Equilibrium shapes of a pair of equal uniform vortices

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The shapes and properties of two equal corotating uniform vortices, rotating steadily about each other, are calculated. An integrodifferential equation for the bounding contour is solved numerically, using Newton's method. The results compare well with those obtained from a simple model. It is shown that steady solutions do not exist if the vortices are too close. The stability to two-dimensional disturbances is discussed qualitatively and the critical separation at which the system becomes unstable is calculated. Some comments are made on the stability of a vortex pair of equal counter rotating uniform vortices.

I. INTRODUCTION

The structure of a linear array of uniform vortices was recently calculated by Saffman and Szeto[1] and the stability to two-dimensional disturbances was considered. Convincing evidence was adduced that the finite size of the vortices does not inhibit the pairing instability of a row of point vortices. A possible model of the continual coalescence of the organized or coherent structures of the turbulent mixing layer[2] is therefore an array of uniform vortices which is unstable to the pairing instability, leading to the coalescence of neighboring vortices as they rotate about each other and the creation of an array with twice the spacing, with the process then repeating.[3] However, as pointed out by Moore and Saffman[4] (see also Saffman and Baker[5]) it is necessary to explain why the vortices coalesce instead of continuing to rotate around one another in symmetric orbits, such as those which can be calculated analytically for an array of point vortices and predict that the distance of closest approach is 0.56 times the initial separation. A possible mechanism for the coalescence is the "tearing" process[1,4,6] related to the nonexistence of steady states when the individual vortices in a configuration are subjected to too large rates of strain; this might predict that as neighboring vortices rotate about each other they approach too close for them to exist as separate entities.

Our purpose here is to investigate this question by studying the equilibrium configurations of a pair of equal, uniform vortices as they rotate about each other, and to demonstrate that there is a minimum separation distance for steady motion. The results also have bearing on the merging of vortices in the wakes of lifting bodies and the jumbo jet trailing vortex alleviation problem.

Roberts and Christiansen[7] studied numerically the unsteady motion of a vortex pair of initially circular vortices using the point vortex approximation with cloud in cell for calculation of the velocity field and found a critical separation of $h/R=1.7$ for merging, where $2h$ is the initial separation of circular vortices and R is their radius. Rossow[8] calculated the same flow using the point vortex approximation with direct summation for calculation of the velocity field and found critical values of 1.7 or 1.9 depending on how the radius was defined. Zabusky *et al.*[9] used a contour dynamics algorithm for the flow and also found the critical value to be 1.7. Since all these calculations are for unsteady flow with a given initial condition, they do not compare exactly with the results found in this paper for steady, noncircular equilibrium shapes, but the agreement is quite good. Our calculations give a critical value of $h/R=1.58$, where $2h$ is the distance between the centroids and πR^2 the area of each vortex, below which no steady solutions exist. A rough calculation[4] based on an elliptical model gave the value 1.43; a more refined calculation using the model gives 1.50.

II. THE ELLIPTICAL MODEL

We first consider the results of a simple model, asymptotically valid as $h/R\to\infty$, which has been found to work remarkably well for $O(1)$ values of h/R for the cases of a linear vortex array[1] and a pair of counter-rotating vortices[10]. It is supposed that each vortex is elliptical and would be in equilibrium in a uniform straining field whose strength is that of the field produced by the other vortices, evaluated at the center of the ellipse. We consider only symmetrical states, in which the major axes of the ellipses are parallel to the line joining the centers (see Fig. 1). It can be shown that there are no solutions of the model with major axes perpendicular to the line joining the centers.

Each vortex is supposed to have circulation Γ, area A, and uniform vorticity $\omega=\Gamma/A$. The system rotates steadily with angular velocity Ω. The velocity and straining field produced by a uniform vortex are easily calculated[11] and we quote the results. Define ξ by the equation

$$c\cosh\xi=2h,\quad c=(a^2-b^2)^{1/2}, \tag{1}$$

where a and b are the semi major and minor axes of the ellipse, so that

$$ab=R^2=A/\pi. \tag{2}$$

Then, the velocity at the center of one vortex induced by the other is $(\Gamma/\pi c)e^{-\xi}$ perpendicular to the line joining the centers and hence

$$\Omega h=(\Gamma/\pi c)e^{-\xi}. \tag{3}$$

The rate of strain at the center of the vortex on the right induced by the other is

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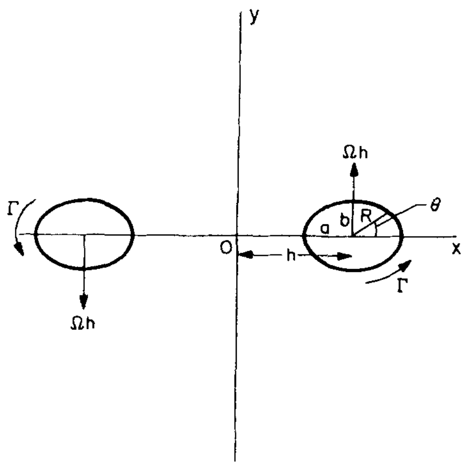

FIG. 1. Sketch of flow configuration for a pair of symmetric counter-rotating vortices.

$$\epsilon = (\Gamma/\pi c^2)e^{-\xi}/\sinh\xi \,, \qquad (4)$$

and the direction of maximum extension makes an angle $-\frac{1}{4}\pi$ with the x axis. The motion can be reduced to rest by superposing an angular velocity $-\Omega$, and we can then apply the result of Moore and Saffman[6] which states that the vortex on the right is in equilibrium in the superposition of a uniform straining field of strength ϵ in the stated direction and a solid body rotation $-\Omega$ if

$$\epsilon(\theta+1)/(\theta-1) + \Omega = (\omega - 2\Omega)\theta/(\theta^2+1) \,, \qquad (5)$$

where $\theta = a/b \geq 1$ is the axis ratio. [If $\epsilon = 0$, this equation gives $\Omega = \theta\omega/(\theta+1)^2$ which is Kirchhoff's formula for the rotation rate of an elliptical vortex.] A form of this equation was used previously by Moore and Saffman[4] with ϵ and Ω replaced by $\Gamma/8\pi h^2$ and $\Gamma/4\pi h^2$, respectively, which are the values in the limit $h \to \infty$, $c \to 0$.

Equations (1)–(5) can easily be solved numerically to determine θ and Ω as functions of h^2/A. The detailed results are presented in Sec. III where they are compared with the numerical solutions of the exact equations. We just mention here the qualitative feature that there are no solutions for $h/R < 1.50$. This is a limit point and there may be two solutions for $h/R > 1.50$. One branch, for which $1.50 > h/R > \infty$, is the continuation of the $h = \infty$ circular vortex limit. The other, for which $1.50 < h/R < 1.51$, describes distorted vortices which overlap when $h/R = 1.51$ and the model loses physical validity.

III. NUMERICAL SOLUTION OF EXACT EQUATION

We take a coordinate system rotating with the vortices (Fig. 1) and suppose that the shapes are given parametrically by

$$z = x + iy = Z_1(s) \,, \quad z = Z_2(s) \,. \qquad (6)$$

We shall only search for symmetrical solutions in which the vortices are mirror images of one another and symmetrical about the line joining the centers. The condition for steady motion with angular velocity Ω is

$$\mathrm{Im}[(u + iv + \Omega y - i\Omega x)\partial Z_1^*(s)/\partial s] = 0 \,, \qquad (7)$$

where $u + iv$ is the velocity of the fluid on the surface of the vortex induced by the vorticity and is given by

$$u + iv = -\frac{\omega}{2\pi}\oint \log|Z_1 - Z_1(\sigma)|\,d\sigma - \frac{\omega}{2\pi}\oint \log|Z_1 - Z_2(\sigma)|\,d\sigma \,. \qquad (8)$$

To solve this equation, we put

$$Z_1 = h + R(\theta)e^{i\theta} \,, \qquad (9)$$

and take the mirror image for Z_2, and use a truncated Fourier expansion in a modified variable $\tilde\theta$ for $R(\theta)$, i.e.,

$$R = \sum_0^N a_n \cos n\tilde\theta \,, \quad \theta = \tilde\theta + \alpha \sin\tilde\theta \,, \qquad (10)$$

where the unknown coefficients a_n are real. We evaluate $u + iv$ by a third order accurate integration formula at the N points $\tilde\theta_j = j\pi/(N+1)$, $j = 1, 2, \ldots, N$, and satisfy Eq. (7) at these N points, thereby obtaining N equations for the $N+2$ unknowns $a_0, \ldots, a_n, \Omega$. Two further equations are provided by requiring that the centroid is at $x = h$ and by normalizing the area to unity. The constant α is chosen between 0 and 1 so that points are concentrated where the vortices are close together and cusps start to form. The set of $N+2$ nonlinear equations was solved by Newton's method to give a family of solutions as functions of h. For $h \to \infty$, the shape is nearly circular and can be calculated by perturbation methods. Euler continuation with respect to arc length was used to follow the branch and go past the limit point where $h = 0.8943$ is a minimum, until the convergence became poor for $h = 0.897$ when the vortices were nearly touching.

Properties of the solution are listed in Table I. Typical shapes are shown in Fig. 2. The results have been normalized on $\Gamma = 1$ and $A = 1$. The quantity $J = -HA/\Gamma$, where H is the angular impulse of the velocity

$$H = -\frac{1}{2}\int\int \omega(x^2 + y^2)\,dx\,dy = -\Gamma(h^2 + K^2) \,, \qquad (11)$$

where K is the radius of gyration of each centroid about its center. The quantity T is the *excess* kinetic energy, defined by

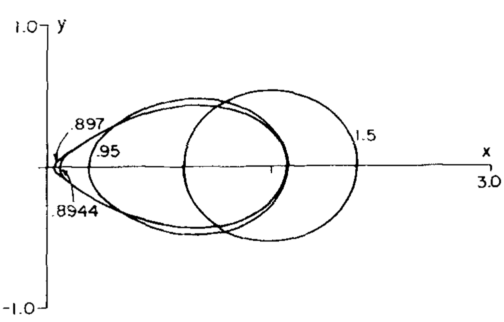

FIG. 2. Shapes of vortices for various values of h. Curves shown are for $h = 1.5$, 0.95, 0.8944 (the minimum value) and 0.897 (closest approach calculated).

TABLE I. Properties of a pair of counter rotating vortices. Results are normalized on $A=1, \Gamma=1$. (Values in parenthesis are from elliptical model of Sec. II.)

h	$2a$	a/b	J	T	$4\pi\Omega$
100.0	1.128	1.0001	10 000	−0.713	0.0001
50.0	1.128	1.0001	2 500	−0.603	0.0004
20.0	1.129	1.0004	400	−0.458	0.0025
10.0	1.129	1.0017	100	−0.350	0.0100
5.0	1.132	1.0066	25	−0.242	0.0400
4.0	1.134	1.0102 (1.01)	16.2	−0.208	0.0625 (0.0625)
3.0	1.140	1.0197 (1.02)	9.16	−0.164	0.1111 (0.11)
2.0	1.153	1.0443 (1.04)	4.16	−0.102	0.2502 (0.25)
1.0	1.293	1.3169 (1.29)	1.165	0.0003	1.0235 (1.01)
0.983	1.306	1.3263 (1.31)	1.132	0.003	1.0621 (1.05)
0.9725	1.315	1.3436 (1.32)	1.112	0.0047	1.0873 (1.07)
0.9628	1.324	1.3612 (1.34)	1.097	0.0063	1.1116 (1.09)
0.9497	1.339	1.3897 (1.36)	1.070	0.0085	1.1464 (1.12)
0.929	1.369	1.4473 (1.40)	1.033	0.0122	1.206 (1.18)
0.9012	1.446	1.5950 (1.47)	0.9895	0.0179	1.3041 (1.26)
0.8943 [b]	1.522	1.7353 (1.50)	0.9845	0.0204	1.3456 (1.28)
0.8944 [a]	1.531	1.7509	0.9855	0.0204	1.3476
0.8946	1.540	1.7663	0.9867	0.0205	1.3491
0.8947	1.542	1.7700	0.9870	0.0205	1.3495
0.895	1.548	1.7801	0.9880	0.0205	1.3501
0.8955	1.557	1.7950	0.9897	0.0205	1.3508
0.896	1.565	1.8078	0.9913	0.0205	1.3509
0.8965	1.572	1.8186	0.9927	0.0205	1.3508
0.897	1.578	1.8279	0.9941	0.0205	1.3506
(0.846) [c]		(2.01)			(1.46)

[a] Vortex of minimum centroid separation.
[b] Vortex of minimum angular momentum.
[c] Minimum separation from elliptical model.

$$T=\frac{1}{2}\int\int \omega\psi\, dx\, dy=\frac{\Gamma}{A}\int\int \psi\, dA\,, \tag{12}$$

where the second integral is over one vortex only and ψ denotes the stream function

$$\psi(z)=-\frac{\Gamma}{2\pi A}\int_A\int \log|z^2-z_1^2|\, dx_1\, dy_1\,. \tag{13}$$

The actual kinetic energy is infinite because the flow is unbounded and behaves like a line vortex of strength 2Γ at infinity. The evaluation of T is rather difficult and is accomplished by transforming to a line integral by integration by parts and repeated use of Green's identities.[1] It can be shown that

$$T\sim(\Gamma^2/4\pi)[-\ln(4J/\pi)+\tfrac{1}{2}] \tag{14}$$

as J and $h/A^{1/2}\to\infty$.

The vortices touch at a finite value of J and $h/A^{1/2}$. The solution branch will then continue into one of the infinite number of families of single rotating noncircular vortices, of which the Kirchhoff vortex is one example, whose existence was uncovered by Deem and Zabusky.[12] Since the vortex will have two-fold symmetry, it probably comes from a bifurcation of the Kirchhoff solution. The important conclusion is that no steady solution exists if J or equivalently $h/A^{1/2}$ is too small, and thus it can be concluded that coalescence is a definite possibility when like vortices approach too closely. (Experimental evidence for tearing in the mixing layer has been given by Dimotakis and Brown.[13]) We now make some remarks on the stability.

IV. STABILITY

The stability of the steady states to two-dimensional infinitesimal disturbances is a straightforward but extremely laborious task and has not yet been calculated. Here, we shall just comment on conclusions which can be reached simply from the properties of the excess kinetic energy using some ideas of Lord Kelvin. For a system of uniform rotating vortices, the steady states have the property that they make

$$\delta T=0 \tag{15}$$

for variations with

$$\delta A=0,\quad \delta H=0\,,\quad \delta\Gamma=0\,. \tag{16}$$

Moreover, the state is stable if T is an absolute maximum or minimum. For our system of two corotating vortices

$$T=\Gamma^2\tilde{T}(J)\,,\quad J=-H/\Gamma A\,, \tag{17}$$

where $\tilde{T}$ is a dimensionless function listed in Table I. It follows that $\tilde{T}$ must be stationary whenever J is, which is consistent with the numerical results. (Unfortunately, the calculation of T is rather difficult and it was not possible without undue expense to obtain sufficient resolution to show clearly the stationary property of T at the minimum of J.) The structure of Eq. (13) implies that variations away from the equilibrium shapes which make the vorticity more compact will increase T and vice versa. For a given J, then T must be bounded below but can be made larger by, for example, amalgamating the two vortices into a single Kirchhoff vortex. Thus we conclude that the branch $0.9845 < J < \infty$ is stable to infinitesimal two-dimensional disturbances and the limit point at $J = 0.9845$, $h/A^{1/2} = 0.8943$, is a point of stability exchange.

We comment now on the Saffman and Szeto[1] model of the turbulent mixing layer, envisaged as a row of finite sized vortices with zero excess energy and values of $h/A^{1/2} = 1.37$. The pairing instability causes the vortices to rotate about each other and at closest approach the value of $h/A^{1/2}$ is $1.37 \times 0.56 = 0.77$ (assuming that the trajectories are approximately those of point vortices). The stability exchange (min$J$) occurs when $h/A^{1/2} = 0.89$. Thus it can be expected that the vortices become unstable as they rotate and coalescence is to be expected.

Finally, we comment on the two-dimensional stability of a horizontal pair of equal and opposite uniform counter rotating vortices moving vertically downwards with speed W. Kelvin's argument here states that the steady shapes are found by making T stationary (in this case T is the actual kinetic energy) for variations which keep the area and strength of each vortex constant as well as the hydrodynamic impulse

$$I = \int \int \omega x \, dx \, dy = 2\Gamma h . \tag{18}$$

For this flow, it is easily shown that as $h \to \infty$, the vortices are approximately circular, the rate of descent $W \sim \Gamma / 4\pi h$, and

$$T \sim (\Gamma^2 / 4\pi)[\log(4h^2\pi / A) + \tfrac{1}{2}] . \tag{19}$$

As h decreases, the vortices become more deformed and the distance of closest approach appears to decrease monotonically. There is no sign of limit point behavior and it appears that on the solution branch the properties behave monotonically. In particular, the kinetic energy decreases monotonically. Pierrehumbert[10] gives results down to $h/A^{1/2} = 0.29$. He also claims there is a limiting vortex at $h/A^{1/2} = 0.26$, which he calculates directly. We believe the numerics are not sufficiently precise to establish this claim and that there are no theoretical reasons why solutions should not exist for arbitrarily small h. It is hoped that future work will resolve this controversy. Assuming the existence of arbitrarily small h solutions we can estimate their kinetic energy by evaluating that of two touching rectangular slabs of vorticity of width $2h$ and height $A/2h$. An elementary calculation, gives $W \sim \Gamma h / A$ and $T \sim (4/3)\Gamma^2 h^2 / A$. Thus whatever the resolution of the controversy may be, the energy is expected to decrease monotonically, either to a finite limit or zero, and there is therefore no reason to expect that the solutions are not unique for all allowable values of $h/A^{1/2}$.

For a given value of I, it is clearly possible to redistribute the vorticity by spreading it out in the vertical direction and thereby reduce T, thus T is not a minimum. If we allow nonsymmetrical or antisymmetrical disturbances, we can make T arbitrarily large by moving one vortex vertically upward while moving the other downward. Thus, T cannot be an absolute maximum or minimum and hence the configuration must be unstable. However, if we restrict displacements to symmetrical disturbances in which each vortex is the mirror image of the other, then the only way to increase T would be to make each vortex more compact. For given size and I, there is a limit to which this is feasible and hence T is bounded above. Since symmetrical solutions appear to be unique, we conclude that T is an absolute maximum for the steady state, which must therefore be stable to symmetrical disturbances. This conclusion that a vortex pair is in general unstable, but is stable if only symmetrical disturbances are allowed, is consistent with observations by Barker[14] who reported that "the lifetime of the vortex pair in the tank was limited by an instability that usually caused one of the cores to burst. We repeated the experiment with a splitter plate. . . each vortex is now interacting with its image across the splitter plate. . . under these conditions the vortices are extremely stable."

The problem of the stability of these configurations to general three-dimensional disturbances appears to be extremely difficult and the available results[6,15] would apply only in the limit $h/A^{1/2} \to \infty$.

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