

EQUIPARTITION OF SPHERE MEASURES BY HYPERPLANES

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ABSTRACT. Measure partition problems are classical problems of geometric combinatorics ([1], [2], [3], [4]) whose solutions often use tools from the equivariant algebraic topology. The potential of the computational obstruction theory approach is partially demonstrated here. In this paper we reprove a result of V.V. Makeev [9] about a 6-equipartition of a measure on S^2 by three planes. The advantage of our approach is that it can be applied on other more complicated questions of the similar nature.

1. STATEMENT OF THE MAIN RESULT

A measure μ is a *proper measure* if

- (A) $\mu([a, b]) = 0$ for any circular arc $[a, b] \subset S^2$, and
- (B) $\mu(U) > 0$ for each nonempty open set $U \subset S^2$.

Three planes H_1, H_2 and H_3 in \mathbb{R}^3 through the origin are in a *fan position* if they intersect along the common line. Planes in the fan position cut the sphere S^2 in six parts $\sigma_1, \dots, \sigma_6$ which can be naturally oriented up to a cyclic permutation.

We are interested in the following measure partition problem.

Problem 1. *Find all six-tuples $(\alpha_1, \dots, \alpha_6) \in \mathbb{N}^6$ that for every proper Borel probability measure μ on the sphere S^2 there exist three planes in the fan*

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position, with angular sectors having the prescribed ration, i.e.

$$(\forall i \in \{1, \dots, 6\}) \mu(\sigma_i) = \frac{\alpha_i}{\alpha_1 + \dots + \alpha_6}.$$

The six-tuples which satisfy these conditions are called *solutions* of the problem. The particular instance is an equipartition case solved by V. V. Makeev in [9].

Theorem 2. *Let μ be a continuous proper Borel probability measure on the sphere S^2 . Then there are three planes in the fan position such that associated angular sectors have the same amount of measure μ .*

In order to reprove the theorem we first formulate a related equivariant problem which is significantly different from the one V. V. Makeev used. That allows us to treat other similar cases in the same manner.

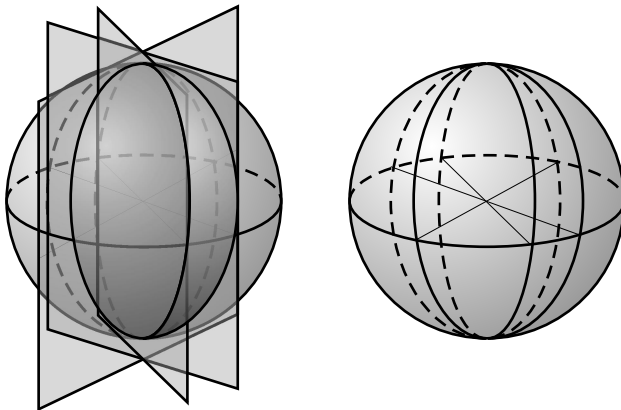


FIGURE 1. Hyperplanes in the fan position and the intersection with S^2 .

2. THE EQUIVARIANT PROBLEM

2.1. From the partition problem to the equivariant problem. We use the configuration space / test map scheme to reduce the partition problem to an equivariant one. The basic idea comes from papers of Imre Bárány and Jiří Matoušek [1], [2].

A *k-fan* $(l; H_1, H_2, \dots, H_k)$ in \mathbb{R}^3 is formed of an oriented line l through the origin and k closed half planes H_1, H_2, \dots, H_k which intersect along the common boundary $l = \partial H_1 = \dots = \partial H_k$. The intersection of the k -fan with the sphere S^2 is equally called. Thus the collection $(x; l_1, \dots, l_k)$ of a point $x \in S^2$ and k great semicircles l_1, \dots, l_k emanating from x is also a k -fan. Sometimes instead of great semicircles we use:

- (A) open angular sectors σ_i between l_i and l_{i+1} , $i = 1, \dots, k$; or
- (B) tangent vectors t_i on l_i , $i = 1, \dots, k$.

We prefer the tangent vector notation $(x; t_1, \dots, t_k)$. The space of all k -fans in \mathbb{R}^3 is denoted by F_k .

Now we are ready to define elements of the target extension scheme.

The configuration space. For a proper Borel probability measure μ on S^2 , the n -configuration space is defined by

$$X_{\mu,n} = \{(x; t_1, \dots, t_n) \in F_n \mid (\forall i = 1, \dots, n) \mu(\sigma_i) = \frac{1}{n}\}.$$

Since every n -fan $(x; t_1, \dots, t_n)$ of the configuration space $X_{\mu,n}$ is completely determined by the pair (x, t_1) , there exists a homeomorphism $X_{\mu,n} \cong V_2(\mathbb{R}^3)$.

The test map. Fix a symmetric six-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^6$ such that $\alpha_1 + \alpha_2 + \alpha_3 = \frac{n}{2}$. For the standard basis e_1, e_2, \dots, e_n in \mathbb{R}^n the associated coordinate functions are denoted by x_1, x_2, \dots, x_n . Denote the hyperplane $W_n = \{x \in \mathbb{R}^n \mid x_1 + x_2 + \dots + x_n = 0\}$. The test map is defined by

$$\Phi : X_{\mu,n} \rightarrow W_n \quad \Phi((x; t_1, \dots, t_n)) = (\theta_1 - \frac{2\pi}{n}, \dots, \theta_n - \frac{2\pi}{n}),$$

where θ_i is an angle between tangent vectors t_i and t_{i+1} in the tangent plane. Here we assume that $t_{n+1} = t_1$.

The action. The dihedral group $\mathbb{D}_{2n} = \langle j, \varepsilon \mid \varepsilon^n = j^2 = 1, \varepsilon j = j \varepsilon^{n-1} \rangle$ acts both on the configuration space $X_{\mu,n}$ and on the hyperplane W_n in the following way

$$X_{\mu,n} : \begin{cases} \varepsilon(x; t_1, \dots, t_n) = (x; t_n, t_1, \dots, t_{n-1}) \\ j(x; t_1, \dots, t_n) = (-x; t_1, t_n, \dots, t_2) \end{cases},$$

$$W_n : \begin{cases} \varepsilon(x_1, \dots, x_n) = (x_2, \dots, x_n, x_1) \\ j(x_1, \dots, x_n) = (x_n, \dots, x_2, x_1) \end{cases},$$

for $(x; t_1, \dots, t_n) \in X_{\mu}$ and $(x_1, \dots, x_n) \in W_n$. It is not hard to check that:

Claim 3. (A) The action of \mathbb{D}_{2n} on $X_{\mu,n}$ is **free**.

(B) The test map $\Phi : X_{\mu,n} \rightarrow W_n$ is \mathbb{D}_{2n} -equivariant.

The test space. The test space in this symmetric problem is the union $\bigcup \mathcal{A} \subset W_n$ of the smallest \mathbb{D}_{2n} -invariant arrangement \mathcal{A} , which contains the linear subspace $L \subset W_n$. The subspace L is defined by linear forms

$$\begin{aligned} \xi_1(x) &= x_1 + \dots + x_{\frac{n}{2}}, & \xi_2(x) &= x_{\alpha_1+1} + \dots + x_{\alpha_1+\frac{n}{2}}, \\ \xi_3(x) &= x_{\alpha_1+\alpha_2+1} + \dots + x_{\alpha_1+\alpha_2+\frac{n}{2}}. \end{aligned}$$

We prove the basic proposition of the configuration space / test map scheme.

Proposition 4. Observe $(\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^6$ such that $\alpha_1 + \alpha_2 + \alpha_3 = \frac{n}{2}$. If there is no \mathbb{D}_{2n} -equivariant map

$$V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A}$$

then for every proper Borel probability measure on the sphere S^2 there exist three planes in the fan position with angular sectors $\sigma_1, \dots, \sigma_6$ such that

$$(\forall i \in \{1, \dots, 6\}) \mu(\sigma_i) = \frac{\alpha_i}{n}.$$

In other words, six-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3)$ is a solution.

Proof. We just illustrate how three planes arise from six halfplanes. Let Φ be the test map for the measure μ and $\Phi((x; t_1, \dots, t_n)) \in \bigcup \mathcal{A}$. Without losing generality we may assume that $\Phi((x; t_1, \dots, t_n)) \in L$, which means that

$$\begin{aligned} \theta_1 - \frac{2\pi}{n} + \dots + \theta_{\frac{n}{2}} - \frac{2\pi}{n} &= 0 & \Rightarrow & \theta_1 + \dots + \theta_{\frac{n}{2}} = \pi \\ \theta_{\alpha_1+1} - \frac{2\pi}{n} + \dots + \theta_{\alpha_1+\frac{n}{2}} - \frac{2\pi}{n} &= 0 & \Rightarrow & \theta_{\alpha_1+1} + \dots + \theta_{\alpha_1+\frac{n}{2}} = \pi \\ \theta_{\alpha_1+\alpha_2+1} - \frac{2\pi}{n} + \dots + \theta_{\alpha_1+\alpha_2+\frac{n}{2}} - \frac{2\pi}{n} &= 0 & \Rightarrow & \theta_{\alpha_1+\alpha_2+1} + \dots + \theta_{\alpha_1+\alpha_2+\frac{n}{2}} = \pi. \end{aligned}$$

Thus, $H_1 \cup H_{\frac{n}{2}}$, $H_{\alpha_1+1} \cup H_{\alpha_1+\frac{n}{2}}$ and $H_{\alpha_1+\alpha_2+1} \cup H_{\alpha_1+\alpha_2+\frac{n}{2}}$ are hyperplanes and they cut μ in the prescribed ration. \square

2.2. The modification of the equivariant problem. This section is a review of methods used in [3] and [4]. The objective is to simplify or alternate the question of the existence of a \mathbb{D}_{2n} -equivariant map $V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A}$. It is done by substituting the Stiefel manifold $V_2(\mathbb{R}^3)$ with the sphere S^3 , but to get the equivalent problem we have to extend the group \mathbb{D}_{2n} . We use the “extension of scalars” equivalence, [5] Section III.3 in the same way as in [3] and [4].

Let X be a left G -space and $H \triangleleft G$ be a normal subgroup. The space of cosets X/H can be seen as a G/H -space by $gH(Hx) = H(gx)$. On the other hand, a G/H -space Z is a G -space via the quotient homomorphism $\pi : G \rightarrow G/H$, i.e. for $g \in G$ and $z \in Z$, $g \cdot z = \pi(g)z$.

Proposition 5. *Now X and Z are G -spaces and $H \triangleleft G$ is a normal subgroup of G that acts trivially on Z . Following maps coexist:*

$$G\text{-map } \alpha : X \rightarrow Z \quad \text{and} \quad G/H\text{-map } \beta : X/H \rightarrow Z,$$

where on the right, X/H and $Z/H = Z$ are interpreted as G/H -spaces.

By the coexistence we mean that one map exists if and only if the other map exists, i.e. that one can't exist without the other.

The proof of this proposition can be found in [3] and [4].

The sphere $S^3 = S(\mathbb{H}) = Sp(1)$ can be seen as the group of all unit quaternions, and let $\eta = \eta_{2n} = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \in S(\mathbb{H})$ be a root of unity. Group generated by η is a subgroup of $S(\mathbb{H})$ of the order $2n$. The generalized quaternion group, [6] p. 253, is the subgroup of the order $4n$ generated by η and j , i.e.

$$\mathbb{Q}_{4n} = \{1, \eta, \dots, \eta^{2n-1}, j, \eta j, \dots, \eta^{2n-1} j\}.$$

The group \mathbb{Q}_{4n} acts on S^3 as a subgroup, and on W_n via the already defined \mathbb{D}_{2n} action by the quotient homomorphism $\mathbb{Q}_{4n} \rightarrow \mathbb{Q}_{4n}/\{1, -1\} \cong \mathbb{D}_{2n}$. The \mathbb{Q}_{4n} action on S^3 is **free**. Also, the \mathbb{Q}_{4n} action on W_n is the restriction of the following \mathbb{Q}_{4n} action on \mathbb{R}^n . Let e_1, \dots, e_n be the standard orthonormal basis in \mathbb{R}^n . The action is defined by

$$\eta \cdot e_i = e_{i \bmod n + 1} \quad \text{and} \quad j \cdot e_i = e_{n-i+1}.$$

If H from the proposition is $\{1, \eta^n\} = \{1, -1\} \subset \mathbb{Q}_{4n}$, then the quotient group \mathbb{Q}_{4n}/H is isomorphic to the dihedral group \mathbb{D}_{2n} of the order $2n$. Also, the group H acts on W_n and \mathbb{R}^n trivially. Thus, the proposition implies.

Corollary 6. *Following maps coexist:*

$$\mathbb{D}_{2n}\text{-map } V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A} \quad \text{and} \quad \mathbb{Q}_{4n}\text{-map } S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}.$$

Remark 7. *Since \mathbb{Q}_{4n} -action on S^3 is free and S^3 is 2-connected, it turns out that the particular \mathbb{Q}_{4n} -action on S^3 is not significant. An exercise in the equivariant obstruction theory says that:*

If \circ and $$ are G -actions on S^3 and \circ is free, then there exists a G -map $f : S^3 \rightarrow S^3$ such that*

$$(\forall g \in G) (\forall x \in S^3) f(g \circ x) = g * f(x).$$

2.3. The new equivariant problem. Applying the results of preceding sections, in light of our combinatorial problem, we study the following problem.

Problem 8. *Prove that there is no \mathbb{Q}_{4n} -map $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}$, where \mathcal{A} is minimal \mathbb{Q}_{4n} (= \mathbb{D}_{2n}) arrangement containing subspace L defined by the linear form:*

$$\xi_1(x) = x_1 + x_2 + x_3, \quad \xi_2(x) = x_2 + x_3 + x_4, \quad \xi_3(x) = x_3 + x_4 + x_5;$$

3. EQUIVARIANT OBSTRUCTION THEORY APPROACH

The basic objective of the equivariant obstruction theory is to define an invariant associated to a question of the extension of the equivariant map in such a way that the nature of the invariant points out whether the extension can or can not be performed. We are going to consider following two basic problems of the (equivariant) obstruction theory. Some of the classical references concerning the obstruction theory are [7], pp.111-122 and [11].

3.1. The Obstruction Theory in a few lines. Let (X, A) be a relative G cellular complex such that G action on $X \setminus A$ is free and let Y be a G space.

The extension problem. Let $f : A \rightarrow Y$ be a G map. Is there a G map $F : X \rightarrow Y$ such that $f = F \circ i$, where $i : A \rightarrow X$ denotes the inclusion.

The homotopy problem. Let $f_0 : X \rightarrow Y$ and $f_1 : X \rightarrow Y$ be G maps such that there is a G homotopy $h : I \times A \rightarrow Y$ from $f_0|_A$ to $f_1|_A$. Is there a G homotopy $H : I \times X \rightarrow Y$ which extends h , i.e. $H|_{\{0\} \times X} = f_0$, $H|_{\{1\} \times X} = f_1$ and $H|_{I \times A} = h$.

The basic theorem of the equivariant obstruction theory can be stated in the following way.

Theorem 9. *Let $n \geq 1$ be a fixed integer and Y path-connected n -simple G space. For every G relative cell complex (X, A) with the free action on $X \setminus A$*

there exists an obstruction exact sequence

$$(1) \quad [X_{n+1}, Y]_G \longrightarrow \text{im}([X_n, Y]_G \rightarrow [X_{n-1}, Y]_G) \xrightarrow{\mathfrak{D}_G^{n+1}} \mathfrak{H}_G^{n+1}(X, A; \pi_n Y)$$

which is natural both in X and Y .

The exactness of the sequence stands for:

(A) Every G -map on $(n-1)$ -skeleton $f : X_{n-1} \rightarrow Y$ which can be equivariantly extended to the n -skeleton $f : X_n \rightarrow Y$ has a unique element $\mathfrak{D}_G^{n+1}(f)$ in $\mathfrak{H}_G^{n+1}(X, A; \pi_n Y)$ called the *obstruction element*;

(B) The exactness of the sequence means that the obstruction element $\mathfrak{D}_G^{n+1}(f)$ is zero if and only if there is a map in the homotopy class of the restriction $f|_{X_{n-1}}$ which can be extended to the $(n+1)$ -skeleton X_{n+1} .

Here $\mathfrak{H}_G^{n+1}(X, A; \pi_n Y)$ denotes the equivariant cohomology defined for example in [7].

Corollary 10. *Let $n \geq 1$ be a fixed integer and Y path-connected, n -simple and $(n-1)$ -connected G space. Then for every G relative cell complex (X, A) with the free action on $X \setminus A$:*

(A) *There exists a G -map $f : X_n \rightarrow Y$.*

(B) *Every two G -maps $f, g : X_n \rightarrow Y$ are G -homotopic on X_{n-1} .*

(C) $\text{im}([X_n, Y]_G \rightarrow [X_{n-1}, Y]_G) = \{*\}$.

Now the obstruction sequence (1) from the preceding theorem becomes

$$(2) \quad [X_{n+1}, Y]_G \longrightarrow \{*\} \xrightarrow{\mathfrak{D}_G^{n+1}} \mathfrak{H}_G^{n+1}(X, \pi_n Y).$$

Thus the possibility for the extension of any G -map $X_n \rightarrow Y$ to the next $(n+1)$ -skeleton X_{n+1} depends only of one element $\mathfrak{D}_G^{n+1}(*) \in \mathfrak{H}_G^{n+1}(X, \pi_n Y)$ which is called the **primary obstruction**.

Conclusion 11. *To compute the primary obstruction $\mathfrak{D}_G^{n+1}(*)$ it is enough to choose a specially convenient G -map f on the n -skeleton X_n and to compute its obstruction $\mathfrak{D}_G^{n+1}(f)$ which must be equal to the primary obstruction $\mathfrak{D}_G^{n+1}(*)$. This method is sometimes called the **general position map scheme**.*

3.2. Our problem in light of the Obstruction Theory.

Computing the obstruction cocycle. Let us discuss the problem 8 in light of Corollary 10 and lay down a methodology for the proof of Theorem 2. The \mathbb{Q}_{4n} -spaces S^3 and $W_n \setminus \bigcup \mathcal{A}$ which participate in our problem have following properties:

(A) sphere S^3 is a 3-dimensional free \mathbb{Q}_{4n} -space,

(B) complement $W_n \setminus \bigcup \mathcal{A}$ is a 3-simple and 2-connected \mathbb{Q}_{4n} -space.

The problem of the existence of a \mathbb{Q}_{4n} -map $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}$ transforms the obstruction sequence (1) in

$$[S^3_{(3)}, Y]_{\mathbb{Q}_{4n}} \longrightarrow \{*\} \xrightarrow{\mathfrak{D}_{\mathbb{Q}_{4n}}^3} \mathfrak{H}_{\mathbb{Q}_{4n}}^3(X, H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})).$$

We used the Hurewicz isomorphism $\pi_2(W_n \setminus \bigcup \mathcal{A}) = H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})$. The equivalence

$$\left(\text{There is a } \mathbb{Q}_{4n}\text{-map } S^3 \rightarrow W_n \setminus \bigcup \mathcal{A} \right) \Leftrightarrow \mathfrak{D}_{\mathbb{Q}_{4n}}^3(*) = 0$$

indicates the necessity of computing $\mathfrak{D}_{\mathbb{Q}_{4n}}^3(*)$.

The computation of the primary obstruction is done by the general position map scheme. The scheme proceeds in following three steps:

(1) Fix the \mathbb{Q}_{4n} cell structures on S^3 , specially the simplicial structure and the cell structure. The cell structure has only one equivariant generator of the maximal dimension. The description of concrete \mathbb{Q}_{4n} simplicial and cell structures of sphere S^3 that we use can be found in [6] pp. 250-254, [4] and [3].

(2) Carefully define a \mathbb{Q}_{4n} -map $f : S^3 \rightarrow W_n$ such that:

(i) the image of the 2-skeleton does not intersect the arrangement $\bigcup \mathcal{A}$, i.e.

$$f(S^3_{(2)}) \cap \bigcup \mathcal{A} = \emptyset,$$

(ii) the set $f(S^3) \cap \bigcup \mathcal{A}$ is finite,

(iii) $(\forall y \in f(S^3) \cap \bigcup \mathcal{A})$ $f(S^3)$ and $\bigcup \mathcal{A}$ intersect transversely along y ,

(iv) $f^{-1}(f(S^3) \cap \bigcup \mathcal{A}) \subset \bigcup_{e \in S^3_{(3)}} \text{relint}(e)$.

(3) The inverse image

$$f^{-1}\left(f(S^3) \cap \bigcup \mathcal{A}\right) \subset \bigcup_{e \in S^3_{(3)}} \text{relint}(e)$$

"enumerates" the obstruction cocycle in the following way

$$(3) \quad \mathfrak{D}_{\mathbb{Q}_{4n}}(f)(e) = \sum_{x \in f^{-1}(f(e) \cap \bigcup \mathcal{A})} \mathbf{I}(e, L_{f(x)}) \|f(x)\|$$

where e is a 3-cell of S^3 . Here $\mathbf{I}(e, L_{f(x)})$ denotes the intersection number of the image $f(e)$ and the appropriate oriented element L_y of the arrangement \mathcal{A} . The class $\|f(x)\|$ can be a point or a broken point class. The notion of point and broken point classes is discussed in greater details in [3] and we rely on it. In general, $\|f(x)\|$ is determined by the tangent space on $f(e)$ at the point $f(x)$. But since we work with a simplicial \mathbb{Q}_{4n} structure on S^3 and require that f is affine on every simplex, the class $\|f(x)\|$ is easier to describe.

The nature of the obstruction cocycle. Let us note two properties about the obstruction cocycle which allow us to narrow our computations. First, we describe what kind of element is the obstruction cocycle $\mathfrak{D}_G^{n+1}(f)$ in the group $\mathfrak{H}_G^{n+1}(X, \pi_n Y)$.

Proposition 12. *Let Y be a path-connected n -simple G space and X a free $(n+1)$ -dimensional G cell complex. If $f : X_n \rightarrow Y$ is a G -map, then the obstruction element $\mathfrak{D}_G^{n+1}(f) \in \mathfrak{H}_G^{n+1}(X, \pi_n Y)$ is a torsion element.*

Second, we point out how to compute the group $\mathfrak{H}_G^{n+1}(X, \pi_n Y)$ in the case when $X = S^3$, $Y = W_n \setminus \bigcup \mathcal{A}$ and $G = \mathbb{Q}_{4n}$.

Proposition 13. *There is an isomorphism*

$$\mathfrak{H}_{\mathbb{Q}_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})) \cong H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}},$$

where $H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}}$ denotes a group of \mathbb{Q}_{4n} -coinvariants of the \mathbb{Q}_{4n} -module $H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})$.

The proofs for preceding two propositions can be found in [3] and [4].

Proving that the obstruction cocycle is or is not zero. With the desire to determine the cohomology class of the primary obstruction $\mathfrak{D}_{\mathbb{Q}_{4n}}(f)$ we have to dive in the topology of the \mathbb{Q}_{4n} -arrangement \mathcal{A} . The knowledge of the nature of the obstruction cocycle suggests the following strategy

(A) Change the \mathbb{Q}_{4n} simplicial structure on S^3 with the \mathbb{Q}_{4n} cell structure which has only one equivariant 3-dimensional generator e in $\mathfrak{C}_{\mathbb{Q}_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z}))$. Express $\mathfrak{D}_{\mathbb{Q}_{4n}}(f)$, computed in the simplicial structure, in terms of the new cell structure. The reason for this change is that the obstruction element is now completely determined by its value on e ,

$$\mathfrak{D}_{\mathbb{Q}_{4n}}(f)(e) \in H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z}).$$

(B) Since $\mathfrak{H}_{\mathbb{Q}_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})) \cong H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}}$ we first compute $H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})$. With a little help of the Poincaré-Alexander duality isomorphism and the Universal coefficient isomorphism, we have (assuming \mathbb{Z} coefficients)

$$(4) \quad H_2(W_n \setminus \bigcup \mathcal{A}) \cong H^{(n-1)-2-1}(\bigcup \hat{\mathcal{A}}) \cong$$

$$(5) \quad \text{Hom}(H_{n-4}(\bigcup \hat{\mathcal{A}}), \mathbb{Z}) \oplus \text{Ext}(H_{n-5}(\bigcup \hat{\mathcal{A}}), \mathbb{Z})$$

where $\hat{\mathcal{A}}$ denotes the one-point compactification of the arrangement \mathcal{A} . The calculations of $H_{n-4}(\bigcup \hat{\mathcal{A}}; \mathbb{Z})$ and $\text{Ext}(H_{n-5}(\bigcup \hat{\mathcal{A}}; \mathbb{Z}), \mathbb{Z})$ can be carried on by the Goresky-MacPherson formula [8]. For example, there is a decomposition (assuming \mathbb{Z} coefficients)

(6)

$$H_{n-4}(\bigcup \hat{\mathcal{A}}) \cong \bigoplus_{V \in P} H_{n-4}(\Delta(P_{<V}) * S^{\dim V}) \cong \bigoplus_{d=0}^{n-4} \bigoplus_{V \in P: \dim V=d} \tilde{H}_{n-5-d}(\Delta(P_{<V}))$$

where P is the intersection poset of the arrangement \mathcal{A} . By convention, $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$. When arrangement does not contain inclusions of codimension one the decomposition is also a decomposition of \mathbb{Q}_{4n} modules.

(C) To compute the coinvariants $H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}}$ we have to keep in mind that Poincaré-Alexander duality isomorphism is not the isomorphism of \mathbb{Q}_{4n} -modules. Fortunately, it is a \mathbb{Q}_{4n} -map up to an orientation

character. On the other hand, Universal coefficient isomorphism is a \mathbb{Q}_{4n} -map. To overcome this difficulty we introduce a new \mathbb{Q}_{4n} -action by

$$g * x = \det(g) g^{-1} \cdot x.$$

where $x \in H_{n-4}(\bigcup \widehat{\mathcal{A}}; \mathbb{Z})$ and $g \in \mathbb{Q}_{4n}$. If we define the relation \sim on $H_{n-4}(\bigcup \widehat{\mathcal{A}}; \mathbb{Z})$ by

$$(\forall x \in H_{n-4}(\bigcup \widehat{\mathcal{A}}; \mathbb{Z})) (\forall g \in \mathbb{Q}_{4n}) g * x \sim x$$

and **assume that** $\text{Ext}(H_{n-5}(\bigcup \widehat{\mathcal{A}}; \mathbb{Z}), \mathbb{Z}) = 0$ (which is often the case), then there exists an isomorphism

$$H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}} \cong H_{n-4}(\bigcup \widehat{\mathcal{A}}; \mathbb{Z}) / \sim.$$

(D) To prove that the cohomology class $[\mathfrak{D}_{\mathbb{Q}_{4n}}(f)(e)]$ is or is not zero we first identify every point class from the sum

$$\mathfrak{D}_{\mathbb{Q}_{4n}}(f)(e) = \sum_{x \in f^{-1}(f(e) \cap (\cup \mathcal{A}))} \mathbb{I}(e, L_{f(x)}) \|f(x)\|$$

along the isomorphism $\varphi : H_2(M(\alpha), \mathbb{Z}) \rightarrow \text{Hom}(H_{n-4}(\bigcup \widehat{\mathcal{A}}(\alpha), \mathbb{Z}), \mathbb{Z})$.

Since this isomorphism is actually a computation of the linking number (when it is correctly defined), then for example

$$\|f(x)\| \longrightarrow \sum_{i \in I} \text{link}(l_i, \|f(x)\|) l_i$$

where $\{l_i | i \in I\}$ is a basis of the group $H_{n-4}(\bigcup \widehat{\mathcal{A}}; \mathbb{Z})$. The final step in this long procedure is to find if

$$\varphi(\mathfrak{D}_{\mathbb{Q}_{4n}}(f)(e)) / \sim \in H_{n-4}(\bigcup \widehat{\mathcal{A}}; \mathbb{Z}) / \sim \cong H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}}$$

is or is not zero.

4. THE PROOF OF THEOREM 2

The proof goes via the Proposition 4 and Corollary 6. Thus we are going to prove that there is no

$$\mathbb{Q}_{24}\text{-map } S^3 \rightarrow W_6 \setminus \bigcup \mathcal{A}$$

where \mathcal{A} denotes the appropriate arrangement defining the test space. We apply the general position map scheme.

(1) *Definition of the general position map* $f : S^3 \rightarrow W_n$. The sphere S^3 is a \mathbb{Q}_{4n} simplicial complex $P_{2n}^{(1)} * P_{2n}^{(2)}$ where $P_{2n}^{(i)}$ represents the sphere S^1 as $2n$ -gon simplicial complex. It is enough to define the image of the single vertex t and everything extends equivariantly. Let $f : S^3 \rightarrow W_6$ be defined on the vertex t by $f(t) = (2, -1, 3, -3, 2, -3)$. Then for example $f(jt) = (-3, 2, -3, 3, -1, 2)$.

(2) *Computation of the singular set*, i.e. the intersection of the image of the maximal cell

$$e = [t, \eta t] * [jt, \eta jt] \cup [\eta t, \eta^2 t] * [jt, \eta jt] \cup \dots \cup [\eta^{n-1} t, \eta^n t] * [jt, \eta jt]$$

and the union of the arrangement $\bigcup \mathcal{A}$. Then

$$\mathfrak{D}_{\mathbb{Q}_{4n}}(f)(e) = \sum_{x \in f^{-1}(f(e) \cap (\bigcup \mathcal{A}))} \mathbf{I}(e, L_{f(x)}) \|f(x)\|.$$

The arrangement \mathcal{A} is the minimal \mathbb{Q}_{24} arrangement containing the subspace L defined by

$$x_1 + x_2 + x_3 = x_2 + x_3 + x_4 = x_3 + x_4 + x_5 = x_4 + x_5 + x_6 = 0.$$

Since $(\forall g \in \mathbb{Q}_{4n}) gL = L$ the arrangement \mathcal{A} "deforms" to just one subspace $\{L\}$. To find the intersection of the f image of the maximal cell

$$e = ([t, \eta t] \cup [\eta t, \eta^2 t] \cup [\eta^2 t, \eta^3 t] \cup [\eta^3 t, \eta^4 t] \cup [\eta^4 t, \eta^5 t] \cup [\eta^5 t, \eta^6 t]) * [jt, \eta jt]$$

with the union of the arrangement $\bigcup \mathcal{A} = L$ we shall intersect L with 6-simplexes which form the maximal cell e . We compute that there is only one simplex whose image intersects L , specially

$$L \cap f([\eta t, \eta^2 t] * [jt, \eta jt]) = \{ \frac{1}{3} f(\eta t) + \frac{1}{6} f(\eta^2 t) + \frac{1}{6} f(jt) + \frac{1}{3} f(\eta jt) \}.$$

If we denote the intersection point by y , then

$$\mathfrak{D}_{\mathbb{Q}_{24}}(f)(e) = \alpha \|y\|$$

where $\alpha \in \{+1, -1\}$ is an associated intersection number.

(3) *Identification of the cohomology class of the obstruction cocycle* $\mathfrak{D}_{\mathbb{Q}_{4n}}(f)(e)$ in the group of coinvariants $H_2(W_n \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{4n}} \cong H_{n-4}(\bigcup \hat{\mathcal{A}}; \mathbb{Z}) / \sim$. First let us observe that $W_6 \setminus \bigcup \mathcal{A} = W_6 \setminus L \simeq S^{6-4}$. Thus $H_2(\bigcup \hat{\mathcal{A}}; \mathbb{Z}) = \mathbb{Z}$ and $\text{Ext}(H_1(\bigcup \hat{\mathcal{A}}; \mathbb{Z}), \mathbb{Z}) = 0$, and consequently

$$H_2(W_6 \setminus \bigcup \mathcal{A}; \mathbb{Z}) \cong H_2(\bigcup \hat{\mathcal{A}}; \mathbb{Z}) = \mathbb{Z}.$$

Since $(\forall g \in \mathbb{Q}_{24}) gL = L$ we check whether some $g \in \mathbb{Q}_{24}$ changes the orientation of L . The element η^5 acts on W_6 by changing its orientation. On the orthogonal complement L^\perp of L the operator η^5 , for the basis $\{e_1 + e_2 + e_3, e_2 + e_3 + e_4, e_3 + e_4 + e_5, e_1 + \dots + e_6\}$ of L^\perp , has the matrix

$$\Xi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\det \Xi = -1$, the element η^5 also changes the orientation of L^\perp and consequently does not change the orientation on L . If $l \in H_2(\bigcup \hat{\mathcal{A}}; \mathbb{Z})$ is the generator associated to the subspace L , then after the discussion about the orientation the set equality $\eta^5 L = L$ implies the homology equality $\eta^5 l = l$. The relation

$$l \sim \eta^{-5} * l = \det(\eta^5) \eta^5 l = -\eta^5 l = -l$$

provides the conclusion that

$$H_2(W_6 \setminus \bigcup \mathcal{A}; \mathbb{Z})_{\mathbb{Q}_{24}} \cong H_2(\bigcup \widehat{\mathcal{A}}; \mathbb{Z}) / \sim \cong \mathbb{Z}_2.$$

From the definition of the point class ([3] and [4]) it is obvious that $\mathfrak{D}_{\mathbb{Q}_{24}}(f)(e) = \alpha \|y\|$ is the Poincare dual of l and consequently not zero. Moreover, after factoring by \sim the element $\varphi(\mathfrak{D}_{\mathbb{Q}_{24}}(f)(e)) / \sim$ is a generator of $H_2(\bigcup \widehat{\mathcal{A}}; \mathbb{Z}) / \sim$ and so

$$[\mathfrak{D}_{\mathbb{Q}_{24}}(f)(e)] \neq 0.$$

Therefore, there is no \mathbb{Q}_{24} -map $f : S^3 \rightarrow W_6 \setminus L$ and the Theorem 2 is proved.

Concluding Remarks. The exposition of the general problem and the eledged techniques are motivated by results jet to come.

The particular result we presented is derived from the fact that there are no \mathbb{Q}_{24} -map $S^3 \rightarrow S^2$. It looks like some type of Borsuk-Ulam theorem could be applied. The reason we have to use the obstruction theory lies in the rather complicated group \mathbb{Q}_{24} and the fact that its action on S^2 is not free. Therefore, there are no short cuts in solving this problem.

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