Discrete Comput Geom 15:147–167 (1996)

Discrete & Computational
Geometry

Equipartition of Mass Distributions by Hyperplanes*

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Abstract. We consider the problem of determining the smallest dimension $d = \Delta(j, k)$ such that, for any j mass distributions in $\mathbf{R}^d$, there are k hyperplanes so that each orthant contains a fraction $1/2^k$ of each of the masses. The case $\Delta(1, 2) = 2$ is very well known. The case $k = 1$ is answered by the ham-sandwich theorem with $\Delta(j, 1) = j$. By using mass distributions on the moment curve the lower bound $\Delta(j, k) \geq j(2^k - 1)/k$ is obtained. We believe this is a tight bound. However, the only general upper bound that we know is $\Delta(j, k) \leq j2^{k-1}$. We are able to prove that $\Delta(j, k) = \lceil j(2^k - 1)/k \rceil$ for a few pairs (j, k) ($(j, 2)$ for $j = 3$ and $j = 2^n$ with $n \geq 0$, and $(2, 3)$), and obtain some nontrivial bounds in other cases. As an intermediate result of independent interest we prove a Borsuk–Ulam-type theorem on a product of balls. The motivation for this work was to determine $\Delta(1, 4)$ (the only case for $j = 1$ in which it is not known whether $\Delta(1, k) = k$); unfortunately the approach fails to give an answer in this case (but we can show $\Delta(1, 4) \leq 5$).

1. Introduction

Let μ be a mass distribution in $\mathbf{R}^d$. The question arises of whether d hyperplanes exist such that each of the 2^d corresponding open orthants contains at most a fraction $1/2^d$ of the mass $\mu(\mathbf{R}^d)$ (point masses are allowed). This is called a *d-partition* or *equipartition* in $\mathbf{R}^d$. The question was posed by Grünbaum in [8] and possibly by others as well. In computational geometry, this question resulted from the problem of designing efficient algorithms for half-space range queries [19]; however, currently the interest is essentially theoretical because better partitioning techniques are known [20], [12], [13] (see [14] for a survey). The case of point masses can be reduced to the case of density functions (see, e.g., [21] for an argument). Thus, without loss of generality we restrict here to this latter case, and then the condition is that each orthant contains a fraction $1/2^d$ of the mass.

* This research was supported by the National Science Foundation under Grant CCR-9118874.

The case $d = 1$ is straightforward. For $d = 2$, the answer is affirmative, the simple argument makes use of the intermediate value theorem. For $d = 3$, the answer is again affirmative [9], [21]; the proof in [21] uses the Borsuk–Ulam theorem on the two-dimensional sphere. For $d \geq 5$, the answer is negative [2]. The open question for $d = 4$ motivated the work presented in this paper and it remains open. We consider the more general problem of determining the smallest dimension $d = \Delta(j, k)$ for which (j, k)-*partitions* exist, that is, given any j mass distributions in $\mathbf{R}^d$, there are k hyperplanes that k-partition each of the mass distributions (a fraction $1/2^k$ of each mass in each of the 2^k orthants). The case $k = 1$ is settled with $\Delta(j, 1) = j$ by the well-known ham-sandwich theorem, and our general question seems a natural generalization of it. Using mass distributions on the moment curve in $\mathbf{R}^d$, the lower bound $\Delta(j, k) \geq j(2^k - 1)/k$ is found. We believe this is a tight bound. However, the only general (any k and j) upper bound we know is $\Delta(j, k) \leq j2^{k-1}$ which is somewhat trivial. For a few cases we can prove $\Delta(j, k) = \lceil j(2^k - 1)/k \rceil$ and obtain some nontrivial bounds for other cases. However, the question of tight bounds remains open in most cases. Thus, we raise a family of problems possibly as hard as the original 4-partition in $\mathbf{R}^4$.

A first step in our approach is to define a certain multivalued function (a direct generalization of [21]) whose zeros are in direct correspondence with equipartitions. Then using an elementary technique we are able to compute the parity of the number of zeros of that function in a subset of the original space of candidate solutions (the *degree* of the function); an odd parity implies the existence of a zero and, hence, an equipartition. This type of parity argument underlies any proof of the Borsuk–Ulam theorem. As an intermediate result of independent interest, we obtain a Borsuk–Ulam type theorem for functions defined on a product of several balls.

The following table shows, for each of $k = 2, \ldots, 5$, the value of $\Delta(j, k)$ up to the first value of j for which the exact value is not known. The entries marked with $\star$ are old results; $\Delta(2, 2) = 3$ is a result of Hadwiger [9], and it implies $\Delta(1, 3) = 3$.

$\Delta(1, 2) = 2^\star$	$\Delta(1, 3) = 3^\star$	$4 \leq \Delta(1, 4) \leq 5$	$7 \leq \Delta(1, 5) \leq 9$
$\Delta(2, 2) = 3^\star$	$\Delta(2, 3) = 5$		
$\Delta(3, 2) = 5$	$7 \leq \Delta(3, 3) \leq 9$		
$\Delta(4, 2) = 6$			
$8 \leq \Delta(5, 2) \leq 9$			

The table below shows some general bounds for $\Delta(j, k)$ with $k = 2, \ldots, 5$: the second column is the lower bound $j(2^k - 1)/k$, and the third column is the upper bound that we prove for $j = 2^m$, $m \geq 1$. The case $k = 1$ is the ham-sandwich theorem (so it is valid for any j).

$k = 1$	j	j
$k = 2$	$3j/2$	$3j/2$
$k = 3$	$7j/3$	$5j/2$
$k = 4$	$15j/4$	$9j/2$
$k = 5$	$31j/5$	$15j/2$

All our upper bounds follow from computing the parity of certain function of a 0–1 matrix (related to the permanent); parity one indicates that equipartition exists, parity zero is inconclusive. As k grows, the size of the matrix and the complexity of computing the function grows rapidly, ruling out its computation even with the help of a computer. We have been unable to compute them analytically for general k. The fact that we do not give results for j not a power of 2 is in part due to the difficulty of computing this function in general, and in part because the resulting parity is zero. For example, we obtain parity zero when trying to establish whether (5, 2)-partitions exist in $\mathbf{R}^8$, but we can still establish that (5, 2)-partitions exist in $\mathbf{R}^9$.

The contents of this paper are as follows. In Section 2 we present the proof technique that allows us to compute the parity of zeros of a function with certain properties. In Section 3 we further elaborate on the technique in the particular case in which the domain of the function is a product of balls, and prove a Borsuk–Ulam-type theorem. In Section 4 we present the reduction of equipartition to a multivalued function, a lower bound and a first upper bound. In Section 5 we present an example, 2-partition, that illustrates all the ideas in a small problem. In Section 6 we present the general case and state the results that we can obtain. Finally, in Section 7 we have some concluding remarks and open problems.

2. Proof Technique

The result for an arbitrary continuous function $f\colon X \to \mathbf{R}^n$, where $X \subseteq \mathbf{R}^n$ is n-dimensional, compact, and equal to the closure of its interior, follows from the result for a continuous piecewise linear function $r = (r_1, \ldots, r_n)\colon |\mathcal{T}| \to \mathbf{R}^n$, where $\mathcal{T}$ is a triangulation of X, that satisfies a nondegeneracy (or general position) property: the zero-set of any m component (coordinate) functions r_i in any m-simplex in $\mathcal{T}$ is either a single point in the interior or empty. The proof technique has two elements, both well known in topology. First, the fact that under the stated conditions (nondegenerate piecewise linearity) for $r = (r', r'')\colon X \to \mathbf{R}^{n-1} \times \mathbf{R}$, the set $r'^{-1}(0)$ is a collection of paths in X with endpoints in the boundary of X (perhaps the earliest reference is [10]). Second, computing the parity of the number of zeros of r in X, as the parity of zeros of r' in $\mathrm{bd}(X)$ where r'' is positive. A parity argument of this type is implicit in any proof of the Borsuk–Ulam theorem (for example, in the combinatorial lemma of Tucker and Fan [17], [6], [3]). Recently, that parity argument was used in [11] to prove the existence of ham-sandwich cuts for point sets and to design an algorithm that finds one.

For completeness we include in an appendix some background definitions and facts. In this and later sections, arguments related to triangulation and nondegeneracy (for which familiarity with the material in the appendix is needed) appear in smaller type; the reader may choose to skip over them.

Nondegeneracy. We need to consider piecewise linear functions with the following *nondegeneracy* (or *general position*) property: $r\colon |\mathcal{T}| \to \mathbf{R}^n$ is nondegenerate if the *zero-set* of any m of its component functions (the set of points where all of them are zero) in any m-simplex is either empty or a single point in the interior, $1 \le m \le n$ (equivalently, any m of its component functions have no (common) zero on an l-simplex where $l < m$, $1 \le m \le n$). Given $f\colon X \to \mathbf{R}^n$, a nondegerate piecewise linear ϵ-approximation is obtained in two steps: (1) Obtain an $(\epsilon/2)$-approximation r^* (then each component r_i^* of r^* can be perturbed by at most $\epsilon/2$ at any vertex and the resulting function is still an ϵ-approximation); (2) *perturb* r^* as follows: Consider the simplices of $\mathcal{T}$ in increasing order of dimension. At each step, for each vertex v, let $\rho_{v,i} > 0$ denote the maximum amount by which the ith component function can be changed while the overall function is still an ϵ-approximation (initially, $\rho_{v,i} \geq \epsilon/2$ for each v, i). A basic step is to consider a k-dimensional simplex σ and any $k+1$ component functions $r^*_{i_0}, \ldots, r^*_{i_k}$, $0 \leq k < n$. By the inductive perturbation, there is at most one point p in the interior of σ which is a zero of $r^*_{i_0}, \ldots, r^*_{i_{k-1}}$. If p is also a zero of $r^*_{i_k}$, then changing by a "small" amount the value of $r^*_{i_k}$ at a vertex v of σ will make the value of $r^*_{i_k}$ at p different from zero. This can be done so that there are new $\rho_{w,i_j} > 0$ for each vertex w of σ and each j (ρ_{w,i_j} must be updated even for $j \neq k$ because the value of $r^*_{i_j}$ may be perturbed in a later step).

Let $X = |\mathcal{T}|$ and $r\colon X \to \mathbf{R}^n$ be nondegenerate piecewise linear. Let $r = (r', r'')$ where r'' is a single component of r, and r' consists of the remaining components of r. Let Z_r be the set of zeros of r in X, let $Z_{r'}$ be the set of zeros of r' in $\mathrm{bd}(X)$, and let $Z^+_{r'}$ and $Z^-_{r'}$ be the sets of zeros of r' in $\mathrm{bd}(X)$ where r'' is positive and negative, respectively. Nondegeneracy implies that Z_r, $Z_{r'}$ are finite sets. Let $P(r; X)$ denote the parity (zero or one) of $|Z_r|$ and let $P^+(r', \underline{r''}; \mathrm{bd}(X))$ denote the parity of $|Z^+_{r'}|$. The following lemma gives a relation between $P(r; X)$ and $P^+(r', \underline{r''}; \mathrm{bd}(X))$ which allows a recursive computation of $P(r; X)$.

Lemma 2.1. *Let $r = (r', r'')\colon X \to \mathbf{R}^{n-1} \times \mathbf{R}$ be a nondegenrate piecewise linear (npl) function. Then $P(r; X) = P^+(r', \underline{r''}; \mathrm{bd}(X))$.*

Proof. Let $\sigma \in \mathcal{T}$ be an n-simplex. Because of nondegeneracy, $r'^{-1}(0) \cap \sigma$ is either empty or a line segment joining the interiors of two $(n-1)$-dimensional faces of σ. It follows that $r'^{-1}(0)$ is a collection of paths (which cannot fork or end abruptly in the interior). Thus a path can be either a closed path in the interior of X, or an open path with endpoints in the boundary of X. An open path can be either (++), (−−), or (+−) depending on the sign of r'' at its endpoints. By nondegeneracy, $Z_{r'} = Z^+_{r'} \cup Z^-_{r'}$. Since the open paths determine a matching on the points in $Z_{r'}$, then $|Z_{r'}|$ is even and, hence, $|Z^+_{r'}| = |Z^-_{r'}| \bmod 2$. Furthermore, the parity of $|Z^+_{r'}|$ is equal to the parity of the number of (+−)-open paths. Closed paths and also (++)- and (−−)-open paths carry an even number of zeros of r, while (+−)-open paths carry an odd number of them. Therefore, the parity of $|Z_r|$ is equal to the parity of (+−)-open paths. Finally, we conclude that $|Z_r| = |Z^+_{r'}| \bmod 2$. □

The following theorem includes the additional elements needed for a recursive computation of $P(r; X)$. $\bigoplus$ denotes sum module 2.

Theorem 2.2. *Let $r = (r', r'')\colon X \to \mathbf{R}^{n-1} \times \mathbf{R}$ be an npl function.*

(i) *Suppose $\mathrm{bd}(X) = \bigcup_{i=1}^{s} Y_i$, where each Y_i is the underlying space of a subcomplex of $\mathcal{T}$, and the Y_i are interior disjoint. Then $P(r; X) = \bigoplus_{i=1}^{s} P^+(r', \underline{r''}; Y_i)$.*
(ii) *Suppose $Y_i = Y_{i,t} \cup Y_{i,b}$ is an interior disjoint union and there is a bijection $\beta\colon Y_{i,t} \cap Z_{r'} \to Y_{i,b} \cap Z_{r'}$ such that, for each $x \in Y_{i,t} \cap Z_{r'}$, $r''(x) = (-1)^a r''(\beta(x))$ where a is zero or one. Then $P^+(r', \underline{r''}; Y_i) = a \cdot P(r'; Y_{i,t})$.*

3. A Borsuk–Ulam-Type Theorem on a Product of Balls

In $\mathbf{R}^n$ the unit *n-ball* is $B^n = \{x \in \mathbf{R}^n \colon \|x\| \le 1\}$ and the unit $(n-1)$-sphere is $S^{n-1} = \{x \in \mathbf{R}^n \colon \|x\| = 1\}$. It is natural to use the sphere S^{n-1} as the space of possible directions. However, it is more convenient for our computations to restrict the space of directions to the ball B^{n-1}, a hemisphere of S^{n-1}. Thus, throughout we work with balls. By gluing balls to obtain spheres, the results of this section can be translated into results for spheres.

The boundary of B^n, $\mathrm{bd}(B^n)$, consists of two copies of B^{n-1} glued by the identification of their boundaries. We consider a product of k balls $\mathcal{B}(n_1, \ldots, n_k) = B^{n_1} \times \cdots \times B^{n_k}$ or just $\mathcal{B}$ when the dimensions are understood. We have $\mathrm{bd}(\mathcal{B}) = \bigcup_{i=1}^k \mathrm{bd}_i(\mathcal{B})$ where $\mathrm{bd}_i(\mathcal{B}) = B^{n_1} \times \cdots \times \mathrm{bd}(B^{n_i}) \times \cdots \times B^{n_k}$. $\mathrm{bd}_i(\mathcal{B})$ consists of two copies of $B^{n_1} \times \cdots \times B^{n_{i-1}} \times B^{n_i - 1} \times B^{n_{i+1}} \times \cdots \times B^{n_k}$ appropriately glued. The ith *antipodal map* $\alpha_i \colon \mathrm{bd}_i(\mathcal{B}) \to \mathrm{bd}_i(\mathcal{B})$ is defined by $\alpha_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_k)$.

Consider $f = (f_1, \ldots, f_l) \colon \mathcal{B}(n_1, \ldots, n_k) \to \mathbf{R}^l$ where $l = \sum_{i=1}^k n_i$. f_i has *positive antipodality* (resp. *negative antipodality*) with respect to the jth ball if, for all $x \in \mathrm{bd}_j(\mathcal{B})$, $f_i(\alpha_j(x)) = f_i(x)$ (resp. $f_i(\alpha_j(x)) = -f_i(x)$). Let $\mathcal{F}_B(n_1, \ldots, n_k)$ be the set of continuous functions $f \colon \mathcal{B}(n_1, \ldots, n_k) \to \mathbf{R}^l$, where $l = \sum_{i=1}^k n_i$, such that, for each $1 \le i \le l$ and $1 \le j \le k$, f_i has either positive or negative antipodality with respect to the jth ball. The *antipodality matrix* $A(f)$ of $f \in \mathcal{F}_B$ is the $l \times k$ 0–1 matrix with $[A(f)]_{ij} \equiv a_{ij}$ equal to zero or one depending on whether the antipodality of f_i with respect to the jth ball is positive or negative. The *antipodality vector* of a component function is the corresponding row of the antipodality matrix. When convenient, we use string notation for this vector, with X representing "do not care"; for example, $0^{i-1}1X^{k-i}$ indicates positive antipodality with respect to the first $i-1$ balls, negative antipodality with respect to the ith ball, and either positive or negative antipodality with respect to the remaining balls.

The classic Borsuk–Ulam antipodal theorem states that if $f \in \mathcal{F}_B(n)$ all of whose components have negative antipodality, then f has a zero. An antipodal theorem for $f \in \mathcal{F}_B(m, n)$ is proved in [22] (with very particular antipodality conditions) using techniques of algebraic topology. Our theorem is much more general and the proof technique is elementary. Rade Živaljević has pointed out that [5] contains a weaker version of a particular case of our theorem, and that in fact the technique of that paper can be used to prove our theorem. (These previous results are particular cases of Lemma 3.3(i) below.) Our elementary proof, however, has the additional advantage that it translates into an algorithm to find a zero. We say a few words about the algorithmic aspects in the last section.

In order to state the main theorem of this section, we need to define a certain 0–1-valued function of a 0–1 matrix, which is related to the permanent. Let A be an $l \times k$ matrix and let $n_1, \ldots, n_k$ be nonnegative integers with $l = \sum_{i=1}^k n_i$; n_i is the *weight* of column i. Let $\mathcal{P}(n_1, \ldots, n_k)$ be the set of functions $\eta \colon [l] \to [k]$ with $|\eta^{-1}(i)| = n_i$ for each $i \in [k]$ ($[m]$ denotes $\{1, \ldots, m\}$). We define

$$\mathrm{perm}_{n_1,\ldots,n_k} A = \sum_{\eta \in \mathcal{P}(n_1,\ldots,n_k)} \prod_{i=1}^{l} a_{i\eta(i)} .$$

In words, $\operatorname{perm}_{n_1,\dots,n_k} A$ is the sum of all products that take one factor per row and n_i factors from the ith column of A. Note that if $l = k$ and $n_1 = \cdots = n_k = 1$, then this corresponds to the usual permanent. Also $\operatorname{perm}_{n_1,\dots,n_k} A$ can be written in terms of the usual permanent as $\operatorname{perm}(A')/(n_1! \cdots n_k!)$ where A' is an $l \times l$ matrix with the ith column of A repeated n_i times for each $i \in [k]$. The following equivalent recursive definition is the form in which $\operatorname{perm}_{n_1,\dots,n_k}$ appears in the proof of the main theorem: $\operatorname{perm}_{0,\dots,0} A = 1$ and

$$\operatorname{perm}_{n_1,\dots,n_k} A = \sum_{n_j \geq 1} a_{ij} \cdot \operatorname{perm}_{n_1,\dots,n_{j-1},n_j-1,n_{j+1},\dots,n_k} A_i,$$

where A_i is A after removing the ith row (this is expansion on the ith row). Note that exchanging rows and columns (together with the corresponding weights) do not affect the value of $\operatorname{perm}_{n_1,\dots,n_k} A$. We write $\operatorname{perm}'_{n_1,\dots,n_k} A$ to denote $\operatorname{perm}_{n_1,\dots,n_k} A \bmod 2$.

We define the *parity* $Q(f) = Q(f; n_1, \dots, n_k)$ of $f \in \mathcal{F}_B(n_1, \dots, n_k)$ as $Q(f) = \operatorname{perm}'_{n_1,\dots,n_k} A(f)$. Now we can state the main theorem of this section.

Theorem 3.1. *Let $f \in \mathcal{F}_B(n_1, \dots, n_k)$. If $Q(f; n_1, \dots, n_k) = 1$, then f has a zero.*

Note that the classical Borsuk–Ulam theorem is a particular case when $k = 1$ and the antipodality matrix is a single column of ones. The proof follows by compactness from the following lemma and an argument for the existence of appropriate approximations.

Lemma 3.2. *Let $r \in \mathcal{F}_B(n_1, \dots, n_k)$ be nondegenerate piecewise linear. Then $P(r; B^{n_1} \times \cdots \times B^{n_k}) = Q(r; n_1, \dots, n_k)$.*

Proof. By induction on $l = \sum_{i=1}^k n_i$. For $l = 0$, both $P(-; B^0 \times \cdots \times B^0) = 1$ and $Q(-; 0, \dots, 0) = 1$. For the inductive step, let $A = [a_{ij}]$ be the antipodality matrix of $r = (r_1, \dots, r_l)$ and let $r' = (r_1, \dots, r_{l-1})$ and $r'' = r_l$ in the notation of Theorem 2.2. Then

$$\begin{aligned} P(r', r''; B^{n_1} \times \cdots \times B^{n_k}) &= \bigoplus_{n_i \geq 1} P^+(r', \underline{r''}; B^{n_1} \times \cdots \times \operatorname{bd}(B^{n_i}) \times \cdots \times B^{n_k}) \\ &= \bigoplus_{n_i \geq 1} a_{li} \cdot P(r'; B^{n_1} \times \cdots \times B^{n_i - 1} \times \cdots \times B^{n_k}) \\ &= \bigoplus_{n_i \geq 1} a_{li} \cdot Q(r'; n_1, \dots, n_i - 1, \dots, n_k) \\ &= Q(r', r''; n_1, \dots, n_k). \end{aligned}$$

The first equality follows by Theorem 2.2(i); the second equality follows by Theorem 2.2(ii) (noting that $B^{n_1} \times \cdots \times \operatorname{bd}(B^{n_i}) \times \cdots \times B^{n_k}$ is the union of two copies of $B^{n_1} \times \cdots \times B^{n_i - 1} \times \cdots \times B^{n_k}$, that a bijection between the zeros of r' in each of the copies exists because of the antipodality of all components with respect to the ith ball, and that a_{li} indicates the antipodality of r'' with respect to the ith ball); the third equality follows by the induction hypothesis; and the last equality follows by the recursive definition of $Q(r)$. □

Existence of Approximations. We use the n-dimensional *cross polytope* C_n as a model for B^n. C_n is the convex hull of the unit vectors $\pm e_i = (0, \dots, 0, \pm 1, 0, \dots, 0)$, where ± 1 is in the ith coordinate. In fact, C_n is the n-ball for the norm $\|x\|_1 = \|(\xi_1, \dots, \xi_n)\|_1 = \sum_{i=1}^{n} |\xi_i|$. Let $V_n = \{e_i, -e_i \colon i = 1, \dots, n\} \cup \{0\}$. The simplicial complex $\mathcal{T} = \mathcal{T}_n = \{\mathrm{conv}(T) \colon T \subseteq V_n$, no two points in T are antipodal$\}$ (i.e., the complex consisting of all cones of the origin with the faces of C_n) is a triangulation of B^n. This triangulation has *antipodal symmetry*: if $\sigma \in \mathcal{T}_n$ and $\sigma \subseteq \mathrm{bd}(B^n)$, then $\alpha(\sigma) \in \mathcal{T}_n$. We need a triangulation $\mathcal{T}$ of $\mathcal{B} = B^{n_1} \times \cdots \times B^{n_k}$ that has *componentwise antipodal symmetry*, that is, if $\sigma \in \mathcal{T}$ and $\sigma \subseteq \mathrm{bd}_i(\mathcal{B})$ for some i, then $\alpha_i(\sigma) \in \mathcal{T}$ (this is important for extending to $\mathcal{B}$ by linearity a vertex map with a certain antipodality matrix so that the resulting function has the same antipodality matrix). First obtain the cell complex $\mathcal{C} = \{\sigma_1 \times \cdots \times \sigma_k \colon \sigma_i \in \mathcal{T}_{n_i}\}$, and then triangulate by taking a barycentric subdivision (note that each cell in $\mathcal{C}$ is a product of simplices). Clearly, the resulting triangulation has componentwise antipodal symmetry; and so does its lth barycentric subdivision $\mathcal{T}^* = \mathrm{Sd}^l(\mathcal{T})$ for any l. Therefore, a piecewise linear approximation defined from the vertex map $\varphi(v) = f(v)$, for $v \in \mathcal{T}^{*(0)}$, is in $\mathcal{F}_{\mathcal{B}}$ and has the same antipodality matrix as f. Finally, we verify that the perturbation procedure indicated in Section 2 can be carried out while preserving the antipodality properties. Consider the *orbits* of the antipodality maps: x and y are in the same orbit if, for some $i_1, \dots, i_s$, $y = \alpha_{i_1} \circ \cdots \circ \alpha_{i_s}(x)$. To preserve the antipodality properties, during the perturbation procedure, when changing the value of a component function at a vertex, it must be changed correspondingly at all the vertices in the same orbit. Since no simplex has two vertices in the same orbit, this procedure works correctly.

There are two cases of interest in which the value $Q(A)$ can be easily computed:

Lemma 3.3.

(i) *If n_i is the number of rows $0^{i-1}1X^{k-i}$ in A, then $Q(A) = 1$.*
(ii) *Let A have two columns, i and j, such that* (a) $n_i = n_j$, (b) *there is a row, t, with $a_{ti} = a_{tj} = 1$ and $a_{ts} = 0$ for $s \neq i, j$, and* (c) *each row k with $a_{ki} = 0$ and $a_{kj} = 1$ can be paired bijectively with a row k' with $a_{k'i} = 1$ and $a_{k'j} = 0$, so that $a_{ks} = a_{k's}$ for $s \neq i, j$. Then $Q(A) = 0$.*

Proof. There is only one nonzero product in the sum defining $\mathrm{perm}_{n_1,\dots,n_k} A$. (ii) Using expansion on row t, $Q(A') + Q(A'')$ is obtained, which is 0 mod 2 because A'' can be transformed into A' by exchanging columns i and j, and each row k with its corresponding row k'. □

Problem 3.4. Is Theorem 3.1 best possible? That is, if A and $n_1, \dots, n_k$ are such that $\mathrm{perm}'_{n_1,\dots,n_k} A = 0$, then does a function $f \in \mathcal{F}_{\mathcal{B}}(n_1, \dots, n_k)$ with $A(f) = A$ and which has no zeros exist?

Problem 3.5. Is there a Tucker–Fan-type combinatorial lemma from which Theorem 3.1 can be derived?[1]

[1] Let C_n be the n-dimensional cross polytope with a barycentric subdivision $\mathcal{T}^*$ of the triangulation $\mathcal{T}_n$ described previously. Let $\xi \colon \mathcal{T}^{*(0)} \to [m] \cup (-[m]) = \{\pm 1, \dots, \pm m\}$ be a labeling of the vertices that has negative antipodality, $\xi(\alpha(v)) = -\xi(v)$. A simplex $\mathrm{conv}(v_0, \dots, v_k)$ of $\mathcal{T}^*$ is alternating if the list $[\xi(v_{i_0}), \dots, \xi(v_{i_k})]$ of the labels of its vertices in increasing order of magnitude, $|\xi(v_{i_0})| \leq \cdots \leq |\xi(v_{i_k})|$, has no repetitions and the signs alternate. The simplex is positive if the first label in that list is positive. The Tucker–Fan combinatorial lemma [17], [6], [3] says that *if no edge in $\mathcal{T}^*$ has opposite labels under ξ, then the number of alternating n-simplices is odd* (*in particular $m > n$*). Therefore, if $m = n$, then there must be an edge with opposite labels. Together with a compactness argument, this implies the Borsuk–Ulam theorem.

4. Equipartition by Hyperplanes

It is sufficient to consider the existence for the case of mass distributions which are density functions with connected open *support* (see, e.g., [21]). We assume the mass distributions are normalized to one. An *oriented hyperplane* $h = (x, t) \in B^{d-1} \times \mathbf{R}$ in $\mathbf{R}^d$ with *normal unit vector* x is the set $\{y \in \mathbf{R}^d\colon x \cdot y = t\}$. h defines two *half-spaces* $h^0 = \{y \in \mathbf{R}^d\colon x \cdot y \geq t\}$ and $h^1 = \{y \in \mathbf{R}^d\colon x \cdot y \leq t\}$ (*positive* and *negative*, respectively). The equipartition problem for $d = 1$ provides the following fact by projecting onto a line.

Fact 4.1. *Let μ be a mass distribution with connected open support. For each direction $x \in B^{d-1}$ in $\mathbf{R}^n$, there is a unique bisecting hyperplane (1-partition) $h^\mu(x)$ normal to x. Furthermore, the function $h^\mu(x)$ is continuous (considering the image space endowed with the natural topology of $B^{d-1} \times \mathbf{R}$) and has negative antipodality.*

4.1. *Reducing Equipartition to a Zero of a Function*

2-Partition. For hyperplanes h_1, h_2 and $i, j \in \{0, 1\}$, let $a_{ij}(h_1, h_2) = \mu(h_1^i \cap h_2^j)$. The goal is to obtain h_1, h_2 so that $a_{ij} = 1/4$ for each pair i, j. Let

$$\begin{aligned} f_{01}(h_1, h_2) &= (a_{00} - a_{01} - a_{11} + a_{10})(h_1, h_2), \\ f_{10}(h_1, h_2) &= (a_{00} + a_{01} - a_{11} - a_{10})(h_1, h_2), \\ f_{11}(h_1, h_2) &= (a_{00} - a_{01} + a_{11} - a_{10})(h_1, h_2). \end{aligned}$$

It is easy to verify that h_1, h_2 form a 2-partition if and only if $f_{01} = f_{10} = f_{11} = 0$. Fix $x_1 \in B^1$. Then, for $x_2 \in B^1$, $h^\mu(x_1), h^\mu(x_2)$ form a 2-partition iff $g(x_2) = f_{11}(h^\mu(x_1), h^\mu(x_2)) = 0$. $g(x_2)$ has negative antipodality. So, by the intermediate value theorem, $x_2 \in B^1$ exists such that $g(x_2) = 0$ (the case $n = 1$ of the Borsuk–Ulam theorem). It can also be verified that the zero is unique under certain conditions (for example, if the support of μ is a ball).

***k*-Partition.** The reduction above was generalized to 3-partition by Yao *et al.* in [21]. From there, the extension to k-partition given next is straightforward. For hyperplanes $h_1, \ldots, h_k$, and for $j_1, \ldots, j_k \in \{0, 1\}$ let

$$a_{j_1 \cdots j_k}(h_1, \ldots, h_k) = \mu\left(\bigcap_{l=1}^{k} h_l^{j_l}\right),$$

the mass of the orthant with indices $j_1, \ldots, j_k$. For $i_1, \ldots, i_k \in \{0, 1\}$, let

$$f_{i_1 \cdots i_k}(h_1, \ldots, h_k) = \sum_{j_1, \ldots, j_k \in \{0,1\}} \varepsilon_{i_1 \cdots i_k}^{j_1 \cdots j_k} a_{j_1 \cdots j_k}(h_1, \ldots, h_k),$$

where $\varepsilon_{i_1 \cdots i_k}^{j_1 \cdots j_k} = (-1)^b$ with $b = \sum_{l=1}^{k} i_l j_l$. We write $f_{i_1 \cdots i_k}^{\mu}$ when there is ambiguity.

$f_{i_1\cdots i_k}(h_1,\ldots,h_k)$ depends only on those hyperplanes h_l with $i_l = 1$, say $h_{l_1},\ldots,h_{l_s}$; more precisely, the mass of an orthant of $h_{l_1},\ldots,h_{l_s}$ appears with positive (resp. negative) sign if that orthant is in the negative half-space for an even (resp. odd) number of the hyperplanes $h_{l_1},\ldots,h_{l_s}$. Thus, $f_{0\cdots 0} = 1$ always. We have the following properties:

Property 4.2. *$f_{i_1\cdots i_k}(h_1,\ldots,h_k)$ is a continuous function of $h_1,\ldots,h_k$.*

Property 4.3.

$$f_{i_1\cdots i_k}(h_1,\ldots,h_{l-1},-h_l,h_{l+1},\ldots,h_k) = (-1)^{i_l} f_{i_1\cdots i_k}(h_1,\ldots,h_{l-1},h_l,h_{l+1},\ldots,h_k).$$

If $i_l = 0$, clearly changing the orientation of h_l does not affect $f_{i_1\cdots i_k}$; if $i_l = 1$, then the signs of all the terms are reversed.

Property 4.4. *$h_1,\ldots,h_k$ form a k-partition of μ if and only if $f_{i_1\cdots i_k}(h_1,\ldots,h_k) = 0$ for $(i_1,\ldots,i_k) \neq (0,\ldots,0)$.*

Proof. If $h_1,\ldots,h_k$ is a k-partition, then it is clear that $f_{i_1\cdots i_k}(h_1,\ldots,h_k) = 0$ for all $(i_1,\ldots,i_k) \neq (0,\ldots,0)$ (and $f_{0\cdots 0} = 1$). To verify the other direction, we show that the matrix of coefficients $\{\varepsilon_{i_1\cdots i_k}^{j_1\cdots j_k}\}$ is orthogonal and, hence, nonsingular:

$$\begin{aligned}\sum_{j_1,\ldots,j_k\in\{0,1\}} \varepsilon_{i_1\cdots i_k}^{j_1\cdots j_k}\varepsilon_{i'_1\cdots i'_k}^{j_1\cdots j_k} &= \sum_{j_1,\ldots,j_k\in\{0,1\}} (-1)^{(j_1,\ldots,j_k)\cdot(i_1+i'_1,\ldots,i_k+i'_k)} \\ &= 0 \qquad \text{unless} \quad (i_1,\ldots,i_k) = (i'_1,\ldots,i'_k).\end{aligned}$$

The last step follows because if $i_k \neq i'_k$, then $i_k + i'_k = 1$, so the sum can be split into the two terms corresponding to $j_k = 0$ and $j_k = 1$, which are equal but of opposite sign; if $i_k = i'_k$, then $i_k + i'_k$ is even, so the sum is twice a sum with one less index, so the claim follows by an inductive argument (the base case being trivial). Since the matrix is nonsingular, the values of the $a_{j_1\cdots j_k}(h_1,\ldots,h_k)$ for which $f_{i_1\cdots i_k}(h_1,\ldots,h_k) = 0$ for $(i_1,\ldots,i_k) \neq (0,\ldots,0)$ and $f_{0\cdots 0}(h_1,\ldots,h_k) = 1$ are unique. That is, they must correspond to a k-partition. □

Note that the functions $f_{i_1\cdots i_k}$ are well defined for any dimension d, but equipartition makes sense only if $d \geq k$ (only then the 2^k orthants are possible). The following property will be useful later in our parity computations.

Property 4.5.

(i) *If $i_1\cdots i_k$ has an even number of ones and if all h_j with $i_j = 1$ are equal, then $f_{i_1\cdots i_k}(h_1,\ldots,h_k) = 1$.*

(ii) *For any j', j'', exchanging simultaneously $i_{j'}$ and $i_{j''}$, and $h_{j'}$ and $h_{j''}$ has no effect on the value of $f_{i_1\cdots i_k}(h_1,\ldots,h_k)$.*

4.2. *Equipartition Problem*

The problem of equipartition by hyperplanes that we consider is the following:

Determine the smallest dimension $d = \Delta(j, k)$ such that (j, k)-partitions exist in $\mathbf{R}^d$, that is, given any j mass distributions in $\mathbf{R}^d$, there are k hyperplanes which form a k-partition for each of the masses.

Let $\mu_1, \ldots, \mu_j$ be mass distributions in $\mathbf{R}^d$. As a first step, the space of candidate solutions for the k-partition problem is $B^{d-1} \times \cdots \times B^{d-1}$ (k factors) where B^{d-1} is a hemisphere of the unit sphere S^{d-1} in $\mathbf{R}^d$. More specifically, if $(x_1, \ldots, x_k) \in B^{d-1} \times \cdots \times B^{d-1}$, then the candidate hyperplanes are $h^{\mu_1}(x_1), \ldots, h^{\mu_1}(x_k)$ where $h^{\mu_1}(x)$ denotes the unique bisecting hyperplane for μ_1 in the direction x. From the discussion above, a solution corresponds to a zero of the function $g = (g_{k,*}^{\mu_1}, g_k^{\mu_2}, \ldots, g_k^{\mu_j})$, where $g_k^{\mu_i}$ consists of the $2^k - 1$ component functions

$$g_{i_1 \cdots i_k}^{\mu_i}(x_1, \ldots, x_k) = f_{i_1 \cdots i_k}^{\mu_i}(h^{\mu_1}(x_1), \ldots, h^{\mu_1}(x_k)),$$

for $(i_1, \ldots, i_k) \neq (0, \ldots, 0)$, and $g_{k,*}^{\mu_1}$ consists of the same components except those with antipodality vector $0^{i-1}10^{k-i}$, which are already equal to zero because of the choice of the hyperplanes as bisectors for μ_1. Thus, g has a total of $l_0 \equiv j(2^k - 1) - k$ components. We use the notation r, $r_k^{\mu_i}$, and $r_{k,*}^{\mu_i}$ for the npl approximations of g, $g_k^{\mu_i}$, and $g_{k,*}^{\mu_i}$, respectively.

A limitation of our approach is that it cannot take advantage, if there is any, of the additional degrees of freedom $(d-1)k - (j(2^k-1)-k) = dk - j(2^k-1)$. More precisely, we limit the space of candidate solutions to $\mathcal{B} = \mathcal{B}(n_1, \ldots, n_k) = B^{n_1} \times \cdots \times B^{n_k}$ where $\max_{i \in [k]} n_i = d - 1$ and $\sum_{i=1}^k n_i = l_0$, so that the dimension of both domain and range of g is l_0. In the following theorem, to simplify the statement, we include rows with antipodality $0^{i-1}10^{k-i}$ and add one to the weight of each column.

Theorem 4.6. *Let $n'_1, \ldots, n'_k$ be such that $\max_{i \in [k]} n'_i = d$ and $\sum_{i=1}^k n'_i = j(2^k - 1)$, and let $A(j, k)$ be the $j(2^k - 1) \times k$ matrix consisting of all 0–1 strings of length k other than 0^k each repeated j times. If $\mathrm{perm}'_{n_1, \ldots, n_k} A(j, k) = 1$, then (j, k)-partitions exist in $\mathbf{R}^d$, and hence $\Delta(j, k) \leq d$.*

Examples. We can now explain how to obtain some of the entries in the first table of the Introduction. In each case the lower bound follows from Theorem 4.7 below. $\Delta(3, 2) \leq 5$ follows from $\mathrm{perm}'_{5,4} A(3, 2) = 1$. $\Delta(3, 3) \leq 9$ follows from $\mathrm{perm}'_{9,8,4} A(3, 3) = 1$; since $\mathrm{perm}'_{7,7,7} A(3, 3) = 0$, $\mathrm{perm}'_{8,7,6} A(3, 3) = 0$ and $\mathrm{perm}'_{8,8,5} A(3, 3) = 0$, we cannot improve this bound. $\Delta(5, 2) \leq 9$ follows from $\mathrm{perm}'_{9,6} A(5, 2) = 1$; $\mathrm{perm}'_{8,7} A(5, 2) = 0$, so we cannot improve the upper bound.

Remark. The second example above illustrates that it is not possible to obtain general tight results (assuming the lower bound is tight) relying only on Theorem 3.1: For this, $n'_i \leq \lceil j(2^k - 1)/k \rceil$ for each i, and then, for some i, j, $n'_i = n'_j$, and Lemma 3.3(ii) implies that $Q(g) = 0$. This is to be expected, because the symmetry of the indices reflects the symmetry of the problem; in fact, exchanging hyperplanes h_i and h_j does not affect a solution, so the number of solutions has even parity. The next section tries to correct this situation.

4.3. *Lower Bound*

Since $k(d-1)$ is the maximum possible dimension of the domain of g, then there is a first constraint $l_0 \leq k(d-1)$, otherwise the existence of zeros cannot be guaranteed (if $l_0 > k(d-1)$, nondegeneracy implies that an npl approximation has no zeros). Rewriting, the necessary condition is $d \geq j(2^k - 1)/k$. As presented, this is a condition for the approach to provide any results. However, this is really a necessary condition for the existence of a k-partition of any j mass distributions in $\mathbf{R}^d$ as discussed next.

Theorem 4.7. $\Delta(j,k) \geq j(2^k - 1)/k$.

Proof. This follows the argument in [2] showing that $\Delta(1,d) > d$ for $d > 4$. Place j mass distributions, each one-dimensional and uniform on an interval, along the d-dimensional moment curve $M^d = \{(t^1, t^2, \ldots, t^d)\colon t \in \mathbf{R}\}$, with no overlap. A simultaneous k-partition of the j masses would need to cut each interval in at least $2^k - 1$ points, for a total of $j(2^k - 1)$ points. On the other hand, since a hyperplane intersects M^d in at most d points, k hyperplanes intersect M^d in at most kd points. It follows that $kd \geq j(2^k - 1)$ is a necessary condition for the existence of simultaneous k-partitions for any j mass distributions in $\mathbf{R}^d$. □

For the moment curve example and $j = 1$, the condition $kd \geq 2^k - 1$ is almost sufficient as we explain next. A k-Gray code C is a list of all k bit strings so that only one bit changes at a time; for C let the *bit wear* of the ith bit, $b_C(i)$, be the number of times that this bit changes in the $2^k - 1$ steps. The *maximum bit wear*, $w(C)$, is $\max_{i \in [k]} b_C(i)$. Now, let μ be a mass distribution in M^d, let $p_1, \ldots, p_{2^k - 1} \in M^d$ be the $2^k - 1$ points that determine 2^k intervals of equal mass, and let C be a k-Gray code with $w(C) \leq d$. Then C determines a k-partition of μ by making the ith hyperplane, $i = 1, \ldots, k$, go through the points $p_{j_1}, \ldots, p_{j_s}$ where $j_1, \ldots, j_s$ are the steps in C where the ith bit changes. Thus, if C with $w(C) \leq \lceil (2^k - 1)/k \rceil$ exists, then the condition $d \geq (2^k - 1)/k$ would be sufficient for the existence of a k-partition. We do not know whether such Gray codes exist for all k; however, constructions that get very close to this are known. Robinson and Cohn [16] have constructed *cyclic* Gray codes (so there are 2^k steps in this case) such that the *wear balance*, $\Delta(C) = \max_{i \neq j} |b_C(i) - b_C(j)|$, is zero or two (zero only in the case that k is a power of 2; Wagner and West [18] have also given a construction for this case). This implies that $w(C) \leq \lceil 2^k/k \rceil + \delta$, where δ is zero or one. Then we obtain a sufficient condition for the existence of a k-partition for a mass distribution in M^d: $d \geq 2^k/k$ for k a power of 2 (which is tight), and $d \geq \lceil 2^k/k \rceil + 1$ otherwise. Thus, if $\Delta(1,k)$ differs from $\lceil (2^k - 1)/k \rceil$ by more than one or two, then one should look elsewhere for a better lower bound construction.

4.4. *A General Upper Bound*

Theorem 4.8. $\Delta(j,k) \leq j2^{k-1}$.

Proof. Set n_i to the number of components of g with antipodality vector $0^{i-1}1X^{k-i}$. Note that $n_1 = j2^{k-1} - 1$ and $n_i < n_1$ for $i > 1$. Therefore, using Lemma 3.3(i) we conclude that (j,k)-partitions exist in R^d with $d = j2^{k-1}$. □

Actually, this result can be obtained in a simple manner. For $d \geq j2^{k-1}$, the ham-sandwich theorem guarantees that the following procedure works: start with the j original masses, in the ith step obtain a 1-partition for the $j2^{i-1}$ masses resulting from the previous step, for $1 \leq i \leq k$. We have given the proof above to illustrate our approach. Better results for a few cases are obtained later.

5. An Example: 2-Partition

According to the moment curve lower bound, $\mathbf{R}^3$ is the lowest dimension we can expect to have 2-partition of two sets. In this section we illustrate our machinery by proving that this is always possible, which is a result of Hadwiger [9]. Our proof, however, is simpler and generalizes to deal with the equipartition problem in general. We also show how it can be extended to $j = 2^m$ masses.

5.1. *2-Partition of Two Masses in* $\mathbf{R}^3$

From the previous section we want to determine the existence of zeros of

$$g = (g_{2,*}^{\mu_1}, g_2^{\mu_2}) = (g_{11}^{\mu_1}, g_{01}^{\mu_2}, g_{10}^{\mu_2}, g_{11}^{\mu_2}).$$

At first, the space of candidate solutions is $B^2 \times B^2$. However, as indicated in the previous section, the parity of zeros of an npl approximation of g in $B^2 \times B^2$ is zero. So, we consider as space of candidate solutions

$$(B^2)^2_{\leq} = \{(x_1, x_2)\colon 0 \leq \|x_1\| \leq \|x_2\| \leq 1\}.$$

The boundary of $(B^2)^2_{\leq}$ consists of two three-dimensional components:

(i) $X_{1,2} = \{(x_1, x_2)\colon x_i \in B^2, 0 \leq \|x_1\| = \|x_2\| \leq 1\}$, and
(ii) $X_{2,3} = \{(x_1, x_2)\colon x_i \in B^2, 0 \leq \|x_1\| \leq \|x_2\| = 1\}$.

$X_{2,3}$ is a copy of $B^2 \times \mathrm{bd}(B^2)$, that is, two copies of $B^2 \times B^1$. Two important subsets of $X_{1,2}$ are the *diagonal* $D_{1,2} = \{(x_1, x_1)\colon x_1 \in B^2\}$ and the *boundary antidiagonal* $A'_{1,2} = \{(x_1, -x_1)\colon x_1 \in \mathrm{bd}(B^2)\}$. (Note that $X_{0,1} = \{(x_1, x_2)\colon x_i \in B^2, 0 = \|x_1\| \leq \|x_2\| \leq 1\}$ is two-dimensional and contributes to the boundary of $(B^2)^2_{\leq}$ only a one-dimensional component.)

$(B^2)^2_{\leq}$ can be triangulated so that $X_{1,2}$, $X_{2,3}$, $D_{1,2}$, and $A'_{1,2}$ are underlying spaces of subcomplexes of the triangulation, and so that the triangulation is symmetric under exchange of x_1 and x_2 in $X_{1,2}$. This is verified for the general case in the next section.

Suppose r is an npl approximation of g in $(B^2)^2_{\leq}$. Then we have

$$\begin{aligned} P(r_{11}^{\mu_1}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; (B^2)^2_{\leq}) &= P^+(\underline{r_{11}^{\mu_1}}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; X_{1,2}) \\ &\quad + P^+(\underline{r_{11}^{\mu_1}}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; X_{2,3}). \end{aligned}$$

The second term is equal to $P(r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; B^2 \times B^1)$ (because of the negative antipodality of $r_{11}^{\mu_1}$ with respect to the second ball and Theorem 2.2(ii)) which is easily evaluated as one. We would like to show that the first term is zero, as follows: For each (x_1, x_2) such that $r_{01}^{\mu_2}(x_1, x_2) = r_{10}^{\mu_2}(x_1, x_2) = r_{11}^{\mu_2}(x_1, x_2) = 0$, we have $r_{01}^{\mu_2}(x_2, x_1) = r_{10}^{\mu_2}(x_2, x_1) = r_{11}^{\mu_2}(x_2, x_1) = 0$ and $r_{11}^{\mu_1}(x_1, x_2) = r_{11}^{\mu_1}(x_2, x_1)$ (using Property 4.5). We say that $(r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}, \underline{r_{11}^{\mu_1}})$ is symmetric for zeros in the boundary. Note that $(x_1, x_2) \neq (x_2, x_1)$ unless $x_1 = x_2$, that is, $(x_1, x_2) \in D_{1,2}$; but in this case (x_1, x_2) cannot be a zero of $(r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2})$ because $D_{1,2}$ is of dimension 2 (and by nondegeneracy cannot have zeros of three component functions). Therefore, using Theorem 2.2(ii),

$$P^+(\underline{r_{11}^{\mu_1}}, r_{01}^{\mu_2}, r_{10}^{\mu_2}, r_{11}^{\mu_2}; X_{1,2}) = 0.$$

For the previous argument to work, the npl approximation r must satisfy the symmetry under exchange of x_1 and x_2 that g satisfies. Specifically, for $(x_1, x_2) \in X_{1,2}$,

$$r_{11}^{\mu_i}(x_1, x_2) = r_{11}^{\mu_i}(x_2, x_1), \qquad r_{01}^{\mu_2}(x_1, x_2) = r_{10}^{\mu_2}(x_2, x_1).$$

However, the second symmetry is not possible while enforcing antipodality and nondegeneracy. For example, in $D_{1,2}$ it would imply $r_{10}^{\mu_2} \equiv r_{01}^{\mu_2}$. Nonetheless, this function has negative antipodality with respect to $D_{1,2}$ as a copy of B^2, so it has a zero in a copy of B^1 on the boundary of $D_{1,2}$. Thus, two component functions have a zero in some simplex of dimension less than 2; a contradiction to nondegeneracy. Similarly, in $A'_{1,2}$ we have $r_{01}^{\mu_2} \equiv -r_{10}^{\mu_2}$ because

$$r_{01}^{\mu_2}(x_1, -x_1) = r_{10}^{\mu_2}(-x_1, x_1) = -r_{10}^{\mu_2}(x_1, x_1) = -r_{10}^{\mu_2}(x_1, -x_1),$$

where the equalities follow by using symmetry, and antipodality with respect to the first and second balls, respectively. Again, the resulting function on $A'_{1,2}$ has negative antipodality and, hence, has a zero. So, again, two component functions have a zero in some simplex of dimension less than 2. In the next section we argue in the general case that these are the only possible conflicts.

To circumvent this problem, we make use of Property 4.5(i), namely, that $r_{11}^{\mu_i} = 1$ in $D_{1,2}$ and $r_{11}^{\mu_i} = -1$ in $A'_{1,2}$. Thus, symmetry can be allowed to fail "close" to $D_{1,2}$ and $A'_{1,2}$, as long as, when it is used in the argument for $X_{1,2}$ above, one of the component functions remaining is $r_{11}^{\mu_i}$. $r_{11}^{\mu_i}$ can be approximated preserving the symmetry everywhere without problem. For $r_{01}^{\mu_2}$ and $r_{10}^{\mu_2}$ do the following: For vertices not in $D_{1,2} \cup A'_{1,2}$ enforce the symmetry, and for vertices in $D_{1,2} \cup A'_{1,2}$ treat the functions as different so that they can be freely perturbed to remove degeneracies. The barycentric subdivision is chosen (previously) so that $r_{11}^{\mu_i}$ is not zero (say $|r_{11}^{\mu_i}| > \frac{1}{2}$) in the simplices incident to $D_{1,2} \cup A'_{1,2}$. With this modification, the parity computation above is correct.

Remark. We solve the conflict between symmetry and nondegeneracy by using the fact that $r_{11}^{\mu_2}$ is different from zero where the conflict appears, and that $r_{11}^{\mu_2}$ is part of the remaining function when expanding on $r_{11}^{\mu_1}$. However, if $j = 1$, when expanding on $r_{11}^{\mu_1}$, there is no other function r_{11} to "shield" the troublesome region. This makes it more difficult to deal with the case $j = 1$. However, with some additional work, the parity can still be computed; at least if the problem size is small. We have done this for the problem of 4-partition in $\mathbf{R}^4$. Unfortunately, the resulting parity is zero.

5.2. *Extension to $j = 2^m$*

We can extend the 2-partition result to $j = 2^m$ masses. We show that the condition $d \geq j(2^k - 1)/k$ is sufficient for $k = 2$ and $j = 2^m$. Replacing values, $d \geq 3j/2$. In this case $l_0 = 2^m(2^2 - 1) - 2 = 3 \cdot 2^m - 2$, so the space of candidate solutions is $(B^{d-1})^2_{\leq}$ where $d - 1 = l_0/2 = 3 \cdot 2^{m-1} - 1$ (thus $d = 3j/2$). What has been said above for $(B^2)^2_{\leq}$ extends to balls of arbitrary dimension; in the next section we deal with the general case. Let r^{μ_i} be the corresponding npl approximations (with symmetry failing "close" to $D_{1,2}$ and $A'_{1,2}$), and note that $(\underline{r_{11}^{\mu_1}}, r_2^{\mu_2}, \ldots, r_2^{\mu_{2m}})$ is symmetric for zeros in the boundary $X_{1,2}$. Then we find

$$P(r_{2,*}^{\mu_1}, r_2^{\mu_2}, \ldots, r_2^{\mu_{2m}}; (B^{d-1})^2_{\leq}) = P(r_2^{\mu_2}, \ldots, r_2^{\mu_{2m}}; B^{d-1} \times B^{d-2}).$$

This last term is equal to $P(r_{11}^{\mu_2}, \ldots, r_{11}^{\mu_{2m}}; B^{2^{m-1}} \times B^{2^{m-1}-1})$ which is equal to $\mathrm{perm}'_{2^{m-1},2^{m-1}-1} A_{2^m-1}$ where A_{2^m-1} is a $(2^m - 1) \times 2$ matrix with all entries equal to one.

Lemma 5.1. $\mathrm{perm}'_{2^{m-1},2^{m-1}-1} A_{2^m-1} = \mathrm{perm}'_{2^{m-2},2^{m-2}-1} A_{2^{m-1}-1}$.

Proof. Let Δ_1 and Δ_2 denote the perm on the left and right, respectively. Since all entries in A_{2^m-1} are one, then Δ_1 is equal to the parity of $|\mathcal{P}(2^{m-1}, 2^{m-1} - 1)|$ (recall the definition from Section 3). Let γ: $[2^m - 1] \to [2^m - 1]$ be defined by $\gamma(i) = 2^m - i$. γ is an *involution*, that is $\gamma \circ \gamma$ is the identity, and it has exactly one fixed point, $i = 2^{m-1}$. $\gamma^*(\eta) = \eta \circ \gamma$ defines an involution on $\mathcal{P}(2^{m-1}, 2^{m-1} - 1)$. Thus, Δ_1 is equal to the parity of the fixed points of γ^*. η is a fixed point of γ^* iff, for each $i \neq 2^{m-1}$, $\eta(i) = \eta(2^m - i)$ and $\eta(2^{m-1}) = 2$ (a fixed point η of γ^* must map its fixed point 2^{m-1} to 2 because only this column has odd size). So if η is a fixed point, η restricted to $[2^{m-1} - 1]$ completely determines η. Furthermore, $|\eta^{-1}(1) \cap [2^{m-1} - 1]| = 2^{m-1}/2 = 2^{m-2}$ and $|\eta^{-1}(2) \cap [2^{m-1} - 1]| = (2^{m-1} - 1 - 1)/2 = 2^{m-2} - 1$. Therefore, the parity of the fixed points of γ^* is precisely Δ_2. □

Applying Lemma 5.1 iteratively, it is found that $\mathrm{perm}'_{2^{m-1},2^{m-1}-1} A_{2^m-1} = \mathrm{perm}'_{1,0} A_1$ which is one. This completes the computation. We summarize this in a theorem.

Theorem 5.2. *For $j = 2^m$ with $m \geq 1$, $\Delta(j, 2) = 3j/2$.*

A similar computation applies for $j = 2^m - 1$. However, it appears that for most of the other values, trying to achieve the lowest dimension possible results in a zero parity, and we cannot conclude the existence. The smallest such case is $j = 5$ for which we know that $(5, 2)$-partitions exist in $\mathbf{R}^9$, but it is open in $\mathbf{R}^8$.

6. General Case

Let $(B^n)^k = B^n \times \cdots \times B^n$ be the usual product of k copies of B^n, and let $(B^n)^k_\leq = \{(x_1, \ldots, x_k)\colon x_i \in B^n, 0 \leq \|x_1\| \leq \cdots \leq \|x_k\| \leq 1\}$.

Triangulation of $(B^n)^k_\leq$. Recall that we regard the cross polytope C_n as our model for B^n under the norm $\|\cdot\|_1$. Let O^n be an *orthant* of B^n (the portion in an orthant of the coordinate system). Note that the *external boundary* of O^n (the portion of the boundary that is also the boundary of B^n) is a copy of O^{n-1}. $(B^n)^k$ consists of $(2^n)^k$ products of k orthants, $(O^n)^k$. For a particular product of orthants, each constraint $\|x_i\| \leq \|x_{i+1}\|$ defines a half-space. Thus, $(B^n)^k_\leq$ consists of $(2^n)^k$ cells, each the intersection of a product of orthants $(O^n)^k$ with $k-1$ half-spaces (hence a cell), denoted $(O^n)^k_\leq$. We need a triangulation of $(B^n)^k_\leq$ with certain symmetries on the boundary; it is sufficient to obtain a cell complex that has those symmetries for a typical cell $(O^n)^k_\leq$, then a barycentric subdivision produces a triangulation. For fixed k, we use induction on n. For $n = 1$, $(O^1)^k_\leq$ is a k-cube intersected with the $k-1$ half-spaces $\|x_i\| \leq \|x_{i+1}\|$, $i = 1, \ldots, k-1$; this cell and its faces is the desired cell complex. For $n > 1$, consider first $(O^n)^k$ (the corresponding product of orthants). Subdivide the product of its external boundaries, a copy of $(O^{n-1})^k$, into $k!$ copies of $(O^{n-1})^k_\leq$ (a copy for each of the orderings of $\|x_1\|, \ldots, \|x_k\|$), and use the inductive construction to obtain a cell complex for each copy of $(O^{n-1})^k_\leq$. This gives a cell complex C for $(O^{n-1})^k$. Then the cells

$$\sigma' = \text{closure}(\{(x_1, \ldots, x_k)\colon x_i \neq 0, (x_1/\|x_1\|, \ldots, x_k/\|x_k\|) \in \sigma\})$$

for $\sigma \in C$ form a cell complex C' for $(O^n)^k$. A cell complex for $(O^{n-1})^k_\leq$ is obtained by intersecting each cell in C' with the $k-1$ half-spaces $\|x_i\| \leq \|x_{i+1}\|$, $i = 1, \ldots, k-1$.

The boundary of $(B^n)^k_\leq$ consists of k full-dimensional components

$$X_{i,i+1} = \{(x_1, \ldots, x_k)\colon x_i \in B^n, 0 \leq \|x_1\| \leq \cdots \leq \|x_i\| = \|x_{i+1}\| \leq \cdots \leq \|x_k\| \leq 1\},$$

for $1 \leq i \leq k$. $X_{k,k+1}$ is a copy of $(B^n)^{k-1}_\leq \times \text{bd}(B^n)$, that is, two copies of $(B^n)^{k-1}_\leq \times B^{n-1}$. Let $X_b = \bigcup_{i=1}^{k-1} X_{i,i+1}$. Other components of interest in $\text{bd}((B^n)^k_\leq)$ are the *diagonals* $D_{ij} = \{(x_1, \ldots, x_k) \in (B^n)^k_\leq\colon x_i = x_j\}$ and the *boundary antidiagonals*

$$A'_{ij} = \{(x_1, \ldots, x_k) \in (B^n)^k_\leq\colon x_i, x_j \in \text{bd}(B^n), x_i = -x_j\}.$$

Let $C_{ij} = \{(x_1, \ldots, x_k) \in (B^n)^k_\leq\colon \|x_i\| = \|x_j\|\}$ (so $C_{i,i+1} = X_{i,i+1}$), and let $\beta_{ij}\colon C_{ij} \to C_{ij}$ be the map that exchanges the ith and jth coordinates, that is,

$$\beta_{ij}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k).$$

A triangulation $\mathcal{T}$ of $(B^n)^k_\leq$ is *symmetric on the boundary* if $\sigma \in \mathcal{T}$ and $\sigma \subseteq C_{ij}$ imply $\beta_{ij}(\sigma) \in \mathcal{T}$. A function $f\colon (B^n)^k_\leq \to \mathbf{R}$ is *symmetric on the boundary* if, for all i, j, $x \in C_{ij}$, then $f(\beta_{ij}(x)) = f(x)$. We have the following.

Lemma 6.1. *The triangulation of $(B^n)^k_{\leq}$ described above is symmetric on the boundary. For each i, j, $X_{i,i+1}$, $D_{i,j}$, $A'_{i,j}$ are underlying spaces of subcomplexes of the triangulation.*

Therefore, the linear extension of a symmetric vertex map is also symmetric.

We say that $r = (r', \underline{r''})$: $(B^n)^k_{\leq} \to \mathbf{R}^{nk}$ is *symmetric for zeros in the boundary* if, for all i, j and $x \in C_{ij}$, $r'(x) = 0$ implies that $r'(\beta_{ij}(x)) = 0$ and $r''(x) = r''(\beta_{ij}(x))$.

Lemma 6.2. *Suppose $r = (r', \underline{r''})$: $(B^n)^k_{\leq} \to \mathbf{R}^{nk}$ is npl and symmetric for zeros in the boundary, and let a be the antipodality of r'' with respect to the kth ball. Then $P(r', r''; (B^n)^k_{\leq}) = a \cdot P(r'; (B^n)^{k-1}_{\leq} \times B^{n-1})$.*

Proof. By Theorem 2.2(i), $P(r', r''; (B^n)^k_{\leq}) = \bigoplus_{i=1}^k P^+(r', \underline{r''}; X_{i,i+1})$. By Theorem 2.2(ii), $P^+(r', \underline{r''}; X_{k,k+1}) = a \cdot P(r'; (B^n)^{k-1}_{\leq} \times B^{n-1})$. Because $(r', \underline{r''})$ is symmetric for zeros in the boundary, all the other terms are zero. □

Nondegeneracy. As in the previous section, in the general case it is not possible to find npl approximations $r_k^{\mu_i}$ satisfying all the required symmetries. Consider a pair $(x; i_1 \cdots i_k)$ where $x \in \mathrm{bd}((B^n)^k_{\leq})$, $i_1, \ldots, i_k \in \{0, 1\}$. An antipodality map α_i, or a symmetry map β_{ij} changes this pair into $(x', i'_1 \cdots i'_k)$ as follows: for α_i, $x' = \alpha_i(x)$ and $i'_j = i_j$ for all j; for β_{st}, $x' = \beta_{st}(x)$ and $i'_s = i_t$, $i'_t = i_s$ and $i'_l = i_l$ for $l \neq s, t$. The *orbit* of $(x, i_1 \cdots i_k)$ is the set of all $(x', i'_1 \cdots i'_k)$ that can be obtained by applying maps α and β. In the perturbation procedure, when perturbing $r_{i_1 \cdots i_k}$ at x, for each $(x', i'_1 \cdots i'_k)$ in the orbit of $(x, i_1 \cdots i_k)$, the function $r_{i'_1 \cdots i'_k}$ must be correspondingly perturbed at x'. A conflict appears if there is a pair $(x'; i'_1 \cdots i'_k)$ in the orbit with either $x \neq x'$ but x, x' vertices of the same simplex, or $x = x'$ and $i_1 \cdots i_k \neq i'_1 \cdots i'_k$. In this case the perturbation may break down. Because of the triangulation, this can only happen, and only in the case $x = x'$, in the the sets D_{ij} and A'_{ij}. However, the function $g_{i_1 \cdots i_k}$, with $i_i = i_j = 1$ and $i_l = 0$ otherwise, is nonzero in D_{ij} and A'_{ij} (we call it a *shield* function). Therefore, the symmetry can be allowed to fail in D_{ij} and A'_{ij}. The perturbation procedure is modified so that symmetry β_{ij} is not enforced in the sets D_{ij} and A'_{ij}, but is enforced elsewhere. Thus, it is correct to assume that the npl approximations have the required symmetry properties as long as, in the expansion in which symmetry is used, a shield function remains for each symmetry used. Thus, the remainder of this section applies only for $j \geq 2$.

The spaces we consider are $\mathcal{B}_{\leq} = \mathcal{B}_{\leq}(m_1, t_1; \ldots; m_s, t_s) = (B^{m_1})^{t_1}_{\leq} \times \cdots \times (B^{m_s})^{t_s}_{\leq}$. A triangulation of $\mathcal{B}_{\leq}$ is easily obtained from the triangulations of its factors as the barycentric subdivision of the product cell complex.

Recall that, for μ_i, $r_k^{\mu_i}$ denotes the npl approximation to $g_k^{\mu_i}$ (each has $2^k - 1$ components), and that $g_{k,*}^{\mu_i}$ and $r_{k,*}^{\mu_i}$ denote the corresponding functions not including the components with antipodality vectors $0^{i-1}10^{k-i}$.

Let $m_1, t_1; \ldots; m_s, t_s$ be such that $\sum_{i=1}^s m_i t_i = l_0$ and $\sum_{i=1}^s t_i = k$. We need to compute

$$P(r_{k,*}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; \mathcal{B}_{\leq}(m_1, t_1; \cdots; m_s, t_s)). \qquad (*)$$

Let $r_{(p,q)}^{\mu_i}$ be the component of $r_k^{\mu_i}$ with antipodality vector

$$v_{p,q} = 0^{t_1} \cdots 0^{t_{p-1}} 1^q 0^{t_p - q} 0^{t_{p+1}} \cdots 0^{t_s},$$

for $1 \leq p \leq s$ and $2 \leq q \leq t_p$, and let $r_{(p)}^{\mu_i}$ consist of the components $r_{(p,q)}^{\mu_i}$ with $2 \leq q \leq t_p$. Let $r_{k,\Delta}^{\mu_i}$ be $r_{k,*}^{\mu_i}$ without the components $r_{(p,q)}^{\mu_i}$, $1 \leq p \leq s$, $2 \leq q \leq t_p$. Thus, $r_{k,*}^{\mu_1} = r_{k,\Delta}^{\mu_1}, r_{(1)}^{\mu_1}, \ldots, r_{(s)}^{\mu_1}$.

Theorem 6.3. *For $j \geq 2$, $(*)$ is equal to*

$$P(r_{k,\Delta}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; (B^{m_1} \times (B^{m_1-1})^{t_1-1}) \times \cdots \times (B^{m_s} \times (B^{m_s-1})^{t_s-1})). \qquad (**)$$

Proof. As noted above, we can guarantee the existence of npl approximations with the required symmetries only for $j \geq 2$. Using Lemma 6.2, we have the following computation:

$$\begin{aligned}
(*) &= P(r_{k,*}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; (B^{m_1})_{\leq}^{t_1} \times \cdots \times (B^{m_s})_{\leq}^{t_s}) \\
&= P(r_{k,\Delta}^{\mu_1}, r_{(1)}^{\mu_1}, \ldots, r_{(s)}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; (B^{m_1})_{\leq}^{t_1} \times \cdots \times (B^{m_s})_{\leq}^{t_s}) \\
&= P(r_{k,\Delta}^{\mu_1}, r_{(2)}^{\mu_1}, \ldots, r_{(s)}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; \\
&\qquad (B^{m_1} \times (B^{m_1-1})^{t_1-1}) \times (B^{m_2})_{\leq}^{t_2} \times \cdots \times (B^{m_s})_{\leq}^{t_s}) \\
&= P(r_{k,\Delta}^{\mu_1}, r_{(3)}^{\mu_1}, \ldots, r_{(s)}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; \\
&\qquad (B^{m_1} \times (B^{m_1-1})^{t_1-1}) \times (B^{m_2} \times (B^{m_2-1})^{t_2-1}) \times (B^{m_3})_{\leq}^{t_3} \times \cdots \times (B^{m_s})_{\leq}^{t_s}) \\
&\vdots \\
&= P(r_{k,\Delta}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; (B^{m_1} \times (B^{m_1-1})^{t_1-1}) \times \cdots \times (B^{m_s} \times (B^{m_s-1})^{t_s-1})). \quad \square
\end{aligned}$$

Thus, the problem has been reduced to the computation of the perm of a matrix. We do not know how to compute this term in general. Certainly, we can compute particular cases by hand or with the help of a computer. There is one case we can do with some generality: a result for j masses can be extended to $j' = j2^m$ masses using a computation similar to the one in the previous section.

Lemma 6.4. *Let $j' = j2^m$ and $m_i' = (m_i + 1)2^m$. Then*

$$\begin{aligned}
&P(r_{k,\Delta}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_j}; (B^{m_1} \times (B^{m_1-1})^{t_1-1}) \times \cdots \times (B^{m_s} \times (B^{m_s-1})^{t_s-1})) \\
&\quad = P(r_{k,\Delta}^{\mu_1}, r_k^{\mu_2}, \ldots, r_k^{\mu_{j'}}; (B^{m_1'} \times (B^{m_1'-1})^{t_1-1}) \times \cdots \times (B^{m_s'} \times (B^{m_s'-1})^{t_s-1})).
\end{aligned}$$

Proof. Let $m_i(l) = (m_i + 1)2^l - 1$, $L(l) = \sum_{i=1}^{s}(m_i(l)t_i + 1)$, and $k = \sum_{i=1}^{s} t_i$. Let

$$\Delta_l = \mathrm{perm}'_{m_1(l)+1, m_1(l), \ldots, m_1(l), \ldots, m_s(l)+1, m_s(l), \ldots, m_s(l)} A_l$$

where A_l is an $L(l) \times k$ matrix that consists of all nonnull 0–1 k-vectors repeated $j2^m$ times, except the vectors $v_{p,q}$, $1 \leq p \leq s$, $2 \leq q \leq t_p$, which are repeated $j2^m - 1$ times (to facilitate the argument we have included all the vectors $0^{i-1}10^{k-i}$ and added one to the size of each column). The P term on the left in the statement is Δ_0 and the P term on the right is Δ_m; so we want to show that $\Delta_0 = \Delta_m$. For this we show that $\Delta_l = \Delta_{l-1}$ by constructing an involution on the terms contributing one to the sum defining Δ_l. Let $A_{l,w}$ be those rows of A_l equal to vector w. Let n_w be the number of rows in $A_{l,w}$. $n_w = j2^m$ for $w \neq v_{p,q}$ nonnull and $n_w = j2^m - 1$ for $w = v_{p,q}$ (in particular, for $m \geq 1$, n_w is even or odd, respectively). Establish an involution γ_w for the rows of $A_{l,w}$ so that γ_w has no fixed point if $w \neq v_{p,q}$, and γ_w has exactly one fixed point $z_{p,q}$ if $w = v_{p,q}$ (like γ of Lemma 5.1). Each γ_w induces an involution γ_w^* on the set of those η in $\mathcal{P}(m_1(l) + 1, m_1(l), \ldots, m_1(l), \ldots, m_s(l) + 1, m_s(l), \ldots, m_s(l))$ that contribute one to Δ_l. Then Δ_l is equal to the parity of the fixed points of all γ_w^*. Let η be such a fixed point. Then η must map $z_{p,q}$ to a column of odd size; the only way this is possible is $\eta(z_{p,q}) = \sum_{i=1}^{p-1} t_i + q$. It follows that $\Delta_l = \Delta_{l-1}$ (after noting that $(m_i(l)+1)/2 = m_i(l-1)+1$ and $(m_i(l)-1)/2 = m_i(l-1)$). □

Lemma 6.4 implies

Theorem 6.5. *For $j \geq 2$, if we conclude that $\Delta(j,k) \leq d$ by computing a corresponding term (**), then $\Delta(j',k) \leq d'$, where $j' = j2^m$ and $d' = d2^m$.*

6.1. *Summary of Results*

In the following table the second column shows the space in which a solution exists for $j = 2$ according to Theorem 6.3, the third column shows the lower bound for $\Delta(j,k)$ from Theorem 4.7, and the fourth column shows the upper bound for $\Delta(j,k)$ that follows for $j = 2^m$ from the $j = 2$ case by Theorem 6.5 (this represents an improvement over the upper bound $\Delta(j,k) \leq j2^{k-1}$). Of course, the first row is just the ham-sandwich theorem, and the case $k = 2$, $j = 2$ was also already known. The results for $k = 4, 5$, $j = 2$, are not tight.

$k=1$	B^1	j	$2j/2 = j$
$k=2$	$(B^2)^2_{\leq}$	$3j/2$	$3j/2$
$k=3$	$(B^4)^2_{\leq} \times B^3$	$7j/3$	$5j/2$
$k=4$	$(B^8)^2_{\leq} \times (B^5)^2_{\leq}$	$15j/4$	$9j/2$
$k=5$	$(B^{14})^3_{\leq} \times B^{11} \times B^4$	$31j/5$	$15j/2$

We can now explain other entries in the first table of the Introduction. Again, lower bounds follow from Theorem 4.7. $\Delta(2,3) \leq 5$ and $\Delta(4,2) \leq 6$ appear in the previous table. The upper bounds that we have for $\Delta(1,k)$ are obtained as follows: A k-partition can be obtained by choosing a bisector (in an arbitrary direction) and then a $(k-1)$-partition for the two resulting masses; thus, $\Delta(1,k) \leq \Delta(2,k-1)$. For example, $\Delta(1,3) \leq \Delta(2,2) = 3$, $\Delta(1,4) \leq \Delta(2,3) = 5$, and $\Delta(1,5) \leq \Delta(2,4) \leq 9$ (which appears in the previous table). For $k = 3$, this results in a tight bound, but possibly not for $k = 4, 5$ (assuming the lower bound is tight).

Finally, we can obtain a small improvement over the bound $\Delta(1,k) \leq 2^{k-1}$ as follows: note that, for $k' < k$, $\Delta(1,k) \leq \max(2^{k-k'-1}, \Delta(2^{k-k'}, k'))$ (first obtain a $(k-k')$-partition of one mass and then a k'-partition of $2^{k-k'}$ masses); thus, using $k' = 5$ above, we have that, for $k \geq 6$, $\Delta(1,k) \leq (15/32)2^{k-1}$.

7. Concluding Remarks

Equipartition by Orthogonal Hyperplanes. Using a trick of Hadwiger [9], equipartition can be obtained by orthogonal hyperplanes. Note that any hyperplanes equipartitioning a uniform mass distribution on a ball are orthogonal. Therefore k-partition of $j+1$ mass distributions implies k-partition by orthogonal hyperplanes of j mass distributions. For example, in $\mathbf{R}^5$, 3-partitions by orthogonal hyperplanes are possible. This is probably not an efficient way to obtain equipartition by orthogonal hyperplanes since, in $\mathbf{R}^2$, 2-partition of a mass distribution by orthogonal lines exists, but 2-partition of two masses does not.

Equipartition by Algebraic Hypersurfaces. In $\mathbf{R}^d$, equipartition of any j mass distributions is always possible by k algebraic hypersurfaces of degree sufficiently large (but the orthants are in general disconnected). For this embed a mass distribution of $\mathbf{R}^d$ into $\mathbf{R}^{d'}$ where d' is such that (j,k)-partitions by hyperplanes exist in $\mathbf{R}^{d'}$. For the embedding, map $(x_1,\dots,x_d)$ to $(m_1(x_1,\dots,x_d),\dots,m_{d'}(x_1,\dots,x_d))$, where the $m_i(x_1,\dots,x_d)$ are different monomials (different from one) in the variables $x_1,\dots,x_d$ of degree at most D. These monomials are linearly independent, so the embedded $\mathbf{R}^d$ does not lie in a hyperplane. D must satisfy the constraint $\binom{D+d}{d} \geq d'+1$, since the term in the left is the number of monomials in d variables of degree at most D. For example, the result that $(1,4)$-partitions by hyperplanes exist in $\mathbf{R}^5$ implies that $(1,4)$-partitions by quadratic curves exist in $\mathbf{R}^2$. In a particular case, more can be said about the type of hypersurface: for the well-known embedding $(x_1,\dots,x_d)$ into $(x_1,\dots,x_d,\sum_{i=1}^{d} x_i^2)$, the hypersurfaces are spheres, therefore (j,k)-partition by hyperplanes in $\mathbf{R}^d$ implies (j,k)-partition by spheres in $\mathbf{R}^{d-1}$. For example, we conclude that $(1,4)$-partitions by spheres are possible in $\mathbf{R}^4$. If $d=1$, the embedding is in the moment curve $M_{d'}$ and the algebraic surfaces become points. For example, it is obtained that given j mass distributions in $\mathbf{R}^1$, there are j points that 2-partition them ($d'=j$ and the j points are the intersections of the moment curve with the ham-sandwich cut hyperplane). This has been investigated by Goldberg and West [7] and by Alon and West [1].

Algorithms. We consider here partitions for point sets. There is a natural procedure to search for a zero of $r=(r',r'')$ when its existence is guaranteed by the parity argument, that is, we know that $P(r;X)=1$. Decompose the domain X into pieces X_i, $i=1,\dots,N$, some N, and compute $P^+(r',\underline{r''};\mathrm{bd}(X_i))$ for each i. This must be one for at least one X_i. Then recurse on one such X_i. This is the approach used in [11] for an algorithm that finds a ham-sandwich cut for point sets (of course, some details need to be worked out). This, however, does not provide a significant improvement over an exhaustive search. It would be interesting to find more efficient algorithms, or to show that computing these partitions is hard (e.g., in the sense of [15] or [4]). Computing approximations to partitions (say each orthant has at most a fraction $1/2^k+\delta$ of the points) can be done in linear time using ϵ-approximations (although with large constants that depend on the dimension).

Open Problems. The obvious open problem is to determine tight bounds for $\Delta(j,k)$. At least it would be interesting to show that the condition $\Delta(j,k) \geq j(2^k-1)/k$ is tight for infinitely many values of k (say k is a power of 2, recall we have results only for small values of k). Two problems related to the Borsuk–Ulam theorem on a product of balls were raised in Section 3. Plus, of course, the question that motivated all this, what about $\Delta(1,4)$?

Acknowledgments

I would like to thank Herbert Edelsbrunner for proposing the 4-partition problem in his lectures, and for discussions and comments on earlier drafts which greatly helped to clarify the ideas. Also thanks to Rade Živaljević for discussions regarding the subject of this paper.

Appendix: Preliminary Definitions

Simplicial Complexes and Triangulations. Consider $X \subseteq \mathbf{R}^n$ compact and full-dimensional. Let bd X denote its boundary. A finite collection C of *cells* (compact convex polyhedra) in $\mathbf{R}^n$ is a *(cell) complex* if (i) $\sigma \in C$ implies $\tau \in C$ for each face τ of σ, and (ii) if $\sigma_1, \sigma_2 \in C$, then $\sigma_1 \cap \sigma_2$ is either empty or a face of both. The *k-skeleton* $C^{(k)}$ of C is the subcomplex of C consisting of all cells in C of dimension at most k. Thus, $C^{(0)}$ is the set of *vertices* of C, and $C = C^{(n)}$ where the *dimension n* of C is the maximum dimensionality of any cell in C. A k-simplex is the convex hull of $k+1$ affinely independent points conv$\{v_0, \dots, v_k\}$. A complex is *simplicial* if it consists of simplices. The *underlying space* of C is $|C| = \bigcup_{\sigma \in C} \sigma$. For $x \in |C|$, the *carrier* of x in C is the unique $\sigma \in C$ such that $x \in \text{int}(\sigma)$, the *interior* of σ. A *triangulation* of X is a simplicial complex $\mathcal{T}$ and a homeomorphism between $|\mathcal{T}|$ and X. The *barycentric coordinates* of $x \in |\mathcal{T}|$ are the unique numbers $t_0, \dots, t_k$ such that $x = \sum_{i=0}^{k} t_i v_i$ where $\sigma = \text{conv}\{v_0, \dots, v_k\} \in C$ is the carrier of x in $\mathcal{T}$ ($0 < t_i < 1$ and $\sum_{i=0}^{k} t_i = 1$).

Barycentric Subdivision. The *barycenter* of a k-simplex $\sigma = \text{conv}\{v_0, \dots, v_k\}$ is the point $\hat{\sigma} = (\sum_{i=0}^{k} v_i)/(k+1)$ (i.e., the point with barycentric coordinates $t_i = 1/(k+1)$). The *barycentric subdivision* $\text{Sd}(\mathcal{T}) = \text{Sd}(\mathcal{T}^{(n)})$ of a simplicial complex $\mathcal{T}$ is defined recursively by $\text{Sd}(\mathcal{T}^{(0)}) = \mathcal{T}^{(0)}$ and $\text{Sd}(\mathcal{T}^{(i)}) = \text{Sd}(\mathcal{T}^{(i-1)}) \cup \{\hat{\sigma}, \hat{\sigma} * \tau : \sigma \in \mathcal{T}^{(i)} - \mathcal{T}^{(i-1)}, \tau \in \text{Sd}(\mathcal{T}^{(i-1)}), \tau \subseteq \text{bd}(\sigma)\}$, where $p * \sigma$ denotes the *cone* conv$(\{p\} \cup \sigma)$ for p a point and σ a simplex (a simplex is replaced by the cones from its barycenter to the simplices triangulating its faces recursively). The lth *barycentric subdivision* $\text{Sd}^l(\mathcal{T})$ of $\mathcal{T}$ is defined by $\text{Sd}^1(\mathcal{T}) = \text{Sd}(\mathcal{T})$ and $\text{Sd}^l(\mathcal{T}) = \text{Sd}(\text{Sd}^{l-1}(\mathcal{T}))$. The "fineness" of a triangulation C is measured by the *mesh* size, $\text{mesh}(\mathcal{T}) = \max_{\sigma \in \mathcal{T}} \text{diam}(\sigma)$. It is verified that $\text{mesh}(\text{Sd}^l(\mathcal{T})) \leq (d/(d+1))^l \, \text{mesh}(\mathcal{T})$, so taking l sufficiently large, $\text{mesh}(\text{Sd}^l(\mathcal{T}))$ can be made arbitrarily small. The barycentric subdivision can also be applied to any (nonsimplicial) cell complex C as long as the barycenter $\hat{\gamma}$ of a cell γ is defined. If γ is the product of simplices $\gamma = \sigma_1 \times \cdots \times \sigma_k$, then the barycenter $\hat{\gamma}$ is the product of the barycenters $\hat{\gamma} = \hat{\sigma}_1 \times \cdots \times \hat{\sigma}_k$. In general, the barycenter is the center of mass of the cell. The barycentric subdivision of a cell complex C gives a triangulation of $|C|$.

Piecewise Linear Approximations. A *vertex map* on a triangulation $\mathcal{T}$ is a map $\varphi: \mathcal{T}^{(0)} \to \mathbf{R}^n$. A *piecewise linear function* r on $|\mathcal{T}|$ is defined from a vertex map by linear extension within each simplex. More precisely, for $x \in |\mathcal{T}|$, $r(x) = \sum_{i=0}^{k} t_i \varphi(v_i)$ where conv$\{v_0, \dots, v_k\}$ is the carrier of x and $t_0, \dots, t_k$ are the barycentric coordinates of x. Given a function $f: X \to \mathbf{R}^n$, a *piecewise linear ϵ-approximation* r of f is a piecewise linear function on a triangulation $\mathcal{T}$ of X so that $\|f(x) - r(x)\|_\infty < \epsilon$ (for simplicity we use the max norm) for all $x \in X$. Since X is compact, f is uniformly continuous. Therefore, given an initial "coarse" triangulation $\mathcal{T}$ of X, there is an integer l such that if r is the piecewise linear extension over $\mathcal{T}^* = \text{Sd}^l(\mathcal{T})$ of the vertex map $\varphi(v) = f(v)$, for $v \in \mathcal{T}^{*(0)}$, then r ϵ-approximates f.

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Received March 14, 1994, *and in revised form August* 1, 1995.