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EQUIVALENCE OF FINITE ELEMENTS FOR NEARLY-INCOMPRESSIBLE ELASTICITY

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Equivalence of Finite Elements for Nearly-Incompressible Elasticity

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1. Introduction

When one passes to the incompressible limit in the theory of elasticity, a special formulation is required. A pressure-like variable is introduced as an unknown and, concomitantly, an additional equation, restricting the motion to be isochoric, must be satisfied. The pressure variable is interpreted as the force which maintains this constraint.

In principle, the usual formulation of elasticity covers all other unconstrained cases. However, it has been discovered in the application of finite element methods that for nearly-incompressible cases numerical problems are encountered with the usual formulation of the theory. These problems have been dealt with in two ways.

The first method is to reformulate the equations for the compressible case in a way reminiscent of the incompressible case (see Herrmann and Toms [1], Herrmann [2], Taylor, Pister and Herrmann [3], Key [4], and Hughes and Allik [5] for background and applications along these lines). What one does is to consider the stress a function of the strain and a pressure variable. The constitutive equation relating the dilatation to the pressure variable then must be satisfied independently. With a judicious choice of shape functions for the displacements and pressure, an effective numerical scheme can be developed. This approach is equally valid for the compressible and incompressible cases. The variational formulation of this theory, due to Herrmann [2], may be viewed as a special case of Reissner's theorem, since only a part of the stress (i.e., the pressure) is considered to be independent. It should be emphasized here that this formulation, although <u>capable</u> of yielding successful numerical algorithms, is no panacea. This fact, although known for some time, does not seem to be widely appreciated. If one is naive in the use of this method it can lead to results equally bad as those obtained by the standard formulation (see [5] for elaboration and numerical examples). However, this method has been used successfully on a wide range of engineering problems (see [1]-[5] and references therein).

Recently, Fried [6] has provided insight into what goes wrong with the usual formulation for the linear isotropic case. As a remedy he suggests underintegrating the troublesome portion of the strain energy. Computations performed by Naylor [7] yield results consistent with Fried's theory. This approach is simpler to implement and more economical than the method involving a pressure variable. However, its use has not yet become widespread in engineering, perhaps due to the fact that it has an ad hoc flavor.

It is the purpose of the present note to show that a certain underintegrated element is in fact identical to an element based upon Herrmann's formulation, which has been used successfully in the past ([5]). The elements in question are a bilinear displacement model, which employs one-point Gaussian quadrature on a portion of the strain energy, and a constant pressure, bilinear displacement model based on Herrmann's formulation.

In Section 2 we establish notations and review the equations of classical elasticity. In Section 3 the finite element equations are derived for the two formulations. The equivalence of the elements is demonstrated in Section 4 and a numerical example which supports the analysis is given in Section 5. A summary and conclusions are contained in Section 6.

2. Equations of Classical Elasticity

Let Ω be a bounded region in $\mathbb{R}^2$, with piecewise smooth boundary $\partial\Omega$. Vectors defined on Ω are written in the standard indicial notation, e.g. $u_\alpha, \alpha = 1,2$, are the cartesian components of the displacement vector. A comma is used to denote partial differentiation and the summation convention is employed, e.g. $\partial u_\alpha/\partial x_\alpha = u_{\alpha,\alpha} = u_{1,1} + u_{2,2}$. A general point in Ω is denoted by x. The equations of classical isotropic elasticity are

$$0 = (\lambda + \mu)\, u_{\beta,\beta\alpha} + \mu\, u_{\alpha,\beta\beta} + f_\alpha \ , \tag{2.1}$$

where λ and μ are the Lamé constants, and f_α denotes the extrinsic body force. The mixed boundary value problem for (2.1) consists of finding functions $u_\alpha(x)$ satisfying (2.1) for all $x \in \Omega$ and

$$\begin{aligned} u_\alpha(x) &= g_\alpha(x) \quad , \quad x \in \partial\Omega_1 \ , \\ n_\beta(x)\ \{\lambda\, u_{\gamma,\gamma}(x)\, \delta_{\alpha\beta} + 2\mu\, u_{(\alpha,\beta)}(x)\} &= h_\alpha(x) \quad , \quad x \in \partial\Omega_2 , \end{aligned} \tag{2.2}$$

where g_α and h_α are the given boundary data, n_β is the unit outward normal vector to $\partial\Omega$, $\delta_{\alpha\beta}$ is the Kronecker delta, $u_{(\alpha,\beta)} = 1/2\,(u_{\alpha,\beta} + u_{\beta,\alpha})$, and $\partial\Omega_1$ and $\partial\Omega_2$ are subregions of $\partial\Omega$ satisfying $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ and $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$.

3. Finite Element Equations

In this section we set up the finite element equations for classical elasticity corresponding to the displacement formulation and Herrmann's formulation. Let $L_2 = L_2(\Omega)$ denote the Hilbert space of Lebesgue square integrable functions defined on Ω. Let $H^1 = H^1(\Omega)$ denote the Sobolev space of functions which are in L_2 and which have generalized derivatives in L_2. Important subspaces of H^1 are $H^1_0 = \{\phi_\alpha \mid \phi_\alpha \in H^1$ and $\phi_\alpha = 0$ on $\partial\Omega_1\}$ and $H^1_{g_\alpha} = \{u_\alpha \mid u_\alpha \in H^1$ and $u_\alpha = g_\alpha$ on $\partial\Omega_1\}$. The finite element equations emanate from the weak forms of the boundary value problem, given as follows:

3.1 Displacement Formulation. Find $u_\alpha \in H^1_{g_\alpha}$ such that for all $\phi_\alpha \in H^1_0$

$$0 = \int_\Omega \{\lambda\, \phi_{\beta,\beta}\, u_{\alpha,\alpha} + 2\mu\, \phi_{\alpha,\beta}\, u_{(\alpha,\beta)} - \phi_\alpha f_\alpha\}\, dx_1\, dx_2 - \int_{\partial\Omega_2} \phi_\alpha\, h_\alpha\, ds, \quad (3.1.1)$$

where $dx_1\, dx_2$ is the area element for Ω and ds is the arc-length element for $\partial\Omega$. If u_α is sufficiently regular (3.1.1) implies (2.1) and $(2.2)_2$ hold.

3.2 Herrmann Formulation. Find $u_\alpha \in H^1_{g_\alpha}$ and $p \in L_2$ such that for all $\phi_\alpha \in H^1_0$ and $q \in L_2$

$$0 = \int_\Omega \{q(\tfrac{1}{\lambda} p + u_{\alpha,\alpha}) + \phi_{\alpha,\beta}(-p\, \delta_{\alpha\beta} + 2\mu\, u_{(\alpha,\beta)}) - \phi_\alpha\, f_\alpha\}\, dx_1\, dx_2$$

$$- \int_{\partial\Omega_2} \phi_\alpha\, h_\alpha\, ds. \quad (3.2.1)^*$$

Satisfaction of (3.2.1) also implies (2.1) and $(2.2)_2$ hold.

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* The variable p, differs by a constant factor from the variable H used in Herrmann's original paper [2], i.e. $p = -\mu\lambda H/(\lambda + \mu)$.

3.3 Displacement Formulation Finite Element Equations. Consider a four-node isoparametric quadrilateral element (see Fig. 1). Let $u_{a\alpha}$ denote the value of the finite element solution at node a for direction α, and let $f_{a\alpha}$ and $h_{a\alpha}$ be the values of f_α and h_α at node a, respectively. Let N_a, $a \in \{1,2,3,4\}$, denote the shape function associated with node a of the e^{th} element. Repeated subscripts a,b,c,d are to be summed over the nodes of the e^{th} element, i.e. over 1,2,3,4. The element equation for node a and direction α corresponding to (3.1.1) is

$$(\lambda\, k^{o}_{a\alpha b\beta} + \mu\, k_{b\alpha a\beta})\, u_{b\beta} + \mu\, k_{ab}\, u_{b\alpha} = S_{ab}\, h_{b\alpha} + m_{ab}\, f_{b\alpha}\,, \tag{3.3.1}$$

where

$$\begin{aligned} k^{o}_{a\alpha b\beta} &= 4(N_{a,\alpha}\, N_{b,\beta}\, J)\big|_{(\xi,\eta)} = (0,0) & S_{ab} &= \int_{\partial\Omega_2} N_a N_b\, ds \\ k_{a\alpha b\beta} &= \int_\Omega N_{a,\alpha}\, N_{b,\beta}\, dx_1\, dx_2 & m_{ab} &= \int_\Omega N_a N_b\, dx_1\, dx_2 \\ k_{ab} &= \int_\Omega N_{a,\alpha}\, N_{b,\alpha}\, dx_1\, dx_2 \end{aligned} \tag{3.3.2}$$

and J is the jacobian determinant of the isoparametric mapping. Observe that the stiffness contribution $k^{o}_{a\alpha b\beta}$ is determined by one-point Gaussian quadrature.

3.4 Herrmann Formulation Finite Element Equations. As in the previous case, the displacements of the finite element equations are given in terms of the shape functions N_a, $a \in \{1,2,3,4\}$. However, the pressure variable p is assumed to be constant over each element. As a result p can be solved for in terms of the corresponding element nodal displacements, viz.

$$p = -\frac{\lambda}{A_e}\, k_{a\alpha}\, u_{a\alpha}\,, \tag{3.4.1}$$

where A_e is the area of the e^{th} element and

$$k_{a\alpha} = \int_\Omega N_{a,\alpha}\, dx_1\, dx_2 \ . \tag{3.4.2}$$

The equilibrium equation for node a and direction α corresponding to (3.2.1) is

$$- p\, k_{a\alpha} + \mu\, (k_{b\alpha a\beta}\, u_{b\beta} + k_{ab}\, u_{b\alpha}) = S_{ab}\, h_{b\alpha} + m_{ab}\, f_{b\alpha} \ . \tag{3.4.3}$$

Substitution of (3.4.1) into (3.4.3) results in

$$(\frac{\lambda}{A_e}\, k_{a\alpha}\, k_{b\beta} + \mu\, k_{b\alpha a\beta})\, u_{b\beta} + \mu\, k_{ab}\, u_{b\alpha} = S_{ab}\, h_{b\alpha} + m_{ab}\, f_{b\alpha} \ . \tag{3.4.4}$$

Comparison of this result with (3.3.1) indicates that the two formulations are the same if and only if $k^o_{a\alpha b\beta}$ and $k_{a\alpha}\, k_{b\beta}/A_e$ are identical.

4. Equivalence of the Elements

To explicitly compute the element stiffness coefficients it is necessary to obtain some preliminary relations. We summarize the pertinent results as follows:

$$N_a = \frac{1}{4}(1 + \xi_a \xi)(1 + \eta_a \eta) \qquad \text{(no sum)}$$

a	ξ_a	η_a
1	1	1
2	-1	1
3	-1	-1
4	1	-1

$$N_{a,\xi} = \frac{1}{4}\xi_a (1 + \eta_a \eta) \qquad \text{(no sum)}$$

$$N_{a,\eta} = \frac{1}{4}\eta_a (1 + \xi_a \xi) \qquad \text{(no sum)}$$

$$\begin{bmatrix} \xi_{,1} & \xi_{,2} \\ \eta_{,1} & \eta_{,2} \end{bmatrix} = \begin{bmatrix} x_{1,\xi} & x_{1,\eta} \\ x_{2,\xi} & x_{2,\eta} \end{bmatrix}^{-1} = \frac{1}{J}\begin{bmatrix} x_{2,\eta} & -x_{1,\eta} \\ -x_{2,\xi} & x_{1,\xi} \end{bmatrix}$$

$$J = x_{1,\xi}\, x_{2,\eta} - x_{1,\eta}\, x_{2,\xi}$$

$$N_{a,\xi} N_{b,\eta} - N_{a,\eta} N_{b,\xi} = \frac{1}{16}\{\xi_a \eta_b (1 + \eta_a \eta + \xi_b \xi) - \xi_b \eta_a (1 + \xi_a \xi + \eta_b \eta)\} \qquad \text{(no sum)}$$

$$J\big|_{(\xi,\eta)} = (0,0) = \frac{1}{16}\, x_{a1}\, x_{b2}\, (\xi_a \eta_b - \xi_b \eta_a)$$

$$= A_e/4$$

$$N_{a,\alpha} = (-1)^{\alpha} y_{b\alpha}(N_{a,\eta}\, N_{b,\xi} - N_{a,\xi}\, N_{b,\eta})/J \qquad \text{(no sum on } \alpha)$$

$$y_{b1} = x_{b2}$$

$$y_{b2} = x_{b1}$$

In the preceding relations, $x_{1,\xi} = \partial x_1/\partial\xi$, etc. and $x_{a\alpha}$ is the x_α coordinate of node a.

4.1 Theorem. $k^{o}_{a\alpha b\beta} = k_{a\alpha} k_{b\beta}/A_e$

Proof. The proof involves straightforward computations which employ the above relations:

$$k^{o}_{a\alpha b\beta} = 4(N_{a,\alpha} N_{b,\beta} J)|_{(\xi,\eta)} = (0,0)$$

$$= 4\{(-1)^{\alpha} y_{c\alpha}(N_{a,\eta} N_{c,\xi} - N_{a,\xi} N_{c,\eta})\ (-1)^{\beta} y_{d\beta}(N_{b,\eta} N_{d,\xi} - N_{b,\xi} N_{d,\eta})/J\}|_{(0,0)}$$

$$= \frac{1}{16\ A_e} (-1)^{\alpha}(-1)^{\beta} y_{c\alpha} y_{d\beta}(\xi_c\eta_a - \xi_a\eta_c)(\xi_d\eta_b - \xi_b\eta_d) \qquad \text{(no sum on } \alpha \text{ and } \beta)$$

$$k_{a\alpha} = \int_{\Omega} N_{a,\alpha}\, dx_1\, dx_2$$

$$= \frac{(-1)^{\alpha}}{4} y_{b\alpha}(\xi_b\eta_a - \xi_a\eta_b) \qquad \text{(no sum on } \alpha)$$

$$k_{a\alpha} k_{b\beta}/A_e = \frac{1}{16\ A_e} (-1)^{\alpha}(-1)^{\beta} y_{c\alpha} y_{d\beta}(\xi_c\eta_a - \xi_a\eta_c)\ (\xi_d\eta_b - \xi_b\eta_d)$$

(no sum on α and β)

5. Numerical Example

To corroborate the analysis a numerical example was run using both the elements described above. The configuration is illustrated in Fig. 2. The beam is fixed at the left end and subjected to a uniform shear applied along the right end. The model consists of 32 plane strain rectangles, E denotes Young's modulus and ν denotes Poisson's ratio. For the rectangular configuration, 2 x 2 Gaussian quadrature is exact. As is clearly seen, the underintegrated displacement model and the element based on Herrmann's formulation give identical results.* Note the stiffening in the exactly integrated displacement model as Poisson's ratio is increased.

* H. Allik and P. Cacciatore provided the results for the constant pressure-bilinear displacement model.

6. Summary and Conclusions

The bilinear displacement model, employing one-point Gaussian quadrature on the λ-term, and the constant pressure variable, bilinear displacement element based upon Herrmann's formulation, have been shown to lead to identical results. A numerical example in support of the analysis has been presented.

This result has considerable practical significance since the underintegrated displacement model can be implemented more simply and economically. In particular, programming the element is simpler and the number of equations in practical problems is reduced by approximately 1/3.

Acknowledgment

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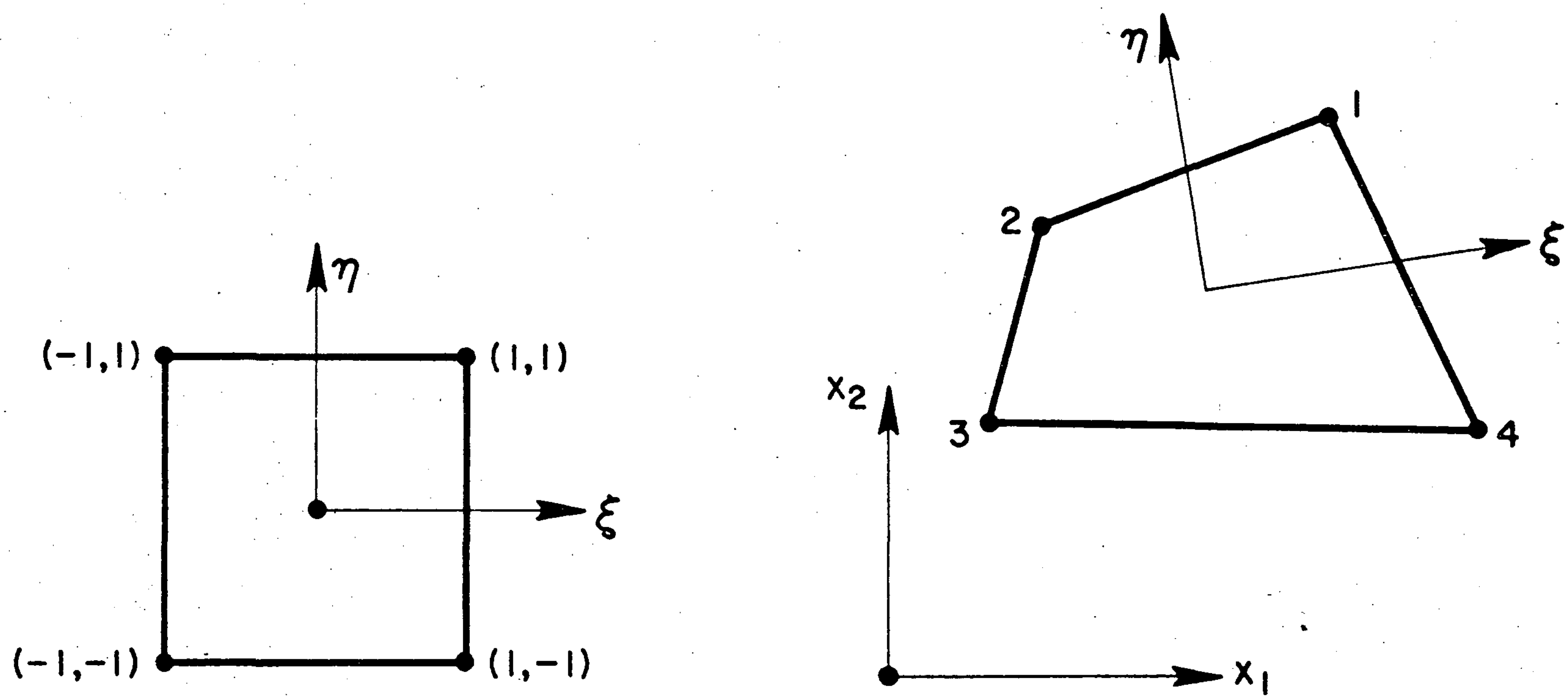
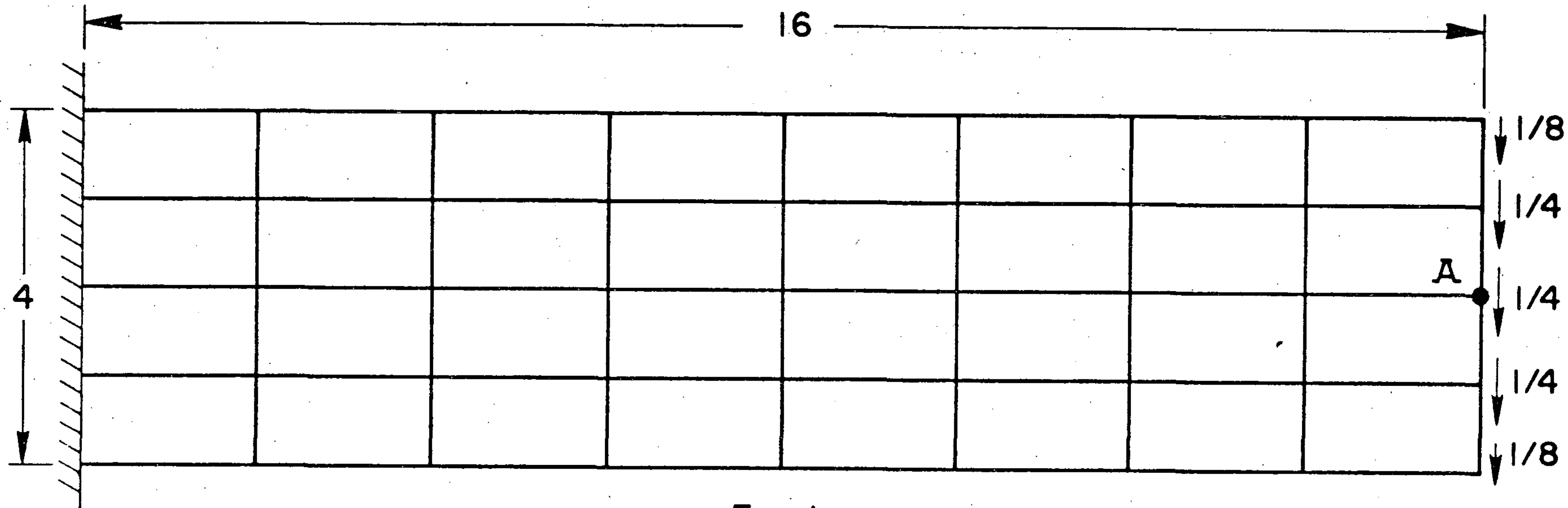


FIG. 1 BILINEAR ISOPARAMETRIC QUADRILATERAL



E = 1

VERTICAL DISPLACEMENT AT POINT A

ELEMENT	QUADRATURE	$\nu = 0.3$	$\nu = 0.499$
BILINEAR DISPLACEMENT MODEL (80)*	EXACT	217.8	26.8
BILINEAR DISPLACEMENT MODEL (80)	1 x 1 λ TERM 2 x 2 μ TERM	224.9	183.3
CONSTANT PRESSURE – BILINEAR DISPLACEMENT MODEL (112)	EXACT	224.9	183.3

*NUMBERS IN PARENTHESES REFER TO THE NUMBER OF EQUATIONS.

FIG. 2 COMPARISON OF FINITE ELEMENTS FOR BEAM SUBJECTED TO END SHEAR

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