

EQUIVALENCE OF BANACH SPACE-VALUED ORNSTEIN-UHLENBECK PROCESSES

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ABSTRACT. Let μ_A be the law of the Ornstein-Uhlenbeck process that solves the equation

$$d\xi_t = A\xi_t dt + dW(t), \quad \xi_0 = 0 \text{ almost surely,}$$

where A is a generator of a C_0 -semigroup on a Banach space B and $W(t)$, $t \geq 0$, is a cylindrical Wiener process on a Hilbert subspace H of B . For two 'drift' operators A_0 and A_1 we study sufficient and necessary conditions for the measures μ_{A_0} and μ_{A_1} to be equivalent.

1. INTRODUCTION

Suppose $\mathcal{S}_0 = \{S_0(t)\}_{t \geq 0}$ and $\mathcal{S}_1 = \{S_1(t)\}_{t \geq 0}$ are two C_0 -semigroups of bounded linear operators on a separable real Hilbert space H , which is continuously embedded into some real Banach space B . Let $\xi^0 = \{\xi_t^0\}_{t \geq 0}$ and $\xi^1 = \{\xi_t^1\}_{t \geq 0}$ be B -valued Ornstein-Uhlenbeck processes with drifts \mathcal{S}_0 and \mathcal{S}_1 , respectively; see Section 4 for the definition. One could think of a B -valued Ornstein-Uhlenbeck process with drift \mathcal{S} as a B -valued solution of the stochastic differential equation (SDE)

$$d\xi_t = A\xi_t dt + dW_t, \quad \xi_0 = 0 \text{ almost surely,}$$

where A is the generator of \mathcal{S} and dW is H -valued white noise.

In this paper we study the problem under which conditions the processes ξ^0 and ξ^1 are 'equivalent'. In order to give the notion of 'equivalence' a precise mathematical meaning, we use the concept of an Ornstein-Uhlenbeck measure. A probability measure ν on the product space $B^{[0,T]}$ will be called a *Ornstein-Uhlenbeck measure* with drift \mathcal{S} if the *canonical process* on $(B^{[0,T]}, \nu)$ defined by

$$\xi_t(\omega) = \omega(t) \quad (t \in [0, T], \omega \in B^{[0,T]}),$$

is a B -valued Ornstein-Uhlenbeck process with drift \mathcal{S} . To every B -valued Ornstein-Uhlenbeck process with drift \mathcal{S} there corresponds a unique canonical Ornstein-Uhlenbeck process with drift \mathcal{S} , and thus a unique Ornstein-Uhlenbeck measure ν on $B^{[0,T]}$. We now define two Ornstein-Uhlenbeck processes ξ^0 and ξ^1 to be *equivalent* if the corresponding Ornstein-Uhlenbeck measures ν_0 and ν_1 are equivalent, i.e. if ν_0 and ν_1 are absolutely continuous with respect to each other.

Denoting by $D(A)$ the domain of an operator A , and by $\|\cdot\|_{\mathcal{L}_2(H)}$ the Hilbert-Schmidt norm of an operator on H , our main results can be formulated as follows:

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Theorem 1.1. *Let ξ^0 and ξ^1 be two B -valued Ornstein-Uhlenbeck processes, with drifts \mathcal{S}_0 and \mathcal{S}_1 respectively, and let ν_0 and ν_1 be the associated Ornstein-Uhlenbeck measures on $B^{[0,T]}$. Consider the following three conditions:*

- (i) $D(A_0) = D(A_1)$, for each $t \in (0, T]$ the linear operator $h \mapsto (A_0 - A_1)S_0(t)h$, defined on the common domain of A_0 and A_1 , extends to a bounded operator on H , and

$$\int_0^T \|(A_0 - A_1)S_0(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty;$$

- (ii) $D(A_0) = D(A_1)$, for each $t \in (0, T]$ the linear operator $h \mapsto (A_1 - A_0)S_1(t)h$, defined on the common domain of A_0 and A_1 , extends to a bounded operator on H , and

$$\int_0^T \|(A_1 - A_0)S_1(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty;$$

- (iii) *The measures ν_0 and ν_1 are equivalent.*

Then (i) and (ii) are equivalent and imply (iii).

In the converse direction, with the same notation as in Theorem 1.1 we have:

Theorem 1.2. *If ν_0 and ν_1 are equivalent, then $D(A_0) = D(A_1)$. If furthermore for each $t \in (0, T]$ the operator $(A_0 - A_1)S_0(t)$, defined on the common domain of A_0 and A_1 , has a bounded extension to H , such that*

$$\int_0^T \|(A_0 - A_1)S_0(t)\| dt < \infty,$$

then for each $t \in (0, T]$ the operators $(A_0 - A_1)S_0(t)$ and $(A_1 - A_0)S_1(t)$ extend to Hilbert-Schmidt operators on H and

$$\int_0^T \|(A_0 - A_1)S_0(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty \quad \text{and} \quad \int_0^T \|(A_1 - A_0)S_1(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty.$$

For certain selfadjoint elliptic operators, equivalence of Hilbert space-valued Ornstein-Uhlenbeck semigroups was first studied by Kozlov [5]. His work was extended by Peszat [9], who considered the case where $B = H$, assuming that

$$\int_0^T \|S_j(t)\|_{\mathcal{L}_2(H)}^2 dt < \infty \quad (j = 0, 1),$$

along with somewhat stronger integrability conditions on the semigroups \mathcal{S}_0 and \mathcal{S}_1 ; see also [3, Chapter 10]. Equivalence of B -valued Ornstein-Uhlenbeck processes corresponding to generators of analytic semigroups has been considered by R\"ockle [11].

Theorems 1.1 and 1.2 are proved in Section 4 and depend on some perturbation results for C_0 -semigroups and equivalence results for Gaussian measures, which are presented in Sections 2 and 3 respectively.

2. A PERTURBATION RESULT

In this preliminary section we will prepare two perturbation results for C_0 -semigroups that will be needed later on. The first of these, which was also used in [9], may be found in [4, Theorem 3.4.1, p. 399; Corollary 1, p. 400, Theorem 13.5.2, p. 409]. We assume the reader to be familiar with the elementary concepts of semigroup theory as can be found in e.g. [8], [4].

Proposition 2.1. *Suppose that A_0 and A_1 are the generators of C_0 -semigroups \mathcal{S}_0 and \mathcal{S}_1 on a Banach space X such that the following three conditions hold:*

- (i) $D(A_0) = D(A_1)$;
- (ii) For each $t \in (0, T]$ the linear operator $(A_0 - A_1)S_0(t)$ has a bounded extension to X ;
- (iii) $\int_0^T \|(A_0 - A_1)S_0(t)\| dt < \infty$,

Then, symmetrically, we have

- (ii)' For each $t \in (0, T]$ the linear operator $(A_1 - A_0)S_1(t)$ has a bounded extension to X ;
- (iii)' $\int_0^T \|(A_1 - A_0)S_1(t)\| dt < \infty$.

Moreover, we have the following 'variation of constants' formula:

$$(2.1) \quad S_1(t)x = S_0(t)x + \int_0^t S_0(t-s)(A_1 - A_0)S_1(s)x ds, \quad x \in X.$$

Let X be a Banach space and let $\mathcal{I} \subset \Lambda(X)$ be an *operator ideal*, i.e. \mathcal{I} is a linear subspace of $\Lambda(X)$ which is a Banach space with respect to a norm $\|\cdot\|_{\mathcal{I}}$ which is stronger than the operator norm on $\Lambda(X)$, such that

$$\|S_0 T S_1\|_{\mathcal{I}} \leq \|S_0\| \|T\|_{\mathcal{I}} \|S_1\|, \quad S_0, S_1 \in \Lambda(X), T \in \mathcal{I}.$$

Proposition 2.2. *Suppose A_0 and A_1 are generators of C_0 -semigroups \mathcal{S}_0 and \mathcal{S}_1 on a Banach space X such that $D(A_0) = D(A_1)$. Let \mathcal{I} be an operator ideal in $\Lambda(X)$. Then the following assertions are equivalent:*

- (i) For each $t \in (0, T]$ the operator $(A_0 - A_1)S_0(t)$ extends to an element of \mathcal{I} and we have

$$\int_0^T \|(A_0 - A_1)S_0(t)\|_{\mathcal{I}}^2 dt < \infty;$$

- (ii) For each $t \in (0, T]$ the operator $(A_0 - A_1)S_1(t)$ extends to an element of \mathcal{I} and we have

$$\int_0^T \|(A_0 - A_1)S_1(t)\|_{\mathcal{I}}^2 dt < \infty.$$

PROOF. We prove that (i) implies (ii); the converse implication follows by symmetry. By (i), the fact that $\|\cdot\|_{\mathcal{I}}$ is stronger than the operator norm and by the Cauchy-Schwartz inequality,

$$\int_0^T \|(A_0 - A_1)S_0(s)\| ds < \infty.$$

Hence by Proposition 2.1,

$$\int_0^T \|(A_0 - A_1)S_1(s)\| ds < \infty.$$

Therefore, by [4, Lemma 13.3.5, p. 395] for all $t \in [0, T]$ and $x \in X$ we have

$$(A_0 - A_1) \int_0^t S_0(t-s)(A_1 - A_0)S_1(s)x ds = \int_0^t (A_0 - A_1)S_0(t-s)(A_1 - A_0)S_1(s)x ds.$$

By this identity, (2.1), and the Young inequality,

$$\begin{aligned} & \int_0^T \|(A_1 - A_0)(S_1(t) - S_0(t))\|_{\mathcal{I}}^2 dt \\ & \leq \int_0^T \left(\int_0^t \|(A_1 - A_0)S_0(t-s)\|_{\mathcal{I}} \|(A_1 - A_0)S_1(s)\| ds \right)^2 dt \\ & \leq \int_0^T \|(A_1 - A_0)S_0(s)\|_{\mathcal{I}}^2 ds \left(\int_0^T \|(A_1 - A_0)S_1(s)\| ds \right)^2 < \infty. \end{aligned}$$

By combining this estimate with (i), condition (ii) immediately follows. \square

3. EQUIVALENCE OF GAUSSIAN MEASURES

In this section we briefly recall some well-known facts concerning (cylindrical) Gaussian measures. For more details we refer to [14], [15], [12], [13], [6].

Let E be a real locally convex topological vector space, with topological dual E' . A subset C of E is said to be a *cylindrical set* if it is contained in the algebra $\mathcal{C}(E)$ of subsets of E generated by all sets of the form $\{x \in E : (\langle x, x'_1 \rangle, \dots, \langle x, x'_n \rangle) \in B\}$ for some $n \geq 1$, $x'_1, \dots, x'_n \in E'$, and Borel set $B \subset \mathbb{R}^n$. A *centered cylindrical Gaussian measure on E* is a finitely additive set function μ on $\mathcal{C}(E)$, whose images under the maps $x \mapsto (\langle x, x'_1 \rangle, \dots, \langle x, x'_n \rangle)$ are σ -additive Gaussian measures on \mathbb{R}^n .

If F is another real locally convex topological vector space, and if $T : E \rightarrow F$ is a continuous linear transformation, then the image $T(\mu) := \mu \circ T^{-1}$ of a centered cylindrical Gaussian measure on E is a centered cylindrical Gaussian measure on F .

Let H be a real Hilbert space. By γ_H we denote the *standard centered cylindrical Gaussian measure* on H , i.e. the centered cylindrical Gaussian measure on H whose image under any of the maps $g \mapsto ([g, h_1]_H, \dots, [g, h_n]_H)$ is the standard Gaussian measure on \mathbb{R}^n .

A continuous linear operator $Q \in \Lambda(E', E)$ is called *positive* if $\langle Qx', x' \rangle \geq 0$ for all $x' \in E'$, and *symmetric* if $\langle Qx', y' \rangle = \langle Qy', x' \rangle$ for all $x' \in E'$ and $y' \in E'$. To every positive symmetric operator $Q \in \Lambda(E', E)$ one can associate a real Hilbert space H_Q in the following way. On the range of Q one has a well-defined inner product $[\cdot, \cdot]_H$ given by

$$[Qx', Qy'] := \langle Qx', y' \rangle \quad (x', y' \in E').$$

Denote by H_Q the Hilbert space completion of range Q with respect to this inner product; this Hilbert space is called the *reproducing kernel Hilbert space* (RKHS) associated with Q . If E is quasi-complete, then the inclusion mapping from range Q into E has a continuous extension to an injective linear map $i : H_Q \rightarrow E$. In this way, the pair (i, H_Q) becomes a Hilbert subspace of E . Moreover, upon identifying H_Q with its dual in the natural way, we then have the operator identity $Q = i \circ i'$.

Conversely, if (i, H) is a real Hilbert subspace of E (i.e. i is a continuous injective linear map from some real Hilbert space H into E), then $Q := i \circ i' \in \mathcal{L}(E', E)$ is positive and symmetric, and its RKHS equals H .

The relationship between centered cylindrical Gaussian measures and positive symmetric operators is described in the following well-known result.

Proposition 3.1. *Let E be a real locally convex topological vector space.*

- (i) *Let H be a real Hilbert space and let $T \in \Lambda(H, E)$. The image cylindrical measure $\mu := T(\gamma_H)$ is a centered cylindrical Gaussian measure on E , whose Fourier transform is given by*

$$\int_E \exp(i\langle x, x' \rangle) d\mu(x) = \exp\left(-\frac{1}{2}\langle T \circ T'x', x' \rangle\right) \quad (x' \in E').$$

Moreover, the RKHS H_Q associated with the positive symmetric operator $Q = T \circ T' \in \Lambda(E', E)$ is the orthogonal complement in H of the kernel of T . As such, H_Q can be identified with range T , the inner product being given by

$$[Tg, Th]_{H_Q} = [Pg, Ph]_H,$$

where P is the orthogonal projection in H onto $(\ker T)^\perp$.

- (ii) If E is quasi-complete and $Q \in \Lambda(E', E)$ is positive and symmetric, and if μ is a centered cylindrical Gaussian measure on E with Fourier transform

$$\int_E \exp(i\langle x, x' \rangle) d\mu(x) = \exp\left(-\frac{1}{2}\langle Qx', x' \rangle\right) \quad (x' \in E'),$$

then $\mu = i(\gamma_H)$, where H is the RKHS of Q and $i : H \hookrightarrow E$ is the natural embedding.

Let E be a real locally convex topological vector space. A measure μ on the σ -algebra $\sigma(\mathcal{C}(E))$ generated by the algebra $\mathcal{C}(E)$ is called a (centered) *Gaussian measure on E* if for all $x' \in E'$ the image measure $\langle \mu, x' \rangle := \mu \circ (x')^{-1}$ is a (centered) Gaussian Borel measure on \mathbb{R} . A continuous linear operator $T \in \Lambda(H, E)$ is said to be γ_H -*radonifying* (or, if no confusion is possible, *radonifying*) if the image cylindrical measure $T(\gamma_H)$ has a (necessarily unique) countably additive extension to a Gaussian measure on E .

If E is separable, then $\sigma(\mathcal{C}(E))$ is just the Borel σ -algebra of E , but in general $\sigma(\mathcal{C}(E))$ is much smaller.

The following three examples of radonifying operators will be of importance:

- If μ is a centered Gaussian measure on E with RKHS H , then the inclusion map $i : H \rightarrow E$ is radonifying, and we have $i(\gamma_H) = \mu$.
- If H and E are Hilbert spaces, then $T \in \Lambda(\mathcal{H}, E)$ is radonifying if and only if T is a Hilbert-Schmidt operator.
- If G and H are Hilbert spaces and $S \in \Lambda(G, H)$ and $T \in \Lambda(H, E)$ are continuous linear operators, then $T \circ S$ is radonifying whenever T is radonifying [1], [10].

Two measures μ_0 and μ_1 defined on the same σ -algebra are said to be *equivalent*, notation $\mu_0 \sim \mu_1$, if they are absolutely continuous with respect to each other, i.e. if $\mu_0 \ll \mu_1$ and $\mu_1 \ll \mu_0$. They are said to be *singular*, notation $\mu_0 \perp \mu_1$, if there exist disjoint measurable sets B_0 and B_1 such that $\mu_0(B_0) = \mu_1(B_1) = 1$. Two Gaussian measures on a real locally convex topological vector space E are either equivalent or singular. This result is due Vakhania and Tarieladze [14], who also give necessary and sufficient conditions for equivalence. In case both μ_0 and μ_1 arise as images under radonifying operator, their characterization assumes the following form.

Proposition 3.2. *Let H be a quasi-complete real Hilbert space, let E be a real locally convex topological vector space E , and let $T_0, T_1 \in \Lambda(H, E)$ be radonifying. Let μ_j be the centered Gaussian measure on E which extends $T_j(\gamma_H)$ ($j = 0, 1$). Let $Q_j = T_j \circ T_j'$ be the covariance operator of μ_j and let H_j denote its RKHS. Then $\mu_0 \sim \mu_1$ if and only if the following two conditions hold:*

- (i) *As subsets of E we have $H_0 = H_1$;*
- (iii) *The map $Q_0x' \mapsto Q_0x' - j_{H_1, H_0}Q_1x'$ ($x' \in E'$) extends to a Hilbert-Schmidt operator on H_0 ; here $j_{H_1, H_0} : H_1 \rightarrow H_0$ is the identity map.*

4. EQUIVALENCE OF ORNSTEIN-UHLENBECK PROCESSES

Throughout the rest of the paper, B is a separable real Banach space, H is real Hilbert subspace of B .

Consider a C_0 -semigroup $\mathcal{S} = \{S(t)\}_{t \geq 0}$ of bounded linear operators on H , with generator A . We assume that for $t > 0$ the linear operator $S(t)$ maps H into B . The resulting operator will be denoted by $S_B(t)$. We suppose that

$$\int_0^T \|S_B^*(t)x^*\|_H^2 dt < \infty, \quad \forall x^* \in B^*.$$

A B -valued stochastic process $\xi = \{\xi_t\}_{t \in [0, T]}$ on some probability space (Ω, \mathbb{P}) is called an B -valued Ornstein-Uhlenbeck process (with drift \mathcal{S}) if for all $t, s \in [0, T]$ and $x^*, y^* \in B^*$ (the Banach space dual of B) we have

$$\mathbb{E}(\langle \xi_t, x^* \rangle \langle \xi_s, y^* \rangle) = \int_0^{t \wedge s} [S^*(t-u) i^* x^*, S^*(s-u) i^* x^*]_H du.$$

Notice that some authors call a process with this property a B -valued Ornstein-Uhlenbeck process with drift A .

Associated with \mathcal{S} is a positive symmetric operator $Q_T \in \Lambda(H)$ defined by

$$Q_T h := \int_0^T S(s) S^*(s) h ds \quad (h \in H).$$

The RKHS associated with Q_T will be denoted by H_T , and the inclusion mapping of H_T into H by i_T . Thus, $Q_T = i_T \circ i_T^*$. We assume that $H_T \subset B$ and denote by j_T the corresponding inclusion map of H_T into B .

It is easy to see (cf. Proposition 2.3 in [2]) that the existence of a B -valued Ornstein-Uhlenbeck process with drift \mathcal{S} implies that the inclusion map $j_T : H_T \hookrightarrow B$ is radonifying. By the results in [2] this implies the existence of an Ornstein-Uhlenbeck measure ν with drift \mathcal{S} on the product σ -algebra of the (quasi-complete) product space $B^{[0, T]}$, i.e. a probability measure ν such that the canonical process $\omega \mapsto \omega(t)$ on the probability space $(B^{[0, T]}, \nu)$ is a B -valued Ornstein-Uhlenbeck process with drift \mathcal{S} . It turns out that the measure ν arises as the image under the (radonifying) map $k \circ \Phi : L^2([0, T]; H) \rightarrow B^{[0, T]}$, where $k : C_0([0, T], B) \rightarrow B^{[0, T]}$ is the natural inclusion map and $\Phi : L^2([0, T]; H) \rightarrow C_0([0, T], B)$ is the convolution operator defined by

$$(4.1) \quad (\Phi f)(t) = \int_0^t S_B(t-s) f(s) ds \quad (t \in [0, T], f \in L^2([0, T]; H)).$$

It is further shown in [2] that the process ξ has a continuous version if \mathcal{S} is an analytic semigroup. In this case, the Ornstein-Uhlenbeck measure ν associated with ξ may be realized on $C_0([0, T]; B)$ as well. Moreover, in this case H_T is equal to the real interpolation space $D_A(\frac{1}{2}, 2) = (D(A), H)_{\frac{1}{2}, 2}$, with equivalent norms.

In order to capture the general case and the analytic case in one framework, throughout the rest of this section we will make the following

Assumption 4.1. Let H and B be as before. Given are two C_0 -semigroups \mathcal{S}_0 and \mathcal{S}_1 on H satisfying the assumptions listed earlier in this section, a real quasi-complete locally convex topological vector space E , and an injective continuous linear map κ from $C_0([0, T]; B)$ into E such that the operators $\kappa \circ \Phi_0$ and $\kappa \circ \Phi_1$ are radonifying.

Here, $\Phi_j : L^2([0, T]; H) \rightarrow C_0([0, T]; B)$ is the convolution operator associated with \mathcal{S}_j , as defined by (4.1).

Example 4.2. From the above discussion we see that Assumption 4.1 is fulfilled in the following situations:

- (i) If the inclusion $j_T : H_T \hookrightarrow B$ is radonifying we may take $E = B^{[0, T]}$ and $\kappa = k$, the natural inclusion map from $C_0([0, T]; B)$ into $B^{[0, T]}$.
- (ii) If \mathcal{S} is analytic and the inclusion $D_A(\frac{1}{2}, 2) \hookrightarrow B$ is radonifying we may take $E = C_0([0, T]; B)$ and $\kappa = I$, the identity operator on $C_0([0, T]; B)$. Alternatively one may take $E = L^2([0, T], B)$ with $\kappa : C_0([0, T]; B) \hookrightarrow L^2([0, T], B)$ the inclusion map.

(iii) If $H = B$ and $S_j(t)$ is Hilbert-Schmidt for all $t \in (0, T]$, with

$$\int_0^T \|S_j(t)\|_{\mathcal{L}_2(H)}^2 dt < \infty, \quad j = 0, 1$$

then Φ_0 and Φ_1 are Hilbert-Schmidt operators on $L^2([0, T]; H)$ and hence radonifying. We then may take $E = L^2([0, T]; H)$ and $\kappa = I$, the identity mapping on $L^2([0, T]; H)$. This is the situation considered by Peszat [9].

We define $\nu_j := (\kappa \circ \Phi_j)(\gamma_{L^2([0, T]; H)})$ ($j = 0, 1$). Being centered Gaussian measures on E , ν_0 and ν_1 are either equivalent or singular. Recalling [3, Appendix B] that Φ_0 and Φ_1 are injective, we have the following necessary and sufficient conditions for equivalence of the measures ν_0 and ν_1 :

Theorem 4.3. *We have equivalence $\nu_0 \sim \nu_1$ if and only if the following two conditions hold:*

- (i) *range $\Phi_0 = \text{range } \Phi_1$;*
- (ii) *The operator $\Phi_1^{-1}\Phi_0 - (\Phi_0^{-1}\Phi_1)^*$ is Hilbert-Schmidt on $L^2([0, T]; H)$.*

PROOF. Let R_j denote the covariance operator of the measure ν_j . By Proposition 3.1 the RKHS H_j of R_j equals $\text{range}(\kappa \circ \Phi_j)$. By the injectivity of κ we therefore have $H_0 = H_1$ if and only if $\text{range } \Phi_0 = \text{range } \Phi_1$.

It remains to prove that, under the assumption $\text{range } \Phi_0 = \text{range } \Phi_1$, the operator V defined by $R_0\phi \mapsto R_0\phi - R_1\phi$ ($\phi \in E'$) is Hilbert-Schmidt on H_0 (note that we identify H_0 and H_1 in order to simplify notation) if and only if $\Phi_1^{-1}\Phi_0 - (\Phi_0^{-1}\Phi_1)^*$ is Hilbert-Schmidt on $L^2([0, T]; H)$. To see this, notice that for all $\phi \in E'$ we have

$$\begin{aligned} V(\kappa\Phi_0\Phi_0^*\kappa^*\phi) &= V(R_0\phi) = R_0\phi - R_1\phi \\ &= \kappa\Phi_0\Phi_0^*\kappa^*\phi - \kappa\Phi_1\Phi_1^*\kappa^*\phi \\ &= \kappa\Phi_0(I - \Phi_0^{-1}\Phi_1(\Phi_0^{-1}\Phi_1)^*)\Phi_0^*\kappa^*\phi \\ &= \kappa\Phi_0(\Phi_0^{-1}\Phi_1(\Phi_1^{-1}\Phi_0 - (\Phi_0^{-1}\Phi_1)^*))\Phi_0^*\kappa^*\phi \end{aligned}$$

Now $\kappa \circ \Phi_0$ is unitary as a map from $L^2([0, T]; H)$ onto $H_0 = \text{range}(\kappa \circ \Phi_0)$. Moreover, $(\kappa \circ \Phi_0)^*$ has weak*-dense range in $L^2([0, T]; H)$ (being the adjoint of an injective operator) and hence dense range in $L^2([0, T]; H)$ (by reflexivity and the Hahn-Banach theorem). From this we see that V is Hilbert-Schmidt if and only if the map

$$f \mapsto (\Phi_0^{-1}\Phi_1(\Phi_1^{-1}\Phi_0 - (\Phi_0^{-1}\Phi_1)^*))f$$

is Hilbert-Schmidt on $L^2([0, T]; H)$. But $\Phi_0^{-1}\Phi_1$ is an isomorphism from $L^2([0, T]; H)$ onto $L^2([0, T]; H)$, so that we finally obtain that V is Hilbert-Schmidt if and only if $\Phi_1^{-1}\Phi_0 - (\Phi_0^{-1}\Phi_1)^*$ is Hilbert-Schmidt on $L^2([0, T]; H)$. \square

In view of this result it will be important to find conditions under which we have $\text{range } \Phi_0 = \text{range } \Phi_1$, and to obtain explicit expressions for the operators $\Phi_1^{-1}\Phi_0$ and $\Phi_0^{-1}\Phi_1$. This will be achieved in a series of lemmas.

The first of these extends [9, p. 272] and [3, Remark 10.9].

Lemma 4.4. *If $\text{range } \Phi_0 \subset \text{range } \Phi_1$, then $D(A_0) \subset D(A_1)$.*

PROOF. By assumption for every $f \in L^2([0, T]; H)$ there exists a function $g \in L^2([0, T]; H)$ such that $\Phi_1 g = \Phi_0 f$ in $C_0([0, T]; B)$. Taking Laplace transforms, this is equivalent to the statement that for every $f \in L^2([0, T]; H)$ there exists a function $g \in L^2([0, T]; H)$ such that

$$(4.2) \quad R(\lambda, A_1) (\Lambda g(\lambda) - e^{-\lambda T} \Phi_1 g(T)) = R(\lambda, A_0) (\Lambda f(\lambda) - e^{-\lambda T} \Phi_0 f(T))$$

for all large enough positive real numbers λ ; here Λ denotes the Laplace transform. Fix $M > 0$ and $\omega \in \mathbb{R}$ such that $\max\{\|S_0(t)\|, \|S_1(t)\|\} \leq M e^{\omega t}$ for all $t \geq 0$, and fix $\lambda_0 > \omega$ such that

$$\frac{M}{\lambda_0 - \omega} < T.$$

For $\alpha \in \mathbb{R}$ and $h \in H$ let $f_{\alpha,h} \in L^2([0, T]; H)$ be defined by $f_{\alpha,h}(t) = e^{\alpha t} \otimes h$. If α satisfies $\omega < \alpha < \lambda_0$, then

$$\begin{aligned} \Lambda f_{\alpha,h}(\lambda_0) - e^{-\lambda_0 T} \Phi_0 f_{\alpha,h}(T) &= \frac{1 - e^{-(\lambda_0 - \alpha)T}}{\lambda_0 - \alpha} \left(h - e^{-(\lambda_0 - \alpha)T} \frac{\lambda_0 - \alpha}{1 - e^{-(\lambda_0 - \alpha)T}} \int_0^T e^{-\alpha t} S_0(t) h dt \right) \\ &=: \frac{1 - e^{-(\lambda_0 - \alpha)T}}{\lambda_0 - \alpha} (I - L_\alpha) h. \end{aligned}$$

Since

$$\left\| \int_0^T e^{-\alpha t} S_0(t) h dt \right\| \leq M \|h\| \int_0^T e^{-(\alpha - \omega)t} dt = M \|h\| \frac{1 - e^{-(\alpha - \omega)T}}{\alpha - \omega},$$

the choice of λ_0 shows that the operator $I - L_\alpha$ is invertible if $\lambda_0 - \alpha$ is small enough.

It follows that

$$\left\{ R(\lambda, A_0) (\Lambda f(\lambda) - e^{-\lambda T} \Phi_0 f(T)) : f \in L^2([0, T]; H), \lambda > \omega \right\} = D(A_0).$$

Hence, in view of 4.2 for all $h \in D(A_0)$ there exists $g \in L^2([0, T]; H)$ such that

$$R(\lambda, A_1) (\Lambda g(\lambda) - e^{-\lambda T} \Phi_1 g(T)) = h.$$

As the left-hand side defines an element of $D(A_1)$, this proves that $D(A_0) \subset D(A_1)$. \square

The following lemma gives a partial representation of the operator $\Phi_1^{-1} \Phi_0$.

Lemma 4.5. *Suppose $D(A_0) \subset D(A_1)$ and let $f \in C([0, T]; D(A_0))$. Then for all $t \in [0, T]$,*

$$\Phi_0 f(t) - \Phi_1 f(t) = \left(\Phi_1 \int_0^\cdot (A_0 - A_1) S_0(\cdot - s) f(s) ds \right) (t).$$

PROOF. A direct computation gives

$$\begin{aligned} & \left(\Phi_1 \int_0^\cdot (A_0 - A_1) S_0(\cdot - s) f(s) ds \right) (t) \\ &= \int_0^t \int_0^\tau S_1(t - \tau) (A_0 - A_1) S_0(\tau - s) f(s) ds d\tau \\ &= \int_0^t \int_s^t S_1(t - \tau) (A_0 - A_1) S_0(\tau - s) f(s) d\tau ds \\ &= \int_0^t \int_s^t \left(\frac{d}{d\tau} S_1(t - \tau) S_0(\tau - s) \right) f(s) d\tau ds \\ &= \int_0^t (S_0(t - s) - S_1(t - s)) f(s) ds \\ &= \Phi_0 f(t) - \Phi_1 f(t). \end{aligned}$$

\square

From this we obtain the following converse of Lemma 4.4:

Lemma 4.6. *Suppose $D(A_0) \subset D(A_1)$. If the operator*

$$f \mapsto \int_0^\cdot (A_0 - A_1)S_0(\cdot - s)f(s) ds, \quad f \in C([0, T]; D(A_0))$$

extends to a bounded linear operator from $L^2([0, T]; H)$ into itself, then $\text{range } \Phi_0 \subset \text{range } \Phi_1$.

PROOF. Denote the bounded extension by Λ . Then by Lemma 4.5, $\Phi_0 = \Phi_1(I + \Lambda)$ on the dense subspace $C([0, T]; D(A_0))$ of $L^2([0, T]; H)$. Therefore this identity extends to all of $L^2([0, T]; H)$, which gives the result. \square

We are now in a position to prove Theorem 1.1:

Theorem 4.7. *The conditions*

$$(i) \ D(A_0) = D(A_1) \text{ and } \int_0^T \|(A_0 - A_1)S_0(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty$$

$$(ii) \ D(A_0) = D(A_1) \text{ and } \int_0^T \|(A_1 - A_0)S_1(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty$$

are equivalent and imply

$$(iii) \ \nu_0 \sim \nu_1.$$

PROOF. The equivalence of (i) and (ii) follows from Proposition 2.2.

Assume now that (i) (and therefore also (ii)) holds. In view of Theorem 4.3, in order to prove that $\nu_0 \sim \nu_1$ it is enough to show that $\text{range } \Phi_0 = \text{range } \Phi_1$ and that both $\Phi_1^{-1}\Phi_0 - I$ and $\Phi_0^{-1}\Phi_1 - I$ are Hilbert-Schmidt on $L^2([0, T]; H)$.

The finiteness of the integrals in (i) and (ii) implies that

$$\int_0^T \|(A_0 - A_1)S_0(s)\| ds < \infty \quad \text{and} \quad \int_0^T \|(A_1 - A_0)S_1(s)\| ds < \infty,$$

and therefore by Young's inequality the operators

$$f \mapsto \int_0^\cdot (A_0 - A_1)S_0(\cdot - s)f(s) ds \quad \text{and} \quad f \mapsto \int_0^\cdot (A_1 - A_0)S_1(\cdot - s)f(s) ds$$

are bounded on $L^2([0, T]; H)$. Now Lemma 4.6 shows that $\text{range } \Phi_0 = \text{range } \Phi_1$.

For all $f \in L^2([0, T]; H)$ and almost all $t \in [0, T]$ we have

$$(\Phi_1^{-1}\Phi_0 f - f)(t) = \int_0^t (A_0 - A_1)S_0(t - s)f(s) ds.$$

Indeed, for $f \in C([0, T]; D(A_0))$ this follows from Lemma 4.5 and the general case follows from this by a density argument. We conclude that the operator $\Phi_1^{-1}\Phi_0 - I$ can be represented as an integral operator on $L^2([0, T]; H)$ with operator-valued kernel $K(s, t) = (A_0 - A_1)S_0(t - s)$. Since by (i),

$$\int_0^T \int_0^T \|K(s, t)\|_{\mathcal{L}_2(H)}^2 ds dt < \infty,$$

the operator-valued version of Mercer's theorem [7] now implies that $\Phi_1^{-1}\Phi_0 - I$ is Hilbert-Schmidt. The same argument shows that also $\Phi_0^{-1}\Phi_1 - I$ is Hilbert-Schmidt. \square

Theorem 4.7 admits a partial converse, for which we need the following simple observation.

Lemma 4.8. *Suppose $\phi : (0, T) \rightarrow \mathbb{R}_+$ is a measurable function with the following property: there exists a constant $M > 0$ such that for all $s, t \in (0, T)$ with $s \leq t$ we have $\phi(s) \geq M\phi(t)$. If*

$$\int_0^T \int_0^t \phi(s) ds dt < \infty,$$

then also

$$\int_0^T \phi(s) ds < \infty.$$

PROOF. By the finiteness of the double integral, there exists $\delta \in (0, T)$ such that $\int_0^\delta \phi(s) ds < \infty$. But from

$$\int_0^T \int_0^t \phi(s) ds dt \geq \int_\delta^T \int_0^t \phi(s) ds dt \geq M \int_\delta^T \int_0^t \phi(t) ds dt \geq \delta M \int_\delta^T \phi(t) dt$$

we conclude that also $\int_\delta^T \phi(t) dt < \infty$. \square

Now we are in a position to prove Theorem 1.2:

Theorem 4.9. *If $\nu_0 \sim \nu_1$, then $D(A_0) = D(A_1)$. If furthermore for each $t \in (0, T]$ the operator $(A_0 - A_1)S_0(t)$ has a bounded extension to H such that*

$$(4.3) \quad \int_0^T \|(A_0 - A_1)S_0(t)\| dt < \infty,$$

then for each $t \in (0, T]$ the operators $(A_0 - A_1)S_0(t)$ and $(A_1 - A_0)S_1(t)$ extend to Hilbert-Schmidt operators on H and

$$(4.4) \quad \int_0^T \|(A_0 - A_1)S_0(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty \quad \text{and} \quad \int_0^T \|(A_1 - A_0)S_1(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty.$$

PROOF. By Theorem 4.3, $\nu_0 \sim \nu_1$ implies that $\text{range } \Phi_0 = \text{range } \Phi_1$. Hence equality of the domains $D(A_0) = D(A_1)$ follows from Lemma 4.4.

By (4.3), Young's inequality and Lemma 4.5, for all $f \in L^2([0, T]; H)$ we have

$$\Phi_1^{-1}\Phi_0 f - f = \int_0^\cdot (A_0 - A_1)S_0(\cdot - s)f(s) ds.$$

Further, by (4.3) and Proposition 2.1, every operator $(A_1 - A_0)S_1(t)$ has a bounded extension to H and we have

$$\int_0^T \|(A_1 - A_0)S_1(t)\| dt < \infty.$$

Using this, it is easy to check that

$$(\Phi_0^{-1}\Phi_1)^* f - f = \int_\cdot^T ((A_1 - A_0)S_1(\cdot - s))^* f(s) ds$$

for all $f \in L^2([0, T]; H)$. Hence,

$$\begin{aligned} & (\Phi_1^{-1}\Phi_0 - (\Phi_0^{-1}\Phi_1)^*) f(t) \\ &= \int_0^T \chi_{(0,t)}(s) (A_0 - A_1)S_1(t - s)f(s) + \chi_{(t,T)}(s) ((A_1 - A_0)S_1(t - s))^* f(s) ds \end{aligned}$$

for almost $t \in [0, T]$ and all $f \in L^2([0, T]; H)$. Since $\Phi_1^{-1}\Phi_0 - (\Phi_0^{-1}\Phi_1)^*$ is Hilbert-Schmidt by Theorem 4.3, Mercer's theorem implies that

$$\int_0^T \int_0^T \chi_{(0,t)}(s) \|(A_0 - A_1)S_1(t - s)\|_{\mathcal{L}_2(H)}^2 + \chi_{(t,T)}(s) \|((A_1 - A_0)S_1(t - s))^*\|_{\mathcal{L}_2(H)}^2 ds dt < \infty.$$

Clearly, this holds if and only if both

$$\begin{aligned} \int_0^T \int_t^T \|((A_1 - A_0)S_1(t-s))^*\|_{\mathcal{L}_2(H)}^2 ds dt &= \int_0^T \int_0^{T-t} \|(A_0 - A_1)S_0(s)\|_{\mathcal{L}_2(H)}^2 ds dt \\ &= \int_0^T \int_0^t \|(A_0 - A_1)S_0(s)\|_{\mathcal{L}_2(H)}^2 ds dt < \infty \end{aligned}$$

and

$$\int_0^T \int_0^t \|(A_0 - A_1)S_1(s)\|_{\mathcal{L}_2(H)}^2 ds dt < \infty.$$

By Lemma 4.8 applied to the functions

$$\phi_j(s) := \|(A_0 - A_1)S_j(s)\|_{\mathcal{L}_2(H)}^2,$$

taking $M_j := \sup_{s \in [0, T]} \|S_j(s)\|^2$, these two conditions are equivalent to (4.4). \square

Theorems 4.7 and 4.9 can be summarized as follows:

Corollary 4.10. *The following are equivalent:*

- (i) $D(A_0) = D(A_1)$ and $\int_0^T \|(A_0 - A_1)S_0(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty$;
- (ii) $D(A_0) = D(A_1)$ and $\int_0^T \|(A_1 - A_0)S_1(s)\|_{\mathcal{L}_2(H)}^2 ds < \infty$;
- (iii) $\nu_0 \sim \nu_1$ and $\int_0^T \|(A_0 - A_1)S_0(s)\| ds < \infty$;
- (iv) $\nu_0 \sim \nu_1$ and $\int_0^T \|(A_1 - A_0)S_1(s)\| ds < \infty$.

If $A_0 = 0$, then this reduces to:

Corollary 4.11. *If $A_0 = 0$ the following assertions are equivalent:*

- (i) A_1 is bounded and Hilbert-Schmidt;
- (ii) $\nu_0 \sim \nu_1$.

Remarks.

- (i) We have only considered equivalence of *centered* Ornstein-Uhlenbeck measures ν on E . These measures can be thought of as representing Ornstein-Uhlenbeck processes *starting at 0*. In a similar way an Ornstein-Uhlenbeck processes starting at some $h \in H$ can be represented by a Gaussian measure ν^h on E with mean $\kappa(S(T)h)$ and whose covariance equals the covariance of the corresponding centered Ornstein-Uhlenbeck measure. As in [9] it is easy to see that equivalence $\nu_0 \sim \nu_1$ implies $\nu_0^{(h)} \sim \nu_1^{(h)}$ for all $h \in H$. In fact, by (2.1) we have

$$S_1(T)h - S_0(T)h = \int_0^T S(T-s)\psi(s) ds,$$

where $\psi := (A_1 - A_0)S_1(\cdot)h \in L^2([0, T]; H)$ by (4.4). Hence by [3, Corollary B.7], $\kappa(S_1(T)h - S_0(T)h)$ belongs to the common RKHS of the measures $\nu_0^{(h)}$ and $\nu_1^{(h)}$, which shows that condition (ii) of Proposition 3.2 is satisfied.

- (ii) Notice that the conditions for equivalence we have obtained are intrinsic, in the sense that they do not refer to the space B but only to H and the semigroups \mathcal{S}_0 and \mathcal{S}_1 . Furthermore they are independent of T .

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