Equity Portfolios Generated by Functions of Ranked Market Weights

Robert Fernholz

INTECH
One Palmer Square
Princeton, NJ 08542

December 14, 1998
Revised July 5, 2000

Abstract

Dynamic equity portfolios can be generated by positive twice continuously differentiable functions of the ranked capitalization weights of an equity market. The return on such a portfolio relative to the market follows a stochastic differential equation that decomposes the relative return into two components: the logarithmic change in the value of the generating function, and a drift process that is of bounded variation. The method can be used to construct broad classes of stock portfolios, and has both theoretical and practical applications. Two applications of the method are presented: one offers an explanation for the size effect, the observed tendency of small stocks to have higher long-term returns than large stocks, and the other provides a rigorous analysis of the behavior of diversity-weighted indices, stock indices with weights that lie between capitalization weights and equal weights.

Key words: Portfolio generating function, local time, size effect, diversity-weighted index.
JEL classification: G11, C62, G19.
MSC (1991) classification: 90A09, 60H30, 60G44.

1 Introduction

Functionally generated portfolios first appeared in Fernholz (1999a) with the entropy-weighted portfolio, and then in general form in Fernholz (1999b) where it was shown that certain functions of the market weights in an equity market generate dynamic portfolios. The return of a functionally generated portfolio relative to the market portfolio follows a stochastic differential equation. This equation decomposes the relative return into two components: the change in the value of the generating function, and a drift process that is of finite variation. By appropriate selection of the generating function, the components can be structured so that the portfolio will have desirable return characteristics. In fact, functionally generated equity portfolios have been used for institutional investment for several years (see Fernholz, Garvy, and Hannon (1998)).

The properties of functionally generated portfolios are somewhat different from the portfolios customarily considered in the literature. Functionally generated portfolios are unlikely to be optimal in the sense of Markowitz (1952,1959), over single periods, or Merton (1969,1971), in continuous time. Nor can they be expected to be asymptotically optimal in the sense of Breiman (1961) or Cover (1991). Functionally generated portfolios are perhaps more like the hedging portfolio corresponding to the Black and Scholes (1973) option pricing formula (see, e.g., Karatzas and Shreve (1998) for a discussion of these concepts). In this case, the option pricing function "generates" the hedging portfolio, and the return on the hedging portfolio must conform exactly to changes in the value of the option, or else arbitrage is possible. In a similar manner, Fernholz (1999a,b) showed that functionally generated portfolios can be used to determine structural market conditions that are compatible with an arbitrage-free market. These conditions are based on observable variables such as market weights rather than on abstract constructions such as an equivalent martingale measure (see Harrison and Kreps (1979) and Harrison and Pliska (1981,1983)).

Fernholz (1999b) assumed that the market is closed, i.e., there are no stocks entering or leaving the market. However, this assumption is not completely satisfactory, especially in practice where the theory has been applied to large-stock indices like the S&P 500 or the Russell 1000 in which stocks frequently enter or leave. In fact, Fernholz, Garvy, and Hannon (1998) observed that an adjustment must be made to the theory to account for "leakage" caused by the systematic attrition of the smaller stocks that are dropped from the S&P 500 Index. To a lesser extent, the same phenomenon will affect even a broad universe such as the market of all publicly traded stocks, since smaller stocks are continually entering the market (as IPOs) or leaving the market (as bankruptcies, buyouts, etc.). Hence, the theory developed by Fernholz (1999b) needs to be extended to account for the movement of small stocks into and out of the market.

In this paper we consider functions of the ranked market weights, and show that under appropriate conditions they also generate portfolios. This will allow us to consider portfolios composed exclusively of large stocks, which are identified by ranked market weights. This problem is more complicated mathematically than Fernholz (1999b) because the ranking of the weights is not a differentiable transformation, and hence Itô's (1951) lemma cannot be applied directly.

We show that a positive twice continuously differentiable function of the ranked market weights generates a dynamic equity portfolio. For such a portfolio the return relative to the market portfolio follows a stochastic differential equation similar to that of Fernholz (1999b). This equation decomposes the relative return into the same two components as in Fernholz (1999b), however now the drift process includes semimartingale local times that account for changes in rank among the market weights. We present two applications of this decomposition. The first application provides an explanation for the size effect, the observed tendency of small stocks to have higher long-term returns than large stocks (see Banz (1981) and Reinganum (1981)), and extends the discrete-time results of Fernholz (1998) to the continuous-time setting. Our result is quite different from alternative explanations for the size effect that have been offered by Roll (1981), Handa, Kothari, and Wasley (1989), and Jegadeesh (1992). The second application provides a rigorous mathematical treatment of diversity-weighted indexing, a new type of passive equity strategy that was introduced by Fernholz, Garvy, and Hannon (1998) and is currently being used for actual investment.

Section 2 of the paper contains some basic definitions and results regarding continuous-time equity portfolios. The main theorem on portfolios generated by functions of ranked market weights is presented in Section 3. In Section 4 we present two applications of the theory, and Section 5 is a summary. An Appendix contains some results that we need on semimartingale local times.

We shall assume that we operate in a continuously-traded, frictionless market in which the stock prices vary continuously and the companies pay no dividends. We assume that companies neither enter nor leave the market, nor do they merge or break up, and that the total number of shares of a company remains constant. Shares of stock are assumed to be infinitely divisible, so we can assume without loss of generality that each company has a single share of stock outstanding.

2 Stocks and portfolios

We shall consider a market $\mathcal{M}$ comprising n stocks represented by their price processes $X_1, \dots, X_n$. We assume that there is a single share of each stock, so $X_i(t)$ represents the total capitalization of the i-th company at time t. The price processes evolve according to the equations

$$X_i(t) = X_0^i \exp\left(\int_0^t \gamma_i(s)\, ds + \int_0^t \sum_{\nu=1}^n \xi_{i\nu}(s)\, dW_\nu(s) \right), \quad t \in [0,T], \tag{2.1}$$

for $i = 1, \dots, n$. Here X_0^i, $i = 1, \dots, n$, are positive constants and $W = \{W(t) = (W_1(t), \dots, W_n(t)), \mathcal{F}_t, t \in [0,T]\}$ is a standard n-dimensional Brownian motion defined on a complete probability space $\{\Omega, \mathcal{F}, P\}$ where $\{\mathcal{F}_t\}$ is the P-augmentation of the natural filtration $\{\mathcal{F}_t^W = \sigma(W(s); 0 \le s \le t)\}$. From (2.1), we see that the stocks satisfy

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t), \quad t \in [0,T], \tag{2.2}$$

for $i = 1, \dots, n$. The *growth rate* processes $\gamma_i = \{\gamma_i(t), \mathcal{F}_t, t \in [0,T]\}$, $i = 1, \dots, n$, are measurable, adapted, and satisfy $\int_0^T |\gamma_i(s)| ds < \infty$, a.s. For $i, \nu = 1, \dots, n$, the *volatility* processes $\xi_{i\nu} = \{\xi_{i\nu}(t), \mathcal{F}_t, t \in [0,T]\}$ are measurable, adapted, and satisfy $\int_0^T \xi_{i\nu}^2(s) ds < \infty$, a.s., with $\xi_{i1}^2(t) + \cdots + \xi_{in}^2(t) > 0$, $t \in [0,T]$, a.s. From (2.2) it follows that each stock is a square-integrable continuous semimartingale.

Remark. The time domain $[0,T]$ is commonly used in mathematical finance due to the need for Girsanov's theorem (see Duffie (1992) or Karatzas and Shreve (1998)). By convention, we shall use the time domain $[0,T]$, but since our results are not dependent on Girsanov's theorem, all of the results remain valid for price processes defined on $[0, \infty)$.

Consider the matrix valued process ξ defined by $\xi(t) = (\xi_{i\nu}(t))_{1\leq i,\nu\leq n}$ and define the *covariance process* σ where $\sigma(t) = \xi(t)\xi^T(t)$. Then

$$\sigma_{ij}(t)\,dt = d\langle \log X_i, \log X_j\rangle_t, \quad t \in [0,T], \quad \text{a.s.}$$

The conditions on the volatility processes ensure that the σ_{ij} are a.s. L^1 functions of t.

Definition 2.1. A *portfolio* in $\mathcal{M}$ is a measurable, adapted process π that is bounded on $[0,T]\times\Omega$ and satisfies

$$\pi_1(t) + \cdots + \pi_n(t) = 1, \quad t \in [0,T], \quad \text{a.s.}$$

The processes π_i represent the respective *proportions*, or *weights*, of each stock in the portfolio. A negative value for $\pi_i(t)$ indicates a short sale.

Suppose $Z_\pi(t)$ represents the value of an investment in π at time t. Then $Z_\pi(t)$ satisfies

$$dZ_\pi(t) = Z_\pi(t)\sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)}, \tag{2.3}$$

for $t \in [0,T]$. This equation and an initial value $Z_\pi(0) > 0$ determine the portfolio value through time (see Fernholz (1999a)), so we shall call the process Z_π the *portfolio value process* for π. Two applications of Itô's lemma transform (2.3) into

$$d\log Z_\pi(t) = \sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \gamma_\pi^*(t)\,dt, \tag{2.4}$$

where

$$\gamma_\pi^*(t) = \frac{1}{2}\Bigg(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)\Bigg), \quad t \in [0,T], \tag{2.5}$$

is called the *excess growth rate.*

Remark. It can be shown that for portfolios without short sales the excess growth rate is nonnegative, and is positive if $\sigma(t)$ is nonsingular and $\pi(t)$ has at least two positive weights (see Fernholz (1999a)).

Definition 2.2. The portfolio μ defined by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \quad t \in [0,T], \tag{2.6}$$

for $i = 1,\ldots,n$, is called the *market portfolio (process).*

It can easily be verified that the weights μ_i defined by (2.6) satisfy the requirements of Definition 2.1, and that they are continuous square-integrable semimartingales. If we let

$$Z(t) = X_1(t) + \cdots + X_n(t), \quad t \in [0,T],$$

then $Z(t)$ satisfies (2.3) with the weights μ_i, so with appropriate initial conditions, $Z_\mu = Z$, and the portfolio value process represents the combined capitalization of all the stocks in the market. Henceforth we shall let μ exclusively represent the market portfolio and Z represent its value process.

The instantaneous *relative return* of X_i with respect to the market at time t is represented by

$$d\log\big(X_i(t)/Z(t)\big) = d\log X_i(t) - d\log Z(t).$$

Since $\mu_i = X_i/Z$, the *relative return process* $\log(X_i/Z)$ can be represented by $\log \mu_i$. The cross variation processes for the relative returns of the stocks in the market generate the (matrix valued) *relative covariance process* $\tau = (\tau_{ij})$, which satisfies

$$\tau_{ij}(t)\,dt = d\langle \log\mu_i, \log\mu_j\rangle_t \tag{2.7}$$

$$= d\langle \log X_i, \log X_j\rangle_t - d\langle \log X_i, \log Z\rangle_t - d\langle \log X_j, \log Z\rangle_t + d\langle \log Z\rangle_t, \tag{2.8}$$

for $1 \le i, j \le n$ and all $t \in [0, T]$, a.s. From (2.8) it follows that for $1 \le i, j \le n$,

$$\tau_{ij}(t) = \sigma_{ij}(t) - \sum_{k=1}^{n} \mu_k(t)\sigma_{ik}(t) - \sum_{l=1}^{n} \mu_l(t)\sigma_{jl}(t) + \sum_{k,l=1}^{n} \mu_k(t)\mu_l(t)\sigma_{kl}(t), \tag{2.9}$$

for $t \in [0, T]$, a.s., and that τ_{ij} is a.s. an L^1 function of t.

Lemma 2.1. *The market portfolio $\mu(t)$ is in the null space of $\tau(t)$, for all $t \in [0, T]$, a.s.*

Proof. From (2.9) it follows that for any portfolio π,

$$\pi(t)\tau(t)\pi^T(t) = \big(\pi(t) - \mu(t)\big)\sigma(t)\big(\pi(t) - \mu(t)\big)^T, \quad t \in [0, T] \quad \text{a.s.},$$

and this expression is zero if $\pi(t) = \mu(t)$. □

By combining (2.5) and (2.9) we obtain

$$\gamma_\pi^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}(t)\right), \quad t \in [0, T], \tag{2.10}$$

which will be useful in the next section.

3 Portfolios generated by functions of ranked market weights

The idea of portfolio generating portfolios was introduced by Fernholz (1999b), where it was shown that a positive twice continuously differentiable function of the market weights will generate a portfolio. Although these functionally generated portfolios had useful theoretical properties, the construction was not sufficiently general to allow for the study of portfolios composed of stocks selected by market capitalization, as occurs in many equity indices. Stocks selected by size depend on the ranks of the stocks in the market, and rank functions are not differentiable. Here we shall extend the results of Fernholz (1999b) to portfolios that are generated by functions of the ranked market weights. We first need a definition for rank processes.

Definition 3.1. Let $X_1, \ldots, X_n$ be processes. For $1 \le k \le n$, the k-th *rank process* of $X_1, \ldots, X_n$ is defined by

$$X_{(k)}(t) = \max_{i_1 < \cdots < i_k} \min\big(X_{i_1}(t), \ldots, X_{i_k}(t)\big), \quad t \in [0, T],$$

where $1 \le i_1$ and $i_k \le n$. We shall adopt the convention that $X_{(0)}$ and $X_{(n+1)}$ are defined such that for $t \in [0, T]$, $X_{(0)}(t) > X_{(1)}(t)$ and $X_{(n+1)}(t) < X_{(n)}(t)$.

According to this definition, for $t \in [0, T]$,

$$\max\big(X_1(t), \dots, X_n(t)\big) = X_{(1)}(t) \geq X_{(2)}(t) \geq \cdots \geq X_{(n)}(t) = \min\big(X_1(t), \dots, X_n(t)\big),$$

so that at any given time, the values of the rank processes represent the values of the original processes arranged in descending order.

We shall study functions of the ranked market weights $\mu_{(1)}(t), \dots, \mu_{(n)}(t)$, and shall use the notation

$$\mu_{(\cdot)}(t) = (\mu_{(1)}(t), \dots, \mu_{(n)}(t)), \quad t \in [0, T].$$

For any $t \in [0, T]$, we can define p_t to be the random permutation of $\{1, \dots, n\}$ such that for $k \in \{1, \dots, n\}$,

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t), \tag{3.1}$$

and

$$p_t(k) < p_t(k+1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t). \tag{3.2}$$

The permutation p_t is uniquely defined by (3.1) and (3.2), and associates each rank process with one of the original market weights that has the same value at time t. For $i, j = 1, \dots, n$, we can define the *relative rank covariance processes* $\tau_{(ij)}$ by

$$\tau_{(ij)}(t) = \tau_{p_t(i)p_t(j)}(t), \quad t \in [0, T]. \tag{3.3}$$

Since for all i and j, τ_{ij} is a.s. an L^1 function of t, the same is true for $\tau_{(ij)}$.

Although a change in a market weight represents the relative return of the corresponding stock, changes in the ranked market weights do not have such a simple interpretation. In order to characterize the changes in the ranked market weights in terms of the relative returns of the stocks, we need to introduce the concept of a semimartingale local time, a measure of the amount of time a process spends near the origin.

Definition 3.2. Let X be a continuous semimartingale. Then the *local time* (at 0) is the process Λ_X defined for $t \in [0, T]$ by

$$\Lambda_X(t) = \frac{1}{2}\Big(|X(t)| - |X(0)| - \int_0^t \operatorname{sgn}(X(s))\, dX(s)\Big), \tag{3.4}$$

where $\operatorname{sgn}(x) = 2\, I_{(0,\infty)}(x) - 1$, with $I_{(0,\infty)}$ the indicator function of $(0, \infty)$.

The asymmetry in sgn induces an asymmetry in the local time: in general Λ_X differs from Λ_{-X}. For general background on local times, see Karatzas and Shreve (1991); the technical results that we shall need can be found in the Appendix. Equation (3.4), which we use as a definition, is one of the *Tanaka-Meyer formulas* (Tanaka (1963), Meyer (1976)). It can be shown that $\Lambda_X(t)$ is almost surely nondecreasing in t, and satisfies

$$I_{\{0\}}(X(t))\, d\Lambda_X(t) = d\Lambda_X(t), \quad t \in [0, T], \quad \text{a.s.} \tag{3.5}$$

(see Karatzas and Shreve (1991), 3.7.1). This implies, for example, that for one-dimensional Brownian motion B, Λ_B is a non-negative random measure on $[0, T]$ that almost surely has support contained in the set $\{t : B(t) = 0\}$, and hence is singular with respect to Lebesgue measure. In order to effectively use local times, we must ensure that the stock price processes we consider exhibit a certain level of nondegeneracy.

Definition 3.3. The processes $X_1, \dots, X_n$ are *pathwise mutually nondegenerate* if:

i) for all $i \neq j$, $\{t : X_i(t) = X_j(t)\}$ has Lebesgue measure zero, a.s.;

ii) for all $i < j < k$, $\{t : X_i(t) = X_j(t) = X_k(t)\} = \varnothing$, a.s.

The components of multidimensional Brownian motion $(W_1, \dots, W_n)$ are pathwise mutually nondegenerate, at least for $t > 0$. Condition *i* is well known for Brownian motion, and condition *ii* follows from the fact that, with probability one, 2-dimensional Brownian motion never returns to the origin (see Karatzas and Shreve (1991)).

The ranked stock price processes $X_{(1)}, \dots, X_{(n)}$ cannot be be represented by individual stocks or by portfolios of stocks. Instead, Proposition A.1 implies that for $1 \leq k \leq n$,

$$dX_{(k)}(t) = \sum_{i=1}^{n} I_{\{0\}}\big(p_t(k) - i\big)\, dX_i(t) + \frac{1}{2} d\Lambda_{X_{(k)} - X_{(k+1)}}(t) - \frac{1}{2} d\Lambda_{X_{(k-1)} - X_{(k)}}(t), \quad t \in [0, T], \quad \text{a.s.}$$

Let us interpret this equation for $X_{(1)}(t)$, the maximum stock price at time t. In this case, when the two largest stocks are momentarily equal in value, the sum on the right-hand side of the equation cannot anticipate which of them will be the maximum in the future. The local time $d\Lambda_{X_{(1)} - X_{(2)}}(t)/2$ adjusts for the difference.

Let us leave local times for now and proceed with the concept of portfolio generating functions and functionally generated portfolios.

Definition 3.4. Let U be an open neighborhood in $\mathbb{R}^n$ of the open simplex

$$\Delta^n = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1, \quad 0 < x_i < 1, \quad i = 1, \dots, n\},$$

and let $\mathbf{S}$ be a positive function defined in U. Then $\mathbf{S}$ *generates* the portfolio π if there exists a measurable, adapted process of bounded variation $\Theta = \{\Theta(t), \mathcal{F}_t, t \in [0, T]\}$ such that

$$\log\big(Z_\pi(t)/Z(t)\big) = \log \mathbf{S}(\mu_{(\cdot)}(t)) + \Theta(t), \quad t \in [0, T], \quad \text{a.s.} \tag{3.6}$$

The function $\mathbf{S}$ in Definition 3.4 is called a *portfolio generating function*, and (3.6) decomposes the relative return process $\log\big(Z_\pi(t)/Z(t)\big)$ into a generating function component $\log \mathbf{S}(\mu_{(\cdot)}(t))$ and a *drift process* $\Theta(t)$. Since the drift process is of bounded variation, the generating function component includes the local martingale part of the relative return process. This decomposition is useful because the variation of the generating function component can be controlled by bounds on $\log \mathbf{S}$, so under certain conditions the drift process will dominate the behavior of the relative return.

Remark. The portfolio π generated by $\mathbf{S}$ in Definition 3.4 is a *hedging portfolio* for $\mathbf{S}(\mu_{(\cdot)}(t))$ in the sense that its relative return hedges out the martingale part of $\log \mathbf{S}$. Definition 3.4 can be extended to include *time-dependent* portfolio generating functions defined on $U \times [0, T]$, with $\log \mathbf{S}(\mu_{(\cdot)}(t), t)$ replacing $\log \mathbf{S}(\mu_{(\cdot)}(t))$ in (3.6). A time-dependent generating function with $\Theta = 0$ can be considered to be an *option pricing function* with hedging portfolio π (see Karatzas and Shreve (1998)).

Portfolio generating functions were introduced in Fernholz (1999b), but only as functions of the market weights, not the *ranked* market weights. For theoretical applications, the use of ranked market weights makes it possible to analyze the effect of company size on portfolio returns. Since in practice one is frequently interested in a specific family of stocks determined by company size, e.g., the S&P 500 Index or the Russell 1000 Index, generating functions based on ranked market weights have much wider applicability. We shall give examples of both of these types of application in the next section.

The next theorem is the main result of this paper. It shows that there exists a broad class of portfolio generating functions that depend on ranked market weights.

Theorem 3.1. *Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ that are pathwise mutually nondegenerate, and let p_t be the random permutation defined by (3.1) and (3.2). Let $\mathbf{S}$ be a positive C^2 function defined on a neighborhood U of Δ^n such that for $i = 1, \ldots, n$, $x_i D_i \log \mathbf{S}(x)$ is bounded on Δ^n. Then $\mathbf{S}$ generates the portfolio π with weights that satisfy*

$$\pi_{p_t(k)}(t) = \Big(D_k \log \mathbf{S}(\mu_{(\cdot)}(t)) + 1 - \sum_{j=1}^{n} \mu_{(j)}(t)\, D_j \log \mathbf{S}(\mu_{(\cdot)}(t))\Big)\mu_{(k)}(t), \quad t \in [0,T], \quad \text{a.s.}, \tag{3.7}$$

for $k = 1, \ldots, n$, and drift process Θ that satisfies

$$d\Theta(t) = \frac{-1}{2\,\mathbf{S}(\mu_{(\cdot)}(t))} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)\,dt \\ + \frac{1}{2}\sum_{k=1}^{n-1}\big(\pi_{p_t(k+1)}(t) - \pi_{p_t(k)}(t)\big)\, d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t), \quad t \in [0,T], \quad \text{a.s.} \tag{3.8}$$

Proof. First we must verify that π defined by (3.7) is a portfolio, and that Θ defined by (3.8) is of bounded variation. If π satisfies (3.7), then $\sum_{i=1}^{n} \pi_i(t) = 1$, and the conditions on $\mathbf{S}$ imply that the processes π_i are bounded on $[0,T] \times \Omega$. Hence, π is a portfolio process. Regarding Θ, let us consider the two expressions on the right hand side of (3.8) separately. The process represented by the first expression is a.s. of bounded variation because $\tau_{(ij)}$ is an L^1 function of t and the rest of the terms are continuous in t. The second expression is a sum of local times multiplied by bounded functions, and hence is also of bounded variation. Therefore Θ is a.s. of bounded variation.

We must show that the portfolio π defined by (3.7) and the drift process Θ defined by (3.8) satisfy (3.6). To accomplish this, we shall analyze the generating function term $\log \mathbf{S}((\mu_{(\cdot)}(t))$ in (3.6) and the relative return process $\log\big(Z_\pi(t)/Z(t)\big)$, and show that the difference of these two terms satisfies (3.8). We first need some preliminary results.

Proposition A.1 implies that the ranked weight processes $\mu_{(k)}$, for $k = 1, \ldots, n$, satisfy

$$d\log\mu_{(k)}(t) = \sum_{i=1}^{n} I_{\{0\}}\big(p_t(k) - i\big)\, d\log\mu_i(t) \\ + \frac{1}{2} d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t) - \frac{1}{2} d\Lambda_{\log\mu_{(k-1)} - \log\mu_{(k)}}(t), \quad t \in [0,T], \quad \text{a.s.} \tag{3.9}$$

This and (2.7) imply that, for $i, j = 1, \ldots, n$,

$$d\langle \log\mu_{(i)}, \log\mu_{(j)}\rangle_t = \tau_{(ij)}(t)\,dt, \quad t \in [0,T], \quad \text{a.s.},$$

so by Itô's Lemma,

$$d\mu_{(i)}(t) = \mu_{(i)}(t)\, d\log\mu_{(i)}(t) + \frac{1}{2}\mu_{(i)}(t)\tau_{(ii)}(t)\,dt, \quad t \in [0,T], \quad \text{a.s.}, \tag{3.10}$$

and

$$d\langle \mu_{(i)}, \mu_{(j)} \rangle_t = \mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)\,dt, \quad t \in [0,T], \quad \text{a.s.} \tag{3.11}$$

Let us also note that for all $t \in [0,T]$, $\sum_{i=1}^n \mu_{(i)}(t) = 1$, and hence, $\sum_{i=1}^n d\mu_{(i)}(t) = 0$.

Let us consider the generating function component of the relative return, $\log \mathbf{S}((\mu_{(\cdot)}(t))$. Itô's lemma, along with (3.11), implies that a.s. for all $t \in [0,T]$,

$$\begin{aligned} d\log \mathbf{S}(\mu_{(\cdot)}(t)) &= \sum_{i=1}^n D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\,d\mu_{(i)}(t) \\ &\quad + \frac{1}{2\,\mathbf{S}(\mu_{(\cdot)}(t))} \sum_{i,j=1}^n D_{ij}\mathbf{S}(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)\,dt \\ &\quad - \frac{1}{2}\sum_{i,j=1}^n D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)\,dt. \end{aligned} \tag{3.12}$$

Now let us consider the relative return process $\log\big(Z_\pi(t)/Z(t)\big)$. From (2.4) we have a.s., for all $t \in [0,T]$,

$$\begin{aligned} d\log\big(Z_\pi(t)/Z(t)\big) &= \sum_{i=1}^n \pi_i(t)\,d\log \mu_i(t) + \gamma_\pi^*(t) \\ &= \sum_{i=1}^n \sum_{k=1}^n I_{\{0\}}\big(p_t(k) - i\big)\pi_{p_t(k)}(t)\,d\log\mu_i(t) + \gamma_\pi^*(t) \\ &= \sum_{k=1}^n \pi_{p_t(k)}(t) \sum_{i=1}^n I_{\{0\}}\big(p_t(k) - i\big)\,d\log\mu_i(t) + \gamma_\pi^*(t) \\ &= \sum_{k=1}^n \pi_{p_t(k)}(t)\,d\log\mu_{(k)}(t) + \frac{1}{2}\sum_{k=1}^{n-1}\big(\pi_{p_t(k)}(t) - \pi_{p_t(k+1)}(t)\big)\,d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t) \\ &\quad + \frac{1}{2}\Bigg(\sum_{i=1}^n \pi_{p_t(i)}(t)\tau_{(ii)}(t) - \sum_{i,j=1}^n \pi_{p_t(i)}(t)\pi_{p_t(j)}(t)\tau_{(ij)}(t)\Bigg)dt \qquad (3.13) \\ &= \sum_{k=1}^n \frac{\pi_{p_t(k)}(t)}{\mu_{(k)}(t)}\,d\mu_{(k)}(t) + \frac{1}{2}\sum_{k=1}^{n-1}\big(\pi_{p_t(k)}(t) - \pi_{p_t(k+1)}(t)\big)\,d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t) \\ &\quad - \frac{1}{2}\sum_{i,j=1}^n \pi_{p_t(i)}(t)\pi_{p_t(j)}(t)\tau_{(ij)}(t)\,dt, \qquad (3.14) \end{aligned}$$

where (3.13) follows from (3.9) and (2.10), and (3.14) follows from (3.10).

Let us simplify the first summation on the right hand side of (3.14). If the weights π_i, $i = 1, \ldots, n$ satisfy (3.7), then

$$\pi_{p_t(k)}(t) = \big(D_k \log \mathbf{S}(\mu_{(\cdot)}(t)) + \varphi(t)\big)\mu_{(k)}(t), \quad t \in [0,T], \tag{3.15}$$

for $k = 1, \ldots, n$, where

$$\varphi(t) = 1 - \sum_{j=1}^n \mu_{(j)}(t) D_j \log \mathbf{S}(\mu_{(\cdot)}(t)), \quad t \in [0,T].$$

In this case, a.s. for all $t \in [0, T]$,

$$\sum_{i=1}^{n} \frac{\pi_{p_t(i)}(t)}{\mu_{(i)}(t)}\, d\mu_{(i)}(t) = \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t) + \varphi(t) \sum_{i=1}^{n} d\mu_{(i)}(t)$$
$$= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t), \tag{3.16}$$

since $\sum_{i=1}^{n} d\mu_{(i)}(t) = 0$.

Now consider the last summation in (3.14). It follows from (3.15) that a.s., for all $t \in [0, T]$,

$$\sum_{i,j=1}^{n} \pi_{p_t(i)}(t)\pi_{p_t(j)}(t)\tau_{(ij)}(t) = \sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)$$
$$+ 2\varphi(t) \sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)$$
$$+ \varphi^2(t) \sum_{i,j=1}^{n} \mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)$$
$$= \sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t), \tag{3.17}$$

since Lemma 2.1 implies that $\mu_{(\cdot)}(t)$ is in the null space of $\big(\tau_{(ij)}(t)\big)$. Equations (3.14), (3.16), and (3.17), imply that a.s. for all $t \in [0, T]$,

$$d \log\big(Z_\pi(t)/Z(t)\big) = \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t)$$
$$- \frac{1}{2} \sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)\, dt$$
$$+ \frac{1}{2} \sum_{k=1}^{n-1} \big(\pi_{p_t(k+1)}(t) - \pi_{p_t(k)}(t)\big)\, d\Lambda_{\log \mu_{(k)} - \log \mu_{(k+1)}}(t).$$

This equation and (3.12) imply (3.8). □

4 Applications

Portfolio generating functions allow us to construct a large class of portfolios, some of which are interesting theoretically, and some of which are useful for actual investment purposes. Our first application is a theoretical explanation of the *size effect*, the observed tendency of small stocks to have higher long-term returns than large stocks (see Banz (1981) and Reinganum (1981)). A number of hypotheses have been proposed to explain the size effect (see, e.g., Roll (1981), Handa, Kothari, and Wasley (1989), and Jegadeesh (1992)); here we present an alternative hypothesis proposed in discrete time by Fernholz (1998).

Our second application is to *diversity-weighted indexing*, a passive indexing methodology that is currently being used for actual investments (see Fernholz, Garvy, and Hannon (1998)). Most stock indices are either capitalization-weighted, as with the S&P 500 Index or the Russell 1000 Index, or equal-weighted, as with the Value Line Index. Diversity weighting introduces a class of portfolios with weights that lie between capitalization weights and equal weights. Moreover, (3.6) allows us to draw conclusions about the performance of diversity-weighted indices.

Example 4.1. (The size effect) We shall use generating functions to generate two portfolios, a large-stock portfolio ξ and a small-stock portfolio η, and then we shall compare their performance.

Let $1 < m < n$, and suppose

$$\mathbf{S}_L(x_1, \ldots, x_n) = x_1 + \cdots + x_m.$$

Then Theorem 3.1 implies that $\mathbf{S}_L$ generates a portfolio ξ with weights

$$\xi_{(k)}(t) = \begin{cases} \dfrac{\mu_{(k)}(t)}{\mathbf{S}_L(\mu_{(\cdot)}(t))}, & k \leq m, \\ 0, & k > m, \end{cases}$$

for $t \in [0, T]$, and drift process that satisfies

$$d\Theta(t) = -\frac{1}{2}\xi_{(m+1)}(t)\, d\Lambda_{\log \mu_{(m)} - \log \mu_{(m+1)}}(t), \quad t \in [0, T], \quad \text{a.s.}$$

Here ξ represents a large-stock index composed of the m largest stocks in the market, and the weights ξ_i, $i = 1, \ldots, n$, represent the capitalization weights of the stocks in this index. The value of the generating function

$$\mathbf{S}_L(\mu_{(\cdot)}(t)) = \mu_{(1)}(t) + \cdots + \mu_{(m)}(t)$$

measures the relative capitalization of the large-stock index compared to the market as a whole.

Similarly

$$\mathbf{S}_S(x) = x_{m+1} + \cdots + x_n$$

generates a portfolio η with weights

$$\eta_{(k)}(t) = \begin{cases} \dfrac{\mu_{(k)}(t)}{\mathbf{S}_S(\mu_{(\cdot)}(t))}, & k > m, \\ 0, & k \leq m, \end{cases}$$

and with drift process that satisfies

$$d\Theta(t) = \frac{1}{2}\eta_{(m+1)}(t)\, d\Lambda_{\log \mu_{(m)} - \log \mu_{(m+1)}}(t) \quad t \in [0, T], \quad \text{a.s.}$$

In this case $\mathbf{S}_S(\mu_{(\cdot)}(t))$ measures the relative capitalization of the small-stock index composed of the $n - m$ smallest stocks in the market. The relative return of the small-stock index versus the large-stock index will satisfy

$$\begin{aligned} d\log\big(Z_\eta(t)/Z_\xi(t)\big) &= d\log\big(\mathbf{S}_S(\mu_{(\cdot)}(t))/\mathbf{S}_L(\mu_{(\cdot)}(t))\big) \\ &\quad + \frac{1}{2}\big(\xi_{(m)}(t) + \eta_{(m+1)}(t)\big)\, d\Lambda_{\log \mu_{(m)} - \log \mu_{(m+1)}}(t), \quad t \in [0, T], \quad \text{a.s.} \end{aligned} \tag{4.1}$$

It can be argued that general market stability imposes bounds on the variation of the relative capitalizations of the small-stock index and the large-stock index. In this case, over a sufficiently long period of time, the relative return will be dominated by the local time term in (4.1), which is a.s. increasing. Hence, the long-term return of the small-stock index will be greater than that of the large-stock portfolio. This phenomenon is structural, and will occur regardless of whether of not small stocks are riskier than large stocks.

Example 4.2. (Diversity-weighted stock indices) The *diversity-weighted* S&P 500 Index was defined in Fernholz, Garvy, and Hannon (1998) to be the portfolio of stocks in the S&P 500 Index generated by

$$\mathbf{D}_p(x) = \Big(\sum_{i=1}^{500} x_i^p\Big)^{1/p},$$

with $p = .76$. ($\mathbf{D}_p$ is called a *measure of diversity.*) That paper was nontechnical, and a rigorous mathematical derivation of the results was not available when it appeared. We shall present the derivation here.

Rather than the S&P 500, let us consider a diversity-weighted version of the large-stock index in Example 4.1. In this case we have the generating function

$$\mathbf{S}(x_1, \dots, x_n) = \Big(\sum_{i=1}^{m} x_i^p\Big)^{1/p},$$

with $1 < m < n$ and $0 < p < 1$. Theorem 3.1 implies that $\mathbf{S}$ generates a portfolio π with

$$\pi_{(k)}(t) = \begin{cases} \dfrac{\mu_{(k)}^p(t)}{\mathbf{S}^p(\mu_{(\cdot)}(t))}, & k \le m, \\ 0, & k > m, \end{cases}$$

for $t \in [0, T]$, and drift process that satisfies

$$d\Theta(t) = (1-p)\gamma_\pi^*(t)\,dt - \frac{1}{2}\pi_{(m)}(t)\,d\Lambda_{\log\mu_{(m)} - \log\mu_{(m+1)}}(t), \quad t \in [0, T], \quad \text{a.s.}$$

We are interested in the performance of π relative to ξ of Example 4.1, and we have a.s., for $t \in [0, T]$,

$$d\log\big(Z_\pi(t)/Z_\xi(t)\big) = d\log\mathbf{D}_p\big(\xi_{(1)}(t), \dots, \xi_{(m)}(t)\big) + (1-p)\gamma_\pi^*(t)\,dt \\ + \frac{1}{2}\big(\xi_{(m)}(t) - \pi_{(m)}(t)\big)\,d\Lambda_{\log\mu_{(m)} - \log\mu_{(m+1)}}(t), \quad (4.2)$$

since $\mathbf{S}(\mu_{(\cdot)}(t))/\mathbf{S}_L(\mu_{(\cdot)}(t)) = \mathbf{D}_p\big(\xi_{(1)}(t), \dots, \xi_{(m)}(t)\big)$. As in Example 4.1, we can argue that market stability implies that $\log\mathbf{D}_p(\xi_{(\cdot)}(t))$ will be stable and mean-reverting over the long term. If this is true, then the long-term relative return of the $\mathbf{D}_p$-weighted index will be dominated by the last two terms in (4.2). Since $p < 1$, the remark following (2.5) implies that $(1-p)\gamma_\pi^*(t)\,dt$ is increasing. The last term, involving the local time, is called *leakage* because it measures the effect on the relative return of stocks that become too small and subsequently are dropped from ("leak" out of) the large-stock index. Since $p < 1$, it follows that $\pi_{(m)}(t) > \xi_{(m)}(t)$, and this implies that the leakage is decreasing. The relative magnitude of these last two terms determines whether the drift process is increasing or decreasing.

To get some idea of the behavior of (4.2) for actual stocks, we ran a simulation using the stock database from the Center for Research in Securities Prices (CRSP) at the University of Chicago. The data included 60 years of monthly values from 1939 to 1998 for all NYSE, AMEX, and NASDAQ stocks after the removal of closed-end funds, REITs, and ADRs not included in the S&P 500 Index. The large-stock portfolio consisted of the largest 1000 stocks in the database for those months that the database contained 1250 or more stocks, and the largest 80% of the stocks those months that the database contained fewer than 1250. We used the parameter value $p = .5$ for $\mathbf{D}_p$, and no trading costs were included.

The results of the simulation are presented in Figure 1: Curve 1 is the change in the generating function, Curve 2 is the drift process, and Curve 3 is the relative return. Each curve shows the cumulative value of the monthly changes induced in the corresponding process by capital gains or losses in the stocks: the curves are unaffected by monthly changes in the composition of the database. As can be seen, Curve 3 is the sum of Curves 1 and 2. The drift process $\Theta(t)$ was the dominant process over the period, with a contribution of 46.4% to the relative return, and was remarkably stable. Of the drift process's contribution, the $(1-p)\gamma^*_\pi(t)\,dt$ term accounted for 88.0%, and the local time term (leakage) accounted for -41.6%. To calculate the total relative return of an investment in the $\mathbf{D}_p$-weighted index versus an investment in the capitalization-weighted index, dividend payments must also be considered. However the difference in dividend payments between the two portfolios was quite small, with a total contribution over the period of only 1.3% in favor of the capitalization-weighted index. (Figure 1 does not include dividend payments.)

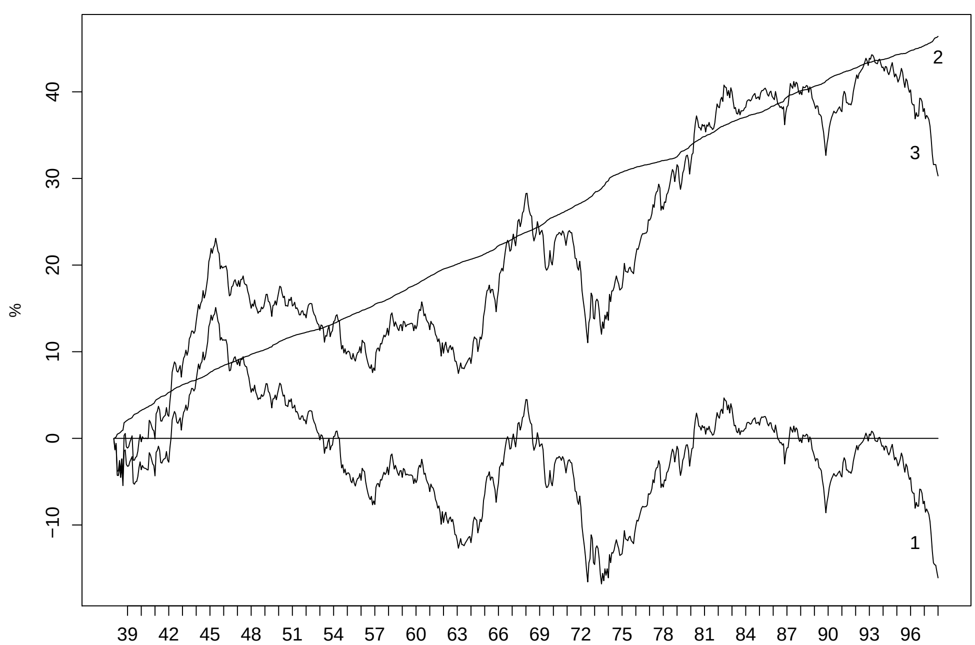

Figure 1: Simulation of the $\mathbf{D}_p$-weighted index, 1939–1998
1: $\log \mathbf{D}_p(\xi_{(\cdot)}(t))$; 2: $\Theta(t)$; 3: $\log\big(Z_\pi(t)/Z_\xi(t)\big)$

If $\log \mathbf{D}_p(\xi_{(\cdot)}(t))$ is mean-reverting and $\Theta(t)$ continues to follow the trend line in Figure 1, then the long-term return of π will be greater than that of ξ. The volatility of the two portfolios will be about the same over periods of the order of the relaxation time of $\log \mathbf{D}_p(\xi_{(\cdot)}(t))$. Hence, π would appear to have higher return than ξ, but with about the same long-term risk. Of course, trading costs and other considerations must be taken into account in any actual investment, however Fernholz, Garvy, and Hannon (1998) indicated that trading costs were not a significant factor for the diversity-weighted S&P 500 Index.

5 Conclusions

We have shown how stock portfolios can be generated by functions of ranked equity market weights. This method offers a wide class of portfolios both for theoretical study and for actual investment. We also have shown how semimartingale local times are involved when the maximum, minimum, and ranks of stock price processes are studied.

The essential feature of functionally generated portfolios is that the relative return of such a portfolio can be decomposed naturally into two components: the change in the value of the generating function and a finite variation drift process. This decomposition can be used to derive results that have both theoretical and practical interest. As a theoretical application, we showed that the size effect may be explained independently of whether or not small stocks have higher risk than larger stocks. As a practical application, we showed that diversity-weighted large-stock indices may have favorable characteristics as long-term investments.

To date only a few functionally generated portfolios have been studied, and it is possible that other interesting classes of applications will be found. Here are a few possibilities: (i) derive testable equilibrium conditions with respect to the relative returns for different strata of stocks ranked by capitalization; (ii) develop connections between functionally generated portfolios and option pricing theory; (iii) develop an optimization theory for functionally generated portfolios.

A Appendix: Rank processes and local times

In this appendix we prove some technical results on semimartingale local times that we need in order to deal with ranked market weights.

Meyer (1976) showed that convex functions of continuous semimartingales are themselves continuous semimartingales, and since Definition 3.1 implies that rank processes are generated by convex functions, it follows that rank processes derived from continuous semimartingales are also continuous semimartingales. However, Meyer (1976) did not provide an explicit representation of the rank processes in terms of the original processes. Representations have been derived for certain cases, but none specifically fits our needs (see, e.g., Carlen and Protter (1992) and Protter and San Martin (1993)). Here we shall provide an explicit representation for rank processes in terms of the original processes from which they are derived. However, to achieve this, we must restrict our consideration to a limited class of continuous semimartingales. Fortunately this class is broad enough to include the market weights that interest us.

Definition A.1. Let X be a continuous semimartingale with canonical decomposition

$$X(t) = X(0) + M_X(t) + V_X(t), \quad t \in [0, T], \quad \text{a.s.,} \tag{A.1}$$

where M_X a continuous local martingale and V_X is a continuous process of bounded variation. Then X is *absolutely continuous* if the random signed measures dV_X and $d\langle X\rangle = d\langle M_X\rangle$ are both almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$.

Note that the sample paths of absolutely continuous semimartingales are usually not absolutely continuous functions of t: consider, e.g., Brownian motion.

Lemma A.1. *Suppose $X_1,\dots,X_n$ are absolutely continuous semimartingales, and f is a real valued C^2 function defined on $\mathbb{R}^n$. Then $f(X_1,\dots,X_n)$ is an absolutely continuous semimartingale.*

Proof. Suppose that $X_1,\dots,X_n$ are absolutely continuous semimartingales and A is a Lebesgue measurable subset of $[0,T]$. Then a.s.,

$$2\left|\int_A d\langle X_i, X_j\rangle_t\right| \le \int_A d\langle X_i\rangle_t + \int_A d\langle X_j\rangle_t,$$

for $1 \le i,j \le n$. Hence, the random signed measure $d\langle X_i, X_j\rangle$ is almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$. The lemma then follows from Itô's lemma applied to $f(X_1(t),\dots,X_n(t))$. □

The semimartingale representation (2.2) and Lemma A.1 imply that stock price processes, the market value process, and market weight processes, are all absolutely continuous semimartingales.

Lemma A.2. *Let X and Y be continuous semimartingales with X absolutely continuous, and suppose that the set $\{t : Y(t) = 0\}$ almost surely has Lebesgue measure 0. Then*

$$\int_0^t I_{\{0\}}(Y(s))\,dX(s) = 0, \quad t \in [0,T], \quad \text{a.s.}$$

Proof. We have

$$dX(t) = dM_X(t) + dV_X(t), \quad t \in [0,T],$$

where M_X and V_X are defined as in the decomposition (A.1). Since dV_X is almost surely absolutely continuous with respect to Lebesgue measure, and the Lebesgue measure of $\{t : Y(t) = 0\}$ is 0, a.s., it follows that

$$\int_0^t I_{\{0\}}(Y(s))\,dV_X(s) = 0, \quad t \in [0,T], \quad \text{a.s.}$$

Therefore,

$$\int_0^t I_{\{0\}}(Y(s))\,dX(s) = \int_0^t I_{\{0\}}(Y(s))\,dM_X(s), \quad t \in [0,T], \quad \text{a.s.}$$

The process U defined by

$$U(t) = \int_0^t I_{\{0\}}(Y(s))\,dM_X(s), \quad t \in [0,T],$$

is a continuous local martingale. Since $\{t : Y(t) = 0\}$ almost surely has Lebesgue measure 0 and $d\langle M_X\rangle$ is almost surely absolutely continuous with respect to Lebesgue measure,

$$\begin{aligned}\langle U\rangle_t &= \int_0^t I_{\{0\}}(Y(s))\,d\langle M_X\rangle_s \\ &= 0, \quad t \in [0,T], \quad \text{a.s.}\end{aligned}$$

It follows that $U(t) = 0$ for all $t \in [0,T]$, a.s. (see Karatzas and Shreve (1991), 1.5.12 and 1.5.21). □

Lemma A.3. *Let X be an absolutely continuous semimartingale such that the set $\{t : X(t) = 0\}$ has Lebesgue measure 0, almost surely. Then*

$$\Lambda_{|X|}(t) = 2\Lambda_X(t), \quad t \in [0,T], \quad \text{a.s.}$$

Proof. By definition,

$$2\Lambda_X(t) = |X(t)| - |X(0)| - \int_0^t \operatorname{sgn}(X(s))\,dX(s), \quad t \in [0,T], \tag{A.2}$$

so

$$d|X(t)| = 2\,d\Lambda_X(t) + \operatorname{sgn}(X(t))\,dX(t), \quad t \in [0,T], \quad \text{a.s.}$$

As in (A.2), a.s., for all $t \in [0,T]$,

$$\begin{aligned}2\Lambda_{|X|}(t) &= |X(t)| - |X(0)| - \int_0^t \operatorname{sgn}|X(s)|\,d|X(s)| \\ &= 2\Lambda_X(t) + \int_0^t \operatorname{sgn}(X(s))\,dX(s) - 2\int_0^t \operatorname{sgn}|X(s)|\,d\Lambda_X(s) - \int_0^t \operatorname{sgn}|X(s)|\operatorname{sgn}(X(s))\,dX(s) \\ &= 4\Lambda_X(t) - 2\int_0^t I_{\{0\}}(X(s))\,dX(s) \qquad \text{(A.3)} \\ &= 4\Lambda_X(t), \qquad \text{(A.4)}\end{aligned}$$

where (A.3) follows from (3.5) and the fact that $\operatorname{sgn}(0) = -1$, and (A.4) follows from Lemma A.2 since X is absolutely continuous and $X(s) = 0$ only on a set of Lebesgue measure 0, almost surely. □

Definition 3.1 constructs rank processes by means of maximum and minimum processes, so we shall consider these processes first.

Lemma A.4. *Let X and Y be pathwise mutually nondegenerate absolutely continuous semimartingales. Then*

$$\begin{aligned}d\max\big(X(t),Y(t)\big) &= I_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t) \\ &\quad + I_{(0,\infty)}\big(Y(t)-X(t)\big)\,dY(t) + d\Lambda_{X-Y}(t), \quad t \in [0,T], \quad \text{a.s.}\end{aligned} \tag{A.5}$$

Proof. We shall do the calculation:

$$\begin{aligned}d\max\big(X(t),Y(t)\big) &= \frac{dX(t)+dY(t)}{2} + \frac{d|X(t)-Y(t)|}{2} \\ &= \frac{dX(t)+dY(t)}{2} + \frac{\operatorname{sgn}\big(X(t)-Y(t)\big)\,d\big(X(t)-Y(t)\big)}{2} + d\Lambda_{X-Y}(t)\end{aligned} \tag{A.6}$$

$$= I_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t) + I_{[0,\infty)}\big(Y(t)-X(t)\big)\,dY(t) + d\Lambda_{X-Y}(t)$$
$$= I_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t) + I_{(0,\infty)}\big(Y(t)-X(t)\big)\,dY(t) + d\Lambda_{X-Y}(t), \quad \text{(A.7)}$$

where (A.6) follows from Definition 3.2, and (A.7) is implied by Lemma A.2. □

A similar result holds for the minimum function since $\min(x,y) = -\max(-x,-y)$.

The following proposition is the result we need in order to prove Theorem 3.1. The proposition shows that rank processes derived from pathwise mutually nondegenerate absolutely continuous semimartingales can be expressed in terms of the original processes, adjusted by local times.

Proposition A.1. *Let $X_1,\dots,X_n$ be pathwise mutually nondegenerate absolutely continuous semimartingales, and for $t \in [0,T]$, let p_t be the random permutation of $\{1,\dots,n\}$ such that for $k=1,\dots,n$,*

$$X_{p_t(k)}(t) = X_{(k)}(t), \quad \text{and} \quad p_t(k) < p_t(k+1) \quad \text{if} \quad X_{(k)}(t) = X_{(k+1)}(t).$$

Then the rank processes $X_{(k)}$, $k=1,\dots,n$, are continuous semimartingales such that a.s., for all $t \in [0,T]$,

$$dX_{(k)}(t) = \sum_{i=1}^{n} I_{\{0\}}\big(p_t(k)-i\big)\,dX_i(t) + \frac{1}{2}d\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{X_{(k-1)}-X_{(k)}}(t). \quad \text{(A.8)}$$

Proof. Since this proof is somewhat technical, it is convenient to explicitly show the dependence of all random variables and processes on $\omega \in \Omega$. By hypothesis and Lemma A.2, we can choose a subset $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that for $\omega \in \Omega'$, and $i,j,k \in \{1,\dots,n\}$, the following conditions will apply:

1. $X_i(t,\omega)$ is continuous in t;
2. for $i \neq j$, the set $\{t : X_i(t,\omega) = X_j(t,\omega)\}$ has Lebesgue measure 0;
3. for $i < j < k$, the set $\{t : X_i(t,\omega) = X_j(t,\omega) = X_k(t,\omega)\} = \varnothing$;
4. for $i \neq j$ and $t \in [0,T]$,
$$d\max\big(X_i(t,\omega),X_j(t,\omega)\big) = I_{(0,\infty)}\big(X_i(t,\omega)-X_j(t,\omega)\big)\,dX_i(t,\omega) + I_{(0,\infty)}\big(X_j(t,\omega)-X_i(t,\omega)\big)\,dX_j(t,\omega) + d\Lambda_{X_i-X_j}(t,\omega);$$
5. for $i \neq j$ and $t \in [0,T]$,
$$d\Lambda_{X_i-X_j}(t,\omega) = \frac{1}{2}d\Lambda_{|X_i-X_j|}(t,\omega).$$
6. for $k=1,\dots,n-1$ and $t \in [0,T]$,
$$d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) = I_{\{0\}}\big(X_{(k)}(t,\omega)-X_{(k+1)}(t,\omega)\big)\,d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega).$$

Conditions 1, 2, and 3 follow directly from Definitions A.1 and 3.3; we can impose condition 4 because of Lemma A.4; condition 5 follows from Lemma A.3; and condition 6 follows from (3.5).

Suppose that $\omega \in \Omega'$, $t_0 \in [0,T]$, and $k \in \{1,\ldots,n\}$, and let $m(\omega) = p_{t_0}(k,\omega)$. Then

$$X_{(k)}(t_0,\omega) = X_{m(\omega)}(t_0,\omega).$$

There are two cases we must consider. In the first case, for all $j \neq m(\omega)$,

$$X_j(t_0,\omega) \neq X_{m(\omega)}(t_0,\omega), \tag{A.9}$$

and in the second case, there is an $r(\omega) \neq m(\omega)$, $1 \leq r(\omega) \leq n$, such that

$$X_{r(\omega)}(t_0,\omega) = X_{m(\omega)}(t_0,\omega). \tag{A.10}$$

Let us consider the first case: Condition 1 implies that (A.9) continues to hold for all t in some neighborhood U of t_0. Hence, for all $t \in U$,

$$X_{(k-1)}(t,\omega) > X_{(k)}(t,\omega) > X_{(k+1)}(t,\omega),$$

so for all $t \in U$, $p_t(k,\omega) = m(\omega)$, and condition 6 implies that (A.8) reduces to

$$dX_{(k)}(t,\omega) = dX_{m(\omega)}(t,\omega).$$

Now consider the second case, so (A.10) holds. Then conditions 1 and 3 imply that there is a neighborhood U of t_0 such that for all $t \in U$, either

$$X_{(k-1)}(t,\omega) > X_{(k)}(t,\omega) \geq X_{(k+1)}(t,\omega) > X_{(k+2)}(t,\omega), \tag{A.11}$$

in which case for $t \in U$,

$$X_{(k)}(t,\omega) = \max(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)), \tag{A.12}$$

or,

$$X_{(k-2)}(t,\omega) > X_{(k-1)}(t,\omega) \geq X_{(k)}(t,\omega) > X_{(k+1)}(t,\omega), \tag{A.13}$$

in which case for $t \in U$,

$$X_{(k)}(t,\omega) = \min(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)). \tag{A.14}$$

Suppose that (A.11) and (A.12) hold. Then for $t \in U$,

$$\begin{aligned}
dX_{(k)}(t,\omega) &= d\max(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)) \\
&= I_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) \\
&\quad + I_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega) + d\Lambda_{X_{r(\omega)}-X_{m(\omega)}}(t,\omega) && \text{(A.15)} \\
&= I_{\{0\}}\big(p_t(k,\omega) - m(\omega)\big)\, dX_{m(\omega)}(t,\omega) \\
&\quad + I_{\{0\}}\big(p_t(k,\omega) - r(\omega)\big)\, dX_{r(\omega)}(t,\omega) + \frac{1}{2} d\Lambda_{|X_{r(\omega)}-X_{m(\omega)}|}(t,\omega) && \text{(A.16)} \\
&= I_{\{0\}}\big(p_t(k,\omega) - m(\omega)\big)\, dX_{m(\omega)}(t,\omega) \\
&\quad + I_{\{0\}}\big(p_t(k,\omega) - r(\omega)\big)\, dX_{r(\omega)}(t,\omega) + \frac{1}{2} d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega)
\end{aligned}$$

$$= \sum_{i=1}^{n} I_{\{0\}}\big(p_t(k,\omega) - i\big)\, dX_i(t,\omega) + \frac{1}{2} d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) - \frac{1}{2} d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega), \tag{A.17}$$

where condition 4 implies (A.15), (A.16) follows from (A.12) and condition 5, and (A.17) follows from condition 6. Hence, if (A.11) and (A.12) hold, (A.8) is valid.

The proof is similar when (A.13) and (A.14) hold, so (A.8) is valid for all $\omega \in \Omega'$. Since $P(\Omega') = 1$, the proposition is proved. □

References

Banz, R. (1981). The relationship between return and market value of common stocks. *Journal of Financial Economics 9*, 3–18.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy 81*, 637–654.

Breiman, L. (1961). Optimal gambling systems for favorable games. *Fourth Berkeley Symposium on Math. Statist. and Probability 1*, 65–78.

Carlen, E. and P. Protter (1992). On semimartingale decompositions of convex functions of semimartingales. *Illinois J. of Mathematics 36*, 420–427.

Cover, T. M. (1991). Universal portfolios. *Mathematical Finance 11*, 1–29.

Duffie, D. (1992). *Dynamic Asset Pricing Theory*. Princeton, NJ: Princeton University Press.

Fernholz, R. (1998, May/June). Crossovers, dividends, and the size effect. *Financial Analysts Journal 54*(3), 73–78.

Fernholz, R. (1999a). On the diversity of equity markets. *Journal of Mathematical Economics 31*(3), 393–417.

Fernholz, R. (1999b). Portfolio generating functions. In M. Avellaneda (Ed.), *Quantitative Analysis in Financial Markets*, River Edge, NJ. World Scientific.

Fernholz, R., R. Garvy, and J. Hannon (1998, Winter). Diversity-weighted indexing. *Journal of Portfolio Management 24*(2), 74–82.

Handa, P., S. P. Kothari, and C. Wasley (1989). The relation between the return interval and betas. *Journal of Financial Economics 23*, 79–100.

Harrison, J. M. and D. Kreps (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory 20*, 381–408.

Harrison, J. M. and S. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications 11*, 215–260.

Harrison, J. M. and S. Pliska (1983). A stochastic calculus model of continuous trading: complete markets. *Stochastic Processes and Their Applications 15*, 313–316.

Itô, K. (1951). On stochastic differential equations. *Memoirs of the American Mathematical Society 4*, 1–51.

Jegadeesh, N. (1992). Does market risk really explain the size effect. *Journal of Financial and Quantitative Analysis 27*, 337–351.

Karatzas, I. and S. Shreve (1991). *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag.

Karatzas, I. and S. Shreve (1998). *Methods of Mathematical Finance*. New York: Springer-Verlag.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance 7*, 77–91.

Markowitz, H. (1959). *Portfolio Selection*. New York: Wiley.

Merton, R. (1969). Lifetime portfolio selection under uncertainty: the continuous-time case. *Review of Economics and Statistics 51*, 247–257.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory 3*, 373–413.

Meyer, P. (1976). Un cours sur les intégrales stochastiques. In Séminaire de Probabilités X*, Lecture Notes in Mathematics*, Volume 511, pp. 245–398. Berlin: Springer-Verlag.

Protter, P. and J. San Martin (1993). General change of variable formulas for semimartingales in one and finite dimensions. *Probability Theory and Related Fields 97*, 363–381.

Reinganum, M. (1981). Misspecification of capital asset pricing: empirical anomalies based on earnings yields and market values. *Journal of Financial Economics 9*, 19–46.

Roll, R. (1981). A possible explanation of the small firm effect. *Journal of Finance 36*, 879–888.

Tanaka, H. (1963). Note on continuous additive functionals of the 1-dimensional Brownian path. *Zeitschrift für Wahrscheinlichkeitstheorie 1*, 251–257.