Equilibrium of a Magnetically Confined Plasma in a Toroid

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A. INTRODUCTION

A static equilibrium of plasma (or of conducting fluid) with scalar pressure p and magnetic field $\mathbf{B}$ is often described by the magnetostatic equations

$$\nabla p = \mathbf{j} \times \mathbf{B}, \tag{A1}$$

$$\nabla \times \mathbf{B} = \mathbf{j}, \tag{A2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{A3}$$

where $\mathbf{j}$ is the electric current density. In particular, these equations apply to many proposed controlled thermonuclear reactors and their prototypes, especially the stellarator [1] and the recently much discussed stabilized pinch. In sections B, C, and D there are derived [2] a variety of properties possessed by solutions of (A1–3).

One of these properties is that if p is constant on the boundary of its region of definition then, under some mild additional assumptions, that boundary must be topologically toroidal. However, prescribing such a boundary surface and the value of p on it by no means determines a unique solution, even though there are as many equations as unknowns (two vector and one scalar). One of our objects is to establish additional conditions which together with the magnetostatic equations (and the boundary prescription) do determine a unique solution. This is achieved in several different ways, the additional conditions always amounting to the specification of two numbers for each surface of constant p.

An experiment is imagined (section E) in which an ideal viscous hydromagnetic fluid exhibits a damped motion until coming to rest in an equilibrium configuration. A number of invariants with respect to any such motion are described in section F. These lead to constraints on the admissible trial states in a variational principle (sections G, H, and I) suggested by the experiment. The quantity varied is the potential energy which is the sum of the magnetic and the internal fluid energies. The variational principle provides a potentially powerful tool for proving the existence of solutions of the magnetostatic equations as well as for obtaining them numerically. It also provides a characterization of solutions by their values of the invariants.

In Section J equations governing a steady state of magnetic field and slowly diffusing plasma are introduced.[3] These amount to the magnetostatic equations together with two auxiliary conditions (K3, 7) for each surface of constant pressure (section K). The system of equations obtained is expected to have a solution which is unique, and this is verified in section L for the limiting case of low pressure.

B. MAGNETIC SURFACES

The magnetostatic equations (A1–3) have the simple consequences

$$\nabla \cdot \mathbf{j} = 0, \tag{B1}$$

$$\mathbf{B} \cdot \nabla p = 0, \tag{B2}$$

$$\mathbf{j} \cdot \nabla p = 0, \tag{B3}$$

$$\mathbf{B} \cdot \nabla \mathbf{B} = \nabla(p + \tfrac{1}{2}\mathbf{B}^2), \tag{B4}$$

$$\nabla \cdot (\mathbf{B} \times \nabla p) = 0, \tag{B5}$$

$$\mathbf{B} \cdot \nabla \mathbf{j} = \mathbf{j} \cdot \nabla \mathbf{B}, \tag{B6}$$

$$\mathbf{B} \cdot \nabla (\mathbf{B} \cdot \mathbf{j}) = \mathbf{j} \cdot \nabla \mathbf{B}^2 \tag{B7}$$

Here (B1) follows from (A2), (B2, 3) from (A1), (B4) from (A1, 2), (B5) from (A2) and (B3), (B6) from the curl of (A1) in view of (A3) and (B1), and finally (B7) from (A1) and (B5) in view of (A3) and (B1).

If p is reasonably smooth and not constant in any (small) region, the equation $p = P$ determines a family of surfaces characterized by their values of the parameter P. By (B2) they are " magnetic surfaces ", in the sense that they are made up of lines of magnetic force, and similarly by (B3) they are " current surfaces ". If such a surface lies in a bounded volume of space and has no edges (because of not intersecting the edge of the region of definition of p) and if either $\mathbf{B}$ or $\mathbf{j}$ nowhere vanishes on it, then by a well-known theorem [4] it must be either a toroid (by which we mean a topological torus) or a Klein's bottle. The latter, however, is not realizable in physical space.

Under normal circumstances each surface $p = P$ (excepting a set of values of P of measure zero) is

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traversed ergodically and consequently determined by any line of force contained in it. Even when this is not so, however, we shall call it a magnetic surface.

As suggested by the foregoing discussion, we now explicitly assume that the magnetic surfaces form a family of nested toroids. The innermost toroid is degenerate, consisting of a single closed curve called the (magnetic) axis. We shall usually also assume that p increases monotonically going inward (as is proved for steady diffusing plasmas in section K) and indeed that $\nabla p \neq 0$ except on the axis.

C. SURFACE QUANTITIES

We now introduce two coordinate functions η and θ. Each is to be multi-valued, its values at any point differing by integers. The function η is to be continuous everywhere and to increase by unity during one traversal of the magnetic axis. The function θ is to be continuous everywhere except at the axis and is to increase by unity during one small loop around the axis. Finally, a pair of values of η and θ is to determine a unique point on each magnetic surface. For definiteness we assume that η, θ, p form a left-handed coordinate system.

For each particular magnetic surface we now define

$$V \equiv \int d\tau \tag{C1}$$

$$U \equiv \int d\tau \mathbf{B}^2 \tag{C2}$$

$$K \equiv \int d\tau \mathbf{B}\cdot\mathbf{j} \tag{C3}$$

$$\psi \equiv \int d\tau \mathbf{B}\cdot\nabla\eta \tag{C4}$$

$$\chi \equiv \int d\tau \mathbf{B}\cdot\nabla\theta \tag{C5}$$

$$I \equiv \int d\tau \mathbf{j}\cdot\nabla\eta \tag{C6}$$

$$J \equiv \int d\tau \mathbf{j}\cdot\nabla\theta, \tag{C7}$$

where $d\tau$ is the volume element and the region of integration is always the interior of the particular surface. (The integrals are well defined, since $\nabla\eta$ and $\nabla\theta$ are single-valued.) We note that V is the enclosed volume and U is twice the enclosed magnetic energy. There seems to be no simple physical interpretation of K, but its vanishing will be seen to be significant (section K). The integrands of (C4–7) can be written as divergences by (A3) and (B1), so we may apply Gauss' theorem; however, since η and θ are not single-valued, it is necessary first to cut the region of integration, say at $\eta = 0$ or at $\theta = 0$ as appropriate. Since by (B2) or (B3) the boundary contribution vanishes except for the double boundary at the cut, we then obtain, for example,

$$\begin{aligned}\psi &= \int_{\eta=1} dS(\nabla\eta/|\nabla\eta|)\cdot\mathbf{B}\eta - \int_{\eta=0} dS(\nabla\eta/|\nabla\eta|)\cdot\mathbf{B}\eta \\ &= \int_{\eta=1} dS(\nabla\eta/|\nabla\eta|)\cdot\mathbf{B},\end{aligned} \tag{C8}$$

where dS is the area element and the integrations are over those parts of the indicated surfaces of constant η which are interior to the particular magnetic surface. Thus ψ is the longitudinal magnetic flux inside the magnetic surface, i.e., the magnetic flux through any cross section of the interior. Similarly, I is the longitudinal current inside the particular surface. On the other hand, χ is what may be called the azimuthal magnetic flux inside the magnetic surface, since it is the flux through any ribbon-like surface of constant θ of which one edge is the magnetic axis and the other lies on the particular surface. Similarly, J is the azimuthal current inside the particular surface.

Functions of position (like p) which are constant on magnetic surfaces will be called surface quantities. The quantities defined by (C1–7) may be interpreted as functions of position in an obvious way and are then surface quantities. Any surface quantity may be considered as a function of any other, and derivatives of one with respect to another are meaningful and are themselves surface quantities.

It may be noted that definitions (C4–7) are invariant under continuous deformation of the coordinate functions η and θ. All functions η with the same direction of increase along the axis are deformable into each other. The analogous statement does not hold for θ, however; two functions θ are continuously deformable into one another if and only if their ribbons of constancy wind around the axis the same number of times. Two functions θ differ by an integral multiple of an acceptable function η, the integral multiplier being the difference of the winding numbers. If θ is increased by an integral multiple of η, χ and J are increased by the corresponding integral multiples of ψ and I, respectively. The results of the next section are manifestly invariant under these changes.

D. RELATIONS AMONG SURFACE QUANTITIES

Let $\mathbf{w}$ be any single-valued vector field satisfying

$$\nabla p\cdot(\nabla\times\mathbf{w}) = 0. \tag{D1}$$

Let $\mathbf{z}$ (P) be a particular point on each surface $p = P$. For each point $\mathbf{x}$ in space define

$$\nu(\mathbf{x}) = \int_{\mathbf{z}(p)}^{\mathbf{x}} d\mathbf{x}\cdot\mathbf{w}, \tag{D2}$$

where the path of integration lies on the surface; in view of (D1) it follows from Stokes' theorem that the value of $\nu(\mathbf{x})$ is independent of the path joining $\mathbf{z}$ to $\mathbf{x}$ for all paths continuously deformable into each other. However, not all paths are deformable into each other, so ν is multi-valued. It can clearly be written

$$\nu = \lambda + \eta \oint_{\substack{\theta=0\\ p=P}} d\mathbf{x}\cdot\mathbf{w} + \theta \oint_{\substack{\eta=0\\ p=P}} d\mathbf{x}\cdot\mathbf{w}, \tag{D3}$$

where λ is some single-valued function and the loop integrals are taken in the direction of increasing η and θ, respectively.

If it were not for the variability of the lower limit in (D2), we would have $\nabla\nu = \mathbf{w}$; as it is we have

$$\nabla p\times(\nabla\nu - \mathbf{w}) = 0 \tag{D4}$$

or equivalently

$$\mathbf{B}\cdot(\nabla\nu - \mathbf{w}) = 0,\ \ \mathbf{j}\cdot(\nabla\nu - \mathbf{w}) = 0. \tag{D5}$$

Now introduce two general surface quantities

$$F \equiv \int d\tau\mathbf{B}\cdot\mathbf{w},\ \ G \equiv \int d\tau\mathbf{j}\cdot\mathbf{w}. \tag{D6}$$

By (D5, 3) and (C4–7) we then obtain

$$\begin{aligned} dF &= d\int d\tau\mathbf{B}\cdot\nabla\nu \\ &= d\psi\oint_{\theta=0} d\mathbf{x}\cdot\mathbf{w} + d\chi\oint_{\eta=0} d\mathbf{x}\cdot\mathbf{w}, \end{aligned} \tag{D7}$$

$$dG = dI\oint_{\theta=0} d\mathbf{x}\cdot\mathbf{w} + dJ\oint_{\eta=0} d\mathbf{x}\cdot\mathbf{w}. \tag{D8}$$

We are now in a position to obtain various relations among the surface quantities by special choices of $\mathbf{w}$ satisfying (D1). In view of (A2) and (B3) we are justified in choosing $\mathbf{w} = \mathbf{B}$. In this case $F = U$ and $G = K$, while by Stokes' theorem applied to that part of the respective surface $\eta = 0$ or $\theta = 0$ which lies inside the magnetic surface we have

$$\oint_{\eta=0} d\mathbf{x}\cdot\mathbf{B} = -I, \tag{D9}$$

$$\oint_{\theta=0} d\mathbf{x}\cdot\mathbf{B} = J + \oint_{\psi=0} d\mathbf{x}\cdot\mathbf{B}, \tag{D10}$$

where the last integral is taken around the magnetic axis ($\psi = 0$) in the direction of increasing η. From (D7, 8) we, therefore, obtain

$$dU = \left(J + \oint_{\psi=0} d\mathbf{x}\cdot\mathbf{B}\right) d\psi - Id\chi, \tag{D11}$$

$$dK = \left(J + \oint_{\psi=0} d\mathbf{x}\cdot\mathbf{B}\right) dI - IdJ. \tag{D12}$$

Another choice for $\mathbf{w}$ is the vector potential $\mathbf{A}$ $(\nabla\times\mathbf{A} = \mathbf{B})$, justified by (B2). This leads analogously to

$$d\int d\tau\mathbf{B}\cdot\mathbf{A} = \left(\chi + \oint_{\psi=0} d\mathbf{x}\cdot\mathbf{A}\right) d\psi - \psi d\chi, \tag{D13}$$

$$d\int d\tau\mathbf{j}\cdot\mathbf{A} = \left(\chi + \oint_{\psi=0} d\mathbf{v}\cdot\mathbf{A}\, dI - \psi dJ. \tag{D14}$$

Our next choice is

$$\mathbf{w} = (\mathbf{B}\times\nabla p)/(\nabla p)^2, \tag{D15}$$

which may be justified by observing that

$$\nabla p\times\mathbf{w} = \mathbf{B} \tag{D16}$$

in view of (B2) and then using (A3). In this case we have $F = 0$ and $G = V$ in view of (A1). Also,

$$\begin{aligned} \oint_{\eta=0} d\mathbf{x}\cdot\mathbf{w} &= \oint_{\eta=1} d\mathbf{x}\cdot\mathbf{w}\eta - \oint_{\eta=0} d\mathbf{x}\cdot\mathbf{w}\eta \\ &= -\int_{p=P} dS(\nabla p/|\nabla p|)\cdot[\nabla\times(\mathbf{w}\eta)] \\ &= \int_{p=P} dS(\nabla p/|\nabla p|)\cdot(\mathbf{w}\times\nabla\eta) \\ &= \int_{p=P} dS(\mathbf{B}\cdot\nabla\eta)/|\nabla p| = -d\psi/dp. \end{aligned} \tag{D17}$$

where the first step is trivial, the second follows from Stokes' theorem applied to the cut magnetic surface, the third follows from (D1), the fourth from (D16), and the last from (C4) and the fact that

$$d\tau = -dSdp/|\nabla p|$$

(since $|dp|/|\nabla p|$ is the distance between two neighboring magnetic surfaces). Similarly

$$\oint_{\theta=0} d\mathbf{x}\cdot\mathbf{w} = d\chi/dp. \tag{D18}$$

Thus (D7) is tautological, but (D8) gives

$$dpdV = d\chi dI - d\psi dJ. \tag{D19}$$

Another possibility is to choose $\mathbf{w}$ to be a gradient, thus satisfying (D1) trivially. Indeed, let $\mathbf{w} = \nabla\mathbf{q}$ be a gradient of a vector field and, therefore, a dyadic, and note that nothing in the derivation of (D7, 8) is invalidated. If $\mathbf{q}$ is single-valued, the loop integrals in (D7, 8) vanish and we may conclude that F and G themselves (now vectors) vanish, which is also obvious from Gauss' theorem. Since $\nabla\mathbf{x}$ is the unit dyadic, taking $\mathbf{q} = \mathbf{x}$ gives

$$\int d\tau\mathbf{B} = 0,\ \ \int d\tau\mathbf{j} = 0. \tag{D20}$$

Taking $\mathbf{q} = \mathbf{B}$ instead and using (B4) gives

$$\begin{aligned} 0 = \int d\tau\nabla(p + \tfrac{1}{2}\mathbf{B}^2) &= \\ &- \int_{p=P} dS(\nabla p/|\nabla p|)(p + \tfrac{1}{2}\mathbf{B}^2), \end{aligned} \tag{D21}$$

and since the first term of the integrand contributes nothing (take p outside the integral and convert back to a volume integral),

$$\int_{p=P} dS(\nabla p/|\nabla p|)\mathbf{B}^2 = 0. \tag{D22}$$

E. AN IMAGINED EXPERIMENT

Suppose that everywhere in a given rigid toroidal tube T with perfectly conducting walls there is a viscous perfectly conducting fluid with an adiabatic equation of state, and also a magnetic field tangent to the tube walls at the walls. Suppose that any heat generated by the viscosity is somehow magically removed so that each element of fluid is isentropic. The system can then lose but not gain energy since there can be no energy flux through the walls.

Let the fluid be initially at rest. In general, it will not be in equilibrium and will start to move. As long as it moves it loses energy, so it must eventually come to rest in a state with less energy than its initial state. Clearly, an initially resting state of minimum energy cannot start moving at all and so must be in equilibrium; i.e., satisfy the magnetostatic equations.

Since we are comparing resting states we are interested in the non-kinetic (i.e., potential) energy W

$$W=\int_T d\tau(\tfrac{1}{2}\mathbf{B}^2+p/[\gamma-1]) \qquad \text{(E1)}$$

where the first and second terms of the integrand are the energy densities of the magnetic field and of the fluid, respectively; γ being the ratio of specific heats of the fluid. (If $\gamma = 1$ the second term should be $p \ln p$.)

F. INVARIANTS

It now appears that minimizing W should provide equilibrium solutions. However, we must be careful. If we minimize W outright we obtain $\mathbf{B} = 0$, $p = 0$, which, although certainly an equilibrium solution, is of no interest and is clearly not a state which will be reached eventually by every initial state. We have neglected to observe that any motion of our fluid is subject to certain constraints. By a constraint here is meant a condition that some quantity be an invariant during any motion, an invariant being a constant of motion which depends only on the instantaneous state and not on the velocity. (All constraints here are holonomic.) Only states with the same invariants can possibly be transformed into each other by a motion. We should therefore not minimize W among all states, but only among states with the same values of the invariants as the initial state.

We must therefore find invariants. Since the fluid is a perfect conductor it carries lines of force with it.[5] Therefore any topological property of the lines of force is an invariant. For instance, if in the initial state there were a line of force ergodic in T then this would have to be carried into a similarly ergodic line in the final state. But the final state has to satisfy (B2), so that p will be constant on the line and hence constant everywhere (if it is to be continuous). Such a state is not of interest. This example shows that we must choose the initial magnetic field to have precisely those topological properties possessed by equilibria of interest.

Accordingly, we choose an initial magnetic field which has a family of nested toroidal magnetic surfaces which are, however, not necessarily surfaces of constant p. The quantities ψ and χ are then defined, and since the lines of force are carried with the fluid, ψ and χ for the magnetic surface formed by some definite set of fluid particles are invariant during a motion. The quantities V, U, K, I and J are also defined but need not be invariant.

The way in which a line of force intertwines with itself as it is continued around its magnetic surface many times is a topological property and therefore another invariant, but it turns out to be describable in terms of ψ and χ and therefore does not provide an independent constraint. Indeed, this intertwining is characterized by the limit of the ratio of the number of loops around the magnetic axis to the number of traversals around the length of the toroid made by a line of force indefinitely prolonged; i.e., the limit of θ/η following the line. This limit is usually denoted by $\iota/2\pi$ and is equal to $d\chi/d\psi$.

Let ϱ be the mass density of the fluid. Then $\varrho d\tau$ is the mass of a little element of fluid and is therefore invariant during a motion. Furthermore, the adiabatic law assumed amounts to requiring that p/ϱ^γ be invariant for a fluid element. We thus have two purely hydrodynamic constraints. But ϱ is of no interest since it enters neither into the magnetostatic equations nor into the potential energy W. Eliminating ϱ we have only one invariant $p^{1/\gamma}d\tau$ for each element of fluid.

The invariants we have found (ψ and χ for magnetic surfaces, $p^{1/\gamma}d\tau$ for fluid elements) apply if we know which fluid element in the final state corresponds to each element in the initial state. However, there is no reference to this correspondence in (E1). We wish to minimize W for all states p, $\mathbf{B}$ which could possibly be reached by a motion from the initial state; i.e., for which there exists some correspondence preserving the value of the invariants. The correspondence not being known ahead of time, it now makes no sense to require that ψ and χ are individually preserved. Nevertheless, the correspondence must be chosen to preserve ψ and that same correspondence must preserve χ. Then χ considered as a function of ψ (for example) must be the same in the final state as in the initial state. In short, the functional relationship between χ and ψ is an invariant which can be specified without knowing the correspondence ahead of time. What has been done, in effect, is to label the surfaces with their values of ψ, after which the only magnetic invariant left is χ, now as a function of the label ψ.

These considerations so far only eliminate consideration of the correspondence of surfaces as a whole. To eliminate consideration of the correspondence of fluid elements within a given magnetic surface, we must use the invariant $p^{1/\gamma}d\tau$ to form a label. Assuming that lines of force are ergodic on almost all surfaces and choosing the correspondence for one particular fluid element as a reference point arbitrarily, we may use the integral of $p^{1/\gamma}d\tau$ along a little flux tube going from some point on the surface to the reference point as a label for that point. This label is clearly invariant and hence extends the correspondence from the arbitrary reference point to the whole surface (since the line of force through the reference point is assumed ergodic on the surface and hence covers a dense set of points).

However, in establishing this labeling we have not exhausted the information available from the invariance of $p^{1/\gamma}d\tau$. There remains the condition that the integral of $p^{1/\gamma}d\tau$ over the shell-like volume bounded by two neighboring surfaces must obviously be invariant. Introducing the surface quantity

$$M=\int d\tau p^{1/\gamma} \tag{F1}$$

(the integral being taken over the interior of the surface), we may require equivalently that M be invariant. (It may be observed that M is just proportional to the mass contained within the surface if ϱ happens to be such that the fluid is isentropic; i.e., if p/ϱ^γ is the same for all fluid elements. The invariance of M then represents conservation of mass.) As with χ, by considering M now to be a function of ψ, we eliminate any reference to the correspondence.

G. STATEMENT OF VARIATIONAL PRINCIPLE

The preceding considerations suggest the following variational principle: a function p and a solenoidal magnetic vector field $\mathbf{B}$ in T, forming nested toroidal magnetic surfaces and having a fixed total longitudinal flux and no normal component at the walls, make W stationary among all such pairs with the same invariant functions $\chi(\psi)$ and $M(\psi)$ if and only if $\nabla p=(\nabla\times\mathbf{B})\times\mathbf{B}$.

Before proving this it is desirable to reformulate it so as to include ψ explicitly in the characterization of a state, since otherwise it is difficult to tell whether a neighboring field $\mathbf{B}$ has nested toroidal surfaces. Correspondingly, we have an additional constraint and an additional variational condition. Thus we propose the following variational principle:

Consider all triples p, $\mathbf{B}$, and ψ in T satisfying the constraints

(a) ψ has toroidal level surfaces, $\psi=C$ at the walls, min $\psi=0$, max $\psi=C$,

(b) $$\nabla\cdot\mathbf{B}=0, \tag{G1}$$

(c) $$\mathbf{B}\cdot\nabla\psi=0, \tag{G2}$$

(d) $$\int_{\psi\leqslant c} d\tau\mathbf{B}\cdot\nabla\eta=c, \tag{G3}$$

(e) $$\int_{\psi\leqslant c} d\tau\mathbf{B}\cdot\nabla\theta=\chi(c), \tag{G4}$$

(f) $$\int_{\psi\leqslant c} d\tau p^{1/\gamma}=M(c); \tag{G5}$$

here C is a constant and $\chi(c)$ and $M(c)$ are arbitrary fixed functions defined for $0\leqslant c\leqslant C$, which is also the range of the last three conditions. Then a particular triple makes W stationary among all such triples if and only if it satisfies the variational conditions

(y) p is a function of ψ alone,

(z) $$\nabla p=(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{G6}$$

It should be noted that in non-degenerate cases in which (almost) any magnetic line of force covers a complete magnetic surface ergodically, (z) implies (y) in view of (B2) and (G2). In degenerate cases (y) is (partly) independent. But in these cases ψ is not itself physically significant, only p and $\mathbf{B}$, which are consistent with a variety of functions ψ. Just in these degenerate cases we could have found additional constraints in our imagined experiment (the integral of $p^{1/\gamma}d\tau$ in each thin closed flux tube) and by omitting these from our formulation of the variational principle we force upon ψ a physical significance (the only constraint on the flow of fluid elements is that they stay on surfaces of constant ψ) which is reflected in the variational condition (y).

H. PROOF

Given a function ψ satisfying (a), it is easily seen by the methods used at the beginning of section D that the most general field $\mathbf{B}$ satisfying (b) and (c) is given by

$$\mathbf{B}=\nabla\psi\times\nabla\nu, \tag{H1}$$

where ν is a multivalued function such that (H1) determines $\mathbf{B}$ uniquely; i.e., on each surface of constant ψ the various branches of ν differ only by constants. Furthermore, (d) and (e) are then satisfied if and only if ν can be written

$$\nu=\lambda+\eta\chi'(\psi)-\theta \tag{H2}$$

with λ single-valued, as can be seen upon comparing (H1) with (D16, 4) and referring to (D3, 17, 18).

Let us first assume that W is stationary and derive (y) and (z). For the moment hold ψ and λ fixed and vary only p. Then

$$\begin{aligned}0=\delta W&=(\gamma-1)^{-1}\int_T d\tau\delta p\\&=(\gamma-1)^{-1}\int_0^C dc\int_{\psi=c}(dS/|\nabla\psi|)\delta p\end{aligned} \tag{H3}$$

for any perturbation δp satisfying

$$\gamma^{-1}\int_{\psi=c}(dS/|\nabla\psi|)p^{1/\gamma-1}\,\delta p=0, \tag{H4}$$

which is obtained from (f) by varying p and also differentiating with respect to c. Picking δp so that the integrand of (H4) approximates to the difference of two Dirac delta functions with peaks at two points of the same surface, we satisfy (H4) and see from (H3) that p must have the same value at the two points. Thus (y) is established. We now have $p=P(\psi)$, whereby (f)

$$P(c)=[M'(c)/\int_{\psi=c}(dS/|\nabla\psi|)]^\gamma. \tag{H5}$$

Next we vary only λ, obtaining

$$\begin{aligned}0=\delta W&=\int_T d\tau\mathbf{B}\cdot\delta\mathbf{B}=\int_T d\tau\mathbf{B}\cdot(\nabla\psi\times\nabla\delta\lambda)\\&=-\int_T d\tau\delta\lambda\nabla\cdot(\mathbf{B}\times\nabla\psi),\end{aligned} \tag{H6}$$

$$(\nabla\times\mathbf{B})\cdot\nabla\psi=0. \tag{H7}$$

Finally we wish to vary only ψ. When we do so, $p=P(\psi)$ varies at a fixed point not only on account of the argument ψ but also because $P(c)$ does. To compute the contribution of $\delta P(c)$ to δW we note that

$$\delta\int_{\psi=c}\frac{dS}{|\nabla\psi|}=\delta\frac{d}{dc}\int_{\psi\leq c}d\tau=-\frac{d}{dc}\int_{\psi=c}dS\frac{\delta\psi}{|\nabla\psi|}, \tag{H8}$$

$$\begin{aligned}\int_T d\tau\delta P(\psi) &= \int_T d\tau P\delta\log P\\ &= \int_0^C dc\int_{\psi=c}\frac{dS}{|\nabla\psi|}P\gamma\left(\frac{d}{dc}\int_{\psi=c}dS\frac{\delta\psi}{|\nabla\psi|}\right)\Big/\int_{\psi=c}\frac{dS}{|\nabla\psi|}\\ &= \gamma\int_0^C dcP(c)\frac{d}{dc}\int_{\psi=c}dS\frac{\delta\psi}{|\nabla\psi|}\\ &= -\gamma\int_0^C dcP'\int_{\psi=c}dS\frac{\delta\psi}{|\nabla\psi|}\\ &= -\gamma\int_T d\tau P'(\psi)\delta\psi,\end{aligned} \tag{H9}$$

in which we have used the fact that $\delta\psi = 0$ at the walls in view of (a). Accordingly we have

$$\begin{aligned}0 = \delta W &= \int_T d\tau[\mathbf{B}\cdot\delta\mathbf{B} + (P'\delta\psi + \delta P)/(\gamma - 1)]\\ &= \int_T d\tau[\mathbf{B}\cdot(\nabla\delta\psi\times\nabla\nu + \nabla\psi\times\nabla\delta\nu) - P'\delta\psi]\\ &= \int_T d\tau\delta\psi[\nabla\cdot(\mathbf{B}\times\nabla\nu) - P'],\end{aligned} \tag{H10}$$

in which we have used (H7); there is no trouble with the discontinuity of $\nabla\theta$ at the magnetic axis because $\delta\psi$ vanishes there (since ψ and $\nabla\psi$ both do). Thus we obtain

$$(\nabla\times\mathbf{B})\cdot\nabla\psi = P'. \tag{H11}$$

Taking the cross-product of (H1) with $\nabla\times\mathbf{B}$ and using the variational conditions (H7, 11) gives

$$(\nabla\times\mathbf{B})\times\mathbf{B} = P'\nabla = \nabla p, \tag{H12}$$

which establishes (z).

It is clear that all the steps can be reversed to show that (y) and (z) imply that $\delta W = 0$ for all perturbations.

I. REMARKS

There is a noteworthy modification of the variational principle. Suppose we omit condition (e). We are then free to vary $\chi(c)$ so we obtain an additional variational condition from

$$\begin{aligned}0 = \delta W &= \int_T d\tau\mathbf{B}\cdot\delta\mathbf{B}\\ &= \int_0^C dc\delta\chi'(c)\int_{\psi=c}(dS/|\nabla\psi|)\mathbf{B}\cdot(\nabla\psi\times\nabla\eta),\end{aligned} \tag{I1}$$

$$\begin{aligned}0 &= \int_{\psi=c}dS(\nabla\psi/|\nabla\psi|)\cdot(\mathbf{B}\times\nabla\eta)\\ &= \int_{\leq c}d\tau\nabla\cdot(\mathbf{B}\times\nabla\eta) = \int d\tau\mathbf{j}\cdot\nabla\eta = I.\end{aligned} \tag{I2}$$

That is, we obtain just the additional condition appropriate for the steady-state of a diffusing plasma (see section K).

Our variational principle characterizes equilibria as stationary states of the potential energy W; i.e., states for which the first variation of W vanishes. The stability of such equilibria has been investigated elsewhere [6] by examining the positive-definiteness of the second variation of W.

Two limiting choices of γ are particularly simple. The first is $\gamma \to \infty$ (incompressibility) for which $M \to V$, so that we prescribe the volume to be enclosed by each surface and vary the magnetic energy alone. The second choice is $\gamma \to 0$ (pressure completely independent of density) for which M'^{γ} approaches the maximum value of p on the surface, so that we prescribe $p(\psi)$ and vary the integral [7] of $\frac{1}{2}\mathbf{B}^2 - p$.

J. THE STEADY SLOWLY DIFFUSING PLASMA

We now wish to obtain a complete set of equations governing a steady-state plasma slowly diffusing through a magnetic field to the containing walls of the toroidal tube T. We assume that the walls are perfect electric conductors with purely tangential magnetic field and also perfect plasma absorbers ($p = 0$ there) and that new plasma is somehow introduced or injected into T (necessary to maintain a steady state) at the source density rate S, which may depend on position. We assume that there is no temperature gradient and ignore a variety of complicating factors, such as nuclear reactions and radiation, which might occur in applications of interest.

Our steady state is almost static so to lowest order in the diffusion velocity $\mathbf{v}$ we have the magnetostatic equations (A1–3). We also have the (first order) equation of continuity

$$\nabla\cdot(p\mathbf{v}) = S, \tag{J1}$$

where we have taken the plasma density to be p in view of the assumed isothermality. In addition we have Maxwell's equation

$$\nabla\times\mathbf{E} = 0 \tag{J2}$$

and Ohm's law [8] which we take in the form

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{j}/\sigma + (\nabla p/p)\alpha, \tag{J3}$$

where $\mathbf{E}$ is the electric field, σ the conductivity (assumed constant and scalar) and α a physical constant.

Now (J2) holds everywhere in space (not just in T) so $\mathbf{E}$ is the gradient of a single-valued scalar. Since $\nabla p/p$ is also such a gradient we can introduce a single-valued scalar ϕ in T satisfying

$$\mathbf{E} = \nabla\phi + (\nabla p/p)\alpha, \tag{J4}$$

so that (J3) becomes

$$\nabla\phi + \mathbf{v}\times\mathbf{B} = \mathbf{j}/\sigma. \tag{J5}$$

Let us consider (J5) as an equation for $\mathbf{v}$. The condition that it have a solution is

$$\mathbf{B}\cdot\nabla\phi = (\mathbf{B}\cdot\mathbf{j})/\sigma \tag{J6}$$

and if this is satisfied the general solution of (J5) is

$$\mathbf{v} = \mathbf{B}^{-2}(\nabla\phi - \mathbf{j}/\sigma) \times \mathbf{B} + a\mathbf{B}, \tag{J7}$$

where a is an arbitrary scalar function. Eliminating $\mathbf{v}$ by (J7) and using (A1), we have for (J1)

$$\nabla\cdot(pa\mathbf{B}) = S + \nabla\cdot[(p/\mathbf{B}^2)(\mathbf{B} \times \nabla\phi + \nabla p/\sigma)], \tag{J8}$$

which may be viewed as a diffusion equation of sorts for p, the diffusion coefficient being $p/\mathbf{B}^2\sigma$.

K. THE TWO AUXILIARY CONDITIONS

By (A3) and (B2) the left-hand side of (J8) may be written $p\mathbf{B}\cdot\nabla a$. Thus (J8) as well as (J6) are what we may call " magnetic differential equations "; namely equations of the type

$$\mathbf{B}\cdot\nabla r = s, \tag{K1}$$

with scalar r and s. Viewed as an equation for r with s given, (K1) determines how r varies along a magnetic line of force. In the nondegenerate case that the line covers a magnetic surface ergodically, (K1) and the assignment of a value to r at one point determine r at a set of points dense in the surface. A necessary condition that the values of r so obtained be expandable to a continuous single-valued function over the whole surface is easily derived by integrating (K1) over the shell volume between two neighboring magnetic surfaces $p = P$ and $p = P + dP$. By (A3), Gauss' theorem, and (B2) the left-hand side then vanishes, while after dividing by dP the right-hand side becomes

$$\int_{p=P} s\,dS/|\nabla p| = 0. \tag{K2}$$

It is plausible to assume (and we shall) that, in the nondegenerate case, (K2) is also a sufficient condition for (K1) to have a continuous single-valued solution r. It is then clear that (K1) determines r up to a surface quantity.

In accordance with the foregoing paragraph, the conditions that (J6) permit a solution ϕ and (J8) a solution a are

$$\int_{p=P} (dS/|\nabla p|)\mathbf{B}\cdot\mathbf{j} = 0, \tag{K3}$$

$$\int_{p=P} (dS/|\nabla p|)\{S + \nabla\cdot[(p/\mathbf{B}^2)(\mathbf{B}\times\nabla\phi + \nabla p/\sigma)]\} = 0. \tag{K4}$$

The latter may be considerably simplified by multiplying by dP and integrating over all magnetic surfaces interior to a particular one; using Gauss' theorem then leads to

$$\int d\tau S \mp \int dS(\nabla p/|\nabla p|)\cdot(\mathbf{B}\times\nabla\phi + \nabla p/\sigma)p/\mathbf{B}^2 = 0, \tag{K5}$$

where the minus or plus sign is to be adopted accordingly as p decreases or increases going outward (so that $\mp\nabla p/|\nabla p|$ is the unit outward normal to the magnetic surface). Eliminating $\nabla\phi$ by (A1) except in the denominator, expanding out the inner products of the cross products, and using (J6) gives

$$\int d\tau S \mp \int (dS/|\nabla p|)(-\mathbf{j}\cdot\nabla\phi + \mathbf{j}^2/\sigma)p = 0. \tag{K6}$$

By (B1, 3) the term involving ϕ can be written as a divergence; converting the surface integral to a shell volume integral, we see by Gauss' theorem and (B3) that the contribution of that term is zero. Since S, σ, and p are essentially positive, we must take the minus sign. Thus p decreases going outward (as assumed at the end of section B) and (K6) may be written

$$\int_{p\geq P} d\tau S - (P/\sigma)\int_{p=P} (dS/|\nabla p|)\mathbf{j}^2 = 0. \tag{K7}$$

Our system of equations now consists of the magnetostatic equations (A1–3) together with the auxiliary conditions (K3, 7). For any solution p, $\mathbf{B}$, $\mathbf{j}$ of this system we can find ϕ and a and therefore $\mathbf{E}$ and $\mathbf{v}$, which together with the solution represent a slowly diffusing equilibrium. (The arbitrariness of a surface quantity each in a and ϕ corresponds to a physical arbitrariness in the total fluid flow along lines of force and in the total charge on a magnetic surface, respectively.)

Condition (K7) can be shown to be equivalent to the energy balance equation, which could have been written down *a priori*. Condition (K3) may be written $-dK/dp = 0$ or, integrating, $K = 0$. When (K3) holds (D12) can be integrated to show that I is proportional to $J + \oint d\mathbf{x}\cdot\mathbf{B}$. Since I and J but not the loop integral vanish at the axis, the constant of proportionality vanishes and $I = 0$. (Conversely the vanishing of I for all surfaces entails that of K.)

Thus (K3) is equivalent to (I2), the extra variational condition obtained by minimizing W without prescribing χ. In other words, in the diffusing plasma the azimuthal magnetic flux adjusts to give the lowest energy; the lines of force associated with the longitudinal magnetic flux are permanently trapped by the perfectly conducting walls, but " untwist " themselves locally as much as possible.

It is not hard to see that the vanishing of dI at a particular magnetic surface implies that the lines of electric current there are closed curves and, indeed, closed curves topologically like (deformable into) curves of constant η. The converse is even easier to see (choose η to be constant on the current lines).

L. THE LOW PRESSURE LIMIT

It is physically quite plausible to suppose that our system of equations has a solution p, $\mathbf{B}$, $\mathbf{j}$, and a unique one for any reasonable general prescription of the tube T, the source function S, the conductivity σ and the total trapped longitudinal magnetic flux C. This supposition is further borne out by the variational principle which shows that two auxiliary conditions for each magnetic surface are just the appropriate number. We now proceed to prove the supposition in the limiting case of low pressure under an assumption on the geometry of T which is apparently necessary if the solution is to behave regularly in the limit. This assumption is that the unique vacuum magnetic field which is purely tangential at the walls and has a prescribed total longitudinal flux C vanishes nowhere and determines a nondegenerate family of nested toroids. (For a wide class of toroidal geometries this is assured to a high approximation by rotational transform theory.[9])

For p small (A1) becomes, in the limit, $\mathbf{j}\times\mathbf{B}=0$ or

$$\mathbf{j}=g\mathbf{B} \tag{L1}$$

with g a scalar function. From (A3) and (B1) we obtain the magnetic differential equation

$$\mathbf{B}\cdot\nabla g = 0, \tag{L2}$$

which implies that g is a surface quantity. But then using (L1) in (K3) and taking g out of the integral gives $g = 0$ and, therefore, $\mathbf{j} = 0$. Thus $\mathbf{j}$ must be small if p is small.

Starting over now with p and $\mathbf{j}$ both small, we see from (A2, 3) that to lowest order $\mathbf{B}$ must be the unique magnetic field of our assumption. Now (A1) is equivalent to the pair of equations obtained by taking the inner and the cross products with $\mathbf{B}$; namely, (B2) and

$$\mathbf{j} = [\mathbf{B}\times\nabla p]/\mathbf{B}^2 + h\mathbf{B}, \tag{L3}$$

with h a scalar function. The only restriction (A2) places on our remaining unknowns p and $\mathbf{j}$ is (B1). We now adopt the point of view that (L3), (B1), and (K3) are conditions on $\mathbf{j}$, while (B2) and (K7) are conditions on p.

Now (L3) expresses $\mathbf{j}$, in terms of what we may consider a new scalar unknown h. But then (B1) is equivalent to the magnetic differential equation

$$\mathbf{B}\cdot\nabla h = -(\mathbf{B}\times\nabla p)\cdot\nabla(1/\mathbf{B}^2) \tag{L4}$$

by (A3) and (B5), while (K3) becomes

$$\int_{p=P} (dS/|\nabla p|)h\mathbf{B}^2 = 0 \tag{L5}$$

and may be viewed as determining the additive surface quantity left arbitrary in h by (L4).

Of course by (B2) p is a surface quantity of the magnetic surfaces of the vacuum field and is therefore a function of ψ only. It is now obvious from (L4, 5) that h is determined completely independently of p except for being proportional to $p'(\psi)$, and by (L3) the same is true for $\mathbf{j}$. Indeed, setting

$$\mathbf{j} = p'\mathbf{y}, \quad h = p'k, \tag{L6}$$

we see that $\mathbf{y}$ and k are uniquely determined independently of p by the three equations obtained from (L3–5) by replacing p, $\mathbf{j}$, and h everywhere by ψ, $\mathbf{y}$ and k, respectively.

We can now write (K7) in the form

$$\int_{\psi\leq c} d\tau S + pp'/\sigma \int_{\psi=c} \mathbf{y}^2 dS/|\nabla\psi| = 0. \tag{L7}$$

(Note that S is of the second order of smallness compared with p and $\mathbf{j}$.) Thus $(p^2)'$ is determined and, since p vanishes at the wall, p is determined, whereupon $\mathbf{j}$ is determined by (L6). This completes the proof.

It may be noted that $(p^2)'$ remains finite at the walls. Thus p' becomes infinite there, p varying as the square root of the distance to the walls.

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