Equilibrium Imitation and Growth

Jesse Perla

University of British Columbia

Christopher Tonetti

Stanford University

The least productive agents in an economy can be vital in generating growth by spurring technology diffusion. We develop an analytically tractable model in which growth is created as a positive externality from risk taking by firms at the bottom of the productivity distribution imitating more productive firms. Heterogeneous firms choose to produce or pay a cost and search within the economy to upgrade their technology. Sustained growth comes from the feedback between the endogenously determined distribution of productivity, as evolved from past search decisions, and an optimal, forward-looking search policy. The growth rate depends on characteristics of the productivity distribution, with a thicker-tailed distribution leading to more growth.

I. Introduction

Productivity growth is a key mechanism for modeling limitless production and technology growth in a resource-constrained economy. Many models capture the economy's technology level as the frontier productivity. However, in any empirical distribution of productivity, there is a large mass of low-productivity firms, and there are very few at the frontier

We would particularly like to thank Boyan Jovanovic, Mike Waugh, the editor Sam Kortum, and an anonymous referee. We would also like to thank Jess Benhabib, Xavier Gabaix, Ricardo Lagos, Bob Lucas, Ben Moll, Tom Sargent, and Gianluca Violante for useful comments and suggestions. A version of this paper was previously distributed as "Endogenous Risk and Growth." Supplementary material is provided online.

[*Journal of Political Economy*, 2014, vol. 122, no. 1]

of the distribution.[1] Hence, there are potentially enormous gains in aggregate output from even marginally increasing the productivity of less productive agents. There are many established models of innovation that investigate the role of a researcher inventing a new frontier technology and mechanisms for the near-frontier firms to adopt it; instead, we develop a model of equilibrium imitation to analyze endogenous changes to the large mass at the lower tail of the productivity distribution.

This paper contributes an analytically tractable endogenous growth mechanism—with an emphasis on the evolution of the productivity distribution—driven by the decisions of the least productive. In the model, heterogeneous firms choose to produce or pay a cost and search within the economy to upgrade their technology. While they may get lucky and draw a high productivity, most will make only modest improvements. The expected benefit from risk taking—that is, the option value of search—will depend on characteristics of the productivity distribution. Search is risky since, ex post, a firm may regret having searched, depending on the draw, as the value of the improved productivity may be less than the cost of search.

With heterogeneous productivities in a search-theoretic growth model, a key question is which distribution to use for new productivity draws. One standard approach is to use an "external" exogenous distribution or stochastic process for idiosyncratic productivity. However, a natural candidate for the distribution of new productivity draws is to sample from the existing distribution of productivities itself. Featuring this "internal" productivity distribution in our model, when a firm searches, it will copy the ideas of an existing firm in the economy. The result of a meeting is that both firms receive the maximum productivity of the pair.[2] So, not only do the unproductive drive growth, but everyone in the economy does, through the diffusion of technology from the more to the less productive. This leads to the key feature of the model: The distribution of productivity evolves endogenously from firms' past, optimal, forward-looking decisions. Aggregate growth is generated as an externality from risk taking by firms at the bottom of the productivity distribution. Not only is growth endogenous, but risk is endogenous, as risk taking through search is an optimal choice.

Section II characterizes the dynamic equilibrium of the model, defines a balanced growth path (BGP) equilibrium, solves for the BGP in closed form, and explores off-BGP dynamics and asymptotic growth rates.

[1] In Gabaix (2009), firm size is shown to empirically fit a Pareto distribution, except for a large mass of small firms. A theoretical relationship between the firm size and productivity distributions is also established. See Aw, Chen, and Roberts (2001) for an estimation of productivity distributions across various industries and over time.

[2] Our modeling of imitation as a meet-and-copy process is similar to that in Jovanovic and Rob (1989) but does not include the potential for additional spillovers beyond the maximum productivity in the pairwise meeting technology.

Because the evolving distribution is endogenously generated by firms' choices, we can analyze the feedback relationship between characteristics of the productivity distribution and growth. We interpret a thicker-tailed productivity distribution with a higher Gini coefficient as having more dispersed opportunities. To understand why the dispersion of opportunities in the productivity distribution changes the growth rate of the economy, consider a firm sampling from a left-truncated distribution. With a fatter right tail of the distribution, firms face higher returns to search—similar to the effect of a mean-preserving spread of the wage offer distribution in a standard McCall search model. Tempering this mechanism, firms have an incentive to wait for others to push the productivity distribution forward before searching since imitation is costly and the value of search is increasing over time. Growth is moderated because infinitesimal firms do not internalize how their technology adoption policies affect the evolution of the distribution.

Solving a planner's problem enables us to investigate this externality, compare the socially efficient economy to the competitive equilibrium, and determine if the forces of the model affect firms and the planner in similar ways. Section III develops this problem, where the planner chooses which firms search. In contrast to the competitive equilibrium, the planner is able to internalize the effect that the search decision has on the evolution of the productivity distribution. Hence, the planner's economy always features higher growth than in the competitive equilibrium. Furthermore, the stronger the degree of inequality in the productivity distribution, the larger the wedge between the planner and the competitive equilibrium's growth rates.

A natural extension is to investigate whether a constrained planner can subsidize search and indirectly adjust the growth rate. In Section IV, we solve the problem of a constrained planner that subsidizes search and satisfies the balanced-budget constraint with linear taxes on productive firms. In the Markov perfect equilibrium, the firms operate in a competitive equilibrium in which the planner recursively chooses optimal taxes and subsidies, subject to a budget constraint. Unlike many tax distortions that affect an agent's elastic labor supply decision, in this economy, proportional taxes impede growth by decreasing the future value of a high productivity draw. However, we find that the constrained planner can overcome this distortion and still achieve the first-best outcome.

Relation to the literature.—Much of the endogenous growth literature, such as Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992), captures the technology level of the economy either as the total number of differentiated products or as a frontier productivity/quality for each good in the economy. This literature then investigates how research expands the technology frontier, how this new technology is adopted across the economy, and the related intertemporal returns to

R&D. Since we do not have a theory of how the frontier expands, our model complements this strain of the growth literature by providing a theory of the evolution of the entire productivity distribution driven by technology diffusion.

In Romer (1990) and many other endogenous growth models, the returns to research are proportional to the current stock of research and, hence, generate geometric growth with a constant research investment. For search-theoretic models interested in the expansion of the technology frontier, the returns to productivity-enhancing investments depend crucially on the sampling distribution. In an early such example, Evenson and Kislev (1976) model applied technological advancement as obtaining, at some cost, higher productivity draws from a distribution over time. However, growth is sustainable only in the long run as a result of exogenous basic research that increases the mean of the distribution.[3] Kortum (1997) incorporates a mechanism by which firms draw from an exogenous distribution of productivities, adjusted for spillovers from the aggregate stock of research.[4]

Building on Kortum (1997) and Eaton and Kortum (1999), Alvarez, Buera, and Lucas (2008) and Lucas (2009) replace the exogenous productivity distribution with the existing cross-sectional distribution of productivity across agents, capturing the idea that each agent learns from surrounding agents. The economy grows by pulling itself up by its bootstraps, as better ideas diffuse across agents through an exogenous process of meeting and learning.

Whether it is exogenous basic research, an aggregate spillover function, or exogenous random meetings, these models depend on the distribution evolving over time and provide an exogenous or semi-endogenous mechanism that improves the distribution. Our paper provides a tractable framework that delivers a shift in the distribution much like that of Evenson and Kislev (1976), Kortum (1997), or Alvarez et al. (2008). Additionally, in our model, both the evolution of the productivity distribution and the technology adoption decision are jointly endogenously determined in equilibrium, as the least productive firms choose to adopt better technologies. Thus, our model is well suited to analyzing the effect that the productivity distribution has on adoption incentives, the effect of adoption behavior in generating the productivity distribution, and the corresponding growth implications of this link.

[3] Similarly, Bental and Peled (1996) have searchers drawing new technologies from a distribution in levels rather than making proportional improvements. In this general equilibrium setting, the frontier grows if an agent gets a lucky draw above the current frontier, and the economy grows as other firms can copy this technology next period.

[4] As in our model, Kortum (1997) and Jones (2005) investigate how a Pareto distribution of productivity/idea draws can be consistent with both a stationary distribution of firm characteristics and constant aggregate productivity growth. Similarly, Eaton and Kortum (1999) derive a BGP with a Pareto distribution for the quality of new ideas.

Our endogenous evolution of the distribution is most similar to that of Lucas and Moll (2014, in this issue), in which heterogeneous agents invest in studying to adopt new ideas and growth is generated as idea adoption evolves the productivity distribution. In contrast to our paper, Lucas and Moll emphasize the intensive margin of time dedicated to learning and develop continuous-time computational solution techniques.

Another complementary approach in the literature, as in the balanced growth models of Luttmer (2007, 2011), is to emphasize the role of selection and entry rather than the adoption of technology by incumbents. Luttmer (2007) studies a model in which both stochastic changes to incumbents' productivity and increased competition due to entering firms that have better technology cause the endogenous destruction of incumbent firms. Entrants draw a productivity and internalize the value of the growing economy, while incumbent firms choose when to exit, given that they must pay a fixed cost to operate. In contrast, in our model, a producing firm's period profits are constant, and the increasing value of technology adoption due to growth leads forward-looking incumbents, who internalize this positive value, to choose to search. While in both papers the existing distribution of technology used in production affects the returns to technology adoption, our paper complements Luttmer's by providing a different perspective on which firms benefit from aggregate growth of the economy.

Finally, this paper emphasizes how the degree of inequality in the economy determines the rate of growth and the strength of the free-riding incentive. An alternative approach, as in Eeckhout and Jovanovic (2002), is to investigate how imperfect technology spillovers can change the degree of inequality in an economy. There, agents do not copy a technology directly; instead their production function includes intratemporal spillovers from the current distribution of technologies. Eeckhout and Jovanovic find that the larger the free-riding incentive, the greater the inequality. Our paper finds that along a BGP, higher inequality increases both the growth rate and the free-riding incentive, while the degree of inequality remains constant over time.

II. The Model Economy

Time is discrete and infinite. There are two types of agents in the economy: consumers and firms. The model admits a representative consumer who simply consumes aggregate production each period. There is a fixed measure of firms with heterogeneous productivity levels, z. The function $F_t(z)$ is the cross-sectional productivity cumulative density function (cdf) in the economy, which will be the aggregate object affecting firms' decisions and will evolve over time endogenously in response to firms' actions. Given their idiosyncratic productivity, firms have a choice to either produce with their existing productivity or search and upgrade their productivity. If a firm chooses to search, it forgoes production and randomly imitates the technology of some producing firm in the existing economy. This new productivity level will be its idiosyncratic state in the period after search. When a firm searches, it samples from the existing productivity distribution. Because in equilibrium the unproductive will choose to search, the distribution of productivities evolves by shifting mass from lower to higher productivity levels. Productivity increases over time, despite the lack of any exogenous forcing process. The model is parsimoniously parameterized by a time discount factor, β, the initial productivity distribution, F_0, and the utility function of the consumer, $u(x)$. See figure 1 for intuition on the evolution of the productivity probability density function (pdf), where $m_t \equiv \min \text{support}\{F_t\}$.

A. Consumers

The lifetime utility of the representative consumer who consumes aggregate output, Y_t, each period is given by

$$\sum_{t=0}^{\infty} \beta^t u(Y_t).$$

The implied intertemporal optimization condition yields

$$\frac{1}{1+r_t} = \beta \frac{u'(Y_{t+1})}{u'(Y_t)}, \tag{1}$$

where r_t is the equilibrium interest rate that ensures that there is no active market for claims to aggregate production. Since there is no aggregate uncertainty, the representative consumer will have a deterministic sequence of consumption.

B. Firms

While consumers are kept trivial and simply eat aggregate output, firms make the important decisions in the economy. There exists a continuum of mass one of risk-neutral, infinitely lived, heterogeneous firms indexed by productivity levels, z. Firms have access to costless linear production technology such that output equals productivity, that is, $y = z$ (maximum scale is one). Since consumers own the firms, a firm's objective is to maximize its discounted stream of output, discounting the future using the interest rate, r_t, determined from the consumer problem.

Agents forecast an equilibrium sequence of $\{r_t, F_t\}_{t\geq 0}$. Given this forecast and idiosyncratic state z, firms choose whether to search and upgrade

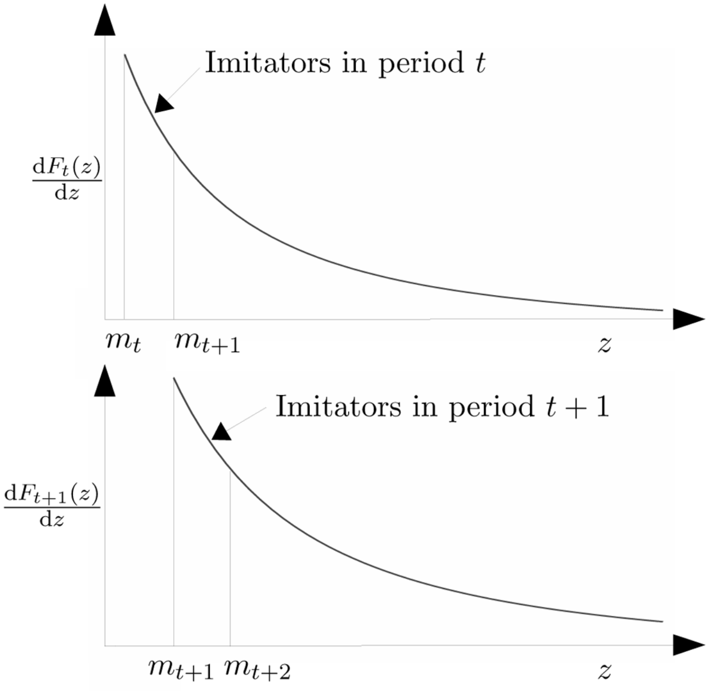

FIG. 1.—Evolution of the productivity pdf

their technology or to produce with their existing technology. The cost of search is forgone production. If the firm chooses to search, it enters the next period with a new productivity value drawn from the existing productivity distribution conditional on being a producer. Even though in equilibrium a firm receives a draw higher than its current productivity with certainty, search is still risky, since the firm may regret the search decision ex post, as the value of improved productivity may be less than the cost of search.[5] The firm weighs the value of producing versus the value of paying the opportunity cost of not producing to upgrade by randomly drawing a new productivity level. The firm solves

$$V_t(z) = \max\left\{z + \frac{1}{1+r_t}V_{t+1}(z),\ \frac{1}{1+r_t}\int V_{t+1}(z')\,dF_t(z'|z' \geq m_{t+1})\right\}. \quad (2)$$

[5] A variation of the model in which firms draw unconditionally from F_t and may reject lower draws is discussed in online App. G. Qualitative features of the model remain the same.

The distribution F_t is endogenous and depends on the decisions of all firms in the economy, requiring the firms to forecast the evolution of the distribution. Given a perceived sequence of distributions, it can be shown—following the arguments of the standard McCall search model—that the solution to this search problem is a reservation value of productivity for each t, $\hat{z}_t$, that solves equation (2) at the indifference point. Firms with productivity z will choose to search in period t if and only if $z \leq \hat{z}_t$.[6] We will restrict focus to a rational expectations equilibrium, in which the infinitesimal agents' forecasts match the evolution of the productivity distribution induced by their actions, that is, $\hat{z}_t = m_{t+1}$. The evolution of F_t is deterministic and perfectly foreseen—made possible by the absence of aggregate risk. Thus, if a firm chooses to search, it expects to and does receive a draw from the conditional distribution $z_{t+1} \sim F_t(z|z \geq m_{t+1})$.

C. *Evolution of the Productivity Distribution*

Given a productivity pdf, $f_t(z)$, the distribution evolves as a mass $F_t(m_{t+1})$ of searching firms draw new productivities from the conditional density of producers:

$$\begin{aligned} f_{t+1}(z) &= f_t(z) + f_t(z|z \geq m_{t+1})F_t(m_{t+1}) \\ &= \frac{f_t(z)}{1 - F_t(m_{t+1})}. \end{aligned} \tag{3}$$

That is, it is a truncation of the distribution F_t at m_{t+1}.[7] Thus, min support$\{F_{t+1}\} = m_{t+1}$ and max support$\{F_{t+1}\} =$ max support$\{F_t\}$. Iterating forward again, we can show that

$$f_{t+2}(z) = \frac{f_t(z)}{1 - F_t(m_{t+2})},$$

demonstrating that only the initial distribution and the last truncation point are necessary to characterize the distribution at any point in time.

[6] Note that the agent forecasts a deterministic evolution of the productivity distribution. Given $\{m_t\}$, the proof that optimal policy is a sequence of reservation productivities follows from the standard solution techniques of search in a nonstationary economy, as in Lippman and McCall (1976).

[7] The assumption that upgrading firms meet only producing firms yields the clean truncation of the distribution, with no mass of firms perpetually left behind. While this simplifying assumption is added for tractability and exposition, economically, it represents directed search toward only the productive firms. The primary mechanism for growth in this paper is from the endogenous selection of who searches rather than from the "selective sampling" of the right-hand tail, although directed search does increase the growth rate. Appendix G solves and analyzes a version of the model with unconditional draws, and Sec. II.G conducts numerical experiments with this model variation.

So, given the initial condition F_0 and a sequence $\{m_{t+1}\}$, the density generated by this law of motion at t is

$$f_t(z) = \frac{f_0(z)}{1 - F_0(m_t)}. \tag{4}$$

With this law of motion, the firm problem can be simplified to

$$V_t(z) = \max\left\{ z + \frac{1}{1+r_t} V_{t+1}(z),\ \frac{1}{1+r_t} \int_{m_{t+1}}^{\infty} V_{t+1}(z') \frac{f_0(z')}{1 - F_0(m_{t+1})} dz' \right\}. \tag{5}$$

D. Equilibrium Concepts

1. Competitive Equilibrium

DEFINITION 1 (Competitive equilibrium). A *competitive equilibrium* consists of an initial distribution F_0 and sequences of reservation productivity levels, value functions, and interests rates, $\{m_t, V_t(\cdot), r_t\}_{t\geq 0}$, such that given $\{r_t\}$, $\{m_{t+1}\}$ are the reservation productivity levels that solve the firm problem, with $\{V_t(\cdot)\}$ the associated value functions and given $\{m_t\}$, $\{r_t\}$ are consistent with the consumer's intertemporal marginal rate of substitution.

A sequence of reservation productivity levels $\{m_t\}$ fully characterizes the equilibrium. Given the law of motion in equation (4), from m_0, the sequence of distributions is characterized by the initial distribution and the sequence of reservation productivities,

$$F_t(z) = \frac{F_0(z) - F_0(m_t)}{1 - F_0(m_t)}.$$

Aggregate production is defined as $Y_t \equiv \int_{m_{t+1}}^{\infty} z dF_t(z)$.[8] The growth factor, $g_t \equiv Y_{t+1}/Y_t$, may diverge or converge depending on the characteristics of F_0. In general, numerical methods are required to solve for $\{m_t\}$. See online Appendix H for details.

As the goal is to fully analyze the growth mechanism and its dependence on model parameters, in this paper we focus on a BGP equilibrium that allows for analytical solutions.

[8] As in Lagos (2006), aggregate production is calculated as an integral of the output of producing firms and, hence, is dependent on properties of the current distribution and searching decisions. Also as in Lagos's paper, more searchers would result in higher mean productivity at the macro level within a given period, while the period's aggregate production would be lower.

2. BGP Distribution Evolution

A definition of a balanced growth path for scalar variables simply requires geometric growth at a constant rate. Defining a BGP for an evolving distribution requires additional restrictions. As the distribution evolves according to sequential truncation, the support of the distribution is certainly changing, and the shape of the distribution could potentially change in an unrestricted dynamic system. To introduce the idea of scale-invariant distributions, we want a concept that removes the scale and maintains the shape of the distribution.[9]

Definition 2 (Scale invariant). A sequence of distributions, $\{F_t\}$, and scales, $\{m_t\}$, are *scale invariant* if

$$F_t(\tilde{z}m_t) \text{ are identical for all } t \geq 0, \tilde{z} \in [1, \infty). \tag{6}$$

Intuitively, scale invariance requires that all quantiles of the initial distribution expand at the same rate each period as the distribution evolves.[10]

3. Balanced Growth Path

Definition 3 (BGP competitive equilibrium). A *BGP competitive equilibrium* is a competitive equilibrium with a constant $g > 1$ such that support $\{F_0\} = [m_0, \infty)$, $\{F_t, m_t\}$ is scale invariant, and $Y_{t+1} = gY_t$ for all $t \geq 0$.

The initial distribution must have infinite right-tailed support or the economy would not be able to grow indefinitely.[11] Requiring production to grow by the constant factor g and requiring scale invariance restricts the BGP equilibrium to be balanced. Restricting $g > 1$ ensures that the BGP equilibrium has growth.

As is well known, not every utility function is compatible with balanced growth. Restricting the lifetime utility function to being homothetic, increasing, and quasi-concave implies that period utility must be of the constant elasticity of substitution form, representing constant relative risk aversion preferences. Thus, for the remainder of the paper, we use a power utility function, $u(Y) = Y^{1-\gamma}/(1 - \gamma)$, ensuring a constant intertemporal marginal rate of substitution (constant r) and consistency with balanced growth.

[9] An alternative approach is to require that the Lorenz curves of the distribution are identical for all t.

[10] Scale invariance is related to the normalization discussed in online App. F.

[11] A similar assumption for the support of potential improvements is used in papers such as Bental and Peled (1996), Kortum (1997), and Eaton and Kortum (1999). This paper does not model or emphasize the technology frontier, and the majority of searchers end up with only minor improvements. These papers, and other research that emphasizes R&D, provide a better description of the limitless growth of the technology frontier.

E. Solution and Analysis

Existence of an equilibrium is proved by construction via a guess and check strategy. It is straightforward to show that a Pareto distribution as the initial condition, F_0, will fulfill the evolution equilibrium requirements.[12] Thus, let F_0 be Pareto: $F_0(z; m_0, \alpha) = 1 - (m_0/z)^\alpha$ and $f_0(z; m_0, \alpha) = \alpha m_0^\alpha z^{-\alpha-1}$.

First, guess that the minimum of support will grow geometrically at the aggregate growth rate, so $m_t = m_0 g^t$. Second, guess that the value of imitation will grow geometrically at the aggregate growth rate, so, for some constant W, $V_t(z) = m_t W$ for $z \leq m_{t+1}$. These guesses will be verified during the solution process.

The solution strategy is to use these guesses to solve for the constants (W and g) as functions of parameters (m_0, α, β, and γ) and then to verify the guesses and that the solution fulfills all BGP equilibrium requirements. The key to the solution strategy is to use the indifference equations obtained by evaluating the value functions at reservation productivities. The problem is analytically tractable because the expectation of a piecewise-linear function of a Pareto random variable can be calculated explicitly.

Given parameter restrictions that guarantee positive and finite growth, the problem is solved analytically with the guesses and BGP requirements verified.

PROPOSITION 1. If the initial productivity distribution, F_0, is Pareto with $m_0 > 0$ and $\alpha > 1$ and

$$\frac{\alpha - 1}{\alpha} < \beta < \min\left\{\left(\frac{\alpha}{\alpha - 1}\right)^{(\gamma-1)/\alpha}, 1\right\},$$

a BGP competitive equilibrium exists with the following properties:

1. The growth factor is

$$g = \left(\beta \frac{\alpha}{\alpha - 1}\right)^{1/(\gamma-1+\alpha)} > 1.$$

2. The reservation productivity level grows geometrically: $m_{t+1} = g m_t$.
3. The interest rate is $r = (g^\gamma/\beta) - 1$.
4. The mass of imitating firms is constant: $S_t = 1 - g^{-\alpha}$.
5. Output grows geometrically: $Y_t = [\alpha/(\alpha - 1)] g^{1-\alpha} m_t$.

[12] See App. C for a proof that the Pareto distribution is the unique distribution that can satisfy the BGP equilibrium requirements.

6. The value function is piecewise linear, with kinks at $\{m_{t+1}\}$. That is, for all $s \in \mathbb{N}$, for $z \in [m_0 g^{t+s}, m_0 g^{t+s+1}]$,

$$V_t(z) = \frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^s\right] z + \left(\frac{1}{1+r}\right)^s m_0 g^{t+s} W,$$

and for $z \leq m_0 g^t$,

$$V_t(z) = m_0 g^t W.$$

Proof. Given the guess that $m_t = m_0 g^t$, the Pareto distribution can be shown to be scale invariant and to generate constant geometric growth of aggregate production and a constant fraction of searching firms. The remaining task is to solve for constants W and g as a function of parameters, such that the reservation productivity, $m_{t+1} = gm_t$, is optimal for firms.

Using equation (5), plug in f_0 and the guess that $m_{t+1} = gm_t$:

$$V_t(z) = \max\left\{z + \frac{1}{1+r} V_{t+1}(z),\ \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} dz'\right\}. \quad (7)$$

Given that the indifference level of productivity is gm_t, using the guess that the value of imitation grows geometrically, $V_t(gm_t) = m_t W$, gives two equalities:

$$m_t W = gm_t + \frac{1}{1+r} gm_t W \quad (8)$$

$$= \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} dz'. \quad (9)$$

Equating the first of the two equalities in equation (8) provides an equation in W and g,

$$W = \frac{g}{1 - g/(1+r)}. \quad (10)$$

Equating the second equality between equation (8) and equation (9) and splitting the integral at the reservation productivity yields

$$gm_t + \frac{1}{1+r} gm_t W = \frac{1}{1+r} \alpha (gm_t)^\alpha \int_{gm_t}^{g^2 m_t} V_{t+1}(z') z'^{-\alpha-1} dz' + \frac{1}{1+r} \alpha (gm_t)^\alpha \int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} dz'. \tag{11}$$

By the decision rule, firms will search at $t + 1$ if $z \leq g^2 m_t$ with value $gm_t W$,

$$\int_{gm_t}^{g^2 m_t} V_{t+1}(z') z'^{-\alpha-1} dz' = gm_t W \int_{gm_t}^{g^2 m_t} z'^{-\alpha-1} dz' \tag{12}$$

$$= \frac{gm_t W}{\alpha} (gm_t)^{-\alpha} (1 - g^{-\alpha}). \tag{13}$$

By the decision rule, firms will produce at $t + 1$ if $z > g^2 m_t$:

$$\int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} dz' = \int_{g^2 m_t}^{\infty} \left[z' + \frac{1}{1+r} V_{t+2}(z') \right] z'^{-\alpha-1} dz' \tag{14}$$

$$= \frac{1}{\alpha - 1} (g^2 m_t)^{1-\alpha} + \frac{1}{1+r} \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} dz'. \tag{15}$$

Using the indifference equation at $t + 1$, where the reservation productivity is $g^2 m_t$,

$$V_{t+1}(g^2 m_t) = g^2 m_t + \frac{1}{1+r} g^2 m_t W = \frac{1}{1+r} \alpha (g^2 m_t)^\alpha \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} dz'. \tag{16}$$

Combining equations (13), (15), and (16) with equation (11) and simplifying yields

$$(1 + r) g^\alpha = -W + \frac{\alpha}{\alpha - 1} g + g\left(1 + \frac{1}{1+r} W\right). \tag{17}$$

As m_t has dropped out of equations (17) and (10), W and g are not functions of time. Hence, the guess of the functional form $V_t(gm_t) = m_t W$ and the equilibrium requirement that g is constant are confirmed.

Combining equation (17) with (10) and solving for g shows that

$$g = \left(\frac{1}{1+r}\frac{\alpha}{\alpha-1}\right)^{1/(\alpha-1)}. \tag{18}$$

Given this fixed g, substituting $1/(1+r) = \beta g^{-\gamma}$ into equation (18) solves for g as a function of parameters

$$g = \left(\beta\frac{\alpha}{\alpha-1}\right)^{1/(\gamma+\alpha-1)}. \tag{19}$$

The restrictions on parameters come from the requirements that both $g > 1$ and $W > 0$. For $W > 0$, it is necessary and sufficient that $g/(1+r) < 1$. The closed-form equation for the value function is derived in Appendix E by using this solution to the firm problem and working with the value function in sequence space. See Section C in Appendix H for a numerical analysis of the magnitude of g in relation to parameter values. QED

The constraint $\beta > (\alpha-1)/\alpha$ guarantees that the discount factor is high enough, compared to the imitation opportunities, to ensure that firms want to search. If $\gamma > 1$, consumer period utility has an Inada condition, and the upper constraint of $\beta < 1$ restrains growth. Otherwise, the constraint $\beta < [\alpha/(\alpha-1)]^{(\gamma-1)/\alpha}$ ensures that patience or inequality is low enough—compared to the curvature of the utility function—to avoid infinite growth.

Model behavior can be summarized by the derivatives of the growth factor with respect to the parameters. As analytical derivatives are attainable, it is easy to show that $dg/d\beta > 0$, $dg/d\alpha < 0$, and $dg/d\gamma < 0$. The growth factor provided in proposition 1 and the above derivatives succinctly capture the connection between all model parameters and the endogenously determined growth rate. This allows a transparent analysis of the relationship of consumers' preferences and the shape of the productivity distribution to growth and the evolution of the productivity distribution.

As β increases and agents become more patient, growth increases because more value is put on higher future consumption, which yields more technology adoption. As γ increases, $1/\gamma$ decreases, and the firms wish to reflect the consumer's lower intertemporal elasticity of substitution by reducing search to produce more earlier.

A decrease in α corresponds to an increase in productivity inequality. As α decreases, the expected value of a draw increases, as the Pareto distribution has more weight in the tail and less weight around the minimum of the support. Thus, a searching firm is more likely to obtain a higher productivity in an economy with more inequality. Fatter tails incentivize more firms to search and generate higher growth.[13]

F. *Asymptotic Growth Rates*

As was just established, the initial distribution essentially determines the BGP growth rate. When the analysis is expanded outside of BGP equilibria, the initial distribution remains essential in determining asymptotic growth rates.

DEFINITION 4 (Power law). A function $L(z)$ is called slowly varying if $\lim_{m\to\infty} L(mz)/L(m) = 1$ for all $z \geq 0$. A distribution F is called a *power law* if, for $\alpha > 0$, the pdf and cdf have the form $f(z) \propto L(z)z^{-\alpha-1}$ and $F(z) \propto 1 - L(z)z^{-\alpha}$.

Section E in Appendix F shows that for all power laws with $\alpha > 1$,

$$\lim_{m\to\infty} \frac{\mathbb{E}[z|z > m]}{m} = \frac{\alpha}{\alpha - 1} > 1.$$

Alternatively, if a distribution has

$$\lim_{m\to\infty} \frac{\mathbb{E}[z|z > m]}{m} = 1,$$

then it is not a power law and intuitively can be thought of as having an infinite tail parameter (and an infinite number of moments).

PROPOSITION 2. Let the expectation of the truncated distribution exist for all $m < \max \text{support}\{F_0\}$. If, for all t and some $\underline{g} > 1$, $g_t \geq \underline{g}$, then F_0 is a power law and an equilibrium exists where $\lim_{t\to\infty} g_t = g(\alpha)$ as determined by proposition 1. Alternatively, if for F_0

$$\lim_{m\to\infty} \frac{\mathbb{E}[z|z > m]}{m} = 1,$$

then $\lim_{t\to\infty} g_t = 1$.

Proof. See Appendix F, Section E.

From proposition 2, if the initial distribution is a power law, the growth rate is ultimately determined by the tail parameter. The condition that $g_t \geq \underline{g} > 1$ for all t is important because there are power law distributions with no asymptotic growth. For non–power law initial distributions, growth can stop as the expectation of the truncated distri-

[13] An informal way to see this is to note that the mean of the Pareto distribution, $m_t\alpha/(\alpha - 1)$, is decreasing in α, while last period's truncation point, m_t, would also be increasing with more search. Also note, if the proportion of firms searching either grew or shrank monotonically with no limit in (0, 1), then there would be no growth in the limit.

bution gets close to its minimum of support, lowering the returns to search below the cost of forgone production. Initial distributions that lead growth rates to converge to zero and the economy to reach a finite asymptotic size include all with finite support, as well as the standard non–power law distributions, such as the normal, lognormal, and the exponential.

G. *Off-BGP Dynamics Example*

Off-BGP dynamics can highlight the impact of the shape of the productivity distribution on the growth of output. Fixing the tail parameter α, figure 2 shows the evolution of growth for both a Fréchet initial distribution and a right-truncated Pareto initial distribution with bounded support.[14]

For the Fréchet distribution, growth is initially high, as many firms are searching. Over time, the growth rate converges from above to the constant growth rate of a BGP equilibrium with a Pareto initial condition with the same tail index. Intuitively, the Fréchet distribution is like a lognormal distribution with a Pareto tail. Initially, for the same minimum of support as a Pareto, the Fréchet distribution has less mass close to the minimum of support and, thus, a higher expectation leading to a higher expected return to search. After repeated truncations, the Fréchet distribution looks more like the Pareto distribution, with more mass close to the minimum of support, leading to less search and slower growth. This confirms that the same forces that lead fatter tails to generate more growth on the BGP are present in the dynamic equilibrium.

For the right-truncated Pareto distribution, initial output is 5 percent lower than the unbounded Pareto since the right truncation of the initial Pareto distribution removes high–productivity tail firms. Growth rates are consequently lower, starting about 0.37 percent below the unbounded case, as the expectation of the truncated distribution is lower than the expectation of the unbounded distribution. However, even though the distribution has finite upper support and growth will eventually stop, as shown in proposition 2, growth rates drop very slowly over time. While the growth rate is asymptotically zero, starting from growth rates of 2.9 percent, after 50 years, growth has dropped by only 0.47 percentage points and, after 100 years, by 1.1 percentage points. This example shows that with bounded support and reasonably cali-

[14] These examples are roughly calibrated to target a 3.28 percent annual growth rate with $\gamma = 1$ and a Pareto tail parameter of $\alpha = 1.5$. In order to better calibrate growth and interest rates, these dynamic examples are calculated allowing for taxation and search costs similar to those in eq. (23); they are also conducted with firms drawing unconditionally from the entire distribution. Search costs and corporate taxes are calibrated such that an average firm takes 5 years to recoup the cost of upgrading its technology. See App. H for details on the bounded Pareto and Fréchet examples, as well as the numerical solution algorithm.

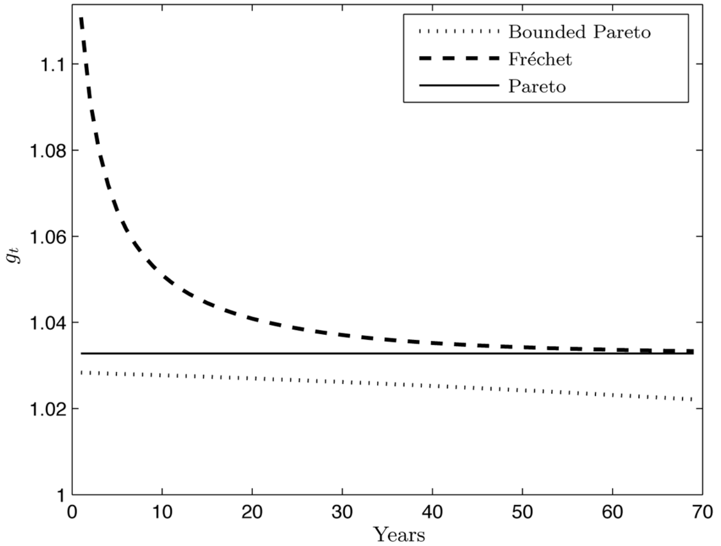

FIG. 2.—Growth transition from Pareto, bounded Pareto, and Fréchet initial conditions

brated growth rates, a far-off finite upper bound on technology can have little impact on short- and medium-term growth dynamics. Economically, at any point in time, a far-off upper bound on technology has very little impact on a firm's choice, as its expected draw is far below the frontier. Thus, for the purposes of studying technology adoption by low-productivity firms, an unbounded Pareto distribution can be a useful approximation of the reality of a finite and growing upper bound on technology.

III. The Planner Problem

As in many other endogenous growth models, in this economy, growth is moderated since agents do not internalize the full social benefit of their private decision to improve. The magnitude of the externality is highlighted by solving a planner problem, where the planner internalizes the social benefit that adoption has on improving the productivity distribution for future adopters.

The planner chooses which firms produce and which upgrade in the economy. For clarity of exposition, the problem is defined recursively, with the distribution $F(z)$ as the aggregate state. It chooses a reservation value, $m'(F)$, below which firms will search. Equivalently, the planner chooses the growth factor $g(F) \geq 1$ such that $m'(F) = g(F)m(F)$, where $m(F) \equiv \min \text{support } \{F\}$. Aggregate production is $Y(F) = \int_{gm(f)}^{\infty} z dF(z)$.

The planner maximizes the lifetime utility of the representative agent, who derives utility from consuming aggregate production:

$$U(F) = \max_{g\geq 1}\left\{\frac{\left[\int_{gm(f)}^{\infty} z dF(z)\right]^{1-\gamma}}{1-\gamma} + \beta U(F')\right\} \tag{20}$$

subject to

$$F'(z) = \frac{F(z) - F(m(F))}{1 - F(gm(F))}.$$

With a particular F_0, the problem can be solved in closed form by guessing and verifying that $U(f) = -A(m(F))^{1-\gamma}$ for some $A > 0$. For a Pareto $F_0(z)$, the optimal choice of g is shown to be independent of the aggregate state, F, and the first-order condition can be used to solve for A and g.

Proposition 3. If the initial productivity distribution, F_0, is Pareto with $m_0 > 0$, $(\alpha - 1)/\alpha < \beta < 1$, $\gamma > 1$, and $\alpha > \gamma/(\gamma - 1)$, then the planner chooses a constant $g > 1$ each period such that

$$g = \left(\beta\frac{\alpha}{\alpha - 1}\right)^{1/(\gamma-1)}. \tag{21}$$

Proof. See Appendix A.

We can summarize the sensitivity of growth to parameter values with the following derivatives: $dg/d\beta > 0$, $dg/d\alpha < 0$, and $dg/d\gamma < 0$. The explanation for $dg/d\beta$ and $dg/d\gamma$ is identical to that of the competitive equilibrium. Analysis of α is given below.

Since a lower α provides a stronger search incentive, the parameter requirement that $\alpha > \gamma/(\gamma - 1)$ ensures that the value of search is not sufficiently high to dominate the curvature of the utility function.[15]

Comparing the planner to the competitive equilibrium.—The growth factors in equations (19) and (21) are similar, but the competitive equilibrium growth factor contains an additional α in the exponent. This α in equation (19) decreases the growth rate and can be seen as capturing the growth penalty from adoption externalities. Firms do not internalize the

[15] Note that the constraints on the planner's problem are more restrictive than those of the competitive equilibrium to prevent infinite growth. For example, the planner's problem solution does not exist for $\gamma \leq 1$.

value of improving future search opportunities when making their search decisions.

The ratio of the planner's first-best g to the competitive equilibrium growth factor is

$$\frac{g_{fb}}{g_{ce}} = \frac{\left(\beta\frac{\alpha}{\alpha-1}\right)^{1/(\gamma-1)}}{\left(\beta\frac{\alpha}{\alpha-1}\right)^{1/(\gamma-1+\alpha)}} = \left(\beta\frac{\alpha}{\alpha-1}\right)^{\alpha/[(\gamma-1)(\gamma-1+\alpha)]}. \tag{22}$$

Note that $g_{fb}/g_{ce} > 1$ and that $d(g_{fb}/g_{ce})/d\alpha < 0$. Hence, the planner always chooses higher growth, as it internalizes the effect of search on the evolution of the productivity distribution. The efficiency wedge between the planner's growth factor and that of the competitive equilibrium is higher when productivity inequality is greater. Since the inequality parameter α also summarizes the strength of the externality, the larger the free-riding incentive, the larger the wedge.

IV. The Constrained Planner: Subsidies and Taxes

A natural alternative to the unconstrained planner's problem is to constrain the planner to operate within a competitive equilibrium and to use taxes and subsidies to indirectly influence the search decision. A trade-off exists for the constrained planner: Subsidies encourage search through a decrease in the search cost, but taxes discourage search by decreasing the future value of higher production. Hence, there exists a potential distortion of growth through the extensive margin.

A. Firm Problem with Linear Taxes

A firm solves its problem given its forecast of a constant taxation and subsidy policy of the planner (τ and ς, respectively):

$$\begin{aligned} V_t(z;\tau,\varsigma) = \max\Bigg\{ &(1-\tau)z + \frac{1}{1+r_t}V_{t+1}(z), \\ &\varsigma z + \frac{1}{1+r_t}\int V_{t+1}(z')\frac{f_t(z')}{1-F_t(m_{t+1})}dz'\Bigg\}. \end{aligned} \tag{23}$$

The resulting growth factor is

$$g(\tau, \varsigma) = \left\{ \left[\frac{1-\tau-\varsigma}{1-\tau-\varsigma\left(\frac{r}{1+r} + \frac{1}{1+r}\frac{\alpha}{\alpha-1}\right)} \right] \frac{1}{1+r}\frac{\alpha}{\alpha-1} \right\}^{1/(\alpha-1)} > 1. \tag{24}$$

The proof closely follows that of the competitive equilibrium without taxation presented in Section II.E and is omitted for brevity (see App. D for a complete derivation). The analysis of this competitive equilibrium is similar to that in Section II.E. As expected, $dg/d\tau < 0$ and $dg/d\varsigma > 0$, capturing the meaningful trade-off between taxes and subsidies.

B. *Constrained Planner Problem with Optimal Linear Taxes*

The planner chooses ς and τ to maximize the utility of the representative consumer subject to a balanced-budget constraint. Additionally, the planner is constrained to work within the existing structure of the economy, and, thus, the growth rate generated must be consistent with the solution to the competitive equilibrium of the firms given in Section IV.A. The solution requires verifying that τ and ς are constant in order to validate the forecasts of the firms. The constrained planner solves

$$U(F) = \max_{0 \le \tau < 1,\, 0 \le \varsigma \le 1-\tau} \left\{ \frac{\left[\int_{gm(f)}^{\infty} z dF(z)\right]^{1-\gamma}}{1-\gamma} + \beta U(F') \right\} \tag{25}$$

subject to

$$g = \left\{ \left[\frac{1-\tau-\varsigma}{1-\tau-\varsigma\left(\frac{r}{1+r} + \frac{1}{1+r}\frac{\alpha}{\alpha-1}\right)} \right] \frac{1}{1+r}\frac{\alpha}{\alpha-1} \right\}^{1/(\alpha-1)}, \tag{26}$$

$$F'(z) = \frac{F(z) - F(m(F))}{1 - F(gm(F))}, \tag{27}$$

$$\frac{1}{1+r} = \beta g^{-\gamma}, \tag{28}$$

$$\tau \int_{gm(F)}^{\infty} z dF(z) = \varsigma \int_{m(F)}^{gm(F)} z dF(z), \tag{29}$$

$$\tau, \varsigma \text{ confirm agent forecasts.} \tag{30}$$

PROPOSITION 4. If the initial productivity distribution, F_0, is Pareto with $m_0 > 0$, $(\alpha - 1)/\alpha < \beta < 1$, $\gamma > 1$, and $\alpha > \gamma/(\gamma - 1)$, then a feasible solution exists such that the constrained planner uses τ and ς to achieve the first-best solution of the unconstrained planner: $g = \{\beta[\alpha/(\alpha - 1)]\}^{1/(\gamma-1)}$.

Proof. See Appendix B.

Thus, subsidizing low-productivity firms to upgrade their productivity increases growth.

V. Conclusion

This paper contributes an analytically tractable mechanism for analyzing growth and the evolution of the productivity distribution, with both the evolution of the productivity distribution and the technology adoption decision jointly endogenously determined in equilibrium. Thus, we can analyze the effect the productivity distribution has on adoption incentives, the effect of adoption behavior in generating the productivity distribution, and the corresponding growth implications of this feedback loop. We develop a solution technique that obtains closed-form expressions for all equilibrium objects—including the growth factor—as a function of intrinsic parameters.

The closed-form solutions for both the distribution and the growth rate clearly illuminate the forces at work in the model that govern the strength of the growth mechanism. In particular, higher productivity inequality leads to more growth by fueling risk taking. The greater the dispersion of opportunities, the more firms are willing to take a risk, even if they might fail and need to continue costly searching.

The competitive equilibrium is inefficient, as firms do not internalize how their search decisions affect the evolution of the distribution. A planner that considers this externality, and chooses which firms search, achieves higher growth. This result suggests a government policy of using a tax-funded program to subsidize the upgrading of unproductive enterprises. Although a trade-off exists for the constrained planner, with subsidies encouraging search and taxation discouraging search, the first best can be achieved.

Appendix A

Solution to Planner's Problem: Proof

To prove proposition 3, guess and verify a functional form for the value function and use the first-order conditions to determine coefficients. To ensure that the growth rate is constant, it must be shown that with the guess, the planner's choice of g is independent of the state F. Finally, given the coefficient values, we need to show that the objective function in the maximization problem is globally concave to confirm that our choice of g is the global maximum.

The current minimum of support, m, summarizes the aggregate state. Substitute the Pareto F_0 into equation (20) and evaluate the integral

$$U(m) = \max_{g \geq 1} \left\{ \frac{\left(\frac{\alpha}{\alpha - 1} g^{1-\alpha} m \right)^{1-\gamma}}{1-\gamma} + \beta U(gm) \right\}. \tag{A1}$$

Guess that the form of the planner value function is $U(m) = -Am^{1-\gamma}$, where $A > 0$.

Substitute the guess into equation (A1) and define constant $Q \equiv [\alpha/(\alpha-1)]^{1-\gamma}$:

$$-Am^{1-\gamma} = \max_{g \geq 1} \left\{ Q \frac{g^{(\alpha-1)(\gamma-1)}}{1-\gamma} m^{1-\gamma} - \beta A g^{1-\gamma} m^{1-\gamma} \right\}. \tag{A2}$$

Using $m > 0$, divide by $m^{1-\gamma}$:

$$-A = \max_{g \geq 1} \left\{ Q \frac{g^{(\alpha-1)(\gamma-1)}}{1-\gamma} - \beta A g^{1-\gamma} \right\}. \tag{A3}$$

Since m has dropped out of the expression, A and g will not be functions of m, confirming our guess on the functional form of $U(m)$ and ensuring that g is constant.

Assume, for now, that g is interior and take the first-order condition

$$0 = g^{\alpha(\gamma-1)}(\alpha - 1)Q - \beta A(\gamma - 1). \tag{A4}$$

Solving for A,

$$A = \frac{Q(\alpha-1)g^{\alpha(\gamma-1)}}{\beta(\gamma-1)}. \tag{A5}$$

Substitute equation (A5) into equation (A3), dropping the max since this g is the argmax. Solving for g,

$$g = \left(\beta \frac{\alpha}{\alpha - 1} \right)^{1/(\gamma-1)}. \tag{A6}$$

Note that $\beta > (\alpha-1)/\alpha$ is necessary and sufficient for $g > 1$.

Define the maximization problem's objective, $\Omega(g)$, from equation (A3) to get

$$\Omega(g) \equiv Q \frac{g^{(\alpha-1)(\gamma-1)}}{1-\gamma} - \beta A g^{1-\gamma}. \tag{A7}$$

Given the parameter restrictions that ensure that g is interior, to show that the optimal g found in equation (A6) is the global maximum, it is sufficient to show global strict concavity of $\Omega(g)$ for any fixed $A > 0$ and variable $g > 1$. Since $\alpha > 1$, $\gamma > 1$, and $A > 0$, a necessary and sufficient condition for $d^2\Omega(g)/dg^2 < 0$ for all $A > 0$ and $g > 1$ is $\alpha > \gamma/(\gamma - 1)$. QED

Appendix B

Constrained Planner Problem: Proof

This proof of proposition 4 shows that the constrained planner can achieve the first-best solution of the unconstrained planner with constant τ, ς. From equation (26),

$$g = \left\{ \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(\frac{r}{1+r} + \frac{1}{1+r}\frac{\alpha}{\alpha - 1}\right)} \right] \frac{1}{1+r}\frac{\alpha}{\alpha - 1} \right\}^{1/(\alpha-1)}. \tag{B1}$$

Substituting in $1/(1 + r) = \beta g^{-\gamma}$ and $r/(1 + r) = 1 - \beta g^{-\gamma}$ from the consumer problem,

$$g = \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(1 + \frac{1}{\alpha - 1}\beta g^{-\gamma}\right)} \beta \frac{\alpha}{\alpha - 1} \right]^{1/(\gamma - 1 + \alpha)}. \tag{B2}$$

The first-best solution, achieved by the planner in equation (21), is

$$g_{fb} = \left(\beta \frac{\alpha}{\alpha - 1} \right)^{1/(\gamma - 1)}. \tag{B3}$$

Assume that the decentralized planner is able to achieve the first best, $g = g_{fb}$, and substitute

$$g = \left[\frac{1 - \tau - \varsigma}{1 - \tau - \varsigma\left(1 + \frac{1}{\alpha} g^{\gamma - 1} g^{-\gamma}\right)} g^{\gamma - 1} \right]^{1/(\gamma - 1 + \alpha)}. \tag{B4}$$

Solving for $\varsigma(\tau)$,

$$\varsigma = \frac{g\alpha(g^{\alpha} - 1)}{g^{\alpha}(1 + g\alpha) - g\alpha}(1 - \tau). \tag{B5}$$

Substituting for $F_t(z)$, the budget constraint of the constrained planner is

$$\tau = \varsigma(g^{\alpha - 1} - 1). \tag{B6}$$

Equations (B5) and (B6) are a system in (τ, ς). Solving yields equations for ς and τ in terms of model intrinsics that satisfy the budget constraint by construction:

$$\varsigma = \frac{g\alpha(1 - g^{-\alpha})}{1 + \alpha(g^{\alpha} - 1)}, \tag{B7}$$

$$\tau = \frac{g\alpha(1 - g^{-\alpha})(g^{\alpha-1} - 1)}{1 + \alpha(g^{\alpha} - 1)}. \tag{B8}$$

Both $\varsigma > 0$ and $\tau > 0$ follow from $g^{\alpha} > 1$. To check the final requirement that $\varsigma + \tau < 1$, add equations (B7) and (B8):

$$\varsigma + \tau = 1 - \frac{1}{\alpha(g^{\alpha} - 1) + 1}, \tag{B9}$$

$g^{\alpha} > 1$ and, hence, both $\varsigma + \tau < 1$ and $\varsigma + \tau > 0$ hold, proving that an interior solution for the subsidies and taxes can achieve the first-best solution. QED

Appendix C

Uniqueness of the Pareto Distribution

The following proposition proves that under the scale invariance requirement, the Pareto distribution is the unique initial condition that can fulfill the BGP equilibrium requirements.

Proposition 5. Given the maintained assumptions, the Pareto distribution is the unique distribution that can satisfy the BGP equilibrium requirements.

Proof. Assume for clarity of exposition that $m_0 = 1 = \min \text{support}\{F_0\}$ (without loss of generality, or use the normalized version of the evolution developed in App. F). Combining the law of motion for the density and the scale invariance equation (in density form), (4) and (6), and rearranging,

$$f_0(\tilde{z}m_t) = \frac{1 - F_0(m_t)}{m_t} f_0(\tilde{z}) \propto f_0(\tilde{z}), \tag{C1}$$

a power law. Differentiate both sides of equation (C1) with respect to m_t:

$$\frac{df_0(\tilde{z}m_t)}{d\tilde{z}}\tilde{z} = \frac{F_0(m_t) - m_t F_0'(m_t) - 1}{m_t^2} f_0(\tilde{z}). \tag{C2}$$

Evaluate at $m_t = 1$ and use that for any initial condition with a minimum of support 1, $F_0'(1) \equiv f_0(1)$ and $F_0(1) = 0$:

$$\frac{df_0(\tilde{z})}{d\tilde{z}}\tilde{z} = [-f_0(1) - 1] f_0(\tilde{z}). \tag{C3}$$

This is an ordinary differential equation in $\tilde{z}$. If we arbitrarily choose the initial condition to be $f_0(1) = \alpha$, then the particular solution of the differential equation matches the parameterization for a Pareto(1, α):

$$f_0(\tilde{z}) = \alpha\tilde{z}^{-\alpha-1}. \tag{C4}$$

Hence, for any initial condition with support $[1, \infty)$, the only solution is the Pareto distribution. This can be renormalized for an arbitrary m_0 to give the Pareto density as the unique initial condition: $f_0(z) = \alpha m_0^{\alpha} z^{-\alpha-1}$. QED

References

Aghion, P., and P. Howitt. 1992. "A Model of Growth through Creative Destruction." *Econometrica* 60 (2): 323–51.

Alvarez, F. E., F. J. Buera, and R. E. Lucas Jr. 2008. "Models of Idea Flows." Working Paper no. 14135, NBER, Cambridge, MA.

Aw, B. Y., X. Chen, and M. J. Roberts. 2001. "Firm-Level Evidence on Productivity Differentials and Turnover in Taiwanese Manufacturing." *J. Development Econ.* 66 (1): 51–86.

Bental, B., and D. Peled. 1996. "The Accumulation of Wealth and the Cyclical Generation of New Technologies: A Search Theoretic Approach." *Internat. Econ. Rev.* 37 (3): 687–718.

Eaton, J., and S. Kortum. 1999. "International Technology Diffusion: Theory and Measurement." *Internat. Econ. Rev.* 40 (3): 537–70.

Eeckhout, J., and B. Jovanovic. 2002. "Knowledge Spillovers and Inequality." *A.E.R.* 92 (5): 1290–1307.

Evenson, R. E., and Y. Kislev. 1976. "A Stochastic Model of Applied Research." *J.P.E.* 84 (2): 265–82.

Gabaix, X. 2009. "Power Laws in Economics and Finance." *Ann. Rev. Econ.* 1: 255–94.

Grossman, G. M., and E. Helpman. 1991. "Quality Ladders in the Theory of Growth." *Rev. Econ. Studies* 58 (1): 43–61.

Jones, C. I. 2005. "The Shape of Production Functions and the Direction of Technical Change." *Q.J.E.* 120 (2): 517–49.

Jovanovic, B., and R. Rob. 1989. "The Growth and Diffusion of Knowledge." *Rev. Econ. Studies* 56 (4): 569–82.

Kortum, S. S. 1997. "Research, Patenting, and Technological Change." *Econometrica* 65 (6): 1389–1420.

Lagos, R. 2006. "A Model of TFP." *Rev. Econ. Studies* 73 (4): 983–1007.

Lippman, S. A., and J. J. McCall. 1976. "Job Search in a Dynamic Economy." *J. Econ. Theory* 12 (3): 365–90.

Lucas, R. E., Jr. 2009. "Ideas and Growth." *Economica* 76:1–19.

Lucas, R. E., Jr., and B. Moll. 2014. "Knowledge Growth and the Allocation of Time." *J.P.E.* 122 (1): xx–xxx.

Luttmer, E. G. J. 2007. "Selection, Growth, and the Size Distribution of Firms." *Q.J.E.* 122 (3): 1103–44.

———. 2011. "Technology Diffusion and Growth." *J. Econ. Theory* 147 (2): 602–22.

Romer, P. M. 1990. "Endogenous Technological Change." *J.P.E.* 98 (5): 71–102.

 DOI: 10.1086/674362

Appendix D from Perla and Tonetti, "Equilibrium Imitation and Growth" (JPE, vol. 122, no. 1, p. 25)

Solving for the BGP Equilibrium

References with no prefix refer to this technical appendix, while those with a prefix of PT refer to equations in the main text.

A. The Firm Problem

This version of the firm problem includes taxes and subsidies and is featured in the constrained planner problem found in Section PT.IV. When taxes and subsidies are zero, that is, $\tau = \varsigma = 0$, this becomes the firm problem developed in Section PT.II.C and throughout the rest of the main paper.

The following extends the firm's problem in equation PT.(2) with a constant proportional tax on production, $0 \leq \tau < 1$, and a constant proportional search subsidy, $0 \leq \varsigma < 1 - \tau$. Although searchers do not produce, they may receive positive period profits from the search subsidy, depending on the value of ς:[16]

$$V_t(z) = \max\left\{(1-\tau)z + \frac{1}{1+r_t}V_{t+1}(z), \varsigma z + \frac{1}{1+r_t}\int_{m_{t+1}}^{\infty} V_{t+1}(z')\frac{f_t(z')}{1-F_t(m_{t+1})}dz'\right\}. \tag{D1}$$

Define the *gross* value of search, before any costs/subsidies, at time t as

$$W_t \equiv \frac{1}{1+r_t}\int_{m_{t+1}}^{\infty} V_{t+1}(z')\frac{f_t(z')}{1-F_t(m_{t+1})}dz'. \tag{D2}$$

Define the *net* value of search as $\varsigma z + W_t$.

B. Guesses to Be Verified

Make the following guesses, which will be verified:

1. The Pareto distribution will fulfill the BGP conditions, including scale invariance as described in Section PT.II.D.2. Given an initial pdf, $f_0(z; m_0, \alpha) = \alpha m_0^{\alpha} z^{-\alpha-1}$, the distribution will evolve according to the truncation law of motion in equation PT.(4). Given this law of motion and initial distribution, $f_t(z) = \alpha m_t^{\alpha} z^{-\alpha-1}$.
2. The reservation productivity will grow geometrically: $m_{t+1} = g m_t$.
3. The *gross* value of search grows geometrically from some constant W. Hence, the *net* value of search is affine in m_t (or linear if $\varsigma = 0$):

$$V_t(z) = m_t W + \varsigma z \quad \text{for } m_t \leq z \leq g m_t. \tag{D3}$$

No guesses or assumptions are made on the structure or linearity of $V_t(z)$ for $z > g m_t$.[17] The solution methodology, following the search literature, solves for firms' optimal policies without needing to solve for V_t.

It is straightforward to prove that the Pareto distribution fulfills the BGP equilibrium requirements, such as scale invariance in equation PT.(6), for any m_0. Moreover, given $m_{t+1} = g m_t$, it can be shown that $Y_{t+1} = g Y_t$ for all m_t.

To verify the last two guesses, it suffices to solve for constants g and W that are not a function of m_t.

[16] Another interpretation of ς is that only a fraction, $1 - \varsigma$, of production is necessary to pay the cost of search. Additionally, the model can be solved with proportional costs rather than subsidies, $-(1-\tau) < \varsigma < 0$, if an external frictionless market is assumed to exist for firms to finance these additional search costs.

[17] In fact, $V_t(z)$ will always be nonlinear because of the z-dependent option value of future search. Appendix E uses the solution to the firm problem to solve for the value function explicitly. Equation (E7) provides an equation and economic interpretation for $V_t(z)$.

C. Detailed Algebra for BGP Proof

Given that $Y_{t+1} = gY_t$, the interest rate is constant: $r = (g^\gamma/\beta) - 1$. Insert the Pareto $f_t(\cdot)$ and $m_{t+1} = gm_t$ into equation (D1) to obtain

$$V_t(z) = \max\left\{(1-\tau)z + \frac{1}{1+r}V_{t+1}(z),\ \varsigma z + \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}dz'\right\}. \tag{D4}$$

Note that with the guess that $m_{t+1} = gm_t$, the indifference level of productivity at time t is gm_t. Thus,

$$V_t(gm_t) = (1-\tau)gm_t + \frac{1}{1+r}V_{t+1}(gm_t) \tag{D5}$$

$$= \varsigma gm_t + \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}dz'. \tag{D6}$$

Using the guess on the affine value of search from equation (D3) with equations (D5) and (D6) gives two equalities:

$$m_tW + \varsigma gm_t = (1-\tau)gm_t + \frac{1}{1+r}(gm_tW + \varsigma gm_t), \tag{D7}$$

$$(1-\tau)gm_t + \frac{1}{1+r}(gm_tW + \varsigma gm_t) = \varsigma gm_t + \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}dz'. \tag{D8}$$

Use equation (D7) to obtain one equation in W and g:

$$W = \frac{\left(1-\tau-\frac{r}{1+r}\varsigma\right)g}{1-g/(1+r)}. \tag{D9}$$

Note that m_t has dropped out of the equation, which is part of the verification that W and g are constants that are independent of the scale of the economy. Rearrange equation (D8) and split the integral at the indifference point for $t+1$:

$$\begin{aligned}\left(1-\tau-\frac{r}{1+r}\varsigma\right)gm_t + \frac{1}{1+r}gm_tW = {} & \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{g^2m_t} V_{t+1}(z')z'^{-\alpha-1}dz' \\ & + \frac{1}{1+r}\alpha(gm_t)^\alpha \int_{g^2m_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}dz'.\end{aligned} \tag{D10}$$

By the decision rule, firms will search at $t+1$ if $z' \le g^2m_t$ with value $gm_tW + \varsigma z'$. Thus,

$$\begin{aligned}\int_{gm_t}^{g^2m_t} V_{t+1}(z')z'^{-\alpha-1}dz' &= \int_{gm_t}^{g^2m_t} (gm_tW + \varsigma z')z'^{-\alpha-1}dz' \\ &= \frac{\varsigma gm_t}{\alpha-1}(gm_t)^{-\alpha}(1-g^{1-\alpha}) + \frac{gm_tW}{\alpha}(gm_t)^{-\alpha}(1-g^{-\alpha}) \\ &= gm_t(gm_t)^{-\alpha}\left[\frac{\varsigma}{\alpha-1}(1-g^{1-\alpha}) + \frac{W}{\alpha}(1-g^{-\alpha})\right].\end{aligned} \tag{D11}$$

By the decision rule, firms will produce at $t+1$ if $z' > g^2 m_t$. Thus,

$$\int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} dz' = \int_{g^2 m_t}^{\infty} \left[(1-\tau) z' + \frac{1}{1+r} V_{t+2}(z') \right] z'^{-\alpha-1} dz' \tag{D12}$$

$$= \frac{1}{\alpha - 1}(1-\tau)(g^2 m_t)^{1-\alpha} + \frac{1}{1+r} \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} dz'. \tag{D13}$$

Using the indifference equation at $t+1$, where the reservation productivity is $g^2 m_t$, yields

$$V_{t+1}(g^2 m_t) = (1-\tau) g^2 m_t + \frac{1}{1+r}(g^2 m_t W + \varsigma g^2 m_t) \tag{D14}$$

$$= \varsigma g^2 m_t + \frac{1}{1+r} \alpha (g^2 m_t)^{\alpha} \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} dz'. \tag{D15}$$

Using the equality between equations (D14) and (D15) and rearranging for the integral yields

$$\frac{1}{1+r} \alpha (g m_t)^{\alpha} \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} dz' = g^{-\alpha} g^2 m_t \left(1 - \tau - \frac{r}{1+r}\varsigma + \frac{1}{1+r} W \right). \tag{D16}$$

Note that equation (D16) gives the second part of the integral in equation (D13). Combining equations (D11), (D13), and (D16) with equation (D10) provides an equation independent of value functions:

$$\begin{aligned} &\left(1 - \tau - \frac{r}{1+r}\varsigma\right) g m_t + \frac{1}{1+r} g m_t W \\ &= \frac{1}{1+r} \alpha (g m_t)^{\alpha} g m_t (g m_t)^{-\alpha} \left[\frac{\varsigma}{\alpha - 1}(1 - g^{1-\alpha}) + \frac{W}{\alpha}(1 - g^{-\alpha}) \right] \\ &+ \frac{1}{1+r} \alpha (g m_t)^{\alpha} \frac{1}{\alpha - 1}(1-\tau)(g^2 m_t)^{1-\alpha} \\ &+ \frac{1}{1+r} g^{-\alpha} g^2 m_t \left(1 - \tau - \frac{r}{1+r}\varsigma + \frac{1}{1+r} W \right). \end{aligned} \tag{D17}$$

Dividing by $g m_t$ and simplifying shows that

$$\begin{aligned} 1 - \tau - \frac{r}{1+r}\varsigma + \frac{1}{1+r} W &= \frac{1}{1+r}\frac{\alpha}{\alpha-1}\varsigma - \frac{1}{1+r}\frac{\alpha}{\alpha-1}\varsigma g g^{-\alpha} \\ &+ \frac{1}{1+r} W - \frac{1}{1+r} W g^{-\alpha} \\ &+ \frac{1}{1+r}\frac{\alpha}{\alpha-1}(1-\tau) g g^{-\alpha} \\ &+ \frac{1}{1+r} g g^{-\alpha}\left(1 - \tau - \frac{r}{1+r}\varsigma\right) + \frac{1}{1+r} g^{-\alpha}\frac{g}{1+r} W. \end{aligned} \tag{D18}$$

Multiplying by $g^{\alpha}(1+r)$ and rearranging yields a second equation in W and g:

$$\begin{aligned} &\left[(1-\tau)(1+r) - \varsigma\left(r + \frac{\alpha}{\alpha - 1} \right) \right] g^{\alpha} \\ &= g\left(1 - \tau + (1-\tau)\frac{\alpha}{\alpha-1} - \varsigma\frac{\alpha}{\alpha-1} - \varsigma\frac{r}{r+1} \right) - \left(1 - \frac{g}{1+r} \right) W. \end{aligned} \tag{D19}$$

Substituting for W from equation (D9) and simplifying gives

$$\left[(1-\tau)(1+r)-\varsigma\left(r+\frac{\alpha}{\alpha-1}\right)\right]g^{\alpha}=g\frac{\alpha}{\alpha-1}(1-\tau-\varsigma). \tag{D20}$$

Solving for g, we have shown that

$$g=\left\{\left[\frac{1-\tau-\varsigma}{1-\tau-\varsigma\left(\frac{r}{1+r}+\frac{1}{1+r}\frac{\alpha}{\alpha-1}\right)}\right]\frac{1}{1+r}\frac{\alpha}{\alpha-1}\right\}^{1/(\alpha-1)}. \tag{D21}$$

As m_t has dropped out of equations (D21) and (D9), the guesses in Section B of the functional form $V_t(z)=Wm_t+\varsigma z$ for $m_t \leq z \leq gm_t$ and $m_{t+1}=gm_t$ for constant W and g are verified. Note that W and g are not functions of time or the minimum of support of f_0. Intuitively, this means that the growth rate is independent of the initial scale of the economy, m_0, and inductively it is independent of the scale of the economy for any t since $m_t=m_0g^t$. Equation (D21) can be compared to the solution without taxes or subsidies presented in equation PT.(18).

To solve for g entirely in terms of model intrinsics, the growth rate needs to be solved as a system of equations with the interest rate given by

$$\frac{1}{1+r}=\beta g^{-\gamma}. \tag{D22}$$

Direct substitution of this interest rate into equation (D21) yields an implicit expression for g in terms of model parameters. For a general α, this implicit equation does not appear to always have an explicit analytical formula for g, but it can be solved explicitly if $\varsigma=0$ or $\gamma=0$. Note that proportional taxes do not distort growth rates in the absence of subsidies. For $\varsigma=0$, as shown in proposition PT.1,

$$g=\left(\beta\frac{\alpha}{\alpha-1}\right)^{1/(\gamma-1+\alpha)}. \tag{D23}$$

Parameter constraints are needed to ensure that $g>1$ and $W>0$, as can be seen in equations (D9) and (D21). Given an equilibrium r, the following are sufficient:[18] (1) $\varsigma+\tau<1$, (2)

$$1-\tau-\varsigma\left(\frac{r}{1+r}+\frac{1}{1+r}\frac{\alpha}{\alpha-1}\right)>0,$$

and (3)

$$\frac{1}{1+r}\frac{\alpha}{\alpha-1}>1.$$

[18] For the specific case of $\varsigma=0$, proposition PT.1 gives, in closed form, the necessary and sufficient parameter constraints in terms of intrinsics.

Appendix E from Perla and Tonetti, "Equilibrium Imitation and Growth" (JPE, vol. 122, no. 1, p. 25)

Solving for the Value Function

The firm's optimal policy is to search at time t if and only if its idiosyncratic productivity is below the reservation productivity threshold m_{t+1}. Define the number of periods a firm with productivity z at time t waits before searching as

$$\xi_t(z) \equiv \arg\min_{s \in \mathbb{N}} \{z \leq m_{t+1+s}\}. \tag{E1}$$

For example, if a firm at time t has productivity $z \leq m_{t+1}$, then the firm searches immediately and $\xi_t(z) = 0$. With a productivity of $m_{t+1} < z \leq m_{t+2}$, then the firm waits one period, and so forth. If, for some s, the firm has productivity $z = m_{t+1+s}$, then the firm is indifferent between waiting for s and $s + 1$ periods before searching.

Recall that the gross value of search at period t (before any costs/subsidies) is

$$W_t = \frac{1}{1 + r_t} \int_{m_{t+1}}^{\infty} V_{t+1}(z') \frac{f_t(z')}{1 - F_t(m_{t+1})} dz'. \tag{E2}$$

In sequential space, given a competitive equilibrium—F_0 and $\{m_t, V_t(\cdot), r_t\}$—and the corresponding $\{W_t, \xi_t(\cdot)\}$, the value function of a firm with productivity z at time t is the discounted sum of production until search plus the net value of search at time $t + \xi_t(z)$.[19] Noting that $W_{t+\xi_t(z)}$ already contains the discount term $1/[1 + r_{t+\xi_t(z)}]$,

$$V_t(z) = (1 - \tau)z \sum_{s=0}^{\xi_t(z)-1} \left(\prod_{i=0}^{s-1} \frac{1}{1 + r_{t+i}} \right) + \varsigma z \left[\prod_{i=0}^{\xi_t(z)-1} \frac{1}{1 + r_{t+i}} \right] + \left[\prod_{i=0}^{\xi_t(z)-1} \frac{1}{1 + r_{t+i}} \right] W_{t+\xi_t(z)}. \tag{E3}$$

If the interest rate is constant (e.g., if $\gamma = 0$ or if the economy is on a BGP), then the expression can be simplified further:

$$V_t(z) = \left\{ (1 - \tau) \frac{1 + r}{r} \left[1 - \left(\frac{1}{1 + r} \right)^{\xi_t(z)} \right] + \varsigma \left(\frac{1}{1 + r} \right)^{\xi_t(z)} \right\} z + \left(\frac{1}{1 + r} \right)^{\xi_t(z)} W_{t+\xi_t(z)} \tag{E4}$$

$$= (1 - \tau) \frac{1 + r}{r} z + \left(\frac{1}{1 + r} \right)^{\xi_t(z)} \left[\varsigma z + W_{t+\xi_t(z)} - (1 - \tau) \frac{1 + r}{r} z \right], \tag{E5}$$

where the first term of (E5) is the value of production and the second term is the option value of search.

At time t, this function is piecewise linear with kinks at $\{m_s\}$ for all $s \geq t$. Where $m_s < z < m_{s+1}$, the slope of the value function in z is the present discounted value of posttax production for $\xi_t(z) - 1$ periods plus the value of the search subsidy discounted $\xi_t(z)$ periods. In equation (E5), this is interpreted as the value of production in perpetuity plus the option value of search. The option value of search is the value of receiving, at $\xi_t(z)$ periods in the future, the subsidy and the expected value of a new productivity draw minus the lost value of producing with z in perpetuity after the $\xi_t(z)$ periods.

It can be shown that for a given t, the option value of search is decreasing in z and asymptotically zero, since for large z the search option is executed far in the future. From equation (E5), as $\xi \to \infty$, $[1/(1 + r)]^{\xi_t(z)} \to 0$. Hence, as long as $W_{t+\xi_t(z)}$ does not grow too fast, the option value of search goes to zero as the waiting time goes to infinity.[20] This condition in the BGP will be fulfilled if in equilibrium $1 + r > g$. Therefore, from (E5), the value function is approximately

[19] The convention used is that for $b < a$, $\sum_a^b = 0$ and $\prod_a^b = 1$.

[20] Or, equivalently, as z goes to infinity since $\xi_t(z)$ is an increasing function.

linear and independent of t for very large z relative to the current minimum of support m_t:

$$V_t(z) \approx (1-\tau)\frac{1+r}{r}z \quad \text{for } z \gg m_t. \tag{E6}$$

Equation (E4) can be simplified further along the BGP, along which $m_t = m_0 g^t$ and $W_t = m_0 g^t W$. Substituting, the value function on the BGP is defined piecewise for all $s \in \mathbb{N}$ as

$$v_t(z) = \begin{cases} \left\{(1-\tau)\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^s\right] + \varsigma\left(\frac{1}{1+r}\right)^s\right\}z + \left(\frac{1}{1+r}\right)^s m_0 g^{t+s} W & \text{for } z \in [m_0\, g^{t+s},\ m_0 g^{t+s+1}] \\ \varsigma z + m_0 g^t W & \text{for } z \le m_0 g^t. \end{cases} \tag{E7}$$

 DOI: 10.1086/674362

Appendix F from Perla and Tonetti, "Equilibrium Imitation and Growth" (JPE, vol. 122, no. 1, p. 25)

Normalization and Stationarity

A normalized version of the problem can be solved numerically for arbitrary initial conditions.[21]

A. Normalization Definitions

Given an initial condition $f_0(z) \equiv f(z)$ and the optimal reservation productivities that characterize optimal firm policies, $\{m_{t+1}\}$, define the following:

Normalized productivity level:

$$\tilde{z} \equiv \frac{z}{m_t} \quad \text{for } \tilde{z} \in [1, \text{ max support }\{f\}/m_t]. \tag{F1}$$

The support follows from the definition $m_t \equiv \text{min support }\{f_t\}$ since the law of motion ensures max support $\{f_t\}$ = max support $\{f\}$.

Growth factor of the minimum of support:

$$g_t \equiv \frac{m_{t+1}}{m_t}. \tag{F2}$$

Normalized value function:

$$V_t(z) \equiv m_t \tilde{V}_t\left(\frac{z}{m_t}\right). \tag{F3}$$

Normalized pdf:

$$f_t(z) \equiv \frac{1}{m_t}\tilde{f}_t\left(\frac{z}{m_t}\right). \tag{F4}$$

Integrating the normalized pdf from m_t to z gives the normalized cdf:

$$F_t(z) \equiv \tilde{F}_t\left(\frac{z}{m_t}\right). \tag{F5}$$

Rearranging the normalization in equation (F4) and using $z = \tilde{z}m_t$ gives an equivalent transformation:

$$\tilde{f}_t(\tilde{z}) = m_t f_t(\tilde{z}m_t). \tag{F6}$$

B. Normalized Law of Motion

Together, the law of motion from equation PT.(4) and equation (F6) generate

$$f_t(\tilde{z}m_t) = \frac{f(\tilde{z}m_t)}{1 - F(m_t)}, \tag{F7}$$

[21] This stationary transformation of the model can also be used to numerically solve for model variations in which the law of motion is not a simple truncation. The algorithm and code described in App. H are intended to support these sorts of extensions.

$$\tilde{f}_t(\tilde{z}) = \frac{m_t f(\tilde{z}m_t)}{1 - F(m_t)}. \tag{F8}$$

Integrating to find the cdf yields

$$\tilde{F}_t(\tilde{z}) = \int_1^{\tilde{z}} \frac{f(\tilde{z}'m_t)}{1 - F(m_t)} m_t d\tilde{z}'. \tag{F9}$$

Doing a change of variables for $z = \tilde{z}m_t$ gives

$$\tilde{F}_t(\tilde{z}) = \frac{F(\tilde{z}m_t) - F(m_t)}{1 - F(m_t)}. \tag{F10}$$

Using the law of motion, $f_{t+1}(z) = f_t(z)/[1 - F_t(m_{t+1})]$, use $z = \tilde{z}m_{t+1}$ and substitute with equations (F4) and (F5) to obtain

$$\tilde{f}_{t+1}(\tilde{z}) = m_{t+1} \frac{\frac{1}{m_t}\tilde{f}_t\left(\frac{z}{m_t}\frac{m_{t+1}}{m_{t+1}}\right)}{1 - \tilde{F}_t\left(\frac{m_{t+1}}{m_t}\right)} = \frac{g_t \tilde{f}_t(\tilde{z}g_t)}{1 - \tilde{F}_t(g_t)}. \tag{F11}$$

Thus, the law of motion in the normalized $\tilde{z}$ space is entirely determined by the initial condition, $\tilde{f}_0(\tilde{z})$, and the sequence of growth factors, $\{g_t\}$.

Using equation (F8), define the asymptotic normalized distribution as

$$\tilde{f}_\infty(\tilde{z}) \equiv \lim_{t\to\infty} \frac{m_t f(\tilde{z}m_t)}{1 - F(m_t)}. \tag{F12}$$

Also, define the asymptotic growth factor of the minimum of support as

$$g_\infty \equiv \lim_{t\to\infty} g_t. \tag{F13}$$

As a check that this normalization is stationary for the BGP, use a Pareto initial condition—$f(z) = \alpha m_0^\alpha z^{-\alpha-1}$—and ensure that it is constant and equal to the normalized Pareto distribution for all t:

$$\tilde{f}_t(\tilde{z}) = m_t \frac{\alpha m_0^\alpha (\tilde{z}m_t)^{-\alpha-1}}{1 - \left[1 - \left(\frac{m_0}{m_t}\right)^\alpha\right]} \tag{F14}$$

$$= \alpha \tilde{z}^{-\alpha-1}, \quad \tilde{z} \in [1, \infty)\, \forall\, t \tag{F15}$$

$$= \tilde{f}_\infty(\tilde{z}) = \tilde{f}_0(\tilde{z}). \tag{F16}$$

C. Normalized Value Functions

Using equation (D1) and substituting the normalized versions of each variable and function gives

$$m_t \tilde{V}_t\left(\frac{z}{m_t}\right) = \max\left\{(1-\tau)z + \frac{1}{1+r_t} m_{t+1}\tilde{V}_{t+1}\left(\frac{z}{m_{t+1}}\right), \right.$$
$$\left. \varsigma z + \frac{1}{1+r_t}\frac{1}{1 - \tilde{F}_t\left(\frac{m_{t+1}}{m_t}\right)} \int_{m_{t+1}}^\infty m_{t+1}\tilde{V}_{t+1}\left(\frac{z'}{m_{t+1}}\right)\frac{1}{m_t}\tilde{f}_t\left(\frac{z'}{m_t}\right)dz' \right\}. \tag{F17}$$

Dividing by m_t and using $\tilde{z} = z/m_t$ yields

$$\tilde{V}_t(\tilde{z}) = \max\Bigg\{(1-\tau)\tilde{z} + \frac{1}{1+r_t}\frac{m_{t+1}}{m_t}\tilde{V}_{t+1}\left(\tilde{z}\frac{m_t}{m_{t+1}}\right), \\ \varsigma\tilde{z} + \frac{1}{1+r_t}\frac{\frac{m_{t+1}}{m_t}}{1-\tilde{F}_t\left(\frac{m_{t+1}}{m_t}\right)}\int_{m_{t+1}}^{\infty}\tilde{V}_{t+1}\left(\frac{z'}{m_{t+1}}\right)\frac{1}{m_t}\tilde{f}_t\left(\frac{z'}{m_t}\right)dz'\Bigg\}. \tag{F18}$$

Using the change of variables formula

$$\int_a^b m(n(q))n'(q)dq = \int_{n(a)}^{n(b)} m(s)ds$$

yields[22]

$$\tilde{V}_t(\tilde{z}) = \max\left\{(1-\tau)\tilde{z} + \frac{1}{1+r_t}g_t\tilde{V}_{t+1}(\tilde{z}/g_t), \varsigma\tilde{z} + \frac{1}{1+r_t}g_t\frac{1}{1-\tilde{F}_t(g_t)}\int_{g_t}^{\infty}\tilde{V}_{t+1}(\tilde{z}'/g_t)\tilde{f}_t(\tilde{z}')d\tilde{z}'\right\}. \tag{F19}$$

The indifference point, g_t, is the root of the following equation:

$$0 = (1-\tau)g_t + \frac{1}{1+r_t}g_t\tilde{V}_{t+1}(1) - \left[\varsigma g_t + \frac{1}{1+r_t}g_t\frac{1}{1-\tilde{F}_t(g_t)}\int_{g_t}^{\infty}\tilde{V}_{t+1}(\tilde{z}'/g_t)\tilde{f}_t(\tilde{z}')d\tilde{z}'\right]. \tag{F20}$$

If the environment is stationary, then the problem can be written recursively. In that case, there should exist a g, $\tilde{V}(\cdot)$, and $1/(1+r) = \beta g^{-\gamma}$ that solve the following fixed-point problem:

$$\tilde{V}(\tilde{z}) = \max\left\{(1-\tau)\tilde{z} + \frac{1}{1+r}g\tilde{V}(\tilde{z}/g), \varsigma\tilde{z} + \frac{1}{1+r}g\frac{1}{1-\tilde{F}(g)}\int_g^{\infty}\tilde{V}(\tilde{z}'/g)\tilde{f}(\tilde{z}')d\tilde{z}'\right\}. \tag{F21}$$

Some systems may become stationary asymptotically or may be stationary after a change of variables.

To find the normalized version of the sequence space formulation in equation (E4), define the normalization of the gross value of search relative to time t as

$$\tilde{W}_{s|t} \equiv \frac{W_{t+s}}{m_t}. \tag{F22}$$

Normalize the optimal waiting time until search in equation (E1) such that

$$\begin{aligned}\tilde{\xi}_t(\tilde{z}) &= \arg\min_{s\in\mathbb{N}}\{\tilde{z}m_t \le m_{t+1+s}\} \\ &= \arg\min_{s\in\mathbb{N}}\{\tilde{z} \le m_{t+1+s}/m_t\}.\end{aligned} \tag{F23}$$

Note that the argmin is the same after this change of variables: $\tilde{\xi}_t(\tilde{z}) = \xi_t(z)$.

[22] For this change of variables, $q = z'$, $n(\cdot) = \cdot/m_t$,

$$m(\cdot) = \tilde{V}_{t+1}\left(\cdot\frac{m_t}{m_{t+1}}\right)\tilde{f}(\cdot).$$

Since $g_t \equiv m_{t+1}/m_t$, m_t can be constructed from the sequence of g_t as

$$m_t = m_0 \prod_{t'=0}^{t-1} g_{t'}, \quad t \geq 1. \tag{F24}$$

Define the normalized future indifference points relative to the minimum of support at t as

$$\tilde{m}_{s|t} \equiv \prod_{t'=0}^{s} g_{t+t'} \quad \forall\, s \geq 0,\ t \geq 0. \tag{F25}$$

Note that $\{\tilde{m}_{s|t} | s \geq 0\}$ are the locations in the $\tilde{z}$ domain of the kinks in the normalized value function at time t (e.g., $\{g_t, g_t, g_{t+1}, \ldots\}$) and

$$\begin{aligned}\tilde{\xi}_t(\tilde{z}) &= \arg\min_{s\in\mathbb{N}} \left\{ \tilde{z} \leq \frac{m_0 \prod_{t'=0}^{t+s} g_{t'}}{m_0 \prod_{t'=0}^{t-1} g_{t'}} \right\} \\ &= \arg\min_{s\in\mathbb{N}} \{\tilde{z} \leq \tilde{m}_{s|t}\}.\end{aligned} \tag{F26}$$

If the interest rate is constant, equation (E4) can be used to derive the normalized value function. Dividing by m_t and using $\tilde{z} = z/m_t$, the normalized value function is

$$\tilde{V}_t(\tilde{z}) = \left\{ (1-\tau)\frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})}\right] + \varsigma\left(\frac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})} \right\}\tilde{z} + \left(\frac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})} \tilde{W}_{\tilde{\xi}_t(\tilde{z})|t}. \tag{F27}$$

This further reduces for $\tilde{z}$ at future normalized indifference points to

$$\tilde{V}_t(\tilde{m}_{t+1+s|t}) = \left\{ (1-\tau)\frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^{s}\right] + \varsigma\left(\frac{1}{1+r}\right)^{s} \right\}\tilde{m}_{t+1+s|t} + \left(\frac{1}{1+r}\right)^{s} \tilde{W}_{s|t}. \tag{F28}$$

Given $\{g_t, W_t\}$, equations (F22), (F26), and (F27) combine to deliver the normalized value function. Along the BGP, g_t is constant. Thus,

$$\tilde{W}_{s|t} = \frac{m_t g^s W}{m_t} = g^s W, \tag{F29}$$

$$\tilde{W}_{0|t} = W \quad \forall t, \tag{F30}$$

$$\tilde{\xi}_t(\tilde{z}) = \arg\min_{s\in\mathbb{N}} \{\tilde{z} \leq g^{s+1}\}, \tag{F31}$$

$$\tilde{V}_t(\tilde{z}) = \left\{ (1-\tau)\frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})}\right] + \varsigma\left(\frac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})} \right\}\tilde{z} + \left(\frac{1}{1+r}\right)^{\tilde{\xi}_t(\tilde{z})} g^{\tilde{\xi}_t(\tilde{z})} W. \tag{F32}$$

Since along the BGP these functions are all independent of t, the normalized value function is constant on the BGP, given by

$$\begin{aligned}\tilde{V}(\tilde{z}) &= \left\{ (1-\tau)\frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^{s}\right] + \varsigma\left(\frac{1}{1+r}\right)^{s} \right\}\tilde{z} + \left(\frac{1}{1+r}\right)^{s} g^s W, \\ &\tilde{z} \in [g^s, g^{s+1}].\end{aligned} \tag{F33}$$

In particular, at the indifference points, the value of a firm is

$$\tilde{V}(g^s) = \left\{(1-\tau)\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^s\right] + \varsigma\left(\frac{1}{1+r}\right)^s\right\}g^s + \left(\frac{1}{1+r}\right)^s g^s W \quad \text{for } s \geq 0. \tag{F34}$$

D. Normalized Production and Interest Rate

Production in period t is

$$Y_t = \int_{m_{t+1}}^{\infty} z f_t(z)dz. \tag{F35}$$

Substituting in the normalized distribution and reorganizing shows

$$Y_t = m_t \int_{m_{t+1}}^{\infty} \frac{z}{m_t}\tilde{f}_t\left(\frac{z}{m_t}\right)\frac{1}{m_t}dz. \tag{F36}$$

Doing a change of variables in the integral yields

$$Y_t = m_t \int_{g_t}^{\infty} \tilde{z}\tilde{f}_t(\tilde{z})d\tilde{z}. \tag{F37}$$

From the consumer's optimization problem, the interest rate satisfies

$$\begin{aligned} \frac{1}{1+r_t} &= \beta\left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma} \\ &= \beta g_t^{-\gamma}\left[\frac{\int_{g_{t+1}}^{\infty}\tilde{z}\tilde{f}_{t+1}(\tilde{z})d\tilde{z}}{\int_{g_t}^{\infty}\tilde{z}\tilde{f}_t(\tilde{z})d\tilde{z}}\right]^{-\gamma}. \end{aligned} \tag{F38}$$

The g_t, defined here as the growth factor of the minimum of support, may not be the growth factor of production off the BGP. Along the BGP, where $\tilde{f}_t$ is stationary and g_t is constant, this yields the BGP interest rate

$$r = \frac{g^\gamma}{\beta} - 1. \tag{F39}$$

E. Asymptotic Growth in Equilibrium: Proof of Proposition PT.2

To show that power laws contradict the condition in proposition PT.2 that $\lim_{m\to\infty}\mathbb{E}[z|z>m]/m = 1$, assume that $F_0(z)$ is a power law with tail parameter $\alpha > 1$. Using the normalizations defined in Section B,

$$\frac{\mathbb{E}[z|z>m]}{m_t} = \int_1^{\infty}\tilde{z}\frac{m_t f_0(m_t\tilde{z})}{1-F_0(m_t)}d\tilde{z}.$$

Simple calculations show that

$$\lim_{m\to\infty}\frac{\mathbb{E}[z|z>m]}{m}=\frac{\alpha}{\alpha-1}>1.$$

To show that an initial distribution that is a power law generates an asymptotic BGP in which the growth rate is a function of the tail parameter α, use equation (F8):[23]

$$\tilde{f}_t(\tilde{z})=\frac{m_t f_0(\tilde{z}m_t)}{1-F_0(m_t)} \tag{F42}$$

$$\propto\frac{m_t L(\tilde{z}m_t)\tilde{z}^{-\alpha-1}m_t^{-\alpha}}{L(m_t)m_t^{-\alpha}} \tag{F43}$$

$$\propto\frac{L(\tilde{z}m_t)}{L(m_t)}\tilde{z}^{-\alpha-1}. \tag{F44}$$

If there is perpetual positive growth, that is, $\lim_{t\to\infty} g_t>1+\epsilon$ for some $\epsilon>0$, then by definition $m_\infty\equiv\lim_{t\to\infty} m_t=\infty$. Thus, using the definition of slowly varying,

$$\lim_{t\to\infty}\tilde{f}_t(\tilde{z})\propto\tilde{z}^{-\alpha-1}. \tag{F45}$$

Therefore, from the stationarity of equation (F21), in any economy with perpetual growth and a power law initial distribution, the asymptotic growth factor is the solution to equation (D21).

Conversely, to derive conditions in which growth stops, define $z_{\max}\equiv\max\text{ support }\{F_0\}$. If $z_{\max}<\infty$, then growth must stop, as eventually $\lim_{t\to\infty} g_t>1+\epsilon$ implies $m_\infty=\infty$, contradicting $m_t\le z_{\max}$ for all t, as must be due to the truncation law of motion.

Finally, consider the case in which $z_{\max}=\infty$ and the initial distribution is not a power law. Note that if there is an equilibrium with no growth at any one point in time, then there is no growth in the limit. At such a point in time, $g=1$, $r=1/\beta-1$, and $\tilde{V}(\tilde{z})=(1-\tau)[(1+r)/r]\tilde{z}$ from equation (F33). Moreover, since no agents choose to search, from equation (F21),

$$(1-\tau)\tilde{z}+\frac{1}{1+r}g\tilde{V}(\tilde{z}/g)\ge\varsigma\tilde{z}+\frac{1}{1+r}g\frac{1}{1-\tilde{F}(g)}\int_g^\infty\tilde{V}(\tilde{z}'/g)\tilde{f}(\tilde{z}')d\tilde{z}'. \tag{F46}$$

To determine whether equation (F46) can hold with equality, evaluate the inequality at the indifference point, assumed above to be $\tilde{z}=g=1$:

$$1-\tau-\varsigma\ge\frac{1}{1+r}\left[\int_1^\infty\tilde{V}(\tilde{z}')\tilde{f}(\tilde{z}')d\tilde{z}'-\tilde{V}(1)\right]. \tag{F47}$$

[23] Alternatively, define a fat-tailed distribution as $F_0(z)$ such that $1-F_0(z)\sim x^{-\alpha}$, where $\alpha>1$ and $\sim$ denotes asymptotic equivalence. From eq. (F10), asymptotically,

$$\tilde{F}_t(\tilde{z})=\frac{1-\tilde{z}^{-\alpha}m^{-\alpha}-(1-m^{-\alpha})}{1-(1-m^{-\alpha})} \tag{F40}$$

$$=1-\tilde{z}^{-\alpha}. \tag{F41}$$

Thus, $\tilde{F}_t(\tilde{z})$ is the normalized Pareto distribution as shown in eq. (F2).

From an initial condition, to find bounds on m_∞, the minimum of support of the asymptotic distribution where firms would choose not to upgrade and growth would stop, substitute equations (F8) and (F33) into (F47) to define

$$\bar{m} \equiv \inf\left\{ m \middle| \left(\frac{1-\tau-\varsigma}{1-\tau}\right)\left(\frac{1}{\beta}-1\right)+1 \geq \int_1^{z_{\max}/m} \tilde{z}' \frac{m f_0(m\tilde{z}')}{1-F_0(m)} d\tilde{z}' \right\}. \tag{F48}$$

From equation (F48), a sufficient condition for the economy to reach an asymptotic maximum size is if a root m exists to the following equation:

$$\left(\frac{1-\tau-\varsigma}{1-\tau}\right)\left(\frac{1}{\beta}-1\right)+1 = \int_1^{z_{\max}/m} \tilde{z}' \frac{m f_0(m\tilde{z}')}{1-F_0(m)} d\tilde{z}'. \tag{F49}$$

Since the distribution is not a power law, equation (F49) has a root if[24]

$$\left(\frac{1-\tau-\varsigma}{1-\tau}\right)\left(\frac{1}{\beta}-1\right)+1 > 1.$$

Given the assumption in proposition PT.2, this is true for any $0 < \beta < 1$ and $0 < \tau - \varsigma < 1$. Thus, when $z_{\max} = \infty$ and F_0 is such that $\lim_{m\to\infty} \mathbb{E}[z|z > m]/m = 1$, there is a terminal, maximum scale of the economy.

The bound is on the current state m for the firm decisions rather than on the actual terminal value of m_∞. Owing to the discreteness of time, in general, $m_\infty \neq \bar{m}$. If $m_\infty < \bar{m}$, then the incentives for technology adoption were insufficient for a final step toward $\bar{m}$. It is conceivable that an equilibrium exists in which $m_{T-1} < \bar{m}$ with sufficient incentives for adoption that generate growth factor $g_{T-1} > 1$ such that $g_{T-1} m_{T-1} = m_T \geq \bar{m}$. However, from T onward, no further growth would occur. QED

[24] Note that if F_0 were a power law, then

$$\lim_{m\to\infty} \frac{\mathbb{E}[z|z > m]}{m} = \frac{\alpha}{\alpha - 1} > 1,$$

independent of m. Taking the limit of $\beta \to 1$ in eq. (F49) gives $1 = \alpha/(\alpha - 1) > 1$. This contradiction shows that there always exist parameters such that power law initial conditions have asymptotic growth.

Appendix G from Perla and Tonetti, "Equilibrium Imitation and Growth" (JPE, vol. 122, no. 1, p. 25)

Unconditional Draws

A. The Firm Problem

Having upgrading firms draw from the conditional distribution of producing firms simplifies the problem and changes the growth rate quantitatively but does not qualitatively change the growth mechanism. If upgrading firms received a draw from the unconditional productivity distribution, the ability to meet low-productivity agents would lower the equilibrium growth rate by allowing congestion effects as firms may take several draws before they successfully upgrade.

In this section, instead of drawing from the distribution $F_t(z|z > m_{t+1})$, firms draw directly from the unconditional $F_t(z)$ distribution. In that case, the firm may choose to reject a draw if it is lower than its current z. The cost or subsidy of search includes forgone production as well as value proportional to the size of the economy or expected draw. For simplicity, assume that it is proportional to the search threshold in the economy, m_{t+1}.[25] Modifying equation (D1) gives

$$V_t(z) = \max\left\{(1-\tau)z + \frac{1}{1+r_t}V_{t+1}(z), \varsigma m_{t+1} + \frac{1}{1+r_t}\int_0^\infty V_{t+1}(\max\{z', z\})dF_t(z')\right\}. \tag{G1}$$

Assume in equilibrium that m_t is increasing (which must be verified). Then firms that search at time t and draw below the current search threshold will search again next period. The probability that the firm draws a z' below the current search threshold is $F_t(m_{t+1})$. Define the gross value of search at time t as

$$W_t \equiv \frac{1}{1+r_t}\int_0^\infty V_{t+1}(\max\{z',\ z\})dF_t(z'). \tag{G2}$$

If firms draw below m_{t+1}, they will search next period. Split the integral into the conditional probability distributions above and below m_{t+1} and substitute the net value of search to yield

$$= \frac{1}{1+r_t}\left\{[1-F_t(m_{t+1})]\int_{m_{t+1}}^\infty V_{t+1}(z')\frac{f_t(z')}{1-F_t(m_{t+1})}dz' + F_t(m_{t+1})(W_{t+1} + \varsigma m_{t+2})\right\}. \tag{G3}$$

To simplify the problem, define $\hat{F}_t(z)$ as the distribution conditional on $z > m_t$:

$$\hat{f}_t(z) \equiv \frac{f_t(z)}{1-F_t(m_t)}, \quad \min \text{support}\{\hat{f}_t\} = m_t, \tag{G4}$$

$$\hat{F}_t(z) \equiv \frac{F_t(z) - F_t(m_t)}{1-F_t(m_t)}. \tag{G5}$$

Recall that the total number of searchers at time t is $S_t = F_t(m_{t+1})$. Define the total number of searchers who were left

[25] An alternative specification is to have the cost/subsidy proportional to the expected draw, conditional on acceptance: $\varsigma\mathbb{E}_t[z|z > m_{t+1}]$. While a cost proportional to z is possible, it complicates this setup since it requires keeping track of the equilibrium distribution of failed searchers. Economically, as all firms have the same expected value of a draw, independent of their z, it makes sense that the cost/subsidy would also be independent of their z beyond forgone production.

behind by obtaining "bad" draws as

$$\bar{S}_t \equiv F_t(m_t). \tag{G6}$$

Note that conditioning the $F_t(z)$ distribution at or above m_t is equal to conditioning the $\hat{F}_t(z)$ distribution at or above m_t, as it is simply two successive truncations. Then, equation (G1) simplifies to

$$\begin{aligned} V_t(z) = \max\Bigg\{&(1-\tau)z + \frac{1}{1+r_t}V_{t+1}(z), \\ &\varsigma m_{t+1} + \frac{1}{1+r_t}(1-S_t)\int_{m_{t+1}}^{\infty} V_{t+1}(z')\frac{\hat{f}_t(z')}{1-\hat{F}_t(m_{t+1})}dz' + \frac{1}{1+r_t}S_t(W_{t+1}+\varsigma m_{t+2})\Bigg\}. \end{aligned} \tag{G7}$$

And the gross value of search is

$$W_t = \frac{1}{1+r_t}\left[(1-S_t)\int_{m_{t+1}}^{\infty} V_{t+1}(z')\frac{\hat{f}_t(z')}{1-\hat{F}_t(m_{t+1})}dz' + S_t(W_{t+1}+\varsigma m_{t+2})\right]. \tag{G8}$$

Using the alternative definition of W_t and cost/subsidy ςm_{t+1}, follow the steps in Appendix E to find the value function on a BGP as

$$V_t(z) = (1-\tau)\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^{\xi_t(z)}\right]z + \left(\frac{1}{1+r}\right)^{\xi_t(z)}\left[W_{t+\xi_t(z)} + \varsigma m_{t+\xi_t(z)+1}\right] \tag{G9}$$

$$\begin{aligned} &= (1-\tau)\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^{s}\right]z + \left(\frac{1}{1+r}\right)^{s} m_0 g^{t+s}(W+g\varsigma), \\ &z \in [m_0 g^{t+s},\ m_0 g^{t+s+1}]. \end{aligned} \tag{G10}$$

B. Law of Motion

The mass of searchers at any point in time, S_t, is now the mass of firms with poor draws in the past plus the new firms that fall below the threshold:

$$S_t = F_t(m_t) + [F_t(m_{t+1}) - F_t(m_t)]. \tag{G11}$$

Using the definitions in equations (G5) and (G6),

$$S_t = \bar{S}_t + (1-\bar{S}_t)\hat{F}_t(m_{t+1}). \tag{G12}$$

The law of motion for $\bar{S}_t$ includes the total number of searchers who draw below m_{t+1} such that

$$\bar{S}_{t+1} = S_t F_t(m_{t+1}) = S_t^2, \tag{G13}$$

$$\bar{S}_{t+1} = [\bar{S}_t + (1-\bar{S}_t)\hat{F}_t(m_{t+1})]^2. \tag{G14}$$

Conditional on drawing above m_{t+1}, the draws are in proportion to the distribution truncated at m_{t+1}, as in equation PT.(3). For this reason, even though the mass of firms below m_{t+1} is not invariant, the truncated distribution is independent of the particular mass in S_t. Hence, the law of motion for the right tail is similar to that in equation PT.(4):

$$\hat{f}_t(z) = \frac{f_0(z)}{1 - F_0(m_t)}. \tag{G15}$$

C. Normalization

The normalization follows Appendix F closely, except that the left-truncated distribution $\hat{F}_t(z)$ is normalized instead of the unconditional distribution $F_t(z)$ (i.e., $\hat{F}_t(z) \equiv \tilde{F}_t(z/m_t)$). From equations (G12), (G14), (F37), and (F38),

$$S_t = \bar{S}_t + (1 - \bar{S}_t)\tilde{F}_t(g_t), \tag{G16}$$

$$\bar{S}_{t+1} = [\bar{S}_t + (1 - \bar{S}_t)\tilde{F}_t(g_t)]^2, \tag{G17}$$

$$Y_t = m_t(1 - S_t)\int_{g_t}^{\infty} \tilde{z}\tilde{f}_t(\tilde{z})d\tilde{z}, \tag{G18}$$

$$\frac{1}{1 + r_t} = \beta\left[g_t \frac{1 - S_{t+1}}{1 - S_t} \frac{\int_{g_{t+1}}^{\infty} \tilde{z}\tilde{f}_{t+1}(\tilde{z})d\tilde{z}}{\int_{g_t}^{\infty} \tilde{z}\tilde{f}_t(\tilde{z})d\tilde{z}}\right]^{-\gamma}. \tag{G19}$$

Substituting the normalizations into equation (G7),[26]

$$\begin{aligned} m_t\tilde{V}_t\left(\frac{z}{m_t}\right) = \max\Bigg\{&(1 - \tau)z + \frac{1}{1 + r_t}m_{t+1}\tilde{V}_{t+1}\left(\frac{z}{m_{t+1}}\right), \\ &\varsigma m_{t+1} + (1 - S_t)\frac{1}{1 + r_t}\frac{1}{1 - \tilde{F}_t\left(\frac{m_{t+1}}{m_t}\right)}\int_{m_{t+1}}^{\infty} m_{t+1}\tilde{V}_{t+1}\left(\frac{z'}{m_{t+1}}\right)\frac{1}{m_t}\tilde{f}_t\left(\frac{z'}{m_t}\right)dz'\Bigg\} \\ &+ S_t\frac{1}{1 + r_t}m_{t+1}\tilde{V}_{t+1}(1). \end{aligned} \tag{G20}$$

[26] For the alternative cost,

$$\begin{aligned} \varsigma\mathbb{E}_t[z|z > m_{t+1}] &= \varsigma\int_{m_{t+1}}^{\infty} z\frac{f_t(z)}{1 - F_t(m_{t+1})}dz \\ &= \varsigma(1 - \bar{S}_t)\int_{m_{t+1}}^{\infty} z\hat{f}_t(z)d = m_t\varsigma(1 - \bar{S}_t)\int_{g_t}^{\infty} \tilde{z}\tilde{f}_t(\tilde{z})d\tilde{z}. \end{aligned}$$

Therefore, on a BGP the costs are a constant proportion of m_t, and the cost term in eq. (G21) for the Pareto distribution and a BGP is $\varsigma(1 - S)g^{-\alpha}$.

Following the same simplifications as in equation (G19),

$$\tilde{V}_t(\tilde{z}) = \max\Big\{(1-\tau)\tilde{z} + \frac{1}{1+r_t} g_t \tilde{V}_{t+1}(\tilde{z}/g_t), \\ \varsigma g_t + (1-S_t)\frac{1}{1+r_t} g_t \int_{g_t}^{\infty} \tilde{V}_{t+1}(\tilde{z}'/g_t)\frac{\tilde{f}_t(\tilde{z}')}{1-\tilde{F}_t(g_t)} d\tilde{z}' + S_t \frac{1}{1+r_t} g_t \tilde{V}_{t+1}(1)\Big\}. \tag{G21}$$

At the indifference point g_t, equate and simplify equation (G21):

$$1-\tau-\varsigma = [1-S_t(g_t)]\frac{1}{1+r_t(g_t)}\left[\int_{g_t}^{\infty} \tilde{V}_{t+1}(\tilde{z}'/g_t)\frac{\tilde{f}_t(\tilde{z}')}{1-\tilde{F}_t(g_t)} d\tilde{z}' - \tilde{V}_{t+1}(1)\right], \tag{G22}$$

where $S_t(g_t)$ and $r_t(g_t)$ are defined in equations (G16) and (G19). Comparing equation (G22) to (F20) shows the role of the congestion in changing the incentives for search.

D. Stationary Equilibrium with Positive Growth

Using equation (G10) to modify equations (F33) and (F34),

$$\tilde{V}(\tilde{z}) = (1-\tau)\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^s\right]\tilde{z} + \left(\frac{1}{1+r}\right)^s g^s(W+g\varsigma), \quad \tilde{z}\in[g^s, g^{s+1}], \tag{G23}$$

$$\tilde{V}(g^s) = (1-\tau)\frac{1+r}{r}\left[1-\left(\frac{1}{1+r}\right)^s\right]g^s + \left(\frac{1}{1+r}\right)^s g^s(W+g\varsigma) \quad \text{for } s \geq 0. \tag{G24}$$

To solve for the BGP, find an implicit function of g by substituting out W, r, S, and $\bar{S}$.

At $\tilde{z} = g$ in equation (G23), equate the $s = 0$ and $s = 1$ cases using continuity of the value function

$$W + g\varsigma = (1-\tau)g + \frac{1}{1+r}g(W+g\varsigma). \tag{G25}$$

Solving for W,

$$W = \frac{\{1-\tau-\varsigma[1-g/(1+r)]\}g}{1-g/(1+r)}. \tag{G26}$$

From equation (G19), on a BGP,

$$r = g^{\gamma}/\beta - 1. \tag{G27}$$

With the normalized Pareto distribution $\tilde{F}(\tilde{z})$ from equation (F15), equations (G16) and (G17) become

$$S = \bar{S} - (1-\bar{S})(1-g^{-\alpha}), \tag{G28}$$

$$\bar{S} = [\bar{S} + (1-\bar{S})(1-g^{-\alpha})]^2. \tag{G29}$$

Solving for $\bar{S}$ and then substituting to solve for S,

$$\bar{S} = (g^{\alpha} - 1)^2, \tag{G30}$$

$$S = g^{\alpha} - 1. \tag{G31}$$

To get the final equation in g, insert the normalized Pareto distribution from equation (F15) into equation (G22) and substitute for $\tilde{V}(1)$ from (G24):

$$1 - \tau - \varsigma = (1 - S)\frac{1}{1 + r}\left[\alpha g^{\alpha}\int_{g}^{\infty}\tilde{V}(\tilde{z}'/g)\tilde{z}'^{-1-\alpha}d\tilde{z}' - W - g\varsigma\right]. \tag{G32}$$

Use the piecewise linearity of equation (G23) to turn this integral into an infinite sum:

$$\int_{g}^{\infty}\tilde{V}(\tilde{z}'/g)\tilde{z}'^{-1-\alpha}d\tilde{z}' \tag{G33}$$

$$= \sum_{s=1}^{\infty}\int_{g^s}^{g^{s+1}}\left\{(1-\tau)\frac{1+r}{r}\left[1 - \left(\frac{1}{1+r}\right)^{s}\right]\tilde{z}/g + \left(\frac{1}{1+r}\right)^{s}g^{s}(W + g\varsigma)\right\}\tilde{z}^{-1-\alpha}d\tilde{z} \tag{G34}$$

$$= (1-\tau)\frac{1+r}{r}\Big/g\sum_{s=1}^{\infty}\left[1 - \left(\frac{1}{1+r}\right)^{s}\right]\frac{g^{s-(s+1)\alpha}(g^{\alpha} - g)}{\alpha - 1} + (W + g\varsigma)\sum_{s=1}^{\infty}\left(\frac{1}{1+r}\right)^{s}g^{s}\frac{g^{-\alpha(s+1)}(g^{\alpha} - 1)}{\alpha} \tag{G35}$$

$$= \frac{g^{-\alpha}\{-g(g\varsigma + W)(\alpha - 1) + g^{\alpha}[g(g\varsigma + W)(\alpha - 1) + (1+r)\alpha - (1+r)\alpha\tau]\}}{[-g + g^{\alpha}(1+r)](\alpha - 1)\alpha}.$$

Substituting this integral, S, W, and r into (G32) and simplifying gives an implicit equation for g:

$$\frac{1 - \tau - \varsigma}{1 - \tau} = \beta g^{\alpha}\frac{(2 - g^{\alpha})\left(\frac{\alpha}{\alpha - 1} - g\right)}{g^{\alpha+\gamma} - \beta g}. \tag{G36}$$

E. Stationary Equilibrium with No Growth

For the distributions, including all with finite support, where $\lim_{t\to\infty} g_t = 1$, follow the steps in Appendix F, Section E, to find the bound m_{∞} at which growth stops:

$$\bar{m} \equiv \inf\left\{m \middle| \left(\frac{1 - \tau - \varsigma}{1 - \tau}\right)\left(\frac{1}{\beta} - 1\right) + 1 \geq \int_{1}^{z_{\max}/m}\tilde{z}'\frac{mf_0(m\tilde{z}')}{1 - F(m)}d\tilde{z}'\right\}. \tag{G37}$$

A root may not exist if there is no initial growth at m_0. Otherwise, $\bar{m}$ is a root to the following equation:

$$\left(\frac{1 - \tau - \varsigma}{1 - \tau}\right)\left(\frac{1}{\beta} - 1\right) + 1 = \int_{1}^{z_{\max}/\bar{m}}\tilde{z}'\frac{\bar{m}f_0(\bar{m}\tilde{z}')}{1 - F_0(\bar{m})}d\tilde{z}'. \tag{G38}$$

For nonmonotone distributions, it is possible for there to be multiple roots, and m_{∞} would be the smallest root.

 DOI: 10.1086/674362

Appendix H from Perla and Tonetti, "Equilibrium Imitation and Growth" (JPE, vol. 122, no. 1, p. 25)

Numerical Algorithm with Unconditional Draws

The following numerically computes a dynamic equilibrium of the economy developed in Appendix G. The use of unconditional rather than conditional draws adds numerical stability to the algorithm when on or very close to the BGP.

A. Setup and Definitions

Set up the following initial and terminal values:

- Given an initial condition for $f(z)$:

  1. Choose an $m_0 > 0$. This is arbitrarily chosen as long as in equilibrium $m_1 > m_0$. If $m_1 \leq m_0$ after the calculations, then lower m_0.
  2. Initialize the number left beyond to be $\bar{S}_0 = F(m_0)$.
  3. Get the normalized version of the truncated pdf from equation (F8): $\tilde{f}(\tilde{z}) = m_0 f(\tilde{z}m_0)/1 - F(m_0)$ by equation (F8).

- Choose a large terminal time T.
- If the system is converging toward a BGP, then the asymptotic value function is $\tilde{V}_T(\tilde{z})$ from equation (G23). If the system is converging toward $g = 1$, then the terminal value will simply be the normalized value of production in perpetuity, $\tilde{V}_T(\tilde{z}) = (1-\tau)[(1+r)/r]\tilde{z}$.

  a. The closed-form value function in equation (G23) can be compared against a naive value function iteration of the stationary version of equation (G21). For the fixed point, as the value function is known to be piecewise linear, cubic splines and other smooth interpolation methods should not be used. Value function iteration is slow but safe.
  b. As a result of the accumulation of numerical errors from integration, these may be different at three to five significant digits. For larger growth rates, this small difference can compound geometrically and change dynamics close to the terminal T. When calculating dynamics in these cases, it often makes sense to use the solution for $\tilde{V}_T(\tilde{z})$ from value function iteration as it is more consistent with the backward induction used in the rest of the algorithm.

- Choose a number M of future indifference points for approximating the value function. At any given t, after M points the value function is assumed to be linear, as shown in equation (E6).
- Choose an initial guess for the sequence of growth factors $\vec{g} \equiv \{g_t\}_{t=0}^{T-1+M}$ where $g_t = g_\infty$ for $T \leq t \leq T-1+M$. This pads the guess of growth factors with the asymptotic growth factor to ensure that there are always M future points in the approximation of the value function (i.e., $M+1$ total points with $\tilde{z} = 1$).

At a particular time, for a particular $\vec{g}$, use equation (F25) with $\vec{g}$ to calculate the set of points to use in the approximation of the value function:

$$\vec{m}_t = \left\{1, \prod_{t'=0}^{0} g_{t+t'}, \prod_{t'=0}^{1} g_{t+t'}, \ldots, \prod_{t'=0}^{M-1} g_{t+t'}\right\}. \tag{H1}$$

For example, if $M = 3$, $\vec{m}_5 = 1, g_5, g_5 g_6, g_5 g_6 g_7$.

B. Iterative Algorithm

Given a current guess $\vec{g}$

1. Calculate the sequence of normalized densities $\tilde{f}_t(\tilde{z})$ for $t = 0, \ldots, T$.

   a. For the baseline model, first calculate the set of (unnormalized) indifference points using equation (F24):

$$\{m_t\}_{t=0}^{T} = \left\{ m_0 \prod_{t'=0}^{t-1} g_{t'} \right\}_{t=0}^{T}. \tag{H2}$$

   b. Use equation (F8) to calculate the normalized sequence of distributions, $\{\tilde{f}_t(z)\}$.[27] For numerical stability in the right tail, it may make sense to use the expression $A/B = e^{\log(A)-\log(B)}$:

$$\tilde{f}_t(\tilde{z}) = \exp(\log(m_t) + \log(f(\tilde{z}m_t)) - \log(1 - F(m_t))), \tag{H3}$$

$$\tilde{F}_t(\tilde{z}) = \exp(\log(F(m_t\tilde{z}) - F(m_t)) - \log(1 - F(m_t))). \tag{H4}$$

   c. Alternatively, for the particular $F_0(z)$, the functions for the truncation at m, $\tilde{f}_m(\tilde{z}; m)$ and $\tilde{F}_m(\tilde{z}; m)$, could be given directly in accordance with equations (F8) and (F10). This method should be used if the expression simplifies to remove m_t from the denominator. Otherwise, for many distributions and large m, $1 - F(m)$ and $f(\tilde{z}m)$ can both reach zero within the computer's level of precision, yielding imprecise or 0/0 values for $\tilde{f}_t(\tilde{z})$ at large t.

   d. From $\bar{S}_0$, use equations (G17) and (G16) to calculate the mass of searchers and those left behind, $\{\bar{S}_t, S_t\}$.

2. Calculate the value function backward for $t = T - 1, \ldots, 0$ in order to solve for the reservation productivities, $\{g_t'\}$. Start with the analytical value $\tilde{V}_T(\tilde{z})$ and use the calculated $\{\tilde{f}_t, r_t, S_t\}$.

   a. Calculate the sequence of points to evaluate as $\vec{m}_t$ using equation (H1).

   b. Calculate the value function for each value in $\vec{m}_t$ using equation (G21).
   - This requires numerical integration over the distribution $\tilde{f}_t(\tilde{z})$ using the previously calculated $\tilde{V}_{t+1}$.[28]
   - Set $\tilde{V}_t(\tilde{z})$ as the piecewise-linear function between the points $\vec{m}_t$ with the calculated values for use in time $t - 1$ calculation. The value function should be extrapolated linearly beyond the M points.

   c. Calculate the indifference point with this value function to find g_t' using equation (G22).
   - For numerical stability when converging toward the BGP, it is crucial when calculating the root that $S_t(g)$ and $r_t(g)$ also move to solve the indifference point.
   - To implement this, for the given $\bar{S}_t$, find a root to equation (G22) using equations (G16) and (G19). In the normalized space, this root is the new g_t.
   - The value function is piecewise linear, so splines and many other approximations are inappropriate. While the kinks should be at m_t, for numerical stability in the backward induction it is helpful to add a finer grid between points.

3. If $\{g_t'\}_{t=0}^{T-1+M}$ is close in norm to $\vec{g}$, stop iterating.[29]

   - Otherwise, using the sequence $\{g_t'\}_{t=0}^{T-1+M}$, update the guess $\vec{g}$. For example, use a linear combination of the two for the new $\vec{g}'$.

[27] For different models that imply different laws of motion, replace (F8) with the new law of motion, and use $\{m_t\}_{t=0}^{T}$ to sequentially calculate an approximation of $\{\tilde{f}_t(z)\}$.

[28] In order to calculate this numerically, a finer grid than $\vec{m}_t$ will be required. Since $\tilde{V}_{t+1}$ is piecewise linear and kinked, Simpson's rule and similar quadrature approaches are not appropriate unless calculated piecewise between each interval in $\vec{m}_{t+1}$. Note that

$$\lim_{\tilde{z}\to\infty} \tilde{V}_{t+1}'(\tilde{z}) = (1 - \tau)\frac{1 + r}{r}$$

and $\lim_{\tilde{z}\to\infty} \tilde{f}_t(\tilde{z}) = 0$. Alternatively, Gauss-Laguerre quadrature can be used for the infinite right tail integral, or adaptive quadrature routines such as Matlab's integral may be used.

[29] At termination, the $\tilde{f}_T(\tilde{z})$ should be checked to be close to $\tilde{f}_\infty(\tilde{z})$ as calculated analytically from eq. (F12).

C. Baseline Calibration

In the baseline model in proposition PT.1, the only parameters are α, β, and γ. The cost of search is in forgone production, so in order to calibrate to a particular growth rate, the length of a time period must be calibrated to change the search costs. To calibrate to reasonable growth rates, this may lead to long period lengths and consequently low β if α is low. In order to establish a yearly period length, negative values of ς in equation (G1) provide multiples of the time periods for lost production. With this variation of the model, the calibrated values are:

- $\gamma = 1$, $\beta = 0.95$: targets annualized interest rates.
- $\alpha = 1.5$: a compromise between estimates of the far right tail of the firm productivity distribution, which is often estimated to be low (e.g., 1.1 if ignoring the lower tail) and the need to fit the Pareto to the whole of the distribution of operating firms.
- $\tau = 0.3$: 30 percent tax rate on earnings.
- $\varsigma = -12$: Note that the mean of the normalized Pareto with $\alpha = 1.5$ is 3. With yearly time periods and the additional cost of lost production for the period, it takes approximately 5 years of profits for an average producer to break even when upgrading its technology. See Perla, Tonetti, and Waugh (2013) for a version of this cost function paid in labor and final goods at equilibrium prices.With the above calibration, the asymptotic annualized growth rate is 3.28 percent.

D. Fréchet Example

For a Fréchet initial condition with pdf and cdf,

$$f(z) = e^{-(z/s)^{-\alpha}} \alpha s^{\alpha} z^{-1-\alpha}, \tag{H5}$$

$$F(z) = e^{-(z/s)^{-\alpha}}. \tag{H6}$$

Using equation (G15), the normalized pdf is

$$\tilde{f}_t(\tilde{z}) = \frac{e^{-[(m/s)\tilde{z}]^{-\alpha}}(m/s)^{-\alpha}}{1 - e^{-(m/s)^{-\alpha}}} \alpha \tilde{z}^{-\alpha-1}. \tag{H7}$$

From Section E in Appendix F, the economy will converge to the BGP with the shape parameter α, independent of s. Hence, the analytical $\tilde{V}_T(\tilde{z})$ is given in equation (F33). Given the calibrations in Section C, with $s = 1$ and $\alpha = 1.5$, the growth dynamics for a Fréchet initial condition is given in figure H1.

To see the connection between the "thickness" of the distribution and the growth rates, figure H1 also plots the normalized right tails of the productivity pdfs, $\tilde{f}_t(\tilde{z})$ for $t = 2, 5, 40$, and the asymptotic limit.[30] Owing to the equivalence of solving a normalized version of the problem from the normalized distributions, as discussed in Appendix F, these distributions are sufficient to interpret the search decision of agents and solve for growth rates of the economy. Note that as t increases, the normalized distribution becomes thinner with a smaller expectation and more mass closer to the minimum of support. This decrease in the expected normalized draw creates decreasing growth rates as the economy converges toward the power law tail. Note that by year 40, the normalized distribution conditional on production is nearly Pareto.

E. Bounded Pareto Example

Since in this model no new technology is ever created through an external R&D process, if the initial distribution has finite support, then eventually growth will stop below the frontier, as discussed in Section E of Appendix G. However, at any point in time a far-off frontier has little effect on the decisions of agents to upgrade their technology. To see this, take the parameters used in Section C, but right-truncate the Pareto distribution. To illustrate a far-off technology frontier, truncate

[30] This diagram, and a discussion of the economics surrounding it, are also in fig. PT.2. While it is not shown here, for other parameter choices, $\tilde{f}_t(\tilde{z})$ could exhibit the increasing left tail of the initial Fréchet distribution for small t.

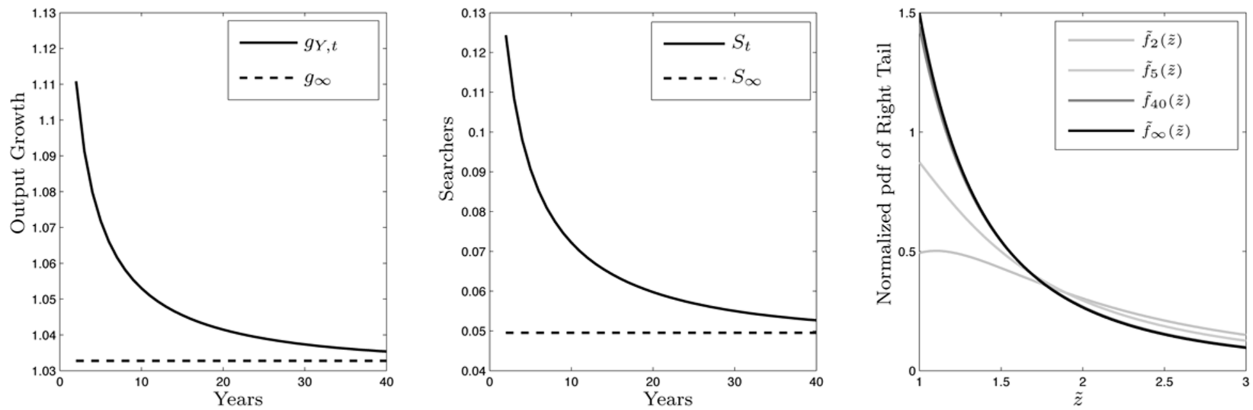

FIG. H1.—Growth and normalized distributions from Fréchet initial condition

the distribution such that $\max\{\tilde{z}\} = 500$, which ends up decreasing production (and hence the expected value of a draw) by about 4.5 percent at year 0. Figure H2 shows the results for 100 years of transition dynamics and compares to the BGP solution with the Pareto initial condition with the same α (denoted g_u and S_u).

With this, the pdf and cdf with a minimum of support m and maximum of support $\bar{z}$ are

$$f(z) = \frac{\alpha m^\alpha z^{-1-\alpha}}{1 - (m/\bar{z})^\alpha}, \tag{H8}$$

$$F(z) = \frac{1 - (m/z)^\alpha}{1 - (m/\bar{z})^\alpha}. \tag{H9}$$

The growth rates start lower than the unbounded case, almost entirely because of the lower expected value in the distribution rather than the bounded support.[31] After about 100 years the growth rate has decreased by less than 1 percent, reflecting the approaching boundary. However, with the calibrated growth rates of 2–3 percent, the technology frontier approaches very slowly and the normalized distribution stays nearly Pareto. Figure H2 shows the productivity distribution after 2 and 400 years. While the normalized maximum of $\tilde{z}$, as calculated by equation (F1), is 500 at $t = 1$ and 6.9197 at $t = 400$, the probability of a large draw changes less drastically. For example, $1 - \tilde{F}_2(3) = 0.1924$ and $1 - \tilde{F}_{400}(3) = 0.1455$.

[31] To see why this drop is significant, note that if $\alpha = 1.5$, the ratio of the expected value of the Pareto distribution bounded below $\tilde{z} = 500$ to the unbounded case is 0.955. When the tail parameter α is higher, this ratio increases toward unity and the initial growth rates are much closer to the unbounded case. For example, with $\alpha = 2.1$, the ratio of expectations for the case bounded below $\tilde{z} = 500$ is 0.998.

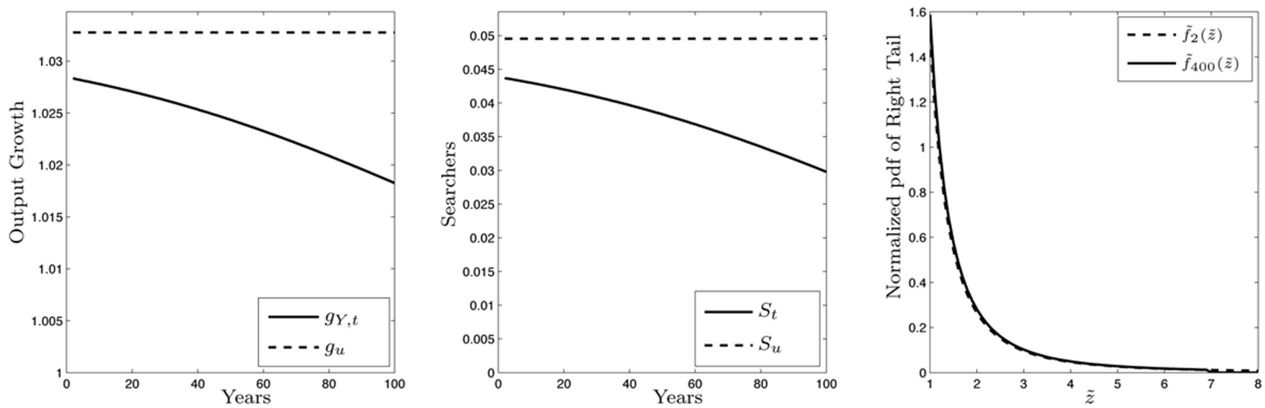

Fig. H2.—Growth and normalized distributions from bounded Pareto initial condition

Additional Reference

Perla, J., C. Tonetti, and M. E. Waugh. 2013. "Equilibrium Technology Diffusion, Trade, and Growth." Manuscript, New York Univ.