EQUIPARTITION OF ENERGY FOR MAXWELL'S EQUATIONS*

BY GEORGE DASSIOS (*National Technical University of Athens*)

1. Introduction. The asymptotic equipartition of the kinetic and the potential energy for the free space wave equation was first stated by Lax and Phillips in [1] (Corollary 2.3, p. 106). Duffin [2] has shown that equipartition of energy is achieved in finite time as long as the initial data have compact support. Both the asymptotic and the finite-time equipartition of energy results have been generalized by Goldstein [3, 4] for abstract wave equations. For abstract equations see also [5, 6]. Costa [7] considered a general first-order symmetric hyperbolic system and proved a partition of energy theorem for uniformly propagative systems.

In the present work, following Duffin [2], the Paley-Wiener theorem is used to prove an equipartition theorem for the electric and the magnetic energy for Maxwell's equations in free space. If the initial electromagnetic wave is in $[L^2(\mathbb{R}^3)]^6$, i.e. each component of the electric and the magnetic field is in $L^2(\mathbb{R}^3)$, then the equipartition of energy is attained asymptotically as $t \to +\infty$. In particular, if the initial data have compact support, then equipartition is attained in finite time R, where R is the Radius of the smallest sphere containing the supports of the six components of the initial electromagnetic disturbance.

2. Maxwell's equations. We consider Maxwell's equations for propagation of electromagnetic waves in a vacuum, in $\mathbb{R}^3$.

The electric field $\mathbf{E}(\mathbf{x}, t) = (E^1, E^2, E^3)$ and the magnetic field $\mathbf{H}(\mathbf{x}, t) = (H^1, H^2, H^3)$ satisfy the following equations:

$$\partial_t \mathbf{E}(\mathbf{x}, t) = \nabla \times \mathbf{H}(\mathbf{x}, t) \tag{1}$$

$$\partial_t \mathbf{H}(\mathbf{x}, t) = -\nabla \times \mathbf{E}(\mathbf{x}, t), \tag{2}$$

$$\nabla \cdot \mathbf{E}(\mathbf{x}, t) = \nabla \cdot \mathbf{H}(\mathbf{x}, t) = 0. \tag{3}$$

The electric energy $e(t)$ and the magnetic energy $m(t)$ of a solution $(\mathbf{E}, \mathbf{H})$ of Eqs. (1)—(3) at time $t \geq 0$ are defined to be

$$e(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\mathbf{E}(\mathbf{x}, t)|^2 dx_1\, dx_2\, dx_3 = \sum_{i=1}^{3} ||E^i(t)||^2, \tag{4}$$

$$m(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\mathbf{H}(\mathbf{x}, t)|^2 dx_1\, dx_2\, dx_3 = \sum_{i=1}^{3} ||H^i(t)||^2, \tag{5}$$

where $||\cdot||$ is the L^2 norm in $\mathbb{R}^3$.

---

* Received November 14, 1978. The author wishes to express his appreciation to Prof. Walter Strauss for introducing him to the subject and to Prof. Constantine Dafermos for continuous help and encouragement. The work here described was performed while the author was a visitor at the Lefschetz Center for Dynamical Systems, Brown University.

The total energy of the electromagnetic wave is defined as

$$\mathscr{E}(t) = e(t) + m(t) = ||\mathbf{E}(t) \cdot \mathbf{E}(t)||^2 + ||\mathbf{H}(t) \cdot \mathbf{H}(t)||^2. \tag{6}$$

It is well known that $\mathscr{E}(t) = \mathscr{E}(0)$ for $t \geq 0$, i.e. the electromagnetic energy in vacuum is conserved.

Consider next the Fourier transform in $\mathbb{R}^3$

$$\hat{f}(\xi) = (2\pi)^{-3/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(i\mathbf{x} \cdot \xi) f(\mathbf{x})\, dx_1\, dx_2\, dx_3\,. \tag{7}$$

Assuming C^1 solutions of (1)–(3) and taking Fourier transform, we obtain the following system of six ordinary differential equations in t for the six scalar fields $\hat{E}^i(t)$, $\hat{H}^i(t)$, $i = 1, 2, 3$:

$$\frac{d}{dt} \hat{\mathbf{E}}(t) = \mathbb{A} \cdot \hat{\mathbf{H}}(t), \tag{8}$$

$$\frac{d}{dt} \hat{\mathbf{H}}(t) = -\mathbb{A} \cdot \hat{\mathbf{E}}(t), \tag{9}$$

supplemented by the algebraic relations

$$\xi \cdot \hat{\mathbf{E}}(t) = \xi \cdot \hat{\mathbf{H}}(t) = 0 \tag{10}$$

where

$$\hat{\mathbf{E}}(t) = \hat{\mathbf{E}}(\xi, t), \qquad \hat{\mathbf{H}}(t) = \hat{\mathbf{H}}(\xi, t),\ \xi = |\xi|\mathbf{n}, \quad |\mathbf{n}| = 1,$$

$$\mathbb{A} = i|\xi|\, \mathbb{I} \times \mathbf{n} \tag{11}$$

is a dyadic and $\mathbb{I}$ is the identity dyadic in $\mathbb{R}^3$. Note that the dot products in (8) and (9) denote contractions of the dyadic $\mathbb{A}$ from the right.

3. The equipartition theorem.

LEMMA 1. The solution of the system (8)–(9) that satisfies the initial conditions

$$\hat{\mathbf{E}}(\xi, 0) = \hat{\mathbf{E}}_0(\xi),\ \hat{\mathbf{H}}(\xi, 0) = \hat{\mathbf{H}}_0(\xi), \tag{12}$$

where $\hat{\mathbf{E}}_0$ and $\hat{\mathbf{H}}_0$ are the Fourier transforms of the initial data, is given by

$$\hat{\mathbf{E}}(t) = \hat{\mathbf{E}}_0 \cos(t|\xi|) + i(\hat{\mathbf{H}}_0 \times \mathbf{n}) \sin(t|\xi|), \tag{13}$$

$$\hat{\mathbf{H}}(t) = \hat{\mathbf{H}}_0 \cos(t|\xi|) - i(\hat{\mathbf{E}}_0 \times \mathbf{n}) \sin(t|\xi|). \tag{14}$$

Proof: Solving (8)–(9) formally, we obtain

$$\hat{\mathbf{E}}(t) = \cos(t\mathbb{A}) \cdot \hat{\mathbf{E}}_0 + \sin(t\mathbb{A}) \cdot \hat{\mathbf{H}}_0\,, \tag{15}$$

$$\hat{\mathbf{H}}(t) = \cos(t\mathbb{A}) \cdot \hat{\mathbf{H}}_0 - \sin(t\mathbb{A}) \cdot \hat{\mathbf{E}}_0\,. \tag{16}$$

We define powers of $\mathbb{A}$ by

$$\mathbb{A}^n = \mathbb{A}^{n-1} \cdot \mathbb{A}, \qquad n = 2, 3, \cdots \tag{17}$$

and show by induction that

$$\mathbb{A}^{2n} = |\xi|^{2n} (\mathbb{I} - \mathbf{n}\mathbf{n}), \tag{18}$$

$$\mathbb{A}^{2n+1} = i|\xi|^{2n+1} (\mathbb{I} \times \mathbf{n}). \tag{19}$$

Therefore we obtain

$$\cos(t\mathbb{A}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \mathbb{A}^{2n} = (\mathbf{I} - \mathbf{nn}) \cos(t|\xi|) \tag{20}$$

and similarly

$$\sin(t\mathbb{A}) = i(\mathbf{I} \times \mathbf{n}) \sin(t|\xi|). \tag{21}$$

Finally we substitute Eqs. (20), (21) into (15), (16) and make use of (10) to obtain the solution of the transformed Maxwell's equations in the form (13), (14). This completes the proof.

LEMMA 2. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a square integrable function of compact support and assume that $f(\mathbf{x}) = 0$, $|\mathbf{x}| > \mathbb{R}$. Let $\hat{f}: \mathbb{R}^3 \to C$ be the three-dimensional Fourier transform of $f(\mathbf{x})$. Define the iterated integral

$$C(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\hat{f}(\xi)|^2 \cos(2t|\xi|)\, d\xi_1\, d\xi_2\, d\xi_3 \int_0^{\infty} = H_c(|\xi|) \cos(2t|\xi|)\, d|\xi|, \tag{22}$$

where

$$H_c(|\xi|) = |\xi|^2 \int_0^{2\pi} d\phi \int_0^{\pi} |\hat{f}|^2 \sin\theta\, d\theta \tag{23}$$

and where $(|\xi|, \theta, \phi)$ are spherical coordinates in the ξ-space.

Then H_c is an even function of $|\xi|$, and $C(t) = 0$ for $t \geq R$.

LEMMA 3: Let f, $\hat{f}$ be as in Lemma 2 and let g, $\hat{g}$ be another such pair. Define the iterated integral

$$S(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \operatorname{Im}(\hat{f}\hat{g}^*)(\xi) \sin(2t|\xi|)\, d\xi_1\, d\xi_2\, d\xi_3 = \int_0^{\infty} H_s(|\xi|) \sin(2t|\xi|)\, d|\xi| \tag{24}$$

where

$$H_s(|\xi|) = |\xi|^2 \int_0^{2\pi} d\phi \int_0^{\pi} \operatorname{Im}(\hat{f}\hat{g}^*) \sin\theta\, d\theta \tag{25}$$

and where * denotes complex conjugate. Then H_s is an odd function of $|\xi|$ and $S(t) = 0$ for $t \geq R$.

The proofs of Lemmas 2 and 3 can be found in the proof of Theorem 1 in [2] and are based on the following corollary of the Paley-Wiener theorem.

PROPOSITION: Let $H(z)$ be an entire function of exponential type c which is absolutely integrable on the real line. Then the support of the Fourier transform of H is contained in $[-c, +c]$.

Proof. See [2].

We are now equipped with the necessary tools to prove the following equipartition theorem for Maxwell's equations.

THEOREM. Let $(\mathbf{E}(\mathbf{x}, t), \mathbf{H}(\mathbf{x}, t))$ be a solution of Maxwell's equations satisfying the initial conditions $\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x})$, $\mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x})$. Then

(i) if $E_0^i \in L^2(\mathbb{R}^3)$ and $H_0^i \in L^2(\mathbb{R}^3)$, $i = 1, 2, 3$, we have

$$e(t) - m(t) \to 0, \quad \text{as} \quad t \to +\infty, \tag{26}$$

$$\lim_{t\to+\infty} e(t) = \lim_{t\to+\infty} m(t) = \frac{1}{2}\mathscr{E}(0); \tag{27}$$

(ii) if $\mathbf{E}_0(\mathbf{x}) = \mathbf{H}_0(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$, we have

$$e(t) = m(t) = \tfrac{1}{2}\,\mathscr{E}(0), \qquad t \geq R. \tag{28}$$

Proof: We first calculate $|\hat{\mathbf{E}}|^2$ and $|\hat{\mathbf{H}}|^2$ using (13) and (14):

$$|\hat{\mathbf{E}}(\xi, t)|^2 = |\hat{\mathbf{E}}_0(\xi)|^2 \cos^2(t|\xi|) + |\hat{\mathbf{H}}_0(\xi)|^2 \sin^2(t|\xi|) + \operatorname{Im}[\mathbf{n}\cdot(\hat{\mathbf{E}}_0(\xi) \times \hat{\mathbf{H}}_0^*(\xi))] \sin(2t|\xi|), \tag{29}$$

$$|\hat{\mathbf{H}}(\xi, t)|^2 = |\hat{\mathbf{H}}_0(\xi)|^2 \cos^2(t|\xi|) + |\hat{\mathbf{E}}_0(\xi)|^2 \sin^2(t|\xi|) - \operatorname{Im}[\mathbf{n}\cdot(\hat{\mathbf{E}}_0(\xi) \times \hat{\mathbf{H}}_0^*(\xi))] \sin(2t|\xi|). \tag{30}$$

Taking the difference of (29) and (30), we obtain

$$|\hat{\mathbf{E}}(\xi, t)|^2 - |\hat{\mathbf{H}}(\xi, t)|^2 = [|\hat{\mathbf{E}}_0(\xi)|^2 - |\hat{\mathbf{H}}_0(\xi)|^2] \cos(2t|\xi|) + 2\operatorname{Im}[\mathbf{n}\cdot(\hat{\mathbf{E}}_0(\xi) \times \hat{\mathbf{H}}_0^*(\xi))] \sin(2t|\xi|), \tag{31}$$

Integrating (31) over the whole space and using Parseval's theorem, we obtain

$$\begin{aligned} e(t) - m(t) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [|\mathbf{E}(\mathbf{x}, t)|^2 - |\mathbf{H}(\mathbf{x}, t)|^2]\, dx_1\, dx_2\, dx_3 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [|\hat{\mathbf{E}}(\xi, t)|^2 - |\hat{\mathbf{H}}(\xi, t)|^2]\, d\xi_1\, d\xi_2\, d\xi_3 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [|\hat{\mathbf{E}}_0(\xi)|^2 - |\hat{\mathbf{H}}_0(\xi)|^2] \cos(2t|\xi|)\, d\xi_1\, d\xi_2\, d\xi_3 \\ &\quad + 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \operatorname{Im}[\mathbf{n}\cdot(\hat{\mathbf{E}}_0(\xi) \times \hat{\mathbf{H}}_0(\xi))] \sin(2t|\xi|)\, d\xi_1\, d\xi_2\, d\xi_3 . \end{aligned} \tag{32}$$

Under the hypotheses of (i), both integrals in the last part of (32) tend to zero as $t \to +\infty$ by the Riemann-Lebesque lemma. Therefore, (26) holds. Eq. (27) is an obvious consequence of (26) and (6).

For the proof of part (ii) we observe that the first integral in the last part of (32) consists of six integrals of the form (22) (23). The evenness of the six functions involved is obvious. Therefore the integral vanishes for $t \geq R$.

After a tedious calculation we obtain

$$I_m(\hat{\mathbf{E}}_0 \times \hat{\mathbf{H}}_0^*)(-|\xi|) = -\operatorname{Im}(\hat{\mathbf{E}}_0 \times \hat{\mathbf{H}}_0^*)(|\xi|); \tag{33}$$

hence all six integrals that appear in the last integral in (32) are odd functions of $|\xi|$. Using Lemma 3, we now obtain that this integral also vanishes for $t \geq R$. This completes the proof of the theorem.

4. Remarks. Zachmanoglou [8] has shown that if u is a solution of the wave equation with initial data that vanish outside a sphere of radius R, then the equipartition of energy theorem implies that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^2\, dx_1\, ds_2\, dx_3 = \text{constant}, \qquad t \geq R. \tag{34}$$

Since each component of the electric and the magnetic field satisfies the scalar wave equation, the results of [2] and [8] can be combined to obtain a componentwise partition of energy for Maxwell's equations. Nevertheless, this result does not show equipartition of electric and magnetic energy. The equipartition result is a consequence of the particular coupling between the electric and magnetic field in Maxwell's equations, while, if we consider the wave equations for each component of the **E** and **H** fields, the equations are completely decoupled.

The componentwise partition result can also be obtained from [7].

REFERENCES

[1] P. D. Lax and R. S. Phillips, *Scattering theory,* Academic Press, New York, 1967

[2] R. S. Duffin, *Equipartition of energy in wave motion,* JMAA **32**, 386–391 (1970)

[3] J. A. Goldstein, *An asymptotic property of solutions of wave equations,* Proc. Amer. Math. Soc. **23**, 359–363 (1969)

[4] J. A. Goldstein, *An asymptotic property of solutions of wave equations II,* JMAA **32**, 392–399 (1970)

[5] M. Shinbrot, *Asymptotic behavior of solutions of abstract wave equations,* Proc. Amer. Math. Soc. **19**, 1403–1406 (1967)

[6] H. A. Levine, *An equipartition of energy theorem for weak solutions of evolutionary equations in Hilbert space: the Lagrange identity method,* J. Diff. Eq. **24**, 197–210 (1977)

[7] D. G. Costa, *On partition of energy for uniformly propagative systems,* JMAA **58**, 56–62 (1977)

[8] E. C. Zachmanoglou, *Integral constants in wave motion,* JMAA **39**, 296–297 (1972)