and } F\in\mathscr{D}_0\,.\tag{6.8}$$

This implies

$$\langle G\,|\,T_{t,n}F\rangle=\langle G\,|\,F\rangle\qquad\text{for each } t\geq0\,.\tag{6.9}$$

Since the diffusion process with generator L_n is ergodic on every hyperplane $\{\sigma\,|\,1/(2n+1)\sum_{k=-n}^{n}\sigma_k=y\}$, we have for each $F\in\mathscr{D}_0$

$$\lim_{t\to\infty} T_{t,n}F = \Gamma_n F \tag{6.10}$$

strongly in $L^2(\boldsymbol{R}^{2n+1}, \mu_\lambda^{(n)})$. Lemma 4.1(1) again implies that (6.10) holds in $\mathscr{H}$. Letting $t\to\infty$ in (6.9) establishes the conclusion. □

For $g\in C_0^\infty(\boldsymbol{R})$ such that $\int g(x)dx = 1$, we define

$$F_0(g)(\sigma) = \int (\sigma_{[x]} - \rho'(\lambda))g(x)dx\,. \tag{6.11}$$

Remark. (1) The definition of $F_0(g)$ is independent of the choice of g, i.e. for $g_1, g_2\in C_0^\infty(\boldsymbol{R})$ such that $\int g_1(x)dx = \int g_2(x)dx = 1$, $F_0(g_1) = F_0(g_2)$ in $\mathscr{H}$.

(2) Particularly, we can take $g = 1/(2n+1)\chi_{[-n,n]}$ in (6.11), although this g is not in $C_0^\infty(\boldsymbol{R})$. We therefore have

$$F_0(g) = \frac{1}{2n+1}\sum_{k=-n}^{n}(\sigma_k - \rho'(\lambda)) \qquad \text{in } \mathscr{H} \tag{6.12}$$

for each $n\in\boldsymbol{Z}^+$ and $g\in C_0^\infty(\boldsymbol{R})$ satisfying $\int g(x)dx = 1$.

The purpose of this section is to show the following.

PROPOSITION 6.2. *Let $G\in P\mathscr{H}$. Then*

$$G = \rho''(\lambda)^{-1}\langle G\,|\,F_0(g)\rangle F_0(g)\,. \tag{6.13}$$

Proof. For every $F\in\mathscr{D}_0\cap\mathscr{F}_{2,[-n,n]}$, we have by Proposition 6.1

$$\langle G\,|\,\Gamma_n F\rangle = \langle G\,|\,F\rangle$$

and, therefore, by (6.12) and Lemma 6.3 which will be stated later

$$\begin{aligned}\langle G\,|\,F\rangle &= \lim_{n\to\infty}\langle G\,|\,\Gamma_n F\rangle \\ &= \lim_{n\to\infty}\Big\langle G\,|\,\rho''(\lambda)^{-1}\frac{d}{d\lambda}\langle F\rangle_\lambda\Big(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k - \rho'(\lambda)\Big)\Big\rangle \\ &= \rho''(\lambda)^{-1}\frac{d}{d\lambda}\langle F\rangle_\lambda\langle G\,|\,F_0(g)\rangle.\end{aligned} \tag{6.14}$$

Here we remind the notation: $[\cdot]_\lambda = \boldsymbol{E}^{\mu_\lambda}[\cdot]$. It is, however, easy to check that

$$\frac{d}{d\lambda}\langle F\rangle_\lambda = \langle F\,|\,F_0(g)\rangle \qquad \text{for every } F\in\mathscr{D}_0\,. \tag{6.15}$$

Combining (6.14) nad (6.15), we obtain

$$\langle G|F\rangle = \rho''(\lambda)^{-1}\langle F|F_0(g)\rangle\langle G|F_0(g)\rangle .$$

Consequently,

$$\langle G - \rho''(\lambda)^{-1}\langle G|F_0(g)\rangle F_0(g)|F\rangle = 0 \qquad \text{for all } F\in\mathscr{D}_0 .$$

Since $\mathscr{D}_0$ is dense in $\mathscr{H}$, we have the conclusion. □

We have used the following lemma for the proof of Proposition 6.2.

LEMMA 6.3. *Let* $F\in\mathscr{D}_0$. *Then*

$$\lim_{n\to\infty}\left\| \Gamma_n F - \rho''(\lambda)^{-1}\frac{d}{d\lambda}\langle F\rangle_\lambda\left(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k - \rho'(\lambda)\right)\right\|_{\mathscr{H}} = 0 . \tag{6.16}$$

Proof. By Lemma 4.2, it is sufficient to show that

$$\lim_{n\to\infty} n\{I_1(n) + I_2(n)\} = 0 , \tag{6.17}$$

where

$$I_1(n) = 2\langle(\Gamma_n F - \langle F\rangle_{h'(1/(2n+1)\sum_{k=-n}^{n}\sigma_k)})^2\rangle$$

$$I_2(n) = 2\left\langle\left(\langle F\rangle_{h'(1/(2n+1)\sum_{k=-n}^{n}\sigma_k)} - \langle F\rangle_\lambda - \rho''(\lambda)^{-1}\frac{d}{d\lambda}\langle F\rangle_\lambda\left(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k - \rho'(\lambda)\right)\right)^2\right\rangle .$$

It will be shown later in Lemmas 6.4 and 6.6 that both $n\cdot I_1(n)$ and $n\cdot I_2(n)$ tend to zero as $n\to\infty$. □

LEMMA 6.4. *Let* $F\in\mathscr{D}_0$. *Then*

$$\lim_{n\to\infty} n\langle(\Gamma_n F - \langle F\rangle_{h'(1/(2n+1)\sum_{k=-n}^{n}\sigma_k)})^2\rangle = 0 . \tag{6.18}$$

In order to prove this lemma, we use the following local central limit theorem:

LEMMA 6.5. *For* $\eta\in\boldsymbol{R}$, *let* $\{X_n\}$ *be a sequence of* $\boldsymbol{R}$*-valued independent random variables with the same distribution* $q_\eta(x+\rho'(\eta))dx$. *Let* $f_n(x,\eta)$ *be the density function of* $1/\sqrt{n}\sum_{k=1}^{n}X_k$. *Then, for* $\lambda\in\boldsymbol{R}$, *there exists* $\varepsilon_0>0$ *such that*

$$f_n(x,\eta) = (2\pi\rho''(\eta))^{-1/2}\exp\left[-\frac{x^2}{2\rho''(\eta)}\right] + r_1(x,\eta)n^{-1/2} + o(n^{-1/2}) \tag{6.19}$$

uniformly in $x\in\boldsymbol{R}$ *and* $\eta\in[\lambda-\varepsilon_0, \lambda+\varepsilon_0]$, *where*

$$(6.20)\quad r_1(x,\eta) = 6^{-1}(2\pi)^{-1/2}\rho''(\eta)^{-7/2}M_3(\eta)(x^3 - 3\rho''(\eta)x)\exp\left[-\frac{x^2}{2\rho''(\eta)}\right],$$

$$(6.21)\quad M_3(\eta) = \int x^3 q_\eta(x+\rho'(\eta))dx.$$

Proof. The proof is essentially given in Petrov [5]. The only different point is that, in our case, we need to check the uniformity in η. But, since ρ and h are smooth functions, one can do it easily. □

We notice the following fact:

$$(6.22)\quad \left\langle\left(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k - \rho'(\lambda)\right)^4\right\rangle_\lambda \sim O(n^{-2})$$

wichh can be established by a direct computation and will be useful for the proofs of Lemmas 6.4 and 6.6.

Proof of Lemma 6.4. We assume that $F = F(\sigma_{-\alpha}, \cdots, \sigma_\alpha)$.

Step 1. We compute for $n \geq \alpha + 1$

$$(6.23)\quad \begin{aligned} \nu_y^{(n)}(F) &= \mu_\lambda^{(n)}\left(F \,\middle|\, \frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k = y\right) \\ &= \hat{Z}(n,y)^{-1}\int_{R^{2n}} d\sigma_{-n}\cdots d\sigma_{n-1}F(\sigma_{-\alpha},\cdots,\sigma_\alpha)\cdot \\ &\qquad \times\exp\Psi((2n+1)y, y, \sigma_{[-n,n-1]}) \\ &= \hat{Z}(n,y)^{-1}M(h'(y))^{-2\alpha-1}\int_{R^{2\alpha+1}} d\sigma_{-\alpha}\cdots d\sigma_\alpha F(\sigma_{-\alpha},\cdots,\sigma_\alpha)\cdot \\ &\qquad \times\exp\left[h'(y)\sum_{k=-\alpha}^{\alpha}\sigma_k - \sum_{k=-\alpha}^{\alpha}U(\sigma_k)\right]\cdot I_{n,y}(\sigma_{-\alpha},\cdots,\sigma_\alpha) \\ &= \hat{Z}(n,y)^{-1}\langle FI_{n,y}\rangle_{h'(y)} \end{aligned}$$

where

$$\Psi(x,y,\sigma_A) = xh'(y) - \sum_{k\in A\cap Z}U(\sigma_k) - U\Big(x - \sum_{k\in A\cap Z}\sigma_k\Big),$$

for $x, y \in \boldsymbol{R}$ and $\sigma_A = \{\sigma_k; k\in A\cap \boldsymbol{Z}\}$, and

$$(6.24)\quad \hat{Z}(n,y) = \int_{R^{2n}} d\sigma_{-n}\cdots d\sigma_{n-1}\exp\Psi((2n+1)y, y, \sigma_{[-n,n-1]}),$$

$$(6.25)\quad \begin{aligned} I_{n,y} &= M(h'(y))^{2\alpha+1}\int_{R^{2n-2\alpha-1}} d\sigma_{-n}\cdots d\sigma_{-\alpha-1}\,d\sigma_{\alpha+1}\cdots d\sigma_{n-1}\cdot \\ &\qquad \times\exp\Psi\left((2n+1)y - \sum_{k=-\alpha}^{\alpha}\sigma_k, y, \sigma_{[-n,-\alpha-1]\cup[\alpha+1,n-1]}\right). \end{aligned}$$

Let $f_n(x, \lambda)$ be the function defined in Lemma 6.5 and put $\eta = h'(y)$. Then it is easy to see by a simple computation

$$(6.26)\quad f_n(x, \eta) = \sqrt{n}\, M(\eta)^{-n} Z(n, \sqrt{n}\, x + n\rho'(\eta)) \exp[\eta(\sqrt{n}\, x + \rho'(\eta)]$$

where

$$(6.27)\quad Z(n, y) = \int_{R^{n-1}} d\sigma_1 \cdots d\sigma_{n-1} \exp\Big[-\sum_{k=1}^{n-1} U(\sigma_k) - U\Big(y - \sum_{k=1}^{n-1} \sigma_k\Big)\Big] .$$

This implies

$$(6.28)\quad Z(n, \sqrt{n}\, x + ny) = \sqrt{n}\, M(\eta)^n e^{-\eta(\sqrt{x}n + ny)} f_n(x, \eta) .$$

Consequently, by (6.24), (6.27) and (6.28)

$$(6.29)\quad \begin{aligned} \hat{Z}(n, y) &= e^{\eta(2n+1)y} Z(2n + 1, (2n + 1)y) \\ &= (2n + 1)^{-1/2} M(\eta)^{2n+1} f_{2+1}(0, \eta) , \end{aligned}$$

and by (6.25), (6.27) and (6.28)

$$(6.30)\quad \begin{aligned} I_{n,y} &= M(\eta)^{2\alpha+1} Z(2(n-\alpha), (2n+1)y - \sum_{k=-\alpha}^{\alpha} \sigma_k) \exp[\eta((2n+1)y - \sum_{k=-\alpha}^{\alpha} \sigma_k)] \\ &= M(\eta)^{2n+1}(2(n-\alpha))^{-1/2} f_{2(n-\alpha)}\Big(\frac{2\alpha+1}{\sqrt{2(n-\alpha)}}\Big(y - \frac{1}{2\alpha+1}\sum_{k=-\alpha}^{\alpha} \alpha_k\Big), \eta\Big) . \end{aligned}$$

Take ε_0 as in Lemma 6.5. Then, by the continuity of $y \to \eta = h'(y)$, there exists $\delta_0 > 0$ such that

$$|\eta - \lambda| = |h'(y) - \lambda| \leq \varepsilon_0$$

for every y: $|y - \rho'(\lambda)| \leq \delta_0$. We set

$$(6.31)\quad y_{n,\alpha} = \frac{2\alpha + 1}{\sqrt{2(n - \alpha)}}\Big(y - \frac{1}{2\alpha + 1}\sum_{k=-\alpha}^{\alpha} \alpha_k\Big) .$$

By (6.29) ~ (6.31) and Lemma 6.5, we have

$$(6.32)\quad \begin{aligned} \hat{Z}(n, y)^{-1} I_{n,y}{}^{-1} &= \Big(\frac{2n + 1}{2(n - \alpha)}\Big)^{1/2} f_{2(n-\alpha)}(y_{n,\alpha}, \eta) f_{2n+1}^{-1}(0, \eta) - 1 \\ &= (1 + o(n^{-1/2}))\{1 + J_{n,y} + \rho''(\eta)^{1/2} o(n^{-1/2})\} \cdot \\ &\qquad \times \{1 + \rho''(\eta)^{1/2} o(n^{-1/2})\}^{-1} - 1 \end{aligned}$$

uniformly in $(\sigma_{-\alpha}, \cdots, \sigma_\alpha) \in R^{2\alpha+1}$ and $y \in [\rho'(\lambda) - \delta_0, \rho'(\lambda) + \delta_0]$, where we denote

$$(6.33)\quad J_{n,y} = \exp\Big[-\frac{y_{n,\alpha}^2}{2\rho''(\eta)}\Big] - 1 + (2\pi\rho''(\eta))^{1/2} r_1(y_{n,\alpha}, \eta)(2(n - \alpha))^{-1/2} ,$$

Since $\rho''(\eta)^{1/2}$ is bounded on $[\lambda-\varepsilon_0, \lambda+\varepsilon_0]$,

$$\rho''(\eta)^{1/2}o(n^{-1/2}) = o(n^{-1/2})\,. \tag{6.34}$$

Combining (6.32) and (6.34),

$$\hat{Z}(n, y)^{-1}I_{n,y} - 1 = J_{n,y} + J_{n,y}o(n^{-1/2}) + o(n^{-1/2})\,.$$

Consequently,

$$\langle(\hat{Z}(n, y)^{-1}I_{n,y} - 1)^2\rangle_{h'(y)} \le 3\langle J^2_{n,y}\rangle_{h'(y)} + \langle J^2_{n,y}\rangle_{h'(y)}o(n^{-1}) + o(n^{-1})\,. \tag{6.35}$$

By (6.20) and (6.31), we have

$$\begin{aligned} J_{n,y} &= -\frac{y^2_{n,\alpha}}{2\rho''(\eta)}e^{-\theta} \\ &\quad + \frac{1}{6}\rho''(\eta)^{-3}M_3(\eta)(y^3_{n,\alpha} - 3\rho''(\eta)y_{n,\alpha})\exp\Big[-\frac{y^2_{n,\alpha}}{2\rho''(\eta)}\Big](2(n-\alpha))^{-1/2} \\ &= \Big\{\Big(y - \frac{1}{2\alpha+1}\sum_{k=-\alpha}^{\alpha}\sigma_k\Big)^2 e^{-\theta} + \Big(y - \frac{1}{2\alpha+1}\sum_{k=-\alpha}^{\alpha}\sigma_k\Big)\cdot \\ &\quad \times\Big(\frac{(2\alpha+1)^2}{2(n-\alpha)}\Big\{y - \frac{1}{2\alpha+1}\sum_{k=-\alpha}^{\alpha}\sigma_k\Big\}^2 - 3\rho''(\eta)\Big)\exp\Big[-\frac{y^2_{n,\alpha}}{2\rho''(\eta)}\Big]\Big\}o(n^{-1/2}) \end{aligned}$$

with some $\theta \in (0, y^2_{n,\alpha}/2\rho''(\eta))$. Set

$$\hat{J}_y = \Big(y - \frac{1}{2\alpha+1}\sum_{k=-\alpha}^{\alpha}\sigma_k\Big)^2\Big\{1 + \Big(y - \frac{1}{2\alpha+1}\sum_{k=-\alpha}^{\alpha}\sigma_k\Big)^4\Big\}\,.$$

Then

$$|J_{n,y}|^2 \le \hat{J}_y o(n^{-1})\,. \tag{6.36}$$

However it is easy to see that $y \to \langle\hat{J}_y\rangle_{h'(y)}$ is continuous and therefore

$$\begin{aligned} &\langle J^2_{n,y}\rangle_{h'(y)} \le \langle\hat{J}_y\rangle_{h'(y)}o(n^{-1}) = o(n^{-1}) \\ &\qquad\text{uniformly in } y \in [\rho'(\lambda)-\delta_0, \rho'(\lambda)+\delta_0]\,. \end{aligned} \tag{6.37}$$

Combining (6.35) and (6.37), we have

$$\begin{aligned} &\langle(\hat{Z}(n, y)^{-1}I_{n,y} - 1)^2\rangle_{h'(y)} = o(n^{-1}) \\ &\qquad\text{uniformly in } y \in [\rho'(\lambda)-\delta_0, \rho'(\lambda)+\delta_0]\,. \end{aligned} \tag{6.38}$$

Step 2. Set

$$A_n = \Big\{\sigma \in \boldsymbol{R}^{\boldsymbol{Z}}: \Big|\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k - \rho'(\lambda)\Big| > \delta_0\Big\} \tag{6.39}$$

By (6.23) and (6.38)

$$\begin{aligned}(6.40)\quad &\langle(\Gamma_n F - \langle F\rangle_{h'(1/(2n+1)\Sigma_{k=-n}^{n}\sigma_k)})^2;\ A_n^c\rangle \\ &= \langle(\langle\hat{Z}(n,y)^{-1}FI_{n,y} - F\rangle_{h'(y)})^2|_{y=1/(2n+1)\Sigma_{k=-n}^{n}\sigma_k};\ A_n^c\rangle \\ &\le \|F\|_\infty^2\langle\langle(\hat{Z}n,y)^{-1}I_{n,y} - 1)^2\rangle_{h'(y)}|_{y=1/(2n+1)\Sigma_{k=-n}^{n}\sigma_k};\ A_n^c\rangle \\ &= o(n^{-1}) \qquad \text{as } n\to\infty\,.\end{aligned}$$

On the other hand, by (6.22)

$$\begin{aligned}(6.41)\quad &\langle(\Gamma_n F - \langle F\rangle_{h'(1/(2n+1)\Sigma_{k=-n}^{n}\sigma_k)})^2;\ A_n\rangle \le 4\|F\|_\infty^2\mu_\lambda(A_n) \\ &\le 4\|F\|_\infty^2\,\delta_0^4\Big\langle\Big(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k - \rho'(\lambda)\Big)^4\Big\rangle = o(n^{-1}) \quad \text{as } n\to\infty\,.\end{aligned}$$

The combination of (6.40) and (6.41) proves the conclusion.

LEMMA 6.6. *Let $F\in\mathscr{D}_0$. Then*

$$\begin{aligned}(6.42)\quad &\lim_{n\to\infty} n\Big\langle\Big\{\langle F\rangle_{h'(1/(2n+1)\Sigma_{k=-n}^{n}\sigma_k)} - \langle F\rangle_\lambda \\ &\qquad - \rho''(\lambda)^{-1}\frac{d}{d\lambda}\langle F\rangle_\lambda\Big(\frac{1}{2n+1}\sum_{k=-}^{n}\sigma_k - \rho'(\lambda)\Big)\Big\}^2\Big\rangle = 0\,.\end{aligned}$$

Proof. Set

$$(6.43)\qquad J_n(y) = \langle F\rangle_{h'(y)} - \langle F\rangle_\lambda - \rho''(\lambda)^{-1}\frac{d}{d\lambda}\langle F\rangle_\lambda(y-\rho'(\lambda))\,.$$

Then, the conclusion follows if we show that

$$(6.44)\qquad \Big\langle J_n\Big(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k\Big)^2\Big\rangle = o(n^{-1}) \qquad \text{as } n\to\infty\,.$$

Let A_n be the set defined by (6.39). Then, we have by (6.15) and (6.22)

$$\begin{aligned}(6.45)\quad &\Big\langle J_n\Big(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k\Big)^2;\ A_n\Big\rangle \\ &\le 6\|F\|_\infty^2(\delta_0^4 + 3(2\alpha+1))^2\delta_0^2)\Big\langle\Big(\frac{1}{2n+1}\sum_{k=-n}^{n}\sigma_k - \rho'(\lambda)\Big)^4\Big\rangle \\ &= o\,(n^{-1})\,.\end{aligned}$$

Noting that

$$\frac{d}{dy}\langle F\rangle_{h'(y)}\Big|_{y=\rho'(\lambda)} = \rho''(\lambda)^{-1}\frac{d}{d\lambda}\langle F\rangle_\lambda\,,$$

we have

$$(6.46)\qquad J_n(y) = \frac{1}{2}\frac{d^2}{dy^2}\langle F\rangle_{h'(y)}\Big|_{y=\theta}(y-\rho'(\lambda))^2$$

with some $\theta \in (\rho'(\lambda) \wedge y, \ \rho'(\lambda) \vee y)$. Notice that $|y - \rho'(\lambda)| \leq \delta_0$ implies $|\theta - \rho'(\lambda)| \leq \delta_0$. Since the function $y \to \langle F \rangle_{h'(y)}$ belongs to $C^2(R)$, there exists $C > 0$ such that

$$(6.47) \qquad \left| \frac{d^2}{dy^2} \langle F \rangle_{h'(y)} \right| \leq C \qquad \text{for } y \in [\rho'(\lambda) - \delta_0, \rho'(\lambda) + \delta_0] .$$

By combining (6.46) with (6.47), and using (6.22), we obtain

$$(6.48) \qquad \begin{aligned} \left\langle J_n \left(\frac{1}{2n+1} \sum_{k=-n}^{n} \sigma_k \right)^2 ; A_n^c \right\rangle &\leq \frac{C}{2} \left\langle \left(\frac{1}{2n+1} \sum_{k=-n}^{n} \sigma_k - \rho'(\lambda) \right)^4 \right\rangle \\ &= o(n^{-1}) . \end{aligned}$$

This establishes (6.44) with the help of (6.45). □

§ 7. Tightness of $\{P_\varepsilon : 0 < \varepsilon \leq 1\}$

The Boltzmann-Gibbs principle has been established by combining the results of Sections 4, 5 and 6. In order to show the tightness of $\{P_\varepsilon : 0 < \varepsilon \leq 1\}$ being defined in Section 2, we first derive the following estimate. The duality between two spaces $\mathscr{E}_r'$ and $\mathscr{E}_r$ will be simply denoted by (,).

LEMMA 7.1. *For $f \in \mathscr{E}_r$ and $F \in L^4(R, q_\lambda(x)dx)$ satisfying $\int_R F(x) q_\lambda(x) dx = 0$, there exists a constant $C = C(F, f) > 0$ such that*

$$(7.1) \qquad \langle (F(\sigma_{[x/\varepsilon]}), \ D_\varepsilon^k f(x))^4 \rangle_\lambda \leq C\varepsilon^2, \quad k = 0, 1, 2$$

where $D_\varepsilon^0 f = f$, $D_\varepsilon^1 f(x) = \nabla_\varepsilon^* f(x) = \varepsilon^{-1}(f(x-\varepsilon) - f(x))$ and $D_\varepsilon^2 f = \Delta_\varepsilon f$.

Proof. Set $g = D_\varepsilon^k f$, $k = 0, 1, 2$. Noting that $\langle F(\sigma_i) \rangle = 0$ for $i \in Z$, we have

$$(7.2) \qquad \begin{aligned} \langle (F(\sigma_{[x/\varepsilon]}), g(x))^4 \rangle_\lambda &= \left\langle \left(\sum_{i=-\infty}^{\infty} F(\sigma_i) \int_{\varepsilon i}^{\varepsilon(i+1)} g(x) dx \right)^4 \right\rangle \\ &= \left\langle \sum_{i=-\infty}^{\infty} \left(F(\sigma_i) \int_{\varepsilon i}^{\varepsilon(i+1)} g(x) dx \right)^4 \right\rangle \\ &\quad + 6 \left\langle \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{i-1} \left(F(\sigma_i) \int_{\varepsilon i}^{\varepsilon(i+1)} g(x) dx \right)^2 \left(F(\sigma_j) \int_{\varepsilon j}^{\varepsilon(j+1)} g(x) dx \right)^2 \right\rangle \\ &\leq 7 \|F\|_{L^4(q_\lambda)}^4 \|g\|_4^4 \varepsilon^2 . \end{aligned}$$

Since $f \in \mathscr{E}_r$ implies $\sup_{0<\varepsilon\leq 1, k} \|D_\varepsilon^k f\|_4 = C_f < \infty$, (7.2) proves the conclusion with $C = 7C_f^4 \|F\|_{L^4(q_\lambda)}^4$. □

PROPOSITION 7.2. *$\{P_\varepsilon : 0 < \varepsilon \leq 1\}$ is tight on $\mathscr{C}$.*

Proof. Since $\mathscr{E}_r$ is a nuclear Fréchet space, by the theorem in [4], $\{P_\varepsilon: 0 < \varepsilon \leq 1\}$ is tight on $\mathscr{C}$ if the family of distribution on $C([0, \infty); \boldsymbol{R})$ of $V_t^\varepsilon(f) = \int V_\varepsilon(t, x)f(x)dx$: $0 < \varepsilon \leq 1$, is tight for each $f \in \mathscr{E}_r$. However, Lemma 7.1 implies $\boldsymbol{E}[V_0^\varepsilon(f)^4] \leq C$ for $\varepsilon \in (0, 1]$ with some $C > 0$. Therefore, noting the stationarity of $V_t^\varepsilon(f)$, it is sufficient for us to show that there exists constant $M > 0$ such that

$$\boldsymbol{E}[(V_t^\varepsilon(f) - V_0^\varepsilon(f))^4] \leq Mt^{3/2}, \qquad \text{for } t \in [0, 1]. \tag{7.3}$$

Set $I(t) = \boldsymbol{E}[(S_t^\varepsilon(f) - S_0^\varepsilon(f))^4]$, where $S_t^\varepsilon(f) = \int S_\varepsilon(t, x)f(x)dx$. Then, by Itô's formula and Hölder's inequality

$$\begin{aligned} I(t) &= 4\int_0^t ds\boldsymbol{E}[(S_s^\varepsilon(f) - S_0^\varepsilon(f))^3(U'(S_\varepsilon(s, x)), \Delta_\varepsilon f)] \\ &\quad + 12\int_0^t ds \sum_{k=-\infty}^{\infty}\left(\int_{\varepsilon k}^{\varepsilon(k+1)} \nabla_\varepsilon^* f(x)dx\right)^2 \boldsymbol{E}[(S_s^\varepsilon(f) - S_0^\varepsilon(f))^2] \\ &\leq 4\int_0^t dsI(s)^{3/4}I_1(s)^{1/4} + 12\varepsilon\int_0^t ds\|\nabla_\varepsilon^* f\|^2 I(s)^{1/2} \end{aligned} \tag{7.4}$$

where

$$I_1(s) = \boldsymbol{E}[(U'(S_\varepsilon(s, x)), \Delta_\varepsilon f)^4] = \langle (U'(\sigma_{[x/\varepsilon]}), \Delta_\varepsilon f)^4 \rangle_\lambda .$$

Notice that

$$\begin{aligned} I(s) &\leq 8\boldsymbol{E}[(S_s^\varepsilon(f) - \rho'(\lambda)^4] + 8\boldsymbol{E}[(S_0^\varepsilon(f) - \rho'(\lambda))^4] \\ &= 16\langle (\sigma_{[x/\varepsilon]} - \rho'(\lambda), f(x))^4 \rangle . \end{aligned}$$

Since $\int (U'(x) - \lambda)q_\lambda(x)dx = 0$ and $\int (x - \rho'(\lambda))q_\lambda(x)dx = 0$, by Lemma 7.1, there exist C_0 and $C_1 > 0$ independent of ε such that

$$I(s) \leq C_0\varepsilon^2 , \tag{7.5}$$

and

$$I_1(s) \leq C_1\varepsilon^2 , \qquad \text{for } 0 < \varepsilon \leq 1 . \tag{7.6}$$

Moreover, from the proof of Lemma 7.1, we know that

$$\|\nabla_\varepsilon^* f\| \leq C_f , \qquad \text{for } 0 < \varepsilon \leq 1 . \tag{7.7}$$

From (7.4)~(7.7), we have

$$I(t) \leq C' t\varepsilon^2 \tag{7.8}$$

where $C' = 4C_0^{3/4}C_1^{1/4} + 12C_0^{1/2}C_f^2$. Therefore, combining (7.4) and (7.6)~

(7.8), we have

$$I(t) \leq C\varepsilon^2(t^{1+3/4} + t^{1+1/2}), \qquad \text{for } t > 0, \tag{7.9}$$

where $C = 7C'^{3/4}C_1^{1/4} + 18C'^{1/2}C_f^2$. The desired estimate (7.3) follows from (7.9). □

§ 8. Proof of main theorem

We are ready to give the Proof of Theorem 2.2. By Proposition 7.2, from every subsequence $\{\varepsilon' \to 0\}$ of $\{\varepsilon\}$, we can find further subsequence $\{\varepsilon'' \to 0\}$ such that $P_{\varepsilon''}$ converges weakly to a certain probability distribution P on $\mathscr{C}$. Define σ-fields $\mathscr{M}_t$ and $\mathscr{M}$ on $\mathscr{C}$ as follows:

$$\mathscr{M}_t = \sigma((V(s), f) \colon 0 \leq s \leq t,\ f \in \mathscr{E}_r,\ V \in \mathscr{C}),$$
$$\mathscr{M} = \sigma(\bigcup_{t \geq 0} \mathscr{M}_t).$$

Here $V(s) \in \mathscr{E}_r'$ is the evaluation of V at time s. For each $f \in \mathscr{E}_r$ and $t \geq 0$, consider a function $M_\varepsilon(t, f)$ on $\mathscr{C}$:

$$\begin{aligned} M_\varepsilon(t, f)(V) &= (V(t), f) - (V(0), f) \\ &\quad - \int_0^t (\varepsilon^{-1/2}(U'(\varepsilon^{1/2}V(s, x) + \rho'(\lambda)), \Delta_\varepsilon f(x))dx,\ V \in \mathscr{C}. \end{aligned} \tag{8.1}$$

Then, from (3.1), we have

$$\begin{aligned} M_\varepsilon(t, f)(V_\varepsilon) &= \sqrt{2} \int \nabla_\varepsilon^* f(x) dw_\varepsilon(t, x) \\ &= \sqrt{2\varepsilon} \sum_{k=-\infty}^{\infty} \int_{\varepsilon k}^{\varepsilon(k+1)} \nabla_\varepsilon^* f(x) dx\, \beta(t/\varepsilon^2, k). \end{aligned} \tag{8.2}$$

This means $M_\varepsilon(t, f)$ is the Brownian motion with variance

$$\frac{2}{\varepsilon} \sum_{k=-\infty}^{\infty} \left(\int_{\varepsilon k}^{\varepsilon(k+1)} \nabla_\varepsilon^* f(x) dx \right)^2$$

defined on the probability space $(\mathscr{C}, \mathscr{M}, P_\varepsilon)$. Consequently,

$$M_\varepsilon(t, f) \quad \text{and} \quad M_\varepsilon(t, f)^2 - \frac{2}{\varepsilon} \sum_{k=-\infty}^{\infty} \left(\int_{\varepsilon k}^{\varepsilon(k+1)} \nabla_\varepsilon^* f(x) dx \right)^2 t$$

are $(P_\varepsilon, \mathscr{M}_t)$-martingales. Therefore,

$$E^{P_\varepsilon}[(M_\varepsilon(t, f) - M_\varepsilon(s, f))\Phi(V)] = 0, \tag{7.3}$$

for $0 < s < t$ and each $\mathscr{M}_s$-measurable bounded and continuous function $\Phi \colon \mathscr{C} \to \boldsymbol{R}$. Let us denote

$$I_1(\varepsilon) = E^{p_\varepsilon}\left[\left\{(V(t), f) - (V(s), f) - \int_s^t \rho''(\lambda)^{-1/2}(V(u), \Delta f)du\right\}\Phi(V)\right],$$

$$I_2(\varepsilon) = E^{p_\varepsilon}\left[\left\{\int_s^t \rho''(\lambda)^{-1/2}(V(u), \Delta f - \Delta_\varepsilon f)du\right\}\Phi(V)\right],$$

$$I_3(\varepsilon) = E^{p_\varepsilon}\left[\left\{\int_s^t (\varepsilon^{-1/2}U'(\varepsilon^{1/2}V(u, x) + \rho'(\lambda)) - \rho''(\lambda)^{-1}V(u, x), \Delta_\varepsilon f(x))du\right\}\Phi(V)\right].$$

Then

(8.4) $$E^{p_{\varepsilon''}}[(M_{\varepsilon''}(t, f) - M_{\varepsilon''}(s, f))\Phi(V)] = I_1(\varepsilon'') + I_2(\varepsilon'') + I_3(\varepsilon'').$$

Now take the limit $\varepsilon'' \to 0$ in (8.4). For I_1, since $p_{\varepsilon''} \to p$ weakly on $\mathscr{C}$, we have

(8.5) $$I_1(\varepsilon'') \to E^p[(M(t, f) - M(s, f))\Phi(V)]$$

where $M(t, f) \equiv M(t, f)(V) = (V(t), f) - (V(0), f) - \rho''(\lambda)^{-1}\int_0^t (V(u), \Delta f)du$. For I_2, it is easy to check that

$$|I_2(\varepsilon)| \le \rho''(\lambda)^{-1}\|\Phi\|_\infty(t - s)\int |x - \rho'(\lambda)|q_\lambda(x)dx\, \varepsilon^{-1/2}\int |\Delta f(x) - \Delta_\varepsilon f(x)|dx.$$

However, since $f \in \mathscr{E}_r$, $\varepsilon^{-1/2}\int |\Delta f(x) - \Delta_\varepsilon f(x)|dx \to 0$ as $\varepsilon \to 0$. Thus

(8.6) $$I_2(\varepsilon) \to 0.$$

For I_3, we have by Proposition 3.1

(8.7) $$I_3(\varepsilon) \to 0.$$

Combining (8.4)~(8.7) with (8.3), we have

$$E^p[(M(t, f) - M(s, f))\Phi(V)] = 0.$$

Hence, $M(t, f)$ is a $(P, \mathscr{M}_t)$-martingale. And by the similar method, we can show that $M(t, f)^2 - 2\|\nabla f\|^2 t$ is a $(P, \mathscr{M}_t)$-martingale, too. These imply that $M(t, f)$ is Brownian motion with variance $2\|\nabla f\|^2$ for each $f \in \mathscr{E}_r$.

For any $(a_1, \cdots, a_m) \in \boldsymbol{R}^m$, $m \in \boldsymbol{N}$, $t_0 = 0 \le t_1 \le \cdots \le t_m, f_1, \cdots, f_m \in \mathscr{E}_r$, a simple computation gives that

$$\sum_{k=1}^m a_k M(t_k, f_k) = \sum_{k=1}^m \{M(t_k, a_k f_k + \cdots + a_m f_m) - M(t_{k-1}, a_k f_k + \cdots + a_m f_m)\}.$$

Noting that the r.h.s. is a sum of independent Gaussian random variables, we know that the linear combination $\sum_{k=1}^m a_k M(t_k, f_k)$ has a Gaussian distribution with respect to P. Therefore, $\{M(t,f)\}_{t\geq 0, f\in\mathscr{E}_r}$ is a Gaussian system and one can check that its mean is zero and covariance is

$$E^P[M(t,f)M(s,g)] = 2(\nabla f, \nabla g)t \wedge s, \qquad \text{for } t, s \geq 0 \text{ and } f, g \in \mathscr{E}_r. \tag{8.8}$$

On the other hand, it is easy to see (cf. [11]) that $V(0)$ is an $\mathscr{E}_r'$-valued Gaussian random variable under P with mean zero and covariance

$$E^P[(V(0),f)(V(0),g)] = \rho''(\lambda)(f,g), \quad f, g \in \mathscr{E}_r. \tag{8.9}$$

For $V \in \mathscr{C}$, define $\tilde{V} \in \mathscr{C}$ such that

$$(\tilde{V}(t),f) = (V(0), e^{t\theta\Delta}f) + M(t,f)(V) + \theta\int_0^t M(s, e)^{(t-s)\theta\Delta}\Delta f)(V)ds, \quad f \in \mathscr{E}_r$$

where $\theta = \rho''(\lambda)^{-1}$. Then, from (8.8) and (8.9), $\{(\tilde{V}_t, f)\}_{t\geq 0, f\in\mathscr{E}_r}$ is a Gaussian system with mean zero and covariance

$$E^P[(\tilde{V}(t),f)(\tilde{V}(s),g)] = \rho''(\lambda)(f, e^{|t-s|\rho''(\lambda)^{-1}\Delta}g). \tag{8.10}$$

However, $\tilde{V} = V$, P-a.s from Lemma 8.1 below and therefore P is independent of the selection of $\{\varepsilon'\}$. This means that P_ε itself converges to P weakly. Since the distribution of the solution of (2.11) coincides with P, we have shown the conclusion of Theorem 2.2.

Finally, we prove the lemma used above.

LEMMA 8.1. $P(\tilde{V} = V) = 1$.

Proof. First we check that $\tilde{V}$ satisfies

$$(\tilde{V}(t),f) = (V(0),f) + \rho''(\lambda)^{-1}\int_0^t (\tilde{V}(u), \Delta f)du + M(t,f)(V).$$

This equality also holds for $V(t)$ instead of $\tilde{V}(t)$. To conclude the proof, it is sufficient to show that

$$E^P[|(\tilde{V}(t),f) - (V(t),f)|] = 0, \qquad \text{for all } t > 0 \text{ and } f \in A, \tag{8.11}$$

where A is a dense subset of $\mathscr{E}_r$. Set $\hat{V}(t) = \tilde{V}(t) - V(t)$. Then $\hat{V}$ satisfies the following equation with probability one:

$$(\hat{V}(t),f) = \rho''(\lambda)^{-1}\int_0^t ds(\hat{V}(s), \Delta f), \qquad \text{for } t > 0,\ f \in \mathscr{E}_r.$$

However, $V(t)$ is stationary under P and also $\tilde{V}(t)$; see (8.10). We therefore have from (8.9)

$$
\begin{aligned}
E^{\nu}[|(\hat{V}(t),f)|] &\leq \rho''(\lambda)^{-1}\int_0^t ds E^{\nu}[|(\hat{V}(s),\Delta f)|] \\
&\leq \sqrt{2}\,\rho''(\lambda)^{-n}\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n E^{\nu}[(V(0),\Delta^n f)^2]^{1/2} \\
&\leq \sqrt{2}\,\rho''(\lambda)^{-n+1/2}t^n\|\Delta^n f\|_{L^2(\mathbf{R})}/n!\,.
\end{aligned}
$$

We take A to be the linear hull of $\{h_m e^{-r\xi(x)}; m\in\mathbf{N}\}$, where $h_m(x)=(2^m m!\sqrt{\pi})^{-1/2}e^{-x^2/2}H_m(x)$, and $H_m(x)$, $m=0,1,2,\cdots$ are the Hermite polynomials. One can check that $C^n\|\Delta^n h_m\|^2/n! = o(1)$ as $n\to\infty$ for all $m\in\mathbf{N}$ with some constant C. This implies (8.11). □

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