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EQUIVALENCE IN OPERATOR ALGEBRAS

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1. Introduction.

In [9; Def. 6.1.1] F. J. Murray and J. von Neumann introduce their well-known notion of "equivalence" between projections E and F in a von Neuman algebra $\mathscr{R}$. If $E=V^*V$ and $F=VV^*$ for some V in $\mathscr{R}$, they say that E is equivalent to F (and write $E\sim F$). Their comparison theory for projections in $\mathscr{R}$, dimension and additive trace functions, and type classification is, then, based on this notion.

In this paper, we introduce another equivalence (cf. Definition A), applicable to all positive operators, and use it to present an alternate development of the comparison theory, additive trace and type classification (cf. § 2 and 3). In § 4 we show that two projections E and F in $\mathscr{R}$, are equivalent (in the sense of Murray and von Neumann) if $E=\sum A_i^*A_i$ and $F=\sum A_iA_i^*$, with A_i in $\mathscr{R}$. This identifies our equivalence with that of Murray and von Neumann (and establishes a new property of Murray–von Neumann equivalence).

Since our equivalence has additivity "built into it" (it is the completely additive extension of Murray–von Neumann equivalence), an especially smooth comparison, trace and type theory can be developed using it. One learns, *a posteriori* (Theorem 4.1), that it coincides with the Murray–von Neumann theory. We emphasize that this approach does *not* banish the difficulties from the fundamentals of the subject; for the identification of the two theories employs the Murray–von Neumann additive trace. It does, however, provide a quick approach to an operative theory intermediate between that of Murray and von Neumann, where the additive trace appears at the end (so to speak), and that of Dixmier [1], [2], where it is assumed at the outset.

Although the main focus of this paper is on von Neumann algebras, the present development provides the opportunity to complete one aspect of the Kaplansky–Rickart program [7], [8], [12] of "algebraization"

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of the theory of von Neumann algebras. We define our equivalence in, and develop the comparison, trace and type theory for, monotone closed C^*-algebras (see § 2). Basically these algebras and Kaplansky's AW^*-algebras are more "algebraic" versions of von Neumann algebras (though they are both strictly larger classes of C^*-algebras). The AW^*-algebras are defined in terms of their idempotents and multiplicative properties — the monotone closed algebras in terms of additional (purely) order-theoretic properties. Since order properties have come to dominate detailed studies of C^*-algebras, the monotone closed algebras would seem to be the natural (current) choice for carrier of an algebraic theory of von Neumann algebras.

We employ the Dixmier books [1], [2], [3] as standard background references for C^*- and von Neumann algebras.

2. An equivalence relation.

Let $\mathfrak{A}$ be a monotone closed C^*-algebra, that is, a C^*-algebra in which every norm-bounded increasing net of self-adjoint elements has a least upper bound. We assume that $\mathfrak{A}$ has a unit, denoted by I (though its existence is a consequence of the remarks following Proposition 2.7).

For an increasing net $\{A_j\}$ in $\mathfrak{A}_{\text{s.a.}}$ with l.u.b. A in $\mathfrak{A}$ we write $A_j \nearrow A$ and $-A_j \searrow -A$. For any net $\{A_j\}$ in $\mathfrak{A}$, we write $A_j \to A$ if there are four decreasing nets $\{A_j^{(k)}\}$ in $\mathfrak{A}_{\text{s.a.}}$, $k=0,1,2,3$, such that (with $i=(-1)^{\frac{1}{2}}$)

$$A_j^{(k)} \searrow A^{(k)}, \quad \sum i^k A_j^{(k)} = A_j, \quad \sum i^k A^{(k)} = A .$$

We note that this is meant as a convenient notation; and we do not assign any topological features to it. If, however, $\mathfrak{A}$ is a von Neumann algebra, then $A_j \to A$ will, of course, imply that $\{A_j\}$ tends strongly to A. (It may be the case that the converse statement is valid.)

In the sequel we shall make repeated use of the polarization formula for the product of S and T in $\mathfrak{A}$:

$$(*) \qquad 4T^*S = \sum_{k=0}^{3} i^k (S+i^kT)^*(S+i^kT) .$$

LEMMA 2.1. *If $A_j \to A$ and $B_j \to B$, then $A_j+B_j \to A+B$. If $A_j \to A$, then $BA_j \to BA$ for any B in $\mathfrak{A}$.*

PROOF. For the first statement, it suffices to prove that if $A_j \searrow A$ and $B_j \searrow B$, then $A+B$ is the g.l.b. of the net $\{A_j+B_j\}$. But if $S \leqq A_j+B_j$, for all j, then, for each fixed j', we have $S \leqq A_j+B_{j'}$ for j larger than

j'. Hence $S-B_{j'}\leqq A$. Since j' is arbitrary, $S-A\leqq B$, whence $A_j+B_j\searrow A+B$.

To prove the second statement, we notice that, from the foregoing, it is enough to consider the case $A_j\searrow A$. Looking instead at the net $\{A_j-A\}$ we may assume $A=0$ (and B self-adjoint). By (*),

$$4BA_j = 4(BA_j^{\frac{1}{2}})A_j^{\frac{1}{2}} = \sum i^k(I+i^kB)^*A_j(I+i^kB)\,.$$

It follows that it is sufficient to prove that $A_j\searrow 0$ implies $C^*A_jC\searrow 0$ for any C in $\mathfrak{A}$. If C is unitary, then this holds because unitary transformations are order isomorphisms. In general we have $C=\sum_{k=0}^{3}a_kU_k$ with U_k unitary and a_k complex numbers. Appealing to the inequality $T^*S+S^*T\leqq S^*S+T^*T$, we have

$$C^*A_jC = (\sum\bar{a}_kU_k{}^*A_j^{\frac{1}{2}})(\sum a_kA_j^{\frac{1}{2}}U_k) \leqq 4\sum|a_k|^2U_k{}^*A_jU_k\,.$$

Hence $C^*A_jC\searrow 0$, completing the proof.

LEMMA 2.2. *If $A_j\to A$ and $\{A_j\}$ is uniformly convergent to B in $\mathfrak{A}$ then $B=A$.*

PROOF. Since $A_j=\sum i^kA_j^{(k)}$ and $A_j^{(k)}\searrow A^{(k)}$, we may assume, using $\{A_j-A\}$ and the nets $\{A_j^{(k)}-A^{(k)}\}$, that $A^{(k)}$ and A are 0. Multiplying by a suitably small positive constant, we may also assume that $\|A_j^{(k)}\|\leqq 1$ for all j and k. With ε positive and j sufficiently large, we have, using $A_j=\sum i^kA_j^{(k)}$, $T^*S+S^*T\leqq S^*S+T^*T$ and Lemma 2.1,

$$B^*B \leqq \varepsilon+A_j{}^*A_j \leqq \varepsilon+4\sum_kA_j^{(k)2} \leqq \varepsilon+4\sum_kA_j^{(k)}\searrow\varepsilon\,.$$

It follows that $B^*B\leqq\varepsilon$, hence $B=0$ $(=A)$.

The next statement establishes an extended version of the Polar Decomposition for a monotone closed $\mathfrak{A}$. With A taken to be $(B^*B)^{\frac{1}{2}}$, one arrives (in essence) at the standard Polar Decomposition.

PROPOSITION 2.3. *If $B^*B\leqq A^*A$, with B and A in $\mathfrak{A}$, there is a C in $\mathfrak{A}$ such that $B=CA$.*

PROOF. Let C_n be $B(n^{-1}I+A^*A)^{-1}A^*$. From (*) we have

$$4C_n = \sum_{k=0}^{3}i^k(A^*+i^kB^*)^*(n^{-1}I+A^*A)^{-1}(A^*+i^kB^*)$$

with $(n^{-1}I+A^*A)^{-\frac{1}{2}}B^*$ as T and $(n^{-1}I+A^*A)^{-\frac{1}{2}}A^*$ as S.

Noting that, with $|\theta|=1$,

$$\begin{aligned}
&\|(A+\theta B)(n^{-1}I+A^*A)^{-1}(A+\theta B)^*\| \\
&\quad = \|(n^{-1}I+A^*A)^{-\frac{1}{2}}(A^*A+\theta A^*B+\bar{\theta}B^*A+B^*B)(n^{-1}I+A^*A)^{-\frac{1}{2}}\| \\
&\quad \leqq \|(n^{-1}I+A^*A)^{-\frac{1}{2}}(A^*A+A^*A+(\theta B)^*\theta B+B^*B)(n^{-1}I+A^*A)^{-\frac{1}{2}}\| \\
&\quad \leqq 4\|(n^{-1}I+A^*A)^{-1}A^*A\| \leqq 4
\end{aligned}$$

(since $B^*B \leqq A^*A$ and $\|H\| \leqq \|K\|$ if $0 \leqq H \leqq K$) and that each of the four summands is increasing with n, there is a C in $\mathfrak{A}$ such that $C_n \to C$. However, $\{C_nA\}$ converges uniformly to B; for, with H_n taken as $I-(n^{-1}I+A^*A)^{-1}A^*A$,

$$\begin{aligned}
\|B-C_nA\|^2 = \|BH_n\|^2 = \|H_nB^*BH_n\| &\leqq \|H_nA^*AH_n\| \\
&= \|H_n(A^*A)^{\frac{1}{2}}\|^2 \to 0
\end{aligned}$$

as $n \to \infty$, from spectral theory. Thus, from Lemmas 2.1 and 2.2, it follows that $B=CA$.

Definition A: For S and T in $\mathfrak{A}^+$ we write $S \approx T$ if there is a set of elements $\{A_i\}$ in $\mathfrak{A}$ such that

$$S = \sum A_i^*A_i \quad \text{and} \quad T = \sum A_iA_i^* .$$

In other words, $S \approx T$ if S and T are the l.u.b. for the nets of finite sums of $A_i^*A_i$'s and $A_iA_i^*$'s respectively.

The following result is needed for further study of the relation $\approx$. It is an adaptation of [10; Prop. 1.1] to the present situation.

Proposition 2.4. *If $\{A_i\}$ and $\{B_j\}$ are sets in $\mathfrak{A}$ such that*

$$\sum A_iA_i^* = \sum B_j^*B_j \in \mathfrak{A}^+ ,$$

then there is a set $\{C_{ij}\}$ in $\mathfrak{A}$ such that

$$A_i^*A_i = \textstyle\sum_j C_{ij}^*C_{ij} \quad \text{and} \quad B_jB_j^* = \textstyle\sum_i C_{ij}C_{ij}^* .$$

Proof. Let S be $\sum A_iA_i^*$ $(=\sum B_j^*B_j)$ and $C_{ij}^{(n)}$ be $B_j(n^{-1}I+S)^{-\frac{1}{2}}A_i$. Note that, for each T in $\mathfrak{A}$,

$$\begin{aligned}
\|B_jTA_i\|^2 = \|B_jTA_iA_i^*T^*B_j^*\| &\leqq \|B_jTST^*B_j^*\| = \|B_jTS^{\frac{1}{2}}\|^2 \\
= \|S^{\frac{1}{2}}T^*B_j^*B_jTS^{\frac{1}{2}}\| &\leqq \|S^{\frac{1}{2}}T^*STS^{\frac{1}{2}}\| = \|S^{\frac{1}{2}}TS^{\frac{1}{2}}\|^2 .
\end{aligned}$$

Thus
$$\|C_{ij}^{(n)} - C_{ij}^{(m)}\| \leqq \|S^{\frac{1}{2}}[(n^{-1}I+S)^{-\frac{1}{2}} - (m^{-1}I+S)^{-\frac{1}{2}}]S^{\frac{1}{2}}\| .$$

The operators $(n^{-1}I+S)^{-\frac{1}{2}}S$ lie in the C^*-algebra (commutative) generated by I and S, and are represented (through the Spectral Theorem) by functions, on the spectrum of S, which are monotone increasing (with n) and tend pointwise to the function representing $S^{\frac{1}{2}}$. From Dini's Theorem $(n^{-1}I+S)^{-\frac{1}{2}}S$ tends uniformly (with n) to $S^{\frac{1}{2}}$.

Hence $\{C_{ij}^{(n)}\}$ is Cauchy convergent (in norm) to some C_{ij} in $\mathfrak{A}$. As $A_iA_i^* \leqq S$ and $B_j^*B_j \leqq S$, there are G_i and H_j in $\mathfrak{A}$ such that $A_i = S^{\frac{1}{2}}G_i$ and $B_j = H_jS^{\frac{1}{2}}$ (from Proposition 2.3). Then

$$C_{ij}^{(n)} = H_j(n^{-1}+S)^{-\frac{1}{2}}SG_i ,$$

which tends uniformly to $H_jS^{\frac{1}{2}}G_i$. Thus $C_{ij} = H_jA_i = B_jG_i$, and

$$A_i^*A_i = G_i^*SG_i = G_i^*(\sum B_j^*B_j)G_i = \sum_j C_{ij}^*C_{ij} ,$$
$$B_jB_j^* = H_jSH_j^* = H_j(\sum A_iA_i^*)H_j^* = \sum_i C_{ij}C_{ij}^* .$$

The foregoing proof produces the C_{ij} for all C^*-algebras; so that the assertion of 2.4 is valid when, for example, the convergences are uniform. It is valid, as well, for concretely represented C^*-algebras when the convergence is strong and the sums lie in the algebra. For the case of monotone closed C^*-algebras, alone, the last paragraph of the proof would suffice.

THEOREM 2.5. *The relation $\approx$ is an equivalence relation in $\mathfrak{A}^+$ which is completely additive, in the sense that $\sum S_i \approx \sum T_i$ when these sums exist and $S_i \approx T_i$.*

PROOF. The complete additivity is clear. To prove the transitivity, we assume $S \approx T$ and $T \approx R$, hence

$$S = \sum A_i^*A_i, \quad T = \sum A_iA_i^* = \sum B_j^*B_j, \quad R = \sum B_jB_j^* .$$

Using Proposition 2.4 on the equality $\sum A_iA_i^* = \sum B_j^*B_j$ we immediately get $S \approx R$.

The next proposition shows that the equivalence classes satisfy a strong Riesz Decomposition property.

PROPOSITION 2.6. *If $\{S_i\}$ and $\{T_j\}$ are sets in $\mathfrak{A}^+$ such that $\sum S_i$ and*

$\sum T_j$ are in $\mathfrak{A}^+$ with $\sum S_i \approx \sum T_j$, then there is a set $\{R_{ij}\}$ in $\mathfrak{A}^+$ such that $S_i \approx \sum_j R_{ij}$ and $T_j = \sum_i R_{ij}$.

PROOF. By assumption we have $\sum S_i = \sum A_k{}^* A_k$ and $\sum T_j = \sum A_k A_k{}^*$. Using Proposition 2.4 on the first equality we get a set $\{B_{ik}\}$ in $\mathfrak{A}$ such that

$$S_i = \sum_k B_{ik}^* B_{ik} \quad \text{and} \quad A_k A_k{}^* = \sum_i B_{ik} B_{ik}^* .$$

Since $\sum_{ik} B_{ik} B_{ik}^* = \sum_j T_j$, we can find a set $\{C_{ijk}\}$ in $\mathfrak{A}$ such that

$$B_{ik}^* B_{ik} = \sum_j C_{ijk}^* C_{ijk} \quad \text{and} \quad T_j = \sum_{ik} C_{ijk} C_{ijk}^* .$$

The elements $(R_{ij} =)\ \sum_k C_{ijk} C_{ijk}^*$ have the desired properties.

DEFINITION B. For S and T in $\mathfrak{A}^+$, we write $S \lesssim T$ if there is an R in $\mathfrak{A}^+$ with $S \approx R \leqq T$.

Note that $\lesssim$ is a partial ordering of the equivalence classes of $\mathfrak{A}^+$; for if $S \approx T_1 \leqq T$ and $T \approx R_0 \leqq R$, using Proposition 2.6,

$$T = T_1 + T_2 \approx R_0 = R_1 + R_2$$

with $T_1 \approx R_1$. Thus $S \approx R_1 \leqq R$.

PROPOSITION 2.7. *If $S \lesssim T$ and $T \lesssim S$, then $S \approx T$.*

PROOF. Since $S \lesssim T$, there is a set $\{A_i\}$ in $\mathfrak{A}$ with $S = \sum A_i{}^* A_i$ and $\sum A_i A_i{}^* \leqq T$. By Proposition 2.3 each A_i has the form $V_i S^{\frac{1}{2}}$ for some V_i in $\mathfrak{A}$. For any R in $\mathfrak{A}^+$ dominated by S, we define φ by:

$$\varphi(R) = \sum V_i R V_i{}^* \ \ (\leqq \sum V_i S V_i{}^* = \sum A_i A_i{}^*) .$$

Then φ is an order preserving affine map into the set of elements of $\mathfrak{A}^+$ dominated by T. Since $R^{\frac{1}{2}}$ is $G S^{\frac{1}{2}}$ for some G in $\mathfrak{A}$ (by Proposition 2.3), we have

$$\sum R^{\frac{1}{2}} V_i{}^* V_i R^{\frac{1}{2}} = \sum G S^{\frac{1}{2}} V_i{}^* V_i S^{\frac{1}{2}} G^* = G S G^* = R ,$$

while $\sum V_i R^{\frac{1}{2}} R^{\frac{1}{2}} V_i{}^* = \varphi(R)$. Thus $\varphi(R) \approx R$.

Since $T \lesssim S$, from the foregoing, there is an order and equivalence preserving affine map ψ from the elements in $\mathfrak{A}^+$ dominated by T into those dominated by S. If

$$S_0 = S, \qquad S_1 = \psi(T) ,$$
$$S_{2n} = (\psi\varphi)^n(S), \quad S_{2n+1} = (\psi\varphi)^n(S_1) ,$$

then $\{S_n\}$ is a decreasing sequence. Hence $S_n \searrow S_\infty$ in $\mathfrak{A}^+$. We have

$$S = \sum_0^\infty (S_{2n} - S_{2n+1}) + \sum_1^\infty (S_{2n-1} - S_{2n}) + S_\infty ,$$
$$S_1 = \sum_1^\infty (S_{2n} - S_{2n+1}) + \sum_1^\infty (S_{2n-1} - S_{2n}) + S_\infty .$$

Since $\psi\varphi(S_{2n} - S_{2n+1}) = S_{2n+2} - S_{2n+3}$, and $\psi\varphi(R) \approx R$, we conclude that $S \approx S_1$ ($\approx T$).

If $0 \leqq f_n(t) \nearrow 1$ for $0 < t \leqq 1$, $f_n(0) = 0$, and f_n is continuous and $0 \leqq S \leqq I$ with S in $\mathfrak{A}$, then $(f_n(S))$ is increasing and bounded by I. Thus $f_n(S) \nearrow [S]$. We refer to $[S]$ as *range projection* of S. From the Spectral Theorem and Dini's Theorem, $f_n(S)f_m(S)$ is increasing and tends in norm to $f_m(S)$, as $n \to \infty$. From Lemmas 2.1 and 2.2, $[S]f_m(S) = f_m(S)$. Thus $[S]^2 = [S]$; and $[S]$ is a projection. Similarly $[S]S = S$. If $AS = 0$, then $Af_n(S) = 0$ for all n since $f_n(S)$ is a uniform limit of polynomials in S without constant term; so that $A[S] = 0$.

For arbitrary A in $\mathfrak{A}$, we take the range projection $[A]$ of A to be that of $(AA^*)^{\frac{1}{2}}\|A\|^{-1}$. From Proposition 2.3, $A = (AA^*)^{\frac{1}{2}} V$, for some V in $\mathfrak{A}$; so that $[A]A = A$. If $GA = A$, for some projection G in $\mathfrak{A}$, then $(I-G)(AA^*)^{\frac{1}{2}} = 0$; and $(I-G)[A] = 0$. Hence $[A]$ can be characterized as the smallest projection in $\mathfrak{A}$ such that $[A]A = A$; and is independent of the sequence (f_n) used.

Having "range projections" in $\mathfrak{A}$, the techniques of [7], [12] produce "spectral resolutions" for self-adjoint elements of $\mathfrak{A}$. In particular, each such element is the norm limit of finite linear combinations of (mutually-orthogonal, spectral) projections in $\mathfrak{A}$.

If E and F are projections in $\mathfrak{A}$, we denote $[E+F]$ by $E \vee F$. Now, if K and H in $\mathfrak{A}$ are such that $0 \leqq K \leqq H$, then $[H]K^{\frac{1}{2}} = [H]H^{\frac{1}{2}} V = H^{\frac{1}{2}} V = K^{\frac{1}{2}}$, from Proposition 2.3; so that $[H]K = K$. Thus $E \vee F$ dominates both E and F. On the other hand, if the projection G in $\mathfrak{A}$ dominates both E and F, then $G(E+F) = E+F$; so that $E \vee F \leqq G$. Thus $E \vee F$ is the smallest projection in $\mathfrak{A}$ dominating E and F. Since $\mathfrak{A}$ is monotone closed, each family $\{E_j\}$ of projections in $\mathfrak{A}$ gives rise to a smallest projection $\bigvee_j E_j$ dominating all E_j — the l.u.b. of the monotone net of finite unions of projections in $\{E_j\}$.

Definition C. The *central carrier* C_A of an operator A in $\mathfrak{A}^+$ is the projection $\bigvee\{U^*[A]U \mid U \text{ unitary in } \mathfrak{A}\}$.

Since the family $\{U^*[A]U \mid U \text{ unitary in } \mathfrak{A}\}$ is a family of projections invariant under the automorphisms $G \to U^*GU$ of the lattice of projections in $\mathfrak{A}$, their l.u.b. C_A in this lattice is invariant under these automorphisms. Thus, since the unitary operators in $\mathfrak{A}$ generate $\mathfrak{A}$ (linearly), C_A lies in the center $\mathscr{C}$ of $\mathfrak{A}$. As $C_A[A]=[A]$ and each central projection which dominates $[A]$ must dominate $U^*[A]U$, for every unitary U in $\mathfrak{A}$, C_A is the smallest central projection such that $C_A A = A$.

LEMMA 2.8. *If E is a projection in $\mathfrak{A}$, $\mathscr{C}E$ is the center of $E\mathfrak{A}E$ and $CE \to CC_E$ is a normal isomorphism of $\mathscr{C}E$ onto $\mathscr{C}C_E$.*

PROOF. As (*-) isomorphisms between C^*-algebras are order isomorphisms, an isomorphism of one monotone closed algebra onto another is normal. Since $E\mathfrak{A}E$ and $\mathfrak{A}C_E$ are monotone closed, their centers are monotone closed (Lemma 2.1). It suffices to observe that $\mathscr{C}E$ and $\mathscr{C}C_E$ are the centers of $E\mathfrak{A}E$ and $\mathfrak{A}C_E$ respectively, and are isomorphic.

With C in $\mathscr{C}$, if $CE=0$, then $CU^*EU=0$, for each unitary U in $\mathfrak{A}$; so that $CC_E=0$ (of course, $CE=CC_EE=0$, if $CC_E=0$). Thus $CE \to CC_E$ is an isomorphism of $\mathscr{C}E$ onto $\mathscr{C}C_E$.

To show that $\mathscr{C}E$ is the center of $E\mathfrak{A}E$ (and, by replacing E with C_E, that $\mathscr{C}C_E$ is the center of $\mathfrak{A}C_E$), we shall make use of the fact that if A and B in $\mathfrak{A}^+$ are such that a projection N in $\mathfrak{A}$ dominates $[EAE]$ and $[EBE]$, then N dominates $E[A+B]E$. Multiplying by a suitable positive scalar, it will suffice to establish this when $A+B \leqq I$. As

$$0 \leqq (A+B)^n \leqq A+B \leqq I ,$$

for each positive integer n, and

$$NE(A+B)EN = E(A+B)E ,$$

N dominates $E(A+B)^nE$. Thus N dominates each $Ef_n(A+B)E$, as well as $E[A+B]E$, where $f_n(t)$ may be taken as $1-(1-t)^n$ for present purposes (cf. remarks following Proposition 2.7).

With N a projection in the center of $E\mathfrak{A}E$, $N \leqq C_NE$. On the other hand, C_N is the l.u.b. of

$$\{[\ldots[[U_1{}^*NU_1+U_2{}^*NU_2]+U_3{}^*NU_3]+\ldots+U_n{}^*NU_n] \mid U_1,\ldots,U_n \text{ unitary in } \mathfrak{A}\} .$$

Successive application of the foregoing observation shows that N dominates

$$E[\ldots[[U_1{}^*NU_1+U_2{}^*NU_2]+U_3{}^*NU_3]+\ldots+U_n{}^*NU_n]E ,$$

when we note that

$$EU^*NUE = EU^*ENEUE = NEU^*EUEN \leqq N,$$

for each unitary U in $\mathfrak{A}$. Thus $EC_NE = C_NE \leqq N$; and $C_NE = N$. Using spectral resolutions, each element of the center of $E\mathfrak{A}E$ has the form CE, with C in $\mathscr{C}$, completing the proof.

With $S \lessapprox T$ and C in $\mathscr{C}^+$, $CS \lessapprox CT$; so that $CS = 0$ if $CT = 0$. A version of the Comparison Theorem (see [1; Lemme 1, p. 227]) for monotone closed algebras follows.

PROPOSITION 2.9. *With S and T in $\mathfrak{A}^+$, there is a central projection P such that $PT \lessapprox PS$ and $(I-P)S \lessapprox (I-P)T$.*

PROOF. Let $\mathscr{F}$ be the family of sets of triples $\langle a, S_a, T_a \rangle$, where a is an ordinal, S_a and T_a are non-zero elements of $\mathfrak{A}^+$ such that

$$S_a \approx T_a, \quad \textstyle\sum_a S_a \leqq S, \quad \sum_a T_a \leqq T,$$

and the ordinals a appearing in a set in $\mathscr{F}$ form an initial segment of ordinals (if $b \leqq a$ and a appears then b appears). Note that the ordinals occurring in a set in $\mathscr{F}$ do not exceed the cardinal of $\mathfrak{A}$; for $\sum_a S_a \leqq S$ and each S_a is non-zero, so that at most a finite number of S_a are equal to a given element of $\mathfrak{A}$.

Zorn's Lemma applies, now, to the sets in $\mathscr{F}$, partially ordered by inclusion, to yield a maximal set (element) $\mathscr{S}_0 (= \{\langle a, S_a, T_a \rangle\})$ in $\mathscr{F}$. Let S_0 and T_0 be $S - \sum S_a$ and $T - \sum T_a$, respectively. Note that, with $\|A\|$ small,

$$B^*B \leqq S_0 \quad \text{and} \quad BB^* \leqq T_0,$$

where $B = T_0^{\frac{1}{2}} A S_0^{\frac{1}{2}}$. Adjoining $\langle a_0, B^*B, BB^* \rangle$ to $\mathscr{S}_0$, where a_0 is the first ordinal not occurring in the triples of $\mathscr{S}_0$, contradicts the maximality of $\mathscr{S}_0$ unless $B = 0$. Thus $T_0 A S_0 = 0$ for each A in $\mathfrak{A}$, and $T_0 A[S_0] = 0$. Since $T_0 U^*[S_0] U = 0$, for each unitary U in $\mathfrak{A}$, $T_0 C_{S_0} = 0$.

Let P be C_{S_0}. Then

$$0 = PT_0 = PT - \textstyle\sum PT_a,$$

$$0 = (I-P)S_0 = (I-P)S - \textstyle\sum (I-P)S_a,$$

$$PT = \textstyle\sum PT_a \approx \sum PS_a \leqq PS,$$

$$(I-P)S = \textstyle\sum (I-P)S_a \approx \sum (I-P)T_a \leqq (I-P)T,$$

completing the proof.

3. Equivalence in semi-finite algebras.

Throughout this section $\mathfrak{A}$ is a monotone closed C^*-algebra.

DEFINITION D. An element S in $\mathfrak{A}^+$ is said to be *finite* if $T \leqq S$ and $T \approx S$ implies $T = S$. When I is finite, we say that $\mathfrak{A}$ is finite.

THEOREM 3.1. *The finite elements of $\mathfrak{A}^+$ form a subcone invariant under equivalence. Moreover if S is finite and $T \lessapprox S$, then T is finite.*

PROOF. If S is finite and $T \approx S$ then T is finite. If namely $R \leqq T$ and $R \approx T$ then $R + (T - R) \approx S$. Hence by Proposition 2.6, we have $S = S_1 + S_2$ with $S_1 \approx R$, $S_2 \approx T - R$. Since $S_1 \leqq S$ and $S_1 \approx S$, we conclude that $S_1 = S$ and $S_2 = 0$. Hence $T = R$.

If S is finite and $T \leqq S$, then T is finite since $R \leqq T$ and $R \approx T$ implies $R + (S - T) \leqq S$ and $R + (S - T) \approx S$, hence $R = T$.

If S is finite, then aS is finite with $a \geqq 0$; and, more generally, AS is finite for each A in $\mathfrak{A}^+$ commuting with S, since $\|A\|^{-1} A\, S \leqq S$.

If S and T are both finite, then with P a central projection chosen as in Proposition 2.9, we have

$$P(S+T) \lessapprox 2PS \quad \text{and} \quad (I-P)(S+T) \lessapprox 2(I-P)T\,.$$

Since $2PS$ and $2(I-P)T$ are finite, we have proved that $P(S+T)$ and $(I-P)(S+T)$ are finite. However, these elements are centrally orthogonal. Hence their sum $S+T$ is also finite.

LEMMA 3.2. *If R in $\mathfrak{A}^+$ is finite and S and T are equivalent elements majorized by R, then $R - S \approx R - T$.*

PROOF. By Proposition 2.9 it suffices to consider the case $R - S \approx G \leqq R - T$.

Then $R = (R - S) + S \approx G + T \leqq R$. Hence $G + T = R$ and $R - S \approx R - T$.

LEMMA 3.3. *If $S_n \nearrow S$, $T_n \nearrow T$ and $S_n \approx T_n$ with all S_n finite, then $S \approx T$.*

PROOF. With $A_n = S_n - S_{n-1}$ and $B_n = T_n - T_{n-1}$ we have $A_n + S_{n-1} \approx B_n + T_{n-1} \approx B_n + S_{n-1}$. Using $A_n + B_n + S_{n-1}$ as R in Lemma 3.2 and $A_n + S_{n-1}$, $B_n + S_{n-1}$ in place of S and T, we conclude that $A_n \approx B_n$. It follows that $S = \sum A_n \approx \sum B_n = T$ (where $S_0 = T_0 = 0$).

PROPOSITION 3.4. *If E and F are finite projections in $\mathfrak{A}$ then $E \vee F$ is finite.*

Proof. Let G_n be $(n^{-1}I+E+F)^{-1}$, so that $G_n(E+F)\nearrow E\vee F$, from the remarks following Proposition 2.7. Then $S_n \approx T_n$, where

$$S_n = G_n(E+F) \;(=G_n^{\frac{1}{2}}EG_n^{\frac{1}{2}}+G_n^{\frac{1}{2}}FG_n^{\frac{1}{2}}) \quad \text{and} \quad T_n = EG_nE+FG_nF .$$

The sequence $\{T_n\}$ is increasing. Since G_n is dominated by both $(n^{-1}I+E)^{-1}$ and $(n^{-1}I+F)^{-1}$ (cf. [3; 1.6.8, p. 15]),

$$T_n \nearrow T \leqq E+F .$$

As $E+F$ is finite (Theorem 3.1),

$$E \vee F \approx T \leqq E+F$$

(Lemma 3.3); so that $E\vee F$ is finite.

Theorem 3.5. *If $\mathfrak{A}$ is finite then every element in $\mathfrak{A}^+$ is equivalent to a unique central element.*

Proof. If C_1 and C_2 are in $\mathscr{C}^+$ with $C_1 \lessapprox C_2$, then with P the range projection of $(C_1-C_2)_+$, we have $PC_2 \leqq PC_1 \lessapprox PC_2$. Hence $PC_2 = PC_1$ and thus $C_1 \leqq C_2$. It follows, in particular, that equivalent central elements are equal (in a finite $\mathfrak{A}$ — though the same is true of central *projections* without a "finiteness" restriction), which proves the uniqueness part of the theorem.

For S in $\mathfrak{A}^+$, if C in $\mathscr{C}^+$ is such that $C \lessapprox S$, then $C \lessapprox \|S\|I$. Hence $\|C\| \leqq \|S\|$. As in the proof of Proposition 2.9, there is a set $\{C_a\}$ in $\mathscr{C}^+$ such that $\sum C_a \lessapprox S$, and such that if $C+\sum C_a \lessapprox S$ with C in $\mathscr{C}^+$, then $C=0$. Thus, with $\sum C_a \approx S-S_0$ we can't have $\varepsilon P \lessapprox PS_0$ for a positive ε and a non-zero central projection P. By comparison (Proposition 2.9), $S_0 \lessapprox \varepsilon I$ (for each positive ε). We can therefore find S_n in $\mathfrak{A}^+$ such that $S_0 \approx S_n$ and $S_n \leqq 2^{-n}I$. But this gives

$$\sum S_n \approx \sum S_{2n} \leqq \sum S_n .$$

Hence $S_n = 0$ for all n, and so $S_0=0$, completing the proof.

The following theorems are best expressed in terms of a larger algebra which we shall describe below.

Let $\overline{\mathfrak{A}}$ be the set of formal sums of the form $\bar{A}=\sum P_nA_n$, where $A_n \in \mathfrak{A}$ and $\{P_n\}$ is a sequence of projections in $\mathscr{C}$ with $\sum P_n = I$. We identify elements $\bar{A}=\sum A_nP_n$ and $\bar{B}=\sum B_mQ_m$ for which $P_nQ_mA_n = P_nQ_mB_m$ for all n and m. For any finite number of elements from $\overline{\mathfrak{A}}$, we can then always arrange a representation with the same central projections. With the obvious definitions

$$\bar{A}+\bar{B} = \sum P_n(A_n+B_n) \quad \text{and} \quad \bar{A}\bar{B} = \sum P_n A_n B_n ,$$

we see that $\bar{\mathfrak{A}}$ becomes an algebra — the algebra of centrally unbounded elements on $\mathfrak{A}$.

We define

$$\bar{A} \geqq 0 \quad \text{if} \quad P_n A_n \geqq 0 \text{ for all } n;$$

$$\bar{A} \approx \bar{B} \quad \text{if} \quad P_n A_n \approx P_n B_n \text{ for all } n;$$

and we say that $\bar{A}$ is finite if all $P_n A_n$ are finite.

Definition E. A monotone closed C^*-algebra $\mathfrak{A}$ is *semi-finite* if each non-zero element of $\mathfrak{A}^+$ dominates some non-zero finite element. It is of type III if it has no non-zero finite elements.

Theorem 3.6. *Each monotone closed C^*-algebra $\mathfrak{A}$ contains a central projection P such that $P\mathfrak{A}$ is semi-finite and $(I-P)\mathfrak{A}$ is of type* III. *There is a finite projection E with central carrier P.*

Proof. Since $U^*[A]U=[U^*AU]$ for unitary U in $\mathfrak{A}$, the union P of the range projections of finite elements is central. If A in $P\mathfrak{A}^+$ is non-zero, $A[B]\neq 0$ for some finite B, since $0\neq A=AP$. Thus

$$0 \neq A^{\frac{1}{2}}BA^{\frac{1}{2}} \leqq \|B\|A;$$

and

$$A^{\frac{1}{2}}BA^{\frac{1}{2}} \approx B^{\frac{1}{2}}AB^{\frac{1}{2}} \leqq \|A\|B .$$

From Theorem 3.1, $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ is finite, and $P\mathfrak{A}$ is semi-finite. By construction of P, $(I-P)\mathfrak{A}$ contains no non-zero finite elements; that is, $(I-P)\mathfrak{A}$ is of type III.

For the second assertion let $\{E_j\}$ be a family of finite projections in $P\mathfrak{A}$ maximal with the property that $\{C_{E_j}\}$ is an orthogonal family. Then $P-\sum C_{E_j}$ $(=Q)$ dominates no finite projection other than 0. Using spectral projections and Theorem 3.1, Q dominates no non-zero finite element. Since $P\mathfrak{A}$ is semi-finite, $Q=0$. Thus $C_E=P$, where $E=\sum E_j$; and E is finite (for if $F\leqq E$ and $F\approx E$, then $C_{E_j}F\leqq E_j$ and $C_{E_j}F\approx E_j$; so that $C_{E_j}F=E_j$ and $F=E$).

If $\mathfrak{A}$ is semi-finite and $C_E=I$ with E a finite projection in $\mathfrak{A}$ (as above), $C\rightarrow CE$ is a normal isomorphism of the center $\mathscr{C}$ of $\mathfrak{A}$ onto the center $\mathscr{C}_E$ of $E\mathfrak{A}E$.

THEOREM 3.7. *If $\mathfrak{A}$ is semi-finite and E is a finite projection with central carrier I, then any finite element in $\overline{\mathfrak{A}}^+$ is equivalent to a unique element in $\overline{\mathscr{C}}_E{}^+$.*

PROOF. If $\bar{S}=\sum P_n S_n$ and $P_n S_n \approx \bar{C}_n E$ for all n, then $\bar{C}=\sum P_n \bar{C}_n \in \overline{\mathscr{C}}^+$ and $\bar{S} \approx \bar{C}E$. It follows that it is enough to prove the theorem for any finite S in $\mathfrak{A}^+$.

For each n there is (by Proposition 2.9) a maximal central projection Q_n such that $Q_n S \approx S_n \leqq n Q_n E$. The sequence $\{Q_n\}$ is increasing hence $Q_n \nearrow Q$. For each $\varepsilon > 0$ we have $(I-Q)E \lessapprox \varepsilon(I-Q)S$. Arguing as in the proof of Theorem 3.5, $(I-Q)E=0$, since S is finite. Thus $Q=I$. Now $S_n \leqq n Q_n E$; and, since $E\mathfrak{A}E$ is finite, we have $S_n \approx C_n E$, from Theorem 3.5. Letting P_n be $Q_n - Q_{n-1}$, we have $\bar{C}$ $(=\sum P_n C_n)$ in $\overline{\mathscr{C}}^+$ and $S \approx \bar{C}E$.

DEFINITION F. A *center-valued trace* on $\mathfrak{A}$ is a completely additive map Φ from the finite elements in $\overline{\mathfrak{A}}^+$ onto $\overline{\mathscr{C}}^+$, for which $\Phi(\bar{S})=\Phi(\bar{T})$ iff $\bar{S} \approx \bar{T}$ and such that $\Phi(\bar{C}\bar{S})=\bar{C}\Phi(\bar{S})$ for each finite $\bar{S}$ in $\overline{\mathfrak{A}}^+$ and each $\bar{C}$ in $\overline{\mathscr{C}}^+$.

THEOREM 3.8. *Each monotone closed, semi-finite C^*-algebra $\mathfrak{A}$ admits a center-valued trace Φ. Every other center-valued trace on $\mathfrak{A}$ is of the form $\bar{S} \to \Phi(\bar{C}\bar{S})$ with $\bar{C}$ invertible in $\overline{\mathscr{C}}^+$. If $\mathfrak{A}$ is finite, then there is a unique normalized $(\Phi(I)=I)$ center-valued trace on $\mathfrak{A}$.*

PROOF. Let E be a finite projection in $\mathfrak{A}$ with central carrier I. Employing Theorem 3.7, define $\Phi(\bar{S})$ to be $\bar{C}$, for each finite $\bar{S}$ in $\overline{\mathfrak{A}}^+$, where $\bar{C} \in \overline{\mathscr{C}}^+$ and $\bar{S} \approx \bar{C}E$. Since $\mathscr{C}_E$ and $\mathscr{C}$ are normally isomorphic (Lemma 2.8), Φ is a center-valued trace on $\mathfrak{A}$.

If Ψ is another center-valued trace, then $\bar{C}=\Psi(E)$ has no central zero-divisors; for if $C\bar{C}=0$, then $\Psi(CE)=0$, so that $CE=0$ and $C=0$. For each n, let Q_n be the maximal central projection such that $Q_n \leqq nQ_n\bar{C}$. Then $\{Q_n\}$ is increasing. Since $\bar{C}$ has no zero-divisors, $Q_n \nearrow I$. Since each $Q_n\bar{C}$ is invertible in $Q_n\mathscr{C}$, $\bar{C}$ is invertible in $\overline{\mathscr{C}}$, with

$$\bar{C}^{-1} = \sum (Q_n - Q_{n-1})(Q_n\bar{C})^{-1}.$$

For each finite $\bar{S}$ in $\overline{\mathfrak{A}}^+$ we have $\bar{S} \approx \Phi(\bar{S})E$. Hence

$$\Psi(\bar{S}) = \Psi(\Phi(\bar{S})E) = \Phi(\bar{S})\Psi(E) = \Phi(\bar{C}\bar{S}).$$

If $\mathfrak{A}$ is finite, then the construction of Φ with $E=I$ yields the normalized center-valued trace, completing the proof.

THEOREM 3.9. *If the center $\mathscr{C}$ of a monotone closed semi-finite C^*-algebra $\mathfrak{A}$ is a von Neumann algebra, then $\mathfrak{A}$ itself is a von Neumann algebra.*

PROOF. From [5; Theorem 1] it suffices to exhibit a separating family of normal functionals on $\mathfrak{A}$. By hypothesis, there is a separating family $\mathscr{F}$ of functionals on $\mathscr{C}$. For each non-zero S in $\mathfrak{A}^+$, there is a finite projection E in $\mathfrak{A}$ such that $ES \neq 0$. Without loss of generality, we may assume that E has central carrier I. Let Φ be the center-valued trace on $\mathfrak{A}$ for which $\Phi(E)=I$ (Theorem 3.8). Then

$$\|S\|I = \|S\|\Phi(E) \geqq \Phi(ESE) \neq 0$$

and $\Phi(ESE) \in \mathscr{C}^+$. By assumption there is an ω in $\mathscr{F}$ such that $\omega(\Phi(ESE)) \neq 0$. We define a normal functional φ on $\mathfrak{A}$ by letting $\varphi(T)$ be $\omega(\Phi(ETE))$. Since $\varphi(S) \neq 0$, the theorem follows.

See [4] for the corresponding result in finite AW^* algebras.

COROLLARY 3.10. *If a monotone closed, semi-finite C^*-algebra is a factor, then it is a von Neumann algebra.*

4. Equivalence of equivalences.

The equivalence relation $\approx$ introduced in § 2 (cf. Definition A) coincides with the Murray–von Neumann equivalence relation $\sim$ [9; Def. 6.1.1] when restricted to the set of projections in a von Neumann algebra. This fact is the substance of the theorem which follows. Throughout the proof of that theorem, the use of terms such as "finite" and "infinite" will refer to their meanings in the theory of von Neumann algebras (as opposed to their definitions in terms of $\approx$). The theorem will establish that the two senses of these terms are the same when applied to von Neumann algebras.

THEOREM 4.1. *If $\{A_i \mid i \text{ in } \mathscr{I}\}$ is a family of operators in the von Neumann algebra $\mathscr{R}$ and $E=\sum A_i^* A_i$, $F=\sum A_i A_i^*$ with E and F projections, then $E \sim F$.*

PROOF. If P is a central projection in $\mathscr{R}$, then $PE=\sum(PA_i)^*PA_i$ and $PF=\sum PA_i(PA_i)^*$. Thus PE and PF are related in $\mathscr{R}P$ as are E and F in $\mathscr{R}$. If we show that $PE \sim PF$ for each central projection P belonging to a family with union I, then $E \sim F$. Using cyclic central projections in place of P, we may assume that the center of $\mathscr{R}$ is countably decomposable. With C_E the central carrier of E in $\mathscr{R}$, $(I-C_E)E$, and hence, $(I-C_E)A_i$ is 0 for all i. Thus $(I-C_E)F=0$; and $C_E=C_F$. Restricting to this common central carrier, we may assume that $C_E=C_F=I$. We may also deal separately with the cases where E is finite and where PE is infinite for each non-zero central projection P (contained in C_E, that is, where E is "purely infinite").

Suppose, first, that E is finite. Since $C_E=I$, $\mathscr{R}$ is semi-finite [1; Prop. 8, p. 97; Prop. 8, p. 245] and admits a normal, faithful, semi-finite trace φ [1; Prop. 9, p. 98]. Let $\mathscr{M}$ be the two-sided ideal in $\mathscr{R}$ on the positive elements of which φ is finite [1; Prop. 1, p. 80]. If PE is in $\mathscr{M}$ for some central projection P, then $(PA_i)^*PA_i$ is in $\mathscr{M}$, for each i, since $0 \leqq (PA_i)^*PA_i \leqq PE$ and $PE(PA_i)^*PA_i=(PA_i)^*PA_i$. As PE and PF are the least upper bounds of the increasing nets of finite sums of $(PA_i)^*(PA_i)$ and $PA_i(PA_i)^*$, respectively, φ is finitely additive and normal, and

$$\varphi((PA_i)^*PA_i) = \varphi(PA_i(PA_i)^*)$$

[1; Cor. 1, p. 81], we conclude that $\varphi(PE)=\varphi(PF)$. Since P_0E and P_0F satisfy the same hypothesis, $\varphi(P_0E)=\varphi(P_0F)$ for each central subprojection P_0 of P. Choosing P_0 such that, say, $P_0E \precsim P_0F$, we have that $P_0E \sim F_0$ for some subprojection F_0 of P_0F. Thus $\varphi(P_0E)=\varphi(F_0)$; and $\varphi(P_0F-F_0)=0$. Since φ is faithful, $PF_0=F_0$; so that $P_0E \sim P_0F$. It follows from the Comparison Theorem [1; Theorem 1, p. 228] that $PE \sim PF$.

To complete the proof that $E \sim F$, under the assumption that E is finite, we produce a family of central projections $\{P_j\}$ such that P_jE is in $\mathscr{M}$ for all j, and such that $\sum P_j=I$. From the foregoing, then, $P_jE \sim P_jF$; so that $E \sim F$. Toward this end, let $\{P_j\}$ be a maximal orthogonal family of projections such that $P_jE \in \mathscr{M}$. If $I-\sum P_j$ $(=P)$ is non-zero, then PE is non-zero as $C_E=I$. Since E is finite we see from [1; Cor. 1, p. 318] that $E\mathscr{R}E$ is a finite algebra [1; Def. 1, p. 241]. The restriction of φ to $E\mathscr{R}E$ being semi-finite, there is [1; Prop. 10, p. 99] a non-zero element T in the center of $E\mathscr{R}E$ such that $T \leqq PE$ and $T \in \mathscr{M}$. With $\mathscr{C}$ the center of $\mathscr{R}$, $\mathscr{C}E$ is the center of $E\mathscr{R}E$. Hence a multiple of T majorizes a non-zero spectral projection of T, of the form QE with Q in $\mathscr{C}$. But QE in $\mathscr{M}$ contradicts the maximality of $\{P_j\}$. It follows that $\sum P_j=I$; and $\{P_j\}$ has the desired properties.

We suppose, now, that E is purely infinite. Then F is purely infinite, for if PF is finite, PE is finite, from the foregoing. If E (or F) is countably decomposable, then, since $C_E=C_F$, we have $E \sim F$ [6; Cor. 5, p.

320]. In the general case we must, however, appeal to a finer classification of infinite projections.

Assume that E and F are not equivalent. Since E and F satisfy the same hypotheses, we may suppose, without loss of generality, that $Q_0E \prec Q_0F$ for each non-zero central subprojection Q_0 of some non-zero central projection Q (employing the Comparison Theorem). Restricting attention to $\mathscr{R}Q$, we may assume that $PE \prec PF$ for each non-zero central projection P.

With $\mathscr{C}$ the center of $\mathscr{R}, \mathscr{C}E$ is the center of $E\mathscr{R}E$. Since $C_E = I$, the mapping $C \to CE$ of $\mathscr{C}$ onto $\mathscr{C}E$ is an isomorphism. From [6; Lemma 4.1.3], there is a (unique) family $\{P_aE\}$ of central projections in $E\mathscr{R}E$ such that either $P_a = 0$ or $E\mathscr{R}EP_a$ has coupling character a, and $\sum P_a = I$. Restricting attention to $\mathscr{R}P_e$, for some non-zero P_e, we may assume that $E\mathscr{R}E$ has coupling character e. Since cyclic projections in $E\mathscr{R}E$ (cyclic under $\mathscr{R}'E$) are cyclic subprojections of E, E is the sum of a family $\{M_j\}$ of equivalent, orthogonal, cyclic projections. The family $\{M_j\}$ has cardinality e, and no such family with smaller cardinality has sum E. As E is purely infinite, a maximal cyclic projection in $E\mathscr{R}E$ is maximal cyclic in $\mathscr{R}$ [6; Remarks on p. 340, Lemmas 3.3.3–3.3.6]; so that the projections M_j may be chosen maximal cyclic in $\mathscr{R}$ [6; Lemma 4.1.4]. Applying these considerations to $F\mathscr{R}F$ and restricting to one of the central projections in $\mathscr{R}$ arising from the coupling character decomposition of $F\mathscr{R}F$ (as we did with P_e), we may assume that F is the sum of an orthogonal family $\{N_k\}$ of maximal cyclic projections. Since maximal cyclic projections are equivalent [6; p. 340], equivalence of projections is additive on orthogonal families [1; p. 225, last paragraph], and $E \prec F$, we have that $e < f$, where f is the cardinality of $\{N_k\}$.

Suppose $E = \bigvee G_h$, where G_h is a cyclic projection in $\mathscr{R}$ with generating vector x_h. If $A_ix_h = 0$, then $A_iA'x_h = 0$, for each A' in $\mathscr{R}'$; so that $A_iG_h = 0$. Since

$$0 \neq A_i = A_iE = A_i(\bigvee G_h)$$

(recall, $0 \leqq A_i^*A_i \leqq E$; so that A_i^* has range projection E_i majorized by E, from which, $EA_i^* = A_i^*$), $A_iG_h \neq 0$ for some h. However, the set $\mathscr{I}_h$ of such i is countable; since

$$\|Ex_h\|^2 = (Ex_h, x_h) = \sum_i \|A_ix_h\|^2 .$$

Since $\mathscr{I} = \bigvee \mathscr{I}_h$ and each $\mathscr{I}_h$ is countable, $b \leqq a\,\aleph_0$, where $\mathscr{I}$ has cardinality b and $\{G_h\}$ has cardinality a. If $\aleph_0 \leqq a$, then $b \leqq a$.

Applying this conclusion to the case where $\{A_i\}$ is an orthogonal family of projections (so that $\sum A_i^* A_i$ is a projection, and, in particular, converges) and $a = \aleph_0$, we see that $b \leqq \aleph_0$. We conclude that a projection which is a countable union of cyclic projections is countably decomposable.

Replacing $\{G_h\}$ by $\{M_j\}$, above, we see that $b \leqq e\,\aleph_0$. If x_j is a generating vector for M_j, then $E_i x_j$ generates a subprojection E_{ij} of E_i. If x is a vector in the range of E_i for which $E_{ij}x = 0$ for all j; then $0 = (A'E_i x_j, x) = (E_i A' x_j, x)$, for each A' in $\mathscr{R}'$. Thus $M_j x = 0$, for all j; and

$$x = E_i x = Ex = \sum M_j x = 0 .$$

It follows that each E_i is the union of e cyclic projections E_{ij} (some, possibly, 0). Since $0 \leqq A_i A_i^* \leqq F$, the range projection F_i of A_i is a subprojection of F. From [1; Prop. 2, p. 226], $E_i \sim F_i$; so that each F_i is the union of e cyclic projections. Now, $E = \bigvee E_i$ and $F = \bigvee F_i$, since $E = \sum A_i^* A_i$ and $F = \sum A_i A_i^*$. Thus F is the union of $b\,e$ cyclic projections. As $F = \sum N_k$ and $\{N_k\}$ has cardinality f, replacing $\{A_i\}$ by $\{N_k\}$, above, we have that $f \leqq b\,e\,\aleph_0$. Moreover $b \leqq e\,\aleph_0$, as noted; so that $f \leqq e^2\,\aleph_0^2$. If e is finite and $f \leqq \aleph_0$, F is a countable union of cyclic projections, and F as well as E are countably decomposable. Since $C_E = C_F = I$ and both E and F are purely infinite and countably decomposable, $E \sim F$, from [6; Lemma 3.3.3]. If e is infinite, $e^2\,\aleph_0^2 = e$; and $f \leqq e$. In either case, e finite or infinite, we contradict our initial arrangement $E \prec F$; and $E \sim F$ completing the proof of Theorem 4.1.

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