EQUILIBRIUM PATHS OF MECHANICAL SYSTEMS WITH UNILATERAL CONSTRAINTS PART I: THEORY

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Abstract

The paper shows that the behaviour of mechanical systems subject to unilateral constraints differs from that of standard systems in subtle and yet important ways; therefore a proper theoretical formulation is required for simulating their behaviour. After showing that the equilibrium equations for a multibody system subject to unilateral constraints have the same form as the standard Kuhn-Tucker conditions in optimisation theory, the first-order equilibrium equations are derived and their integration is discussed. At a general integration step one has to distinguish between constraints that are strongly active, weakly active, and inactive. Whereas strongly active constraints can be treated like bilateral constraints and inactive constraints can be neglected, weakly active constraints need to be constantly re-analysed to determine if they switch to a different state. The outcome is that, in addition to the well-known limit points and bifurcation points, a new type of limit point can exist, where the path is non-smooth and the first-order equilibrium equations —after elimination of any strongly active constraints— non-singular. Such points are called corner limit points. In analogy with common limit points, the degree of instability of the system changes by one at a corner limit point.

(Key words: Multibody Systems, Unilateral Constraint, Equilibrium Path, Limit Point, Snap-through, Bifurcation.)

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Nomenclature

E, E_0, E_+	constraint matrices of bilateral, weakly and strongly active unilateral constraints
G	matrix defined in Equation 50
H, H_+	reduced Hessian matrices
H_+/H	Schur complement of H in H_+
I, I^*, I^*_+	index sets of unilateral constraints, and of active and strongly active constraints
$\mathcal{L}$	Lagrangian function
T, T_+	matrices of vectors that span the nullspace of E^T or E_+^T
U	potential energy
W	matrix of eigenvectors of reduced Hessian
W_+	matrix defined in Equation 74
c_1, c_2, c_3, c_4	coefficients
d_{ii}	elements of diagonalized reduced Hessian
g_j, g, g_0	constraint function, column vector of constraint functions, corresponding column vector for weakly active constraints
$\hat{g}$	vector defined in Equation 51
m	number of bilateral or unilateral constraints
m_+	number of unilateral constraints that are strongly active
n	number of generalized coordinates
p	control parameter
s	path parameter
t_i	i-th row of T
t_0	column vector defined Equation 55
w_i	i-th column of W
x, y, z	vectors
φ_j, φ	generalized coordinates, column vector of generalized coordinates
λ_j, λ, λ_0, λ_+	Lagrange multipliers, column vector of Lagrange multipliers, corresponding column vectors for weakly (subscript 0) and strongly active (subscript +) constraints

1 Introduction

A common problem in structural mechanics is the determination of the static equilibrium path of a structure for varying values of a certain parameter, which may be an external load, a displacement, or the arc-length of the path itself. The reason why it is useful to know the equilibrium path, and also which of its parts are stable or unstable, is that it characterises the behaviour of the structure. Designers are, of course, interested in any special features of a structure that might cause it to suddenly collapse. Special points of the equilibrium path are very important. For example, in a shallow arch or dome a catastrophic change of configuration occurs at the limit point, and the load-carrying capacity of a thin-walled cylindrical shell in compression suddenly decreases at the bifurcation point. Therefore, much work has been done on the computation of the equilibrium paths of all kinds of structures, and also on robust solution procedures to handle special features that are encountered in certain cases. See, for example, Crisfield (1991) for more details.

More recently, there has been a growing interest in mechanical systems that consist of rigid, or elastically deformable elements, such as rods or shells, connected together by passive joints, such as hinges or sliders, driven by motors or other kinds of actuators. These are known as multibody systems and are covered by a now extensive literature (Garcia de Jalon and Bayo 1994, Shabana 1989). Multibody systems have a variety of applications, including robotic arms and automotive suspensions.

The particular type of system that we are interested in is a linkage of flexible elements connected by mechanical joints that undergo large changes of configuration. This is a model for a type of deployable structure that is packaged for storage and/or transportation, and can be automatically deployed into its operational configuration (Pellegrino and Guest 1999). Many such structures have a small mobility and hence can be analysed by a purely kinematic approach, see for example Kumar and Pellegrino (1997). However, there is an increasing tendency to design systems with a mobility of zero, whose flexible elements are able to store strain energy during folding and to release it during deployment (Hayman, Hedgepeth and Park 1994).

To simulate such systems, it is useful to think in terms of an equilibrium path in a configuration space that includes a set of independent coordinates augmented with a parameter that controls the configuration of the system. Standard commercial multibody packages are currently unable to handle the complex behaviour of these structures. Recently, a general computational approach to systems with bilateral constraints has been proposed by Crisfield (1997), and some simple examples have been analysed by Cardona and Huespe (1998).

The research that is presented in this paper originated from the need to develop a realistic simulation of a solid surface deployable antenna, recently developed in the Deployable Structures Laboratory at Cambridge University (Guest and Pellegrino 1996). A physical model of the antenna had shown that the antenna follows different paths during deployment and retraction; in Schulz and Pellegrino (1998) we described in detail this behaviour. We showed, by analysing a simple multibody system, that the observed behaviour can be explained by the existence of two limit points on the equilibrium path; the first would cause a snap during deployment and the second a snap during retraction. Thus, it would be possible for a deployable structure to be in a particular configuration at the beginning of a deployment-retraction cycle and in a different configuration at the end of it. Further work on systems that model more closely the real antenna has shown, however, that the two limit points that are required to produce this behaviour exist only for a small range of system parameters.

An important effect that was not included in our previous study and, as far as we know, has not been previously considered in multibody systems, is the effect of unilateral constraints on the equilibrium path. We have discovered that there are some important differences between systems of this type and the better-known type of systems subject to bilateral constraints. Thus, the aim of this paper is to develop a full understanding of the type of behaviour that can occur in systems with unilateral constraints and, based on this understanding, to develop a proper formulation for simulating such systems. Simple examples are used to illustrate key points, and a full application is presented in the companion paper (Schulz and Pellegrino 1999).

The paper is laid out as follows. Section 2 derives the first-order equilibrium equations for a multibody system that is subject to bilateral constraints, by generalising a formulation in terms of independent coordinates in Thompson and Hunt (1973). A way of switching the integration variable near a limit point is discussed.

Section 3 derives the first-order equilibrium equations for a conservative multibody system subject to unilateral constraints. In addition to requiring that the Lagrangian of the system be stationary in an equilibrium configuration, it is also required that the reactions of all unilateral constraints be either zero, if the constraint is not active, or non-negative, if the constraint is active. Mathematically, these conditions have the same form as the standard Kuhn-Tucker conditions in optimisation theory.

It is important to distinguish between strongly active constraints, which in a particular configuration are active with non-zero reaction, and weakly active, whose reaction is zero. When integrating the first-order equilibrium equations, at a general step all strongly active constraints can be treated like bilateral constraints until they become weakly active or inactive, but weakly active constraints need to be re-analysed to determine if they switch to a different state.

Section 4 shows that, in addition to the well-known type of limit points, there is a new type where the path is non-smooth and the first-order equilibrium equations non-singular. Such points are named *corner limit points*; they are associated with the change of state of a weakly active constraint.

In Section 5 it is shown that, in analogy with common limit points, at a corner limit point the degree of instability of a system also changes by one. A simple example is presented in Section 6. Section 7 analyses the different cases that can be encountered when the first-order equilibrium equations become singular.

A discussion, including a brief comparison of bilaterally and unilaterally constrained systems, concludes the paper.

2 Systems with bilateral constraints

We consider a conservative multibody system which is described by a set of n generalized position coordinates $\varphi = [\varphi_1 \quad \varphi_2 \quad \dots \quad \varphi_n]^T$ and m scleronomic, bilateral constraints

$$g_j(\varphi) = 0, \quad j = 1, \dots, m \tag{1}$$

The potential energy U of the system is assumed to be a function of the generalized coordinates and a single control parameter p. It is assumed that the constraint functions, as well as the energy function are C^2-continuous. A standard procedure for enforcing m constraints is to introduce the Lagrangian function

$$\mathcal{L}(\varphi, p, \lambda) = U(\varphi, p) - g^T \lambda \tag{2}$$

where λ is a vector of Lagrange multipliers and

$$g = [g_1 \quad g_2 \quad \dots \quad g_m]^T \tag{3}$$

The equations of equilibrium for this system can be obtained by differentiating Equation 2 with respect to φ and λ, which gives

$$\nabla_\varphi \mathcal{L} = \nabla_\varphi U - E\lambda = 0 \quad , \tag{4}$$

$$\nabla_\lambda \mathcal{L} = -g = 0 \quad . \tag{5}$$

Here E, of size $n \times m$, is a constraint matrix, defined by

$$E = [\nabla g_1 \quad \nabla g_2 \quad \dots \quad \nabla g_m] \tag{6}$$

It will be assumed that the gradients of the constraint functions are independent in all configurations of interest, and hence E has full rank (regularity assumption). The term $E\lambda$ represents the generalized constraint reactions.

Our aim is to determine equilibrium configurations of the system for varying values of the control parameter p. Excluding bifurcational behaviour, for the moment, the tangent to the equilibrium path in the φ-p space is uniquely defined at every point of the path. The equilibrium path can be expressed with the aid of a parameter s:

$$\varphi = \varphi(s), \quad \lambda = \lambda(s), \quad p = p(s) \tag{7}$$

A suitable choice for s would be the arc-length of the path itself or —away from limit points— the control parameter p. Note that the symbol φ is being used to denote both a general configuration of the system and an equilibrium configuration.

Starting from a given equilibrium configuration ($s = 0$) the equilibrium path can be traced by applying a continuation method, i.e. the parameter s is incremented in finite steps and the nonlinear algebraic equations, Equations 4-5 are solved iteratively for the current value of s. Efficient techniques for doing this exist (Riks 1979, Crisfield 1980).

A different approach will be followed in this study. The idea is to define the equilibrium path in terms of a set of differential equations in φ and λ that are numerically integrated with a standard solver. This is conceptually similar to using a continuation algorithm that evaluates derivatives in order to obtain improved initial estimates in each step. These differential equations are the *first-order equilibrium equations* for the system (Thompson and Hunt 1973), obtained by differentiating Equations 4-5 with respect to s

$$(\nabla^2_\varphi \mathcal{L})\dot{\varphi} + (\nabla_\varphi U')\dot{p} - E\dot{\lambda} = 0 \tag{8}$$

$$E^T \dot{\varphi} = 0 \tag{9}$$

where $(\dot{\ })$ denotes total differentiation with respect to s; $(\ )'$ denotes partial differentiation with respect to p; and the Hessian of the Lagrangian function

$$\nabla^2_\varphi \mathcal{L} = \nabla^2_\varphi U - \sum_{j=1}^{m} \lambda_j \nabla^2 g_j = \nabla^2_\varphi U - [E_{,\varphi_1}\lambda \quad E_{,\varphi_2}\lambda \quad \dots] \tag{10}$$

is not necessarily regular.

Equation 9 is equivalent to stating that $\dot{\varphi}$ is in the nullspace of the constraint matrix transposed. Hence, considering a matrix T whose columns span the nullspace

$$\dot{\varphi} = Tx \tag{11}$$

where $x \in R^{n-m}$. Substituting Equation 11 into Equation 8 and premultiplying by T^T yields

$$Hx + T^T(\nabla_\varphi U')\dot{p} = 0 \tag{12}$$

where

$$H = T^T(\nabla^2_\varphi \mathcal{L})T \tag{13}$$

is the reduced Hessian.

If the reduced Hessian is regular we can solve Equation 12 for x and substitute the result into Equation 11

$$\dot{\varphi} = -TH^{-1}T^T(\nabla_\varphi U')\dot{p} \tag{14}$$

Then, the path derivative of λ is obtained by pre-multiplying Equation 8 by E^T and solving the resulting equation

$$\dot{\lambda} = (E^T E)^{-1}E^T\{(\nabla^2_\varphi \mathcal{L})\dot{\varphi} + (\nabla_\varphi U')\dot{p}\} \tag{15}$$

Note that the matrix $E^T E$ is positive definite due to the regularity assumption.

If we choose the control parameter as the path parameter, $s = p$, Equations 14 and 15 become

$$\begin{aligned} \frac{d\varphi}{dp} &= -TH^{-1}T^T\nabla_\varphi U' && (16)\\ \frac{d\lambda}{dp} &= (E^T E)^{-1}E^T\{(\nabla^2_\varphi \mathcal{L})\frac{d\varphi}{dp} + \nabla_\varphi U'\} && (17) \end{aligned}$$

2.1 Singularities

In some equilibrium configurations the reduced Hessian H may become singular. Let $W = [w_1 \dots w_m]$ be the matrix of the eigenvectors of H; the diagonalised form of H is

$$W^T H W = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \dots & \\ & & & d_{mm} \end{bmatrix} . \tag{18}$$

Introducing the transformation

$$x = Wy \tag{19}$$

the diagonalized form of Equation 12 is

$$W^T HWy + W^T T^T \nabla_\varphi U' \dot{p} = 0 \quad . \tag{20}$$

Assuming that only the first eigenvalue is zero, the first equation in Equation 20, which corresponds to this eigenvalue, reads

$$0y_1 + w_1^T T^T \nabla_\varphi U' \dot{p} = 0 \tag{21}$$

If $w_1^T T^T \nabla_\varphi U' = 0$, it is likely that the system is at a *bifurcation point*, in which case a higher-order analysis is required (Thompson and Hunt 1973). A corresponding equation has been derived, for the case of unconstrained systems, by Riks (1979). The post-critical behaviour can be determined by applying a branch switching procedure (Stein, Wagner and Wriggers 1990).

If $w_1^T T^T \nabla_\varphi U' \neq 0$, then Equation 21 requires

$$\dot{p} = 0 \tag{22}$$

and the system is likely to be at a *limit point* (Thompson and Hunt 1973), in which case the solution of Equation 20 is

$$y = [y_1 \quad 0 \quad \ldots \quad 0]^T \tag{23}$$

and hence, from Equations 19 and 11

$$x = w_1 y_1 \tag{24}$$

$$\dot{\varphi} = T w_1 y_1 \tag{25}$$

Because $\dot{p} = 0$, the control parameter p cannot be chosen as path parameter and hence an alternative choice is required. Any generalised coordinate such that $\dot{\varphi}_i \neq 0$ can be chosen and, considering Equation 25, this requires that the i-th component of Tw_1 be not equal to zero.

In practice, it is best to switch parameters before getting too close to a limit point, and a reasonable strategy is to choose the particular generalized coordinate that corresponds to the largest magnitude component of Tw_1. Letting φ_i be the new path parameter, Equation 12 is supplemented by

$$\dot{\varphi}_i = 1 \tag{26}$$

Hence, Equations 11 and 12 yield

$$\begin{bmatrix} H & T^T \nabla_\varphi U' \\ t_i & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{27}$$

where t_i denotes the i-th row of T. After solving Equation 27 we can compute the path derivatives of the generalized coordinates according to Equation 11.

3 Systems with unilateral constraints

This section extends the formulation presented in Section 2 to systems with unilateral constraints, representing frictionless contacts between different parts of the system. Consider a system with m unilateral constraints

$$g_j(\varphi) \geq 0, \quad j \in I \tag{28}$$

where $I = \{1, 2, \dots, m\}$ identifies the full set of constraints. For simplicity, it is assumed that there are no bilateral constraints, and that the coordinates φ are a set of independent coordinates when all unilateral constraints are inactive.

As in the previous section, the potential energy U of this conservative system depends on the generalized coordinates φ plus a single control parameter p. The same assumptions on the continuity of U and g_j are made, and an analogous Lagrangian function is defined

$$\mathcal{L}(\varphi, p) = U - \sum_{j \in I} \lambda_j g_j \quad . \tag{29}$$

The differentiation of this function with respect to φ is unchanged from Section 2 and yields an equation equivalent to Equation 4. Equation 5 is now replaced by inequality constraints and, additionally, the Lagrange multipliers must be non-negative, and cannot be non-zero together with the corresponding constraint function (complementarity condition). That is, a constraint is either inactive ($g_j > 0$) or it is active ($g_j = 0$) and its reaction is positive ($\lambda_j \geq 0$). See Pfeiffer and Glocker (1996) for further details. Mathematically

$$\nabla_\varphi \mathcal{L} = \nabla_\varphi U - \sum_{j \in I} \lambda_j \nabla g_j = 0 \quad , \tag{30}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = -g_j \leq 0, \quad \lambda_j \geq 0, \quad \lambda_j g_j = 0, \quad j \in I \quad . \tag{31}$$

In optimization theory Equations 30-31 are known as Kuhn-Tucker conditions (Fletcher 1987). It is well-known that their validity is subject to a regularity condition, see below.

We distinguish between *active constraints*, for which $g_j = 0$, whose index set is

$$I^* = \{j \in I | g_j(\varphi) = 0\} \tag{32}$$

and *inactive constraints*, for which $g_j > 0$. Active constraints can be further subdivided into *strongly active*, $\lambda_j > 0$, whose index set is

$$I_+ = \{j \in I | g_j = 0, \quad \lambda_j > 0\} \tag{33}$$

and *weakly active constraints*, $\lambda_j = 0$, whose set is simply $I^* \backslash I_+$. The regularity condition requires that the gradient vectors ∇g_j of all active constraints be linearly independent.

We are interested in tracing the equilibrium path of this system following an approach similar to Section 2. However, the path in the φ-p space is in general non-smooth, due to the change of state of unilateral constraints from active to inactive, or viceversa. Therefore, whereas for systems with bilateral constraints the equilibrium path is smooth and the path derivatives are continuous, now we will have to consider left-hand side and right-hand side derivatives, which may not coincide.

In analogy with Section 2, we have to differentiate Equation 30 with respect to s. Due to the non-smoothness of the path, we replace ordinary derivatives with right-hand side derivatives with respect to s, denoted by $(\dot{\ })$

$$(\nabla^2_\varphi \mathcal{L})\dot{\varphi} + (\nabla_\varphi U')\dot{p} - E_+\dot{\lambda}_+ - E_0\dot{\lambda}_0 = 0 \quad , \tag{34}$$

$$\nabla^2_\varphi \mathcal{L} = \nabla^2_\varphi U - \sum_{j \in I_+} \lambda_j \nabla^2 g_j \tag{35}$$

where the constraint matrices for the strongly and weakly active constraints, E_+ and E_0 respectively, and the corresponding column vectors of path derivatives of Lagrange multipliers, are

$$E_+ = [\nabla g_j], \quad \dot{\lambda}_+ = \{\dot{\lambda}_j\}, \quad j \in I_+ \quad , \tag{36}$$

$$E_0 = [\nabla g_j], \quad \dot{\lambda}_0 = \{\dot{\lambda}_j\}, \quad j \in I^* \backslash I_+ \quad . \tag{37}$$

The reason why the last two terms in Equation 34 include only the active constraints is because for every point $\tilde{\varphi}$ of the equilibrium path, excluding end points, there exists a neighbourhood on the path where the strict inequality $g_j > 0$ holds for all constraints which are inactive at $\tilde{\varphi}$. In this neighbourhood, the corresponding Lagrange multipliers λ_j are equal to zero, due to the complementarity condition $g_j\lambda_j = 0$, and thus their path derivatives $\dot{\lambda}_j$ are also zero at $\tilde{\varphi}$.

Next, we turn to Equation 31. Since $\nabla_\varphi U$ and ∇g_j are continuous on the equilibrium path in the φ-p space, the regularity assumption and Equation 30 imply that λ_j is also continuous. For this reason, at every equilibrium point $\tilde{\varphi}$ there exists a neighbourhood on the equilibrium path —excluding end points— where the strict inequality $\lambda_j > 0$ holds for all constraints which are strongly active at $\tilde{\varphi}$. The corresponding constraint functions are equal to zero in this neighbourhood because of the complementarity condition, and hence also their path derivatives at $\tilde{\varphi}$. In conclusion,

$$\dot{g}_j = 0, \quad j \in I_+ \quad . \tag{38}$$

The corresponding equation for systems with bilateral constraints would be identical. In analogy with Section 2, we replace Equation 38 with

$$\dot{\varphi} = T_+ x \quad , \tag{39}$$

where the columns of T_+ span the nullspace of E_+^T.

For weakly active constraints the path derivatives of the constraint functions and of the corresponding Lagrange multipliers are greater than or equal to zero. Moreover, complementarity must hold for these path derivatives

$$\dot{g}_j \geq 0, \quad \dot{\lambda}_j \geq 0, \quad \dot{\lambda}_j \dot{g}_j = 0, \quad j \in I^* \backslash I_+ \quad . \tag{40}$$

Otherwise, the constraint functions and the Lagrange multipliers could become negative and there would exist points where the complementarity condition, Equation 31, would be violated.

At this point, we eliminate the strongly active constraints by introducing a reduced Hessian, as in Section 2

$$H_+ = T_+^T (\nabla_\varphi^2 \mathcal{L}) T_+ \quad . \tag{41}$$

Defining

$$\dot{g}_0 = \{\dot{g}_j\}, \quad j \in I^* \backslash I_+ \tag{42}$$

we can write Equations 34 and 40 as

$$H_+ x + T_+^T (\nabla_\varphi U') \dot{p} - T_+^T E_0 \dot{\lambda}_0 = 0 \quad , \tag{43}$$

$$\dot{g}_0 \geq 0, \quad \dot{\lambda}_0 \geq 0, \quad \dot{\lambda}_0^T \dot{g}_0 = 0 \quad , \tag{44}$$

where x and $\dot{g}_0$ are linked by

$$\dot{g}_0 = E_0^T T_+ x \quad . \tag{45}$$

The inequalities in Equation 44 are to be interpreted in the sense that all elements of a vector are greater than or equal to zero.

Equations 43 and 44 are the first-order equilibrium equations of a multibody system with unilateral constraints. Similar equilibrium equations were derived by Björkman (1992) for structures that come into contact with a single rigid obstacle. Björkman made use of the mathematical concept of B-differentiability. Our derivations are more straightforward, as no advanced mathematical concepts are required.

Consider the case of systems with bilateral constraints. The standard stability condition for a solution φ, λ of Equations 4-5 is

$$z^T(\nabla^2_\varphi \mathcal{L})z > 0 \tag{46}$$

for any $z \neq 0$ in the nullspace of the constraint matrix transposed. This is a sufficient condition for uniqueness of the solution and regularity of the reduced Hessian. In the case of systems with unilateral constraints the stability condition would have to be satisfied only for those elements $z \neq 0$ of the nullspace of the matrix of strongly active constraints transposed that also satisfy the condition

$$z^T \nabla g_j \geq 0 \quad \forall j \in I^* \backslash I_+ \quad . \tag{47}$$

However, this will not guarantee the reduced Hessian H_+ to be positive definite, or indeed Equations 43 and 44 to have a unique solution.

The situation can be illustrated with the aid of the example depicted in Figure 1. The two generalized coordinates are subjected to two linear, unilateral constraints. The potential energy U is a quadratic, positive semidefinite function of the generalized coordinates, whose contours are shown in the figure. For $s = 0$, Figure 1(a), the intersection of the constraint functions corresponds to a stable equilibrium point and Equation 46 holds for $z \neq 0$ in the nullspace of E_+^T and satisfying Equation 47. There is no other feasible equilibrium point. Both constraints are weakly active at the point of equilibrium.

Let us now consider a continuous transformation of the potential energy from $s = 0$, Figure 1(a), to $s = \epsilon > 0$, Figure 1(b). Obviously, all points on the segment AB correspond to equilibrium solutions and, thus, the path derivative of φ at the equilibrium point of Figure 1(a) is not unique. There are path derivatives with

1. $\dot{g}_1 > 0$, $\dot{g}_2 > 0$,

2. $\dot{g}_1 > 0$, $\dot{g}_2 = \dot{\lambda}_2 = 0$, and

3. $\dot{g}_1 = \dot{\lambda}_1 = 0$, $\dot{g}_2 > 0$.

If, instead, the continuous transformation from $s = 0$, Figure 1(a), to $s = \epsilon > 0$ produces the result shown in Figure 1(c), the path derivative is unique and $\dot{\lambda}_0 > 0$. In the next sections we discuss the solution of the first-order equilibrium equations depending on the features of the reduced Hessian.

4 Regular H_+

If the reduced Hessian is non-singular Equation 43 can be solved for x and the solution substituted into Equation 45. Together with Equation 44, this yields

$$\dot{g}_0 = G\dot{\lambda}_0 + \dot{p}\hat{g} \quad , \tag{48}$$

$$\dot{g}_0 \geq 0, \quad \dot{\lambda}_0 \geq 0, \quad \dot{\lambda}_0^T \dot{g}_0 = 0 \tag{49}$$

where

$$G = E_0^T T_+ H_+^{-1} T_+^T E_0 \quad , \tag{50}$$

$$\hat{g} = -E_0^T T_+ H_+^{-1} T_+^T \nabla_\varphi U' \quad . \tag{51}$$

This is a Linear Complementarity Problem (LCP) in standard form, which has been dealt with by, e.g. Murty (1988), Lemke (1970), and Cottle (1977). Contact problems, both of a static and dynamic nature, invariably lead to complementarity problems, in some cases of the linear type. Examples of LCP in dynamic mechanical systems can be found in Pfeiffer and Glocker (1996) and Klarbring (1988); structural mechanics examples can be found in Björkman (1991).

In tracing the equilibrium path of a system with unilateral constraints, when one or more of these constraints are weakly active we have to determine in each step which of the weakly active constraints become inactive and which remain active, in order to update the index sets I^* and I_+. In general, $\dot{\lambda}_j$ is discontinuous at a point where an inactive constraint becomes active, or viceversa. Such an event can induce the transformation of other weakly active constraints into strongly active or inactive.

For this reason, treating unilateral constraints as either bilateral if they are active, or altogether neglecting them when their Lagrange multiplier becomes negative, without solving the LCP can lead in some cases to non-convergence and errors.

After solving the LCP, the right-hand side path derivatives of φ can be computed by substituting $\dot{\lambda}_0$ into Equation 43, solving for x, and substituting into Equation 39

$$\dot{\varphi} = T_+ H_+^{-1} T_+^T \left(E_0 \dot{\lambda}_0 - (\nabla_\varphi U') \dot{p} \right) \tag{52}$$

The right-hand side path derivatives of λ_+ are obtained by solving Equation 34

$$\dot{\lambda}_+ = \left(E_+^T E_+ \right)^{-1} E_+^T \left((\nabla_\varphi^2 \mathcal{L}) \dot{\varphi} + (\nabla_\varphi U') \dot{p} - E_0 \dot{\lambda}_0 \right) \tag{53}$$

Equations 52 and 53 are integrated. At each step the constraint function of the inactive constraints and the Lagrange multipliers of the strongly active constraints are checked for zeros. In this way the index sets I^* and I_+ can be updated.

4.1 Positive definite H_+

Due to the regularity assumption the matrix G is also positive definite and, therefore, the LCP in Equations 48 and 49 has a unique solution both for $\dot{p} < 0$ and for $\dot{p} > 0$ (Murty 1988). The two solutions that are obtained correspond to the two directions in which we can move on the path, Figure 2.

It can be shown that Equations 48 and 49 represent the Kuhn-Tucker conditions for a solution of the associated Quadratic Programming Problem (QP)

$$\min_{\dot{\lambda}_0} \left(\frac{1}{2} \dot{\lambda}_0^T G \dot{\lambda}_0 + \hat{g}^T \dot{\lambda}_0 \right), \quad \dot{\lambda}_0 \geq 0 \tag{54}$$

which can be solved instead of the LCP.

4.2 Non-positive definite H_+

In this case, the uniqueness of the solution of the LCP is not guaranteed and hence *bifurcational behaviour* can occur despite the fact that the reduced Hessian is not singular. This is an important difference from systems with bilateral constraints.

If H_+ is, for example, negative definite G is also negative definite. In fact, G can be negative definite even in the case of an indefinite H_+.

In practical situations often only one constraint is weakly active. In this case, if G is negative, the possible solutions of the LCP are illustrated in Figure 3. The three lines plotted have the same slope G (< 0). For $\hat{g} = 0$ the LCP has the same unique solution for $\dot{p} > 0$ and $\dot{p} < 0$, as $\dot{p}\hat{g} = 0$. The weakly active constraint remains weakly active and the left-hand side and right-hand side path derivatives of φ and λ coincide, see Equations 52 and 53.

If $\hat{g} \neq 0$ the LCP has two solutions for $\dot{p}\hat{g} > 0$ and no solution for $\dot{p}\hat{g} < 0$. Thus, if there are two solutions for $\dot{p} < 0$ ($\dot{p} > 0$), there is no solution for $\dot{p} > 0$ ($\dot{p} < 0$). This means that the equilibrium path has reached a maximum (minimum) in the φ-p space, which will be called a *corner limit point* because of the non-smoothness of the path at this point.

At a *common limit point* the condition $\dot{p} = 0$ holds, but at a corner point it does not necessarily hold, Figure 4(b). The reduced Hessian, H_+, is singular at a common limit point, whereas at a corner point it can be regular but not positive definite.

Note that the equilibrium path may also have reached a minimum or maximum in the φ-p space if $G = 0$ (H_+ regular but indefinite) and $\hat{g} \neq 0$. In this case, the LCP has a unique solution for $\dot{p} = 0$, corresponding to a horizontal tangent, and a unique solution for $\dot{p}\hat{g} > 0$, corresponding to a non-horizontal tangent. In order to find out if the path has indeed reached a maximum or minimum we have to derive the second order or possibly even higher order equilibrium equations.

If the number of weakly active constraints is greater than one, it is still possible to find corner limit points. In addition, one can find mathematical examples where there is a unique solution of the LCP for $\dot{p} < 0$ ($\dot{p} > 0$) and no solution for $\dot{p} > 0$ ($\dot{p} < 0$). In such cases the equilibrium path has reached an *end point*, Figure 4(a). This is not possible with a single active constraint, as in this case the plot in Figure 3 shows that there are either two solutions or none.

5 Stability at corner limit points

The stability of a mechanical system that is subject only to bilateral or strongly active constraints can be analysed by considering the reduced Hessian, H_+. The number of negative eigenvalues of this matrix is equal to the *degree of instability of the system* (Thompson and Hunt 1973).

At a common limit point the degree of instability changes by one if a single stability coefficient changes sign, and an obvious question is whether the behaviour at a corner limit point is the same. Let us consider the case of a corner limit point, $s = s_P$, where a single constraint is weakly active. It is assumed that, in advancing on the equilibrium path with increasing s, this constraint is transformed from strongly active into inactive. Therefore, the dimension of H_+ increases by one. Next, it will be shown that in this case the number of negative stability coefficients also increases by one, and hence the degree of instability increases.

At such a corner limit point the basis of the nullspace of the matrix of strongly active constraints transposed is obtained by augmenting with a vector t_0 the basis of the nullspace of

$[E_+ \quad E_0]^T$

$$T_+ = [T \quad t_0] \quad . \tag{55}$$

Note that E_0 is a column vector and that t_0 has the property

$$t_0^T E_0 \neq 0 \tag{56}$$

because of the regularity assumption. Next, we partition the reduced Hessian

$$H_+ = \begin{bmatrix} H & T^T(\nabla_\varphi^2 \mathcal{L})t_0 \\ t_0^T(\nabla_\varphi^2 \mathcal{L})T & t_0^T(\nabla_\varphi^2 \mathcal{L})t_0 \end{bmatrix} \tag{57}$$

where

$$H = T^T(\nabla_\varphi^2 \mathcal{L})T \tag{58}$$

can be interpreted as the reduced Hessian to the left of s_P. Since H_+ is regular, assuming that H and $t_0^T(\nabla_\varphi^2 \mathcal{L})t_0$ are non-singular, the inverse matrix has the form

$$(H_+)^{-1} = \begin{bmatrix} \cdots & \cdots \\ \cdots & (H_+/H)^{-1} \end{bmatrix} \tag{59}$$

where

$$H_+/H = t_0^T(\nabla_\varphi^2 \mathcal{L})t_0 - t_0^T(\nabla_\varphi^2 \mathcal{L})TH^{-1}T^T(\nabla_\varphi^2 \mathcal{L})t_0 \tag{60}$$

is the Schur complement of H in H_+ (Cottle 1974). Substituting Equation 59 into Equation 50 and evaluating

$$T_+^T E_0 = \begin{bmatrix} 0 \\ t_0^T E_0 \end{bmatrix} \tag{61}$$

we arrive at

$$G = (t_0^T E_0)^2 (H_+/H)^{-1} \tag{62}$$

Since G is negative at the corner limit point, the Schur complement must also be negative. Using the inertia formula (Cottle 1974)

$$\mathrm{In}(H_+) = \mathrm{In}(H) + \mathrm{In}(H_+/H) \quad , \tag{63}$$

where the inertia In of a real symmetric matrix is the triple

$$[\text{no. eigenvalues} > 0, \text{no. eigenvalues} < 0, \text{no. eigenvalues} = 0] \quad ,$$

we can conclude that the number of negative eigenvalues of H_+ increases by one at the corner limit point. Thus, if the equilibrium path is stable for $s = s_P - \epsilon$ it is unstable for $s = s_P + \epsilon$, where ϵ is a small positive number. This result agrees completely with the behaviour at a common limit point.

6 Example of corner limit point

A simple system exhibiting a corner limit point is shown in Figure 5. This system consists of a rigid bar AB of length L, connected by a frictionless hinge A to a rigid foundation, and an extensional spring of stiffness k, whose undeformed length is zero. The spring is attached to the bar at B and to a vertical slider at D. A unilateral constraint on point B requires it to lie above a horizontal line through C.

The position coordinate is the angle φ between the bar and the horizontal, and the control parameter p is the displacement of the end of the spring, measured from the line AC and positive upwards.

In the initial configuration, shown in the figure, $\varphi = 0$ and $p = -L$. The bar is in contact with C and the unilateral constraint $g(\varphi) = \sin\varphi \geq 0$ is strongly active.

The equilibrium path is shown in Figure 6 for initially increasing p. The initial configuration corresponds to $(0, -L)$. As p increases from $-L$ to 0 there is no change in φ. At $(0, 0)$ the constraint becomes weakly active, and the path derivatives of the gap, $\dot{g}_0$, and of the reaction, $\dot{\lambda}_0$, are found by solving the corresponding LCP

$$\dot{g}_0 = -(kL^2)^{-1}\dot{\lambda}_0 - \dot{p}L^{-1} \tag{64}$$

$$\dot{g}_0 \geq 0, \quad \dot{\lambda}_0 \geq 0, \quad \dot{\lambda}_0\dot{g}_0 = 0 \tag{65}$$

Because the complementarity condition requires at least one of $\dot{\lambda}_0$, $\dot{g}_0$ to be zero, this LCP can be solved by trial and error. Thus, first we set $\dot{g}_0 = 0$ and find $\dot{\lambda}_0 = -kL\dot{p} \geq 0$, which requires $\dot{p} \leq 0$. Second, we set $\dot{\lambda}_0 = 0$ and obtain $\dot{g}_0 = -\dot{p}L^{-1} \geq 0$, which also requires $\dot{p} \leq 0$. So, there are two non-trivial solutions for $\dot{p} < 0$ and no solution for $\dot{p} > 0$, as shown in Figure 6.[1]

So, from the corner point the system can move in two different directions on the equilibrium path. If the system goes back towards the initial configuration, bar AB remains horizontal and in contact with the constraint at B; this part of the path is stable. Alternately, AB can rotate by an amount such that AB and BD are always parallel; this part of the path is unstable.

In practice, from O the system will jump to a stable configuration with $\varphi = 180°$, from which φ can be increased to follow a stable path that is not shown in the figure.

[1]Note that $\dot{p}$ is the right-hand side derivative, which is equal to the infinitesimal change of p when *moving away* from the corner point.

7 Singular H_+

We introduce a transformation analogous to Equation 19, where W contains the eigenvectors of H_+ at the singularity $s = s_0$, and diagonalise Equation 43

$$W^T H_+ W y + W^T T_+^T (\nabla_\varphi U')\dot{p} - W^T T_+^T E_0 \dot{\lambda}_0 = 0 \tag{66}$$

If the set of weakly active constraints is empty, the last term disappears and the problem becomes identical to one involving only bilateral constraints, see Section 2.1.

If the set of weakly active constraints is not empty, the situation becomes more complex. Assuming that only one eigenvalue of H_+ is zero, the solution of the diagonalised equations is

$$c_1 \dot{p} - c_2^T \dot{\lambda}_0 = 0 \quad , \tag{67}$$

$$y_i = \frac{1}{d_{ii}} w_i^T T_+^T \left(E_0 \dot{\lambda}_0 - \dot{p} \nabla_\varphi U' \right), \quad i = 2, \ldots, m_+ \tag{68}$$

where

$$c_1 = w_1^T T_+^T \nabla_\varphi U' |_{s=s_0} \tag{69}$$

$$c_2 = E_0^T T_+ w_1 |_{s=s_0} \tag{70}$$

and m_+ is the number of constraints that are strongly active in the configuration $s = s_0$.

We have analysed the case in which the number of weakly active constraints is one, in which case E_0 is a column vector and c_2, $\dot{\lambda}_0$ and $\dot{g}_0$ are scalar quantities, and have found that bifurcational behaviour can occur only for

$$c_1 = 0 \tag{71}$$

This condition corresponds to Equation 22 for systems with bilateral constraints, but recall that systems with unilateral constraints can show bifurcational behaviour also when H_+ is regular, Section 4.2.

Next, we summarise the results of a detailed study of the various cases that can occur.

If $c_1 \neq 0$ and $c_2 \neq 0$, it is likely that we are at a *corner limit point*. The first order equilibrium equations have two solutions, corresponding to two directions on the equilibrium path. For one of these directions $\dot{p} = 0$ and a more detailed classification of the different cases that are possible, see Figure 7, requires the introduction of

$$c_3 = W_+^T T_+^T \left(E_0 - \frac{c_2}{c_1} \nabla_\varphi U' \right) \tag{72}$$

$$c_4 = E_0^T W_+ c_3 \tag{73}$$

where

$$W_+ = \left[\begin{array}{cccc} \frac{w_2}{\sqrt{d_{22}}} & \frac{w_3}{\sqrt{d_{33}}} & \cdots & \frac{w_{m_+}}{\sqrt{d_{m_+ m_+}}} \end{array} \right] \tag{74}$$

If $c_1 \neq 0$ and $c_2 = 0$ it is likely that we are at a *common limit point*, where the equilibrium path is smooth and the weakly active constraint does not change state.

8 Discussion

The first-order equilibrium equations of a conservative multibody system with unilateral constraints that are functions of the coordinates, but not of the control parameter of the system, have been derived in a straightforward manner, without recourse to advanced mathematical concepts. If the reduced Hessian matrix of the system is regular a standard Linear Complementarity Problem (LCP) is obtained, whose dimension is equal to the number of weakly active constraints.

During the integration of the equilibrium path equations, when the index set of weakly active constraints is non-empty we have to determine at every step of the integration which of the weakly active constraints become inactive and which remain active. This requires the LCP to be solved. Strongly active unilateral constraints can be treated in a straightforward way, like bilateral constraints.

For systems with bilateral constraints a sufficient condition for stability can be formulated in terms of second-order derivatives. If this condition holds the path derivatives are uniquely defined. A corresponding second-order condition exists for systems with unilateral constraints, but it does not ensure that the path derivatives are unique.

In contrast to bilaterally constrained systems, systems subject to unilateral constraints can be at a limit or bifurcation point even if the reduced Hessian matrix is regular. Therefore, a new type of limit point, called a corner limit point, exists for such systems. At corner limit points the equilibrium path is non smooth.

At an end point the LCP has a unique solution when the control parameter changes in one direction, but no solution in the other direction. It has been shown that, if the constraint functions are independent of the control parameter, end points can exist only if at least two unilateral constraints are weakly active.

A simple example system exhibiting a corner limit point has been discussed. However, finding a good example of a system with constraints independent of the control parameter and which has an end point is more difficult. This difficulty can be understood by recalling that, under the assumptions made above, an end point requires at least two constraints to be weakly active whereas a corner limit point requires only one.

Finally, note that a corner limit point may look exactly like an end point in a plot of the control parameter vs. a particular coordinate that happens not to vary in the vicinity of the limit point. Such plots can be misleading, and care should be taken before reaching general conclusions. We believe that it is important that behind any simulation software for the analysis of multibody systems of the type described in this paper there should be a solid theoretical framework that can address the many subtle issues discussed in this paper.

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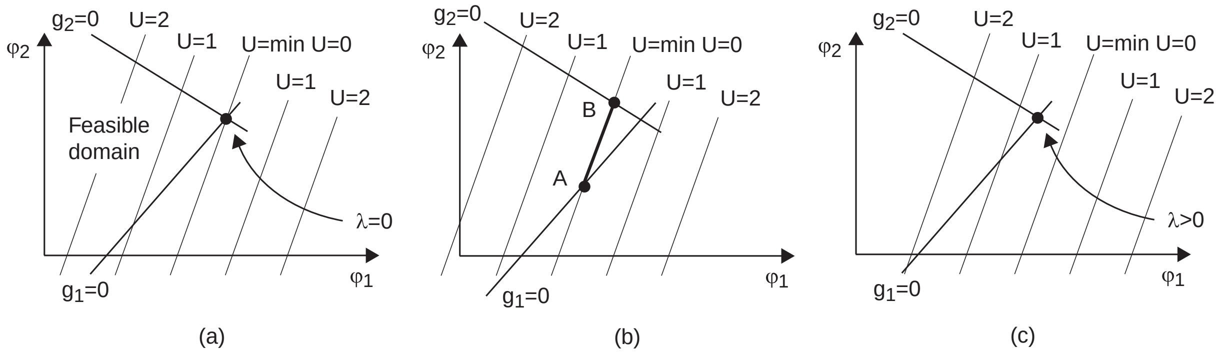

Figure 1: Solution of first order equilibrium equations (a) $s = 0$, (b) $s = \epsilon > 0$ the solution is non-unique, (c) $s = \epsilon > 0$ but the solution is unique.

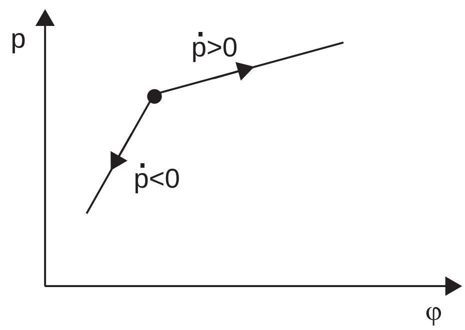

Figure 2: Solution of LCP for positive definite H_+.

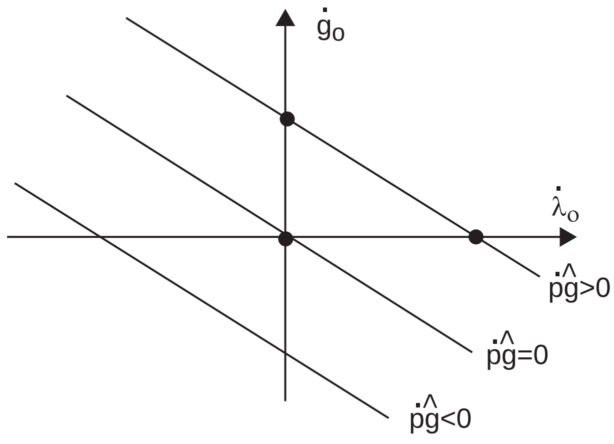

Figure 3: Solution of LCP for one weakly active constraint and negative G.

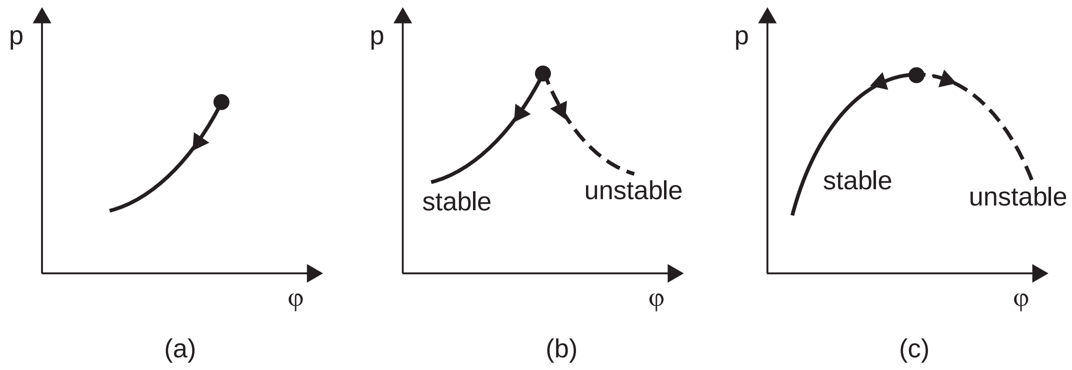

Figure 4: (a) End point (number of weakly active constraints > 1); (b) Corner limit point; (c) Common limit point.

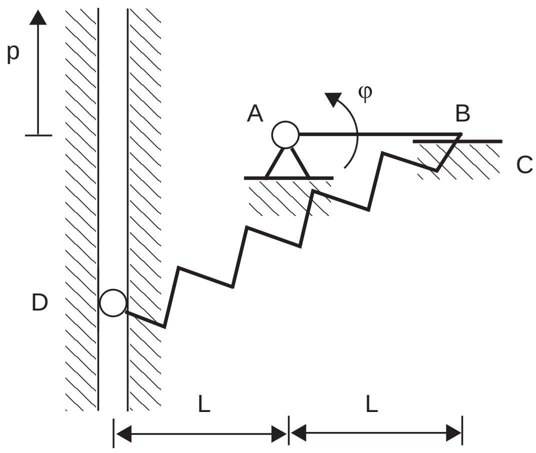

Figure 5: Example of system with corner limit point; in the initial configuration $p = -L$, $\varphi = 0$.

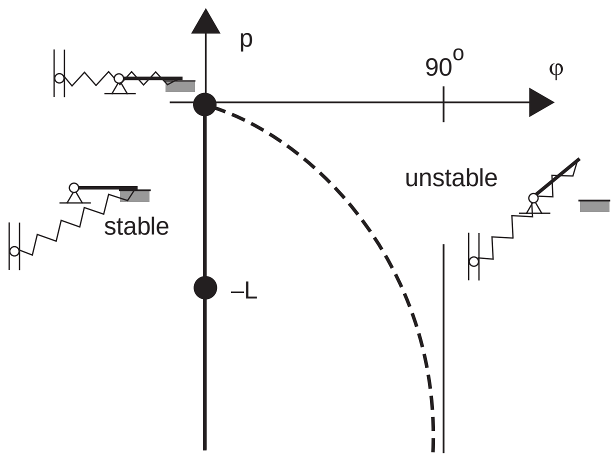

Figure 6: Equilibrium path of simple example.

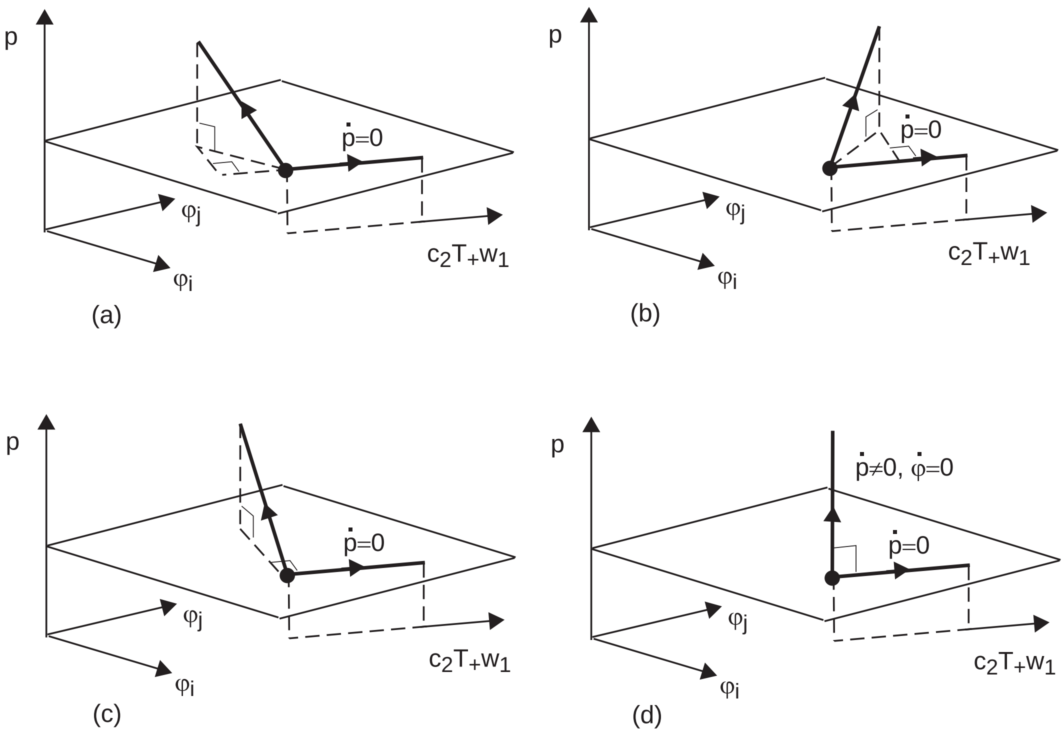

Figure 7: Different shapes of equilibrium path at a corner point, for $c_1 \neq 0$ and $c_2 \neq 0$; (a) $c_3 \neq 0$, $c_4 > 0$; (b) $c_3 \neq 0$, $c_4 < 0$; (c) $c_3 \neq 0$, $c_4 = 0$; (d) $c_3 = 0$.

List of Figures