

Equivalence of Linear MMSE Detection in DS-CDMA and MC-CDMA Systems over Time and Frequency Selective Channels

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The goal of this paper is to compare the performance of the linear minimum mean square error (MMSE) detector for a class of code division multiple access (CDMA) systems in time and frequency selective channels. Specifically, we consider direct sequence (DS)-CDMA, multicarrier (MC)-CDMA, and the MC-DS-CDMA systems. Two key tools are used in our development. First, a general time-frequency framework that includes the different CDMA systems as special cases. Second, the duality between time and frequency domains that is used to derive equivalences between the different CDMA systems operating over purely frequency selective and purely time selective channels. We then combine the insights obtained from these special cases to assess the performance of CDMA systems over time and frequency selective channels. We provide sufficient conditions for the codes employed by the CDMA systems for the equivalences to hold. Numerical results are presented to illustrate the results.

Keywords and phrases: CDMA, multicarrier system, time-selective channels, frequency selective channels, multiuser detection, diversity.

1. INTRODUCTION

CDMA has emerged as a promising wireless technology for meeting the physical layer challenges of modern communication networks. This is due to the multiple access capability of CDMA systems as well as their robustness against fading. Many signaling schemes have been proposed for CDMA systems. Three important CDMA systems are DS-CDMA, MC-CDMA, and the hybrid MC-DS-CDMA. The key to understanding and relating these systems is how these signaling schemes interact with the channel.

Denote the symbol duration with T and its essential two-sided bandwidth with B . The time-bandwidth product is denoted by $N \approx TB$ which is also the dimension of the signal space. A CDMA system with such time-bandwidth product can support up to N users by employing signature codes of length N . In general, the signature code is transmitted over a set of basis waveforms, each of which has a duration T_o and essential two-sided bandwidth B_o . The different signaling schemes differ in how T_o and B_o are chosen. In DS-CDMA [1], the basis waveforms are generated via time shifts of a *nar-*

row time pulse with duration $T_o = T/N$ and bandwidth $B_o = B$. The transmitter and the distributions of signature code in time and frequency domains are shown in Figures 1a and 2a, respectively. In MC-CDMA [2, 3, 4], the basis waveforms are generated via frequency shifts of a narrowband pulse with duration $T_o = T$ and bandwidth $B_o = B/N$. The transmitter and the distributions of signature code in time and frequency domains for this system are shown in Figures 1b and 2b, respectively. MC-DS-CDMA [2, 5, 6] is a hybrid system where $N_t \times N_f$ basis waveforms ($N = N_t N_f$) are used. The basis functions are generated via the time-frequency shifts of a pulse of duration $T_o = T/N_t$ and bandwidth $B_o = B/N_f$. Notice that the DS-CDMA (MC-CDMA) system is a special case of the MC-DS-CDMA system when $N_f = 1$, $N_t = N$ ($N_f = N$, $N_t = 1$). The MC-DS-CDMA transmitter and the distributions of signature code in time and frequency domains are shown in Figures 1c and 2c, respectively.

The effect of the channel on the three CDMA systems is determined by two factors: *channel selectivity* and *channel dispersion*. The channel is frequency (time) *selective* if the transmitted signal exhibits different fading coefficients across

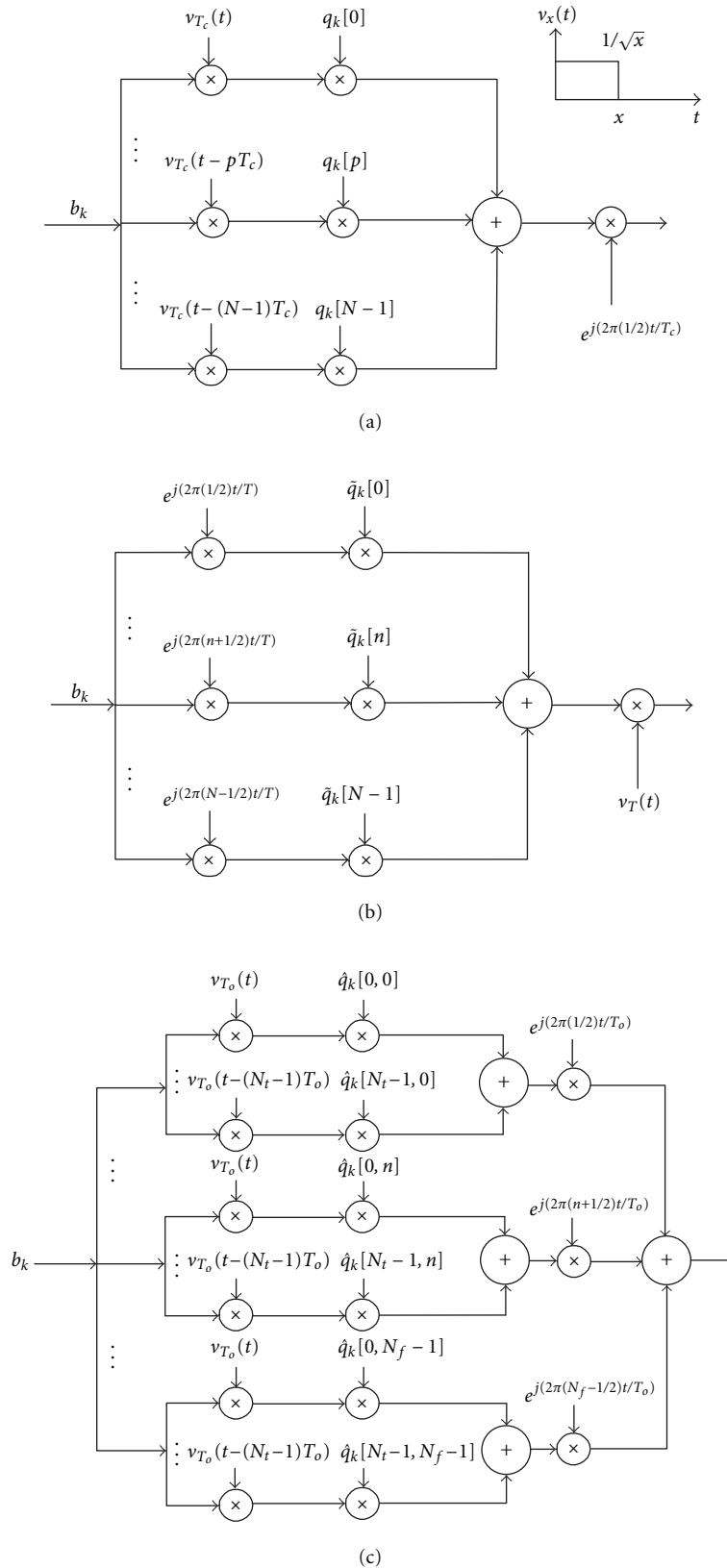


FIGURE 1: The transmitter structure for different CDMA systems. (a) DS-SS system where $q_k[p]$ is the p th entry of the k th user signature code. (b) MC-SS system where $\tilde{q}_k[n]$ is the n th entry of the k th user signature code. (c) MC-DS-SS system where $\hat{q}_k[p, n]$ is the (p, n) th entry of the k th user signature code.

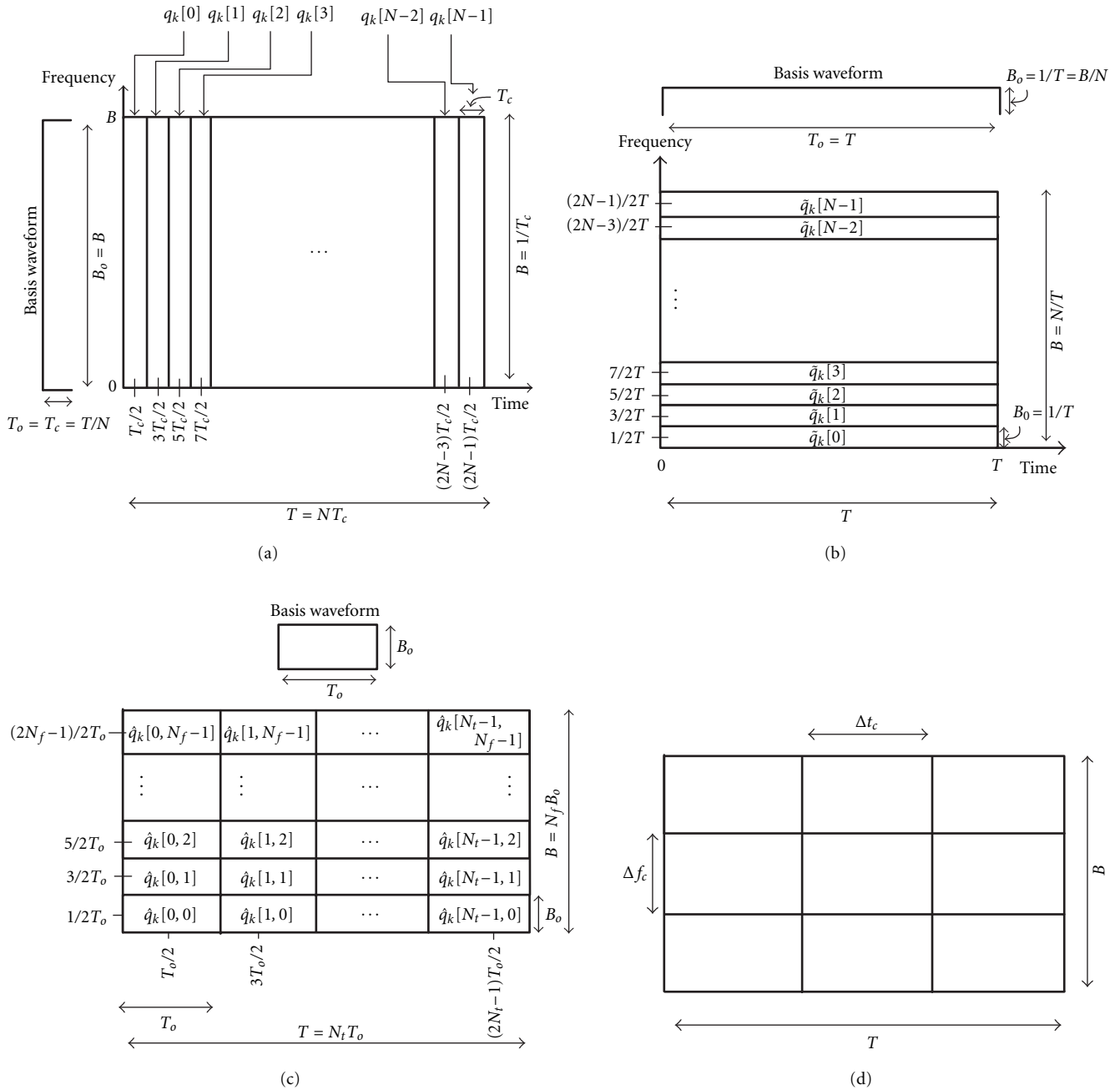


FIGURE 2: Time-frequency distribution of (a) signature code in DS-CDMA system, (b) signature code in MC-CDMA system, (c) signature code in MC-DS-CDMA system, and (d) channel coherence time and bandwidth.

its bandwidth (duration). More specifically, for any signaling scheme, the channel is frequency (time) selective if the overall bandwidth (duration) $B > \Delta f_c$ ($T > \Delta t_c$).¹ The channel is time and frequency selective (TFS) if $T > \Delta t_c$ and $B > \Delta f_c$

¹As clarified in Section 2.2, Δf_c (Δt_c) is the coherence bandwidth (time)—that is, the frequency (time) span over which the channel is strongly correlated.

(see Figure 2d). The channel is *dispersive in time* if it is *frequency selective (FS) per basis waveform* (i.e., $B_o > \Delta f_c$), *dispersive in frequency* if it is *time selective (TS) per basis waveform* (i.e., $T_o > \Delta t_c$), and *dispersive in time and frequency* if it is TFS per basis waveform (i.e., $B_o > \Delta f_c$ and $T_o > \Delta t_c$). If the channel is nondispersive for a particular signaling scheme, we say that this scheme diagonalizes the channel—that is, the basis functions serve as eigenfunctions of the channel such that when they are transmitted through the channel they do

not interfere (this will be elaborated later). In this paper, we consider purely FS channels, purely TS channels, and general TFS channels. The basis functions in different CDMA signaling schemes serve as eigenfunctions for certain types of channels. The DS-CDMA basis waveforms are the eigenfunctions of TS channels. On the other hand, the MC-CDMA basis waveforms are the eigenfunctions of FS channels. For proper choice of T_o and B_o , the MC-DS-CDMA basis waveforms diagonalize TFS channels [7, 8]. The performance of these signaling schemes for single-user scenario has been analyzed in several papers (see, e.g., [2] and the references therein). However, a unified treatment of the performance of these systems in multiuser scenarios and for different types of channels is lacking.

In this paper, we compare the performance of the above three systems in the context of multiuser detection in the up-link case. The comparison is based on linear MMSE receivers. We show that, for certain channel conditions, the different systems can perform in a near identical fashion via appropriate choice of signature codes. To develop these results, we first introduce a general framework that includes the three CDMA systems as special cases. Furthermore, we use the notion of duality between time and frequency domains to develop equivalences between the different systems that allow us to relate their performance under different channel conditions.

Section 2 introduces the notations used in this paper as well as the channel model considered. In Section 3, we discuss CDMA systems in TFS channel as well as the effect of channel selectivity on the received signal. In Section 4, we lay the foundation for the analysis in the following sections by discussing the duality between time and frequency domains. In Section 5, we derive expressions for signal-to-interference-and-noise ratio (SINR)² and probability of error (P_e) for the different system receivers which are used to derive sufficient conditions for the different systems to perform equivalently in FS, TS, and TFS channels. In Section 6, we illustrate our results through some numerical examples. Concluding remarks are provided in Section 7.

2. PRELIMINARIES

This section introduces the notation adopted throughout as well as the channel model under consideration.

2.1. Notation

Boldface lower case letters are used to denote (column) vectors and boldface upper case letters are used to denote matrices. The following conventions are used throughout the paper:

- $\lfloor x \rfloor$ denotes the largest integer smaller than x ;
- $\lceil x \rceil$ denotes the smallest integer larger than x ;
- \mathbf{x}^T is the transpose of vector \mathbf{x} ;
- \mathbf{x}^H is the conjugate transpose of vector \mathbf{x} ;

²The SINR is defined to be the ratio of the energy in the desired signal to the energy in noise plus interference in the received decision statistics.

$\mathbf{x}(l)$ is an l -shifted version of the vector \mathbf{x} , that is,

$$\mathbf{x}(l) = \left[\underbrace{0 \ \cdots \ 0}_l \ x[0] \ x[1] \ \cdots \ x[N-l-1] \right]^T \quad (1)$$

and $x[n]$ is the n th entry of vector \mathbf{x} . If $l < 0$, the shift is to the left,

$\mathbf{A} \odot \mathbf{B}$ is the Hadamard product between \mathbf{A} and \mathbf{B} , that is, point-wise multiplication between the different elements of \mathbf{A} and \mathbf{B} ;

$\text{diag}\{\mathbf{x}\}$ is a diagonal matrix whose entries are those of vector \mathbf{x} ;

$\mathbf{A}(n_1 : n_2)$ is the submatrix built by the (n_1) th through n_2 th column of \mathbf{A} ;

\mathbf{I}_N is the identity matrix of dimension N ;

$\mathbf{1}_N$ is an $N \times 1$ vector with unity entries;

$\mathbf{0}_N$ is an $N \times 1$ vector with zero entries;

$E[\cdot]$ is the expected value;

$\mathbf{R}_{\mathbf{x},\mathbf{y}}$ is $E[\mathbf{x}\mathbf{y}^H]$.

2.2. Channel model

We assume that all users have identical channel *statistics*; however, different users encounter independent channel realizations. We adopt the wide sense stationary uncorrelated scattering (WSSUS) channel model [1, 9] which is characterized by a randomly time-varying impulse response $c_k(t, \tau)$ for user k . For fixed τ , $c_k(t, \tau)$ is a complex WSS Gaussian process in t . The process for different values of τ is uncorrelated (independent). The time-varying transfer function of the channel is

$$\tilde{c}_k(t, f) = \int_{-\infty}^{\infty} c_k(t, \tau) e^{-j2\pi f\tau} d\tau. \quad (2)$$

Under the WSSUS assumption, $\tilde{c}_k(t, f)$ is a WSS process in both t and f . The statistics of $\tilde{c}_k(t, f)$ are characterized by the *spaced-time, spaced-frequency* correlation function defined as $\phi(\Delta t, \Delta f) = E[\tilde{c}_k(t, f)\tilde{c}_k^*(t - \Delta t, f - \Delta f)]$ and is assumed the same for all users. Under the assumption that the spaced-time correlation function $\phi_{\Delta t}(\Delta t) = E[\tilde{c}_k(t, \cdot)\tilde{c}_k^*(t - \Delta t, \cdot)]$ is fixed for every path (i.e., independent of τ or equivalently independent of f), $\phi(\Delta t, \Delta f)$ becomes separable [10], that is,

$$\phi(\Delta t, \Delta f) = \phi_{\Delta t}(\Delta t)\psi_{\Delta f}(\Delta f), \quad (3)$$

where $\psi_{\Delta f}(\Delta f) = E[\tilde{c}_k(\cdot, f)\tilde{c}_k^*(\cdot, f - \Delta f)]$ is the spaced-frequency correlation function which is independent of time.

The duration over which the channel is strongly correlated is called the *coherence time*, Δt_c . Define the *Doppler power spectrum* function to be

$$\Phi(\theta) = \int_{-\infty}^{\infty} \phi_{\Delta t}(\Delta t) e^{-j2\pi\Delta t\theta} d\Delta t. \quad (4)$$

The support of $\Phi(\theta)$ is called the *Doppler spread* $2B_d$ of the channel; $\Phi(\theta) = 0$, $|\theta| > B_d$. The coherence time and Doppler spread are inversely related, $\Delta t_c \approx 1/2B_d$.

The channel power at different values of τ is called the

multipath intensity profile, $\Psi(\tau) = E[|c_k(\cdot, \tau)|^2]$. The support of $\Psi(\tau)$ is called the *multipath spread* of the channel and is denoted by T_m ; $\Psi(\tau) = 0$ for $\tau < 0$ or $\tau > T_m$. The spaced-frequency correlation function is the Fourier transform of $\Psi(\tau)$, that is, $\psi_{\Delta f}(\Delta f) = \int_{-\infty}^{\infty} \Psi(\tau) e^{-j2\pi\Delta f\tau} d\tau$. The frequency span over which the channel is strongly correlated is called the *coherence bandwidth*, Δf_c . Coherence bandwidth and multipath spread are inversely related, $\Delta f_c \approx 1/T_m$.

The FS channel corresponds to $B > \Delta f_c$ and $T \ll \Delta t_c$, the TS channel corresponds to $B \ll \Delta f_c$ and $T > \Delta t_c$, and the general TFS channel corresponds to $B > \Delta f_c$ and $T > \Delta t_c$. The diversity order due to frequency selectivity (multipath diversity) is $L + 1$ where $L = \lceil BT_m \rceil$, while the diversity order due to time selectivity (Doppler diversity) is $2M + 1$ where $M = \lceil B_d T \rceil$ [11].

3. CDMA SYSTEMS

In this section, we briefly describe the DS-CDMA, MC-CDMA, and the MC-DS-CDMA systems in a general TFS channel (the different systems in FS and TS channels are special cases) under the assumption of synchronous reception for simplicity of exposition.

3.1. DS-CDMA system

In a DS-CDMA system, the prototype basis waveform is the chip waveform $v_{T_c}(t) = 1/\sqrt{T_c}$, $0 \leq t \leq T_c$ (assumed rectangular for simplicity) which has a duration $T_c = T_o = T/N$ and a bandwidth $B_o = B = 1/T_c$. Each user is assigned a signature waveform that is generated from time shifts of the prototype basis waveform as

$$q_k(t) = \sum_{p=0}^{N-1} q_k[p] u_p(t) = \sum_{p=0}^{N-1} q_k[p] v_{T_c}(t - pT_c), \quad (5)$$

where $\{u_p(t) = v_{T_c}(t - pT_c)$, $p = 0, 1, \dots, N-1\}$ are the set of basis functions used in transmission and the length N signature code of user k $\{q_k[p]\}$ serves as the expansion coefficients. The transmitted signal for user k can be written as

$$s_k(t) = \sum_{i=-\infty}^{\infty} b_{k,i} q_k(t - iT), \quad (6)$$

where $b_{k,i}$ is the i th bit corresponding to user k . We assume that $T \gg T_m$ and $B \gg B_d$ so that the intersymbol interference (ISI) is negligible and a one shot detector suffices. Hence, without loss of generality, we focus on the detection of $b_{1,0}$ (0th bit for user 1) and drop the index 0 for simplicity.

Now we derive the form for the sampled received signal for a TFS channel (the FS and TS channels are special cases). A DS-CDMA system over a TFS channel is dispersive in time ($B_o = B > \Delta f_c$) but typically nondispersive in frequency since $T_o < \Delta t_c$ (see Figures 2a and 2d). The noise-free received signal for user k in TFS channel can be written as

$$x_k(t) = b_k \sum_{p=0}^{N-1} \sum_{l=0}^L h_k[p, l] q_k[p] v_{T_c}(t - pT_c - lT_c), \quad (7)$$

where

$$h_k[p, l] = \int_0^{T_m} c_k\left(\left(p + \frac{1}{2}\right)T_c, \tau\right) \text{sinc}(l - B\tau) e^{j\pi(l - B\tau)\tau} d\tau \quad (8)$$

is the channel coefficient corresponding to the l th path and p th chip of the k th user. In (7), the k th user-transmitted power is absorbed in $h_k[p, l]$, a convention that is used throughout the paper. The time dispersion (multipath) is clear in (7) since the information transmitted over a particular chip is dispersed into the adjacent chips. The overall multiuser received signal is

$$r(t) = \sum_{k=1}^K x_k(t) + v(t), \quad (9)$$

where $v(t)$ is complex additive white Gaussian noise (AWGN) with power spectral density σ^2 . Front-end processing at the receiver corresponds to projecting onto the basis waveforms. Assuming the receiver to be synchronized to the first path ($l = 0$) in (7), projecting over $u_p(t)$, we get

$$\begin{aligned} r[p] = \langle r, u_p \rangle &= \frac{1}{\sqrt{T_c}} \int_{pT_c}^{(p+1)T_c} r(t) dt \\ &= \sum_{k=1}^K b_k \sum_{l=0}^L h_k[p-l, l] q_k[p-l] + v[p], \end{aligned} \quad (10)$$

where $\langle r, u_p \rangle = \int r(t) u_p^*(t) dt$ and $v[p]$ is the p th noise sample. For convenience of notation, we express $\{r[p]\}$ in a vector form as

$$\begin{aligned} \mathbf{r} &= [r[0] \ r[1] \ \dots \ r[N-1]]^T \\ &= b_1 \mathbf{Q}_1 \mathbf{h}_1 + \sum_{k=2}^K b_k \mathbf{Q}_k \mathbf{h}_k + \mathbf{v} \end{aligned} \quad (11)$$

$$= b_1 \mathbf{g}_1 + \sum_{k=2}^K b_k \mathbf{g}_k + \mathbf{v}, \quad (12)$$

where $\mathbf{Q}_k = [\mathbf{Q}_k(0) \ \dots \ \mathbf{Q}_k(L)]$ and has a dimension $N \times N(L+1)$, $\mathbf{Q}_k(l) = \text{diag}\{\mathbf{q}_k(l)\}$ and has a dimension $N \times N$, \mathbf{q}_k is the $N \times 1$ signature code vector of k th user, $\mathbf{h}_k = [\mathbf{h}_{k,0}^T \ \dots \ \mathbf{h}_{k,L}^T]^T$ and has a dimension $N(L+1) \times 1$, $\mathbf{h}_{k,l} = [h_k[-l, l] \ \dots \ h_k[N-1-l, l]]^T$ and has a dimension $N \times 1$,³ \mathbf{v} is zero mean N -dimensional Gaussian noise vector with covariance matrix $\sigma^2 \mathbf{I}_N$, and $\mathbf{g}_k = \mathbf{Q}_k \mathbf{h}_k$. Note that, in (11) and (7) due to time selectivity, the channel varies over the symbol duration—for a given path l , the different chips may encounter different fading coefficients.

For FS channels (no time selectivity $T < \Delta t_c$), $h_k[p, l]$ is constant over p within a symbol duration, that is,

³Negative l indices correspond to those channel coefficients from previous symbols due to ISI. These samples are ignored in the analysis since typically $N \gg L$.

$h_k[p, l] \approx h_k[0, l]$, $p = 0, 1, \dots, N - 1$. Hence, $\mathbf{Q}_k(l)\mathbf{h}_{k,l} = h_k[0, l]\mathbf{Q}_k(l)\mathbf{1}_N = h_k[0, l]\mathbf{q}_k(l)$. Now, we can rewrite (11) replacing $\mathbf{Q}_k(l)$ by $\mathbf{q}_k(l)$ in the definition of \mathbf{Q}_k , and $\mathbf{h}_{k,l}$ with $\tilde{h}_k[0, l]$ in the definition of \mathbf{h}_k . Notice that, in this case, \mathbf{Q}_k is Toeplitz due to multipath dispersion. On the other hand, for TS channels (no frequency selectivity, $B_o = B < \Delta f_c$), $L = 0$. Hence, \mathbf{Q}_k reduces to $\mathbf{Q}_k(0)$ and \mathbf{h}_k reduces to $\mathbf{h}_{k,0}$. The diagonal structure of the system is evident in this case form (11).⁴

3.2. MC-CDMA system

In an MC-CDMA system, the prototype basis waveform is the complex exponential $v_T(t)e^{j(2\pi(1/2)/T)t}$ which has a duration $T_o = T$ and bandwidth $B_o = B/N = 1/T$. Each user is assigned a signature waveform that is generated from frequency shifts of the prototype basis waveform as

$$\tilde{q}_k(t) = \sum_{n=0}^{N-1} \tilde{q}_k[n] \tilde{u}_n(t), \quad (13)$$

where $\{\tilde{u}_n(t) = v_T(t)e^{j(2\pi(n+1/2)/T)t}$, $n = 0, 1, \dots, N - 1\}$ are the set of basis functions used in transmission and $\tilde{q}_k[n]$ is the n th entry of the k th user signature code. The transmitted signal for user k is

$$\tilde{s}_k(t) = b_k \sum_{n=0}^{N-1} \tilde{q}_k[n] v_T(t) e^{j(2\pi(n+1/2)t/T)}. \quad (14)$$

Comparing Figures 2b and 2d, we can see that the typically MC-CDMA system is frequency dispersive over a TFS channel since $T_o = T > \Delta t_c$ but nondispersive in time since $B_o < \Delta f_c$. Using the model in [12], the noise-free received signal for user k in a TFS channel can be written as

$$\tilde{x}_k(t) = b_k \sum_{n=0}^{N-1} \sum_{m=-M}^M \tilde{q}_k[n] \tilde{h}_k[m, n] \times v_T(t) e^{j2\pi mt/T} e^{j(2\pi(n+1/2)t/T)}, \quad (15)$$

where $M = \lceil B_d T \rceil$. The frequency dispersion is clear in (15) since the information transmitted over a particular subcarrier is dispersed into the adjacent $2M$ subcarriers due to Doppler dispersion. In (15),

$$\tilde{h}_k[m, n] = \int_{-B_d}^{B_d} \tilde{C}_k\left(\theta, \frac{n+1/2}{T}\right) \text{sinc}(m - \theta T) e^{-j\pi(m-\theta T)} d\theta \quad (16)$$

is the channel coefficient corresponding to the n th subcarrier and m th Doppler shift of the k th user, and $\tilde{C}_k(\theta, f) = \int \tilde{c}_k(t, f) e^{-j2\pi\theta t} dt$ where $\tilde{c}_k(t, f)$ is defined in (2). The overall received signal for all users is

$$\tilde{r}(t) = \sum_{k=1}^K \tilde{x}_k(t) + v(t). \quad (17)$$

Projecting over $\tilde{u}_n(t)$, we get

$$\begin{aligned} \tilde{r}[n] &= \langle \tilde{r}, \tilde{u}_n \rangle \\ &= \frac{1}{\sqrt{T}} \int_0^T r(t) e^{-j(2\pi(n+1/2)t/T)} dt \\ &= \sum_{k=1}^K b_k \sum_{m=-M}^M \tilde{h}_k[m, n-m] \tilde{q}_k[n-m] + \tilde{v}[n]. \end{aligned} \quad (18)$$

The sampled received vector can be written as

$$\tilde{\mathbf{r}} = b_1 \tilde{\mathbf{Q}}_1 \tilde{\mathbf{h}}_1 + \sum_{k=2}^K b_k \tilde{\mathbf{Q}}_k \tilde{\mathbf{h}}_k + \tilde{\mathbf{v}} \quad (19)$$

$$= b_1 \tilde{\mathbf{g}}_1 + \sum_{k=2}^K b_k \tilde{\mathbf{g}}_k + \tilde{\mathbf{v}}, \quad (20)$$

where $\tilde{\mathbf{Q}}_k = [\tilde{\mathbf{Q}}_k(-M) \cdots \tilde{\mathbf{Q}}_k(M)]$ and has a dimension $N \times N(2M+1)$, $\tilde{\mathbf{Q}}_k(m) = \text{diag}\{\tilde{q}_k(m)\}$ and has a dimension $N \times N$, $\tilde{\mathbf{q}}_k$ is the $N \times 1$ signature code of the k th user, $\tilde{\mathbf{h}}_k = [\tilde{h}_{k,-M}^T \cdots \tilde{h}_{k,M}^T]^T$ and has a dimension $N(2M+1) \times 1$, $\tilde{\mathbf{h}}_{k,m} = [\tilde{h}_k[m, 0] \cdots \tilde{h}_k[m, N-1]]^T$ and has a dimension $N \times 1$, and $\tilde{\mathbf{g}}_k = \tilde{\mathbf{Q}}_k \tilde{\mathbf{h}}_k$. Note that, in (19) due to frequency selectivity for a given Doppler shift m , fading coefficients change over different subcarriers.

In FS channels (no time selectivity, $T_o = T < \Delta t_c$), $M = 0$. Hence, in this case, $\tilde{\mathbf{Q}}_k$ reduces to $\tilde{\mathbf{Q}}_k(0)$ and $\tilde{\mathbf{h}}_k$ reduces to $\tilde{h}_{k,0}$. The diagonal structure of the system is evident in this case.⁵ On the other hand, in TS channel (no frequency selectivity, $B, \Delta f_c$), $\tilde{h}_k[m, n] \approx \tilde{h}_k[m, 0]$, $n = 0, 1, \dots, N - 1$. Parallel to the argument in Section 3.1, in this case, $\tilde{\mathbf{h}}_k = [\tilde{h}_k[-M, 0] \cdots \tilde{h}_k[M, 0]]^T$ and $\tilde{\mathbf{Q}}_k(m)$ reduces to $\tilde{q}_k(m)$ so that $\tilde{\mathbf{Q}}_k$ becomes Toeplitz (due to Doppler dispersion).

3.3. MC-DS-CDMA system

In an MC-DS-CDMA system, the prototype basis waveform is $v_{T_o}(t)e^{j(2\pi(1/2)/T_o)t}$ which has a duration $T_o = T/N_t$ and bandwidth $B_o = B/N_f$. Each user is assigned a signature waveform that is generated from time-frequency shifts of the prototype basis waveform as

$$\hat{q}_k(t) = \sum_{n=0}^{N_f-1} \sum_{p=0}^{N_t-1} \hat{q}_k[p, n] \hat{u}_{p,n}(t), \quad (21)$$

where $\{\hat{u}_{p,n}(t) = v_{T_o}(t - pT_o)e^{j(2\pi(n+1/2)/T_o)t}\}$ are the set of basis functions used in transmission and $\hat{q}_k[p, n]$ is the (p, n) th entry of the k th user's spreading code transmitted over $\hat{u}_{p,n}(t)$. The transmitted signal for user k can be written as

$$\hat{s}_k(t) = b_k \sum_{n=0}^{N_f-1} \sum_{p=0}^{N_t-1} \hat{q}_k[p, n] v_{T_o}(t - pT_o) e^{j(2\pi(n+1/2)t/T_o)}. \quad (22)$$

⁴Recall that a DS-CDMA system diagonalizes TS channels.

⁵Recall that MC-CDMA system diagonalizes FS channels.

TABLE 1: Summary of the possible MC-DS-CDMA cases in regard to interaction with the channel.

Case 1	Case 2	Case 3	Case 4
Dispersion in time only $B_o > \Delta f_c, T_o \ll \Delta t_c$	Dispersion in frequency only $B_o \ll \Delta f_c, T_o > \Delta t_c$	No dispersion $B_o \ll \Delta f_c, T_o \ll \Delta t_c$	Dispersion in time and frequency $B_o > \Delta f_c, T_o > \Delta t_c$

Comparing Figures 2c and 2d, depending on the choice of N_t and N_f , the effect of the channel can be categorized into four cases that are summarized in Table 1.

3.3.1 Case 1

In this case, the channel affects each time-frequency basis function in an FS fashion—that is, there is dispersion in time but not in frequency. The noise-free received signal for user k is

$$\hat{x}_k(t) = b_k \sum_{n=0}^{N_f-1} \sum_{p=0}^{N_t-1} \sum_{l=0}^{L_o} h_k[p, n; l] \hat{q}_k[p, n] v_{T_o} \times (t - pT_o - lT_o) e^{j(2\pi(n+1/2)t/T_o)}, \quad (23)$$

where $L_o = \lceil T_m B_o \rceil$ and analogous to (8),

$$h_k[p, n; l] = \int_0^{T_m} c\left(\left(p + \frac{1}{2}\right)T_o, \tau\right) \times \text{sinc}(l - B_o \tau) e^{j2\pi(n+1/2)(l - B_o \tau) \tau} d\tau. \quad (24)$$

The overall received signal for all users is

$$\hat{r}(t) = \sum_{k=1}^K \hat{x}_k(t) + \nu(t). \quad (25)$$

The projection on $\hat{u}_{p,n}(t)$ is

$$\begin{aligned} \hat{r}[p, n] &= \langle \hat{r}, \hat{u}_{p,n} \rangle = \frac{1}{\sqrt{T_o}} \int_{pT_o}^{(p+1)T_o} \hat{r}(t) e^{-j(2\pi(n+1/2)t/T_o)} dt \\ &= \sum_{k=1}^K b_k \sum_{l=0}^{L_o} h_k[p - l, n; l] \hat{q}_k[p - l, n] + \hat{\nu}[p, n], \end{aligned} \quad (26)$$

and the sampled received vector is

$$\hat{\mathbf{r}} = b_1 \hat{\mathbf{Q}}_1^{(f)} \hat{\mathbf{h}}_1^{(f)} + \sum_{k=2}^K b_k \hat{\mathbf{Q}}_k^{(f)} \hat{\mathbf{h}}_k^{(f)} + \hat{\mathbf{v}} \quad (27)$$

$$= b_1 \hat{\mathbf{g}}_1^{(f)} + \sum_{k=2}^K b_k \hat{\mathbf{g}}_k^{(f)} + \hat{\mathbf{v}}, \quad (28)$$

where

$$\hat{\mathbf{Q}}_k^{(f)} = \begin{bmatrix} \hat{\mathbf{Q}}_k^{(f)}(0) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_k^{(f)}(1) & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{\mathbf{Q}}_k^{(f)}(N_f - 1) \end{bmatrix} \quad (29)$$

and has a dimension $N \times N(L_o + 1)$,

$$\hat{\mathbf{Q}}_k^{(f)}(n) = \left[\hat{\mathbf{Q}}_k^{(f)}(n, 0) \cdots \hat{\mathbf{Q}}_k^{(f)}(n, L_o) \right] \quad (30)$$

and has a dimension $N_t \times N_t(L_o + 1)$,

$$\hat{\mathbf{Q}}_k^{(f)}(n, l) = \text{diag} \left\{ \hat{\mathbf{q}}_{k,n}^{(f)}(l) \right\} \quad (31)$$

and has a dimension $N_t \times N_t$,

$$\begin{aligned} \hat{\mathbf{q}}_{k,n}^{(f)} &= \left[\hat{q}_k[0, n] \cdots \hat{q}_k[N_t - 1, n] \right]^T, \\ \hat{\mathbf{h}}_k^{(f)} &= \left[\hat{\mathbf{h}}_{k,0}^{(f)T} \cdots \hat{\mathbf{h}}_{k,N_f-1}^{(f)T} \right]^T \end{aligned} \quad (32)$$

and has a dimension $N(L_o + 1) \times 1$,

$$\hat{\mathbf{h}}_{k,n}^{(f)} = \left[\hat{\mathbf{h}}_{k,n,0}^{(f)T} \cdots \hat{\mathbf{h}}_{k,n,L_o}^{(f)T} \right]^T \quad (33)$$

and has a dimension $N_t(L_o + 1) \times 1$,

$$\hat{\mathbf{h}}_{k,n,l}^{(f)} = \left[h_k[-l, n; l] \cdots h_k[N_t - 1 - l, n; l] \right]^T \quad (34)$$

and has a dimension $N_t \times 1$, and

$$\hat{\mathbf{g}}_k^{(f)} = \hat{\mathbf{Q}}_k^{(f)} \hat{\mathbf{h}}_k^{(f)}. \quad (35)$$

The superscript f stands for frequency domain since $\hat{\mathbf{q}}_{k,n}^{(f)}$ corresponds to part of the signature code corresponding to all time shifts of the n th frequency (n th row in Figure 2c).

Note that the DS-CDMA system in TFS channels is a special case of this system when $N_f = 1$ (hence, $N_t = N$ and $L_o = L$).

3.3.2 Case 2

In this case, the channel affects each time-frequency basis function in a TS fashion—that is, there is dispersion in frequency but not in time. The noise-free received signal is

$$\begin{aligned} \hat{x}_k(t) &= b_k \sum_{n=0}^{N_f-1} \sum_{p=0}^{N_t-1} \sum_{m=-M_o}^{M_o} \tilde{h}_k[p, n; m] \hat{q}_k[p, n] v_{T_o} \\ &\quad \times (t - pT_o) e^{j(2\pi mt/T_o)} e^{j(2\pi(n+1/2)t/T_o)}, \end{aligned} \quad (36)$$

where $M_o = \lceil B_d T_o \rceil$ and

$$\begin{aligned} \tilde{h}_k[p, n; m] &= \int_{-B_d}^{B_d} \tilde{C}_k\left(\theta, \frac{n+1/2}{T_o}\right) \text{sinc}(m - \theta T_o) \\ &\quad \times e^{-j2\pi(p+1/2)(m - \theta T_o)} d\theta. \end{aligned} \quad (37)$$

In this case, $\hat{r}[p, n]$ becomes

$$\hat{r}[p, n] = \sum_{k=1}^K b_k \sum_{m=-M_o}^{M_o} \tilde{h}_k[p, n - m; m] \hat{q}_k[p, n - m] + \hat{v}[p, n], \quad (38)$$

and the received vector is

$$\hat{\mathbf{r}} = b_1 \hat{\mathbf{Q}}_1^{(t)} \hat{\mathbf{h}}_1^{(t)} + \sum_{k=2}^K b_k \hat{\mathbf{Q}}_k^{(t)} \hat{\mathbf{h}}_k^{(t)} + \hat{\mathbf{v}} \quad (39)$$

$$= b_1 \hat{\mathbf{g}}_1^{(t)} + \sum_{k=2}^K b_k \hat{\mathbf{g}}_k^{(t)} + \hat{\mathbf{v}}, \quad (40)$$

where

$$\hat{\mathbf{Q}}_k^{(t)} = \begin{bmatrix} \hat{\mathbf{Q}}_k^{(t)}(0) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Q}}_k^{(t)}(1) & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{\mathbf{Q}}_k^{(t)}(N_t - 1) \end{bmatrix} \quad (41)$$

and has a dimension $N \times N(2M_o + 1)$,

$$\hat{\mathbf{Q}}_k^{(t)}(p) = [\hat{\mathbf{Q}}_k^{(t)}(p, -M_o) \cdots \hat{\mathbf{Q}}_k^{(t)}(p, M_o)] \quad (42)$$

and has a dimension $N_f \times N_f(2M_o + 1)$,

$$\hat{\mathbf{Q}}_k^{(t)}(p, m) = \text{diag} \{ \hat{\mathbf{q}}_{k,p}^{(t)}(m) \} \quad (43)$$

and has a dimension $N_f \times N_f$,

$$\hat{\mathbf{q}}_{k,p}^{(t)} = [\hat{q}_k[p, 0] \cdots \hat{q}_k[p, N_f - 1]]^T, \quad (44)$$

$$\hat{\mathbf{h}}_k^{(t)} = [\hat{\mathbf{h}}_{k,0}^{(t)T} \cdots \hat{\mathbf{h}}_{k,N_f-1}^{(t)T}]^T$$

and has a dimension $N(2M_o + 1) \times 1$,

$$\hat{\mathbf{h}}_{k,p}^{(t)} = [\hat{\mathbf{h}}_{k,p,-M_o}^{(t)T} \cdots \hat{\mathbf{h}}_{k,p,M_o}^{(t)T}]^T \quad (45)$$

and has a dimension $N_f(2M_o + 1) \times 1$,

$$\hat{\mathbf{h}}_{k,p,m}^{(t)} = [\tilde{h}_k[p, 0; m] \cdots \tilde{h}_k[p, N_f - 1; m]]^T \quad (46)$$

and has a dimension $N_f \times 1$, and

$$\hat{\mathbf{g}}_k^{(t)} = \hat{\mathbf{Q}}_k^{(t)} \hat{\mathbf{h}}_k^{(t)}. \quad (47)$$

The superscript t stands for time domain since $\hat{\mathbf{q}}_{k,p}^{(t)}$ corresponds to all frequency shifts of the p th time shift (p th column in Figure 2c).

Note that MC-CDMA system over TFS channel is a special case of this system when $N_t = 1$ (hence, $N_f = N$ and $M_o = M$).

3.3.3 Case 3

In this case, the channel affects each time-frequency basis function in a nonselective fashion in both time and frequency domains. That is, the effective channel matrix is diagonal in this case. This interesting signaling scheme is discussed in details in [7, 12], and we call it a TF-CDMA system. It was shown that the conditions on N_f and N_t necessary for this case to hold can be met for underspread channels, that is, $T_m B_d \ll 1$ [13]. Basically, the conditions in this case are equivalent to having $T_m \ll T_o$ and $B_d \ll B_o$. That is, $T_m B_d \ll T_o B_o = 1$ which is the condition for underspread channels. The noise-free received signal in this case is given by [7]

$$\hat{x}_k(t) = b_k \sum_{n=0}^{N_f-1} \sum_{p=0}^{N_t-1} \hat{h}_k[p, n] \hat{q}_k[p, n] \times v_{T_o}(t - pT_o) e^{j(2\pi(n+1/2)t/T_o)}, \quad (48)$$

where

$$\hat{h}_k[p, n] = \int_{pT_o}^{(p+1)T_o} \int_{nB_o}^{(n+1)B_o} \tilde{c}_k(t, f) df dt \approx \tilde{c}_k\left(\left(p + \frac{1}{2}\right)T_o, \left(n + \frac{1}{2}\right)B_o\right). \quad (49)$$

The approximation in (49) is due to the fact that, for proper choice of T_o and B_o , $\tilde{c}_k(t, f)$ is almost flat over the basis function's time and frequency support. In this case, $\hat{r}[p, n]$ in (26) is

$$\hat{r}[p, n] = \sum_{k=1}^K b_k \hat{h}_k[p, n] \hat{q}_k[p, n] + \hat{v}[p, n] \quad (50)$$

from which the diagonal nature of the system is evident. The received vector is given by (27) with the following definitions:

$$\begin{aligned} \hat{\mathbf{Q}}_k^{(f)} &= \text{diag} \{ \hat{\mathbf{q}}_k^{(f)} \}, \\ \hat{\mathbf{q}}_k^{(f)} &= [\hat{\mathbf{q}}_{k,0}^{(f)T} \cdots \hat{\mathbf{q}}_{k,N_f-1}^{(f)T}]^T, \\ \hat{\mathbf{h}}_k^{(f)} &= [\hat{\mathbf{h}}_{k,0}^{(f)T} \cdots \hat{\mathbf{h}}_{k,N_f-1}^{(f)T}]^T, \end{aligned} \quad (51)$$

where $\hat{\mathbf{h}}_{k,n}^{(f)} = [\hat{h}_k[0, n] \cdots \hat{h}_k[N_t - 1, n]]^T$. The sampled received vector can be also written as (39) with the following definitions:

$$\begin{aligned} \hat{\mathbf{Q}}_k^{(t)} &= \text{diag} \{ \hat{\mathbf{q}}_k^{(t)} \}, \\ \hat{\mathbf{q}}_k^{(t)} &= [\hat{\mathbf{q}}_{k,0}^{(t)T} \cdots \hat{\mathbf{q}}_{k,N_t-1}^{(t)T}]^T, \\ \hat{\mathbf{h}}_k^{(t)} &= [\hat{\mathbf{h}}_{k,0}^{(t)T} \cdots \hat{\mathbf{h}}_{k,N_t-1}^{(t)T}]^T, \end{aligned} \quad (52)$$

where $\hat{\mathbf{h}}_{k,p}^{(t)} = [\hat{h}_k[p, 0] \cdots \hat{h}_k[p, N_f - 1]]^T$.

The MC-CDMA system over an FS channel is a special case of this system when $N_t = 1$ (hence $N_f = N$) and $B_d = 0$.

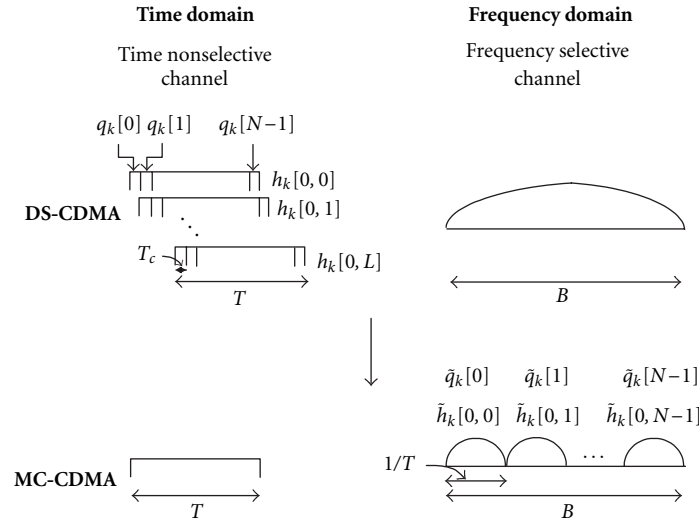


FIGURE 3: DS-CDMA and MC-CDMA systems in an FS channel. The DS-CDMA basis waveforms experience multipath dispersion in time due to frequency selectivity. The MC-CDMA system is diagonal since each subcarrier encounters nonselective fading in the frequency domain.

Also, the DS-CDMA system over a TS channel is a special case of this system when $N_f = 1$ (hence $N_t = N$) and $T_m = 0$.

3.3.4 Case 4

In this case, the channel affects each basis function in a TFS fashion—that is, there is dispersion in time and in frequency. The conditions above imply that $T_o < T_m$ and $B_o < B_d \Rightarrow T_m B_d > T_o B_o = 1$. This is the case of overspread channels and is not addressed in this paper since typical wireless channels are underspread [1].

3.3.5 Some remarks on special cases

Cases 1, 2, 3, and 4 discuss MC-DS-CDMA system behavior over TFS channels. The MC-DS-CDMA system in an FS (TS) channel experiences Case 1 (Case 2) since the channel is FS (TS) per basis waveform (so that the basis functions encounter multipath (Doppler) dispersion), and Case 3 where the channel is nonselective per basis waveform. The system cannot experience Case 2 (Case 1) in an FS (TS) channel due to the absence of time (frequency) selectivity. In the sequel, whenever we address the MC-DS-CDMA system over an FS (TS) channel, we only consider Cases 1 and 3 (2 and 3). The following special cases hold.

(1) Case 1 in the MC-DS-CDMA system over an FS channel is a special case of that in Section 3.3.1 since $B_d = 0$, and thus, $T \ll \Delta t_c$ (as opposed to the less stringent condition $T_o \ll \Delta t_c$ in Section 3.3.1). In this case, $h_k[p, n; l] \approx h_k[0, n; l]$, $p = 0, 1, \dots, N_t - 1$, $\hat{\mathbf{Q}}_k^{(f)}(n)$ becomes a Toeplitz matrix constituted from $\hat{\mathbf{q}}_{k,n}^{(f)}$, and $\hat{\mathbf{h}}_{k,n}^{(f)} = [h_k[0, n; 0] \cdots h_k[0, n; L_o]]^T$.

(2) Case 2 in the MC-DS-CDMA system in TS channel is a special case of that in Section 3.3.2 when $T_m = 0$, and thus, $B \ll \Delta f_c$ (as opposed to $B_o \ll \Delta f_c$ in Section 3.3.2). In this case, $\hat{h}_k[p, n; m] \approx \hat{h}_k[p, 0; m]$, $n = 0, 1, \dots, N_f - 1$,

$\hat{\mathbf{Q}}_k(p)$ becomes Toeplitz matrix constituted from $\hat{\mathbf{q}}_{k,p}$, and $\hat{\mathbf{h}}_{k,p} = [\hat{h}_k[p, 0, -M_o] \cdots \hat{h}_k[p, 0; M_o]]^T$.

(3) Case 3 in the MC-DS-CDMA system in an FS channel is a special case of that in Section 3.3.3 when $B_d = 0$, that is, $\hat{h}_k[p, n] \approx \hat{h}_k[0, n]$, $p = 0, 1, \dots, N_t - 1$. In this case,

$$\hat{\mathbf{Q}}_k^{(f)} = \begin{bmatrix} \hat{\mathbf{q}}_{k,0}^{(f)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{q}}_{k,1}^{(f)} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{\mathbf{q}}_{k,N_f-1}^{(f)} \end{bmatrix}, \quad (53)$$

$$\mathbf{h}_k^{(f)} = [\hat{h}_k[0, 0] \cdots \hat{h}_k[0, N_f - 1]]^T.$$

(4) Case 3 in the MC-DS-CDMA system in a TS channel is a special case of that in Section 3.3.3 when $T_m = 0$, that is, $\hat{h}_k[p, n] \approx \hat{h}_k[p, 0]$, $n = 0, 1, \dots, N_f - 1$. In this case,

$$\hat{\mathbf{Q}}_k^{(t)} = \begin{bmatrix} \hat{\mathbf{q}}_{k,0}^{(t)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{q}}_{k,1}^{(t)} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{\mathbf{q}}_{k,N_f-1}^{(t)} \end{bmatrix}, \quad (54)$$

$$\mathbf{h}_k^{(t)} = [\hat{h}_k[0, 0] \cdots \hat{h}_k[N_t - 1, 0]]^T.$$

4. TIME-FREQUENCY DUALITY

As shown in Section 3 (and illustrated in Figure 3), in an FS channel, each basis function in a DS-CDMA system encounters FS fading since $B_o > \Delta f_c$. Consequently, the transmitted signal encounters multipath dispersion and a RAKE receiver is needed for multipath combining. To avoid multipath dispersion, the MC-CDMA system uses basis waveforms that

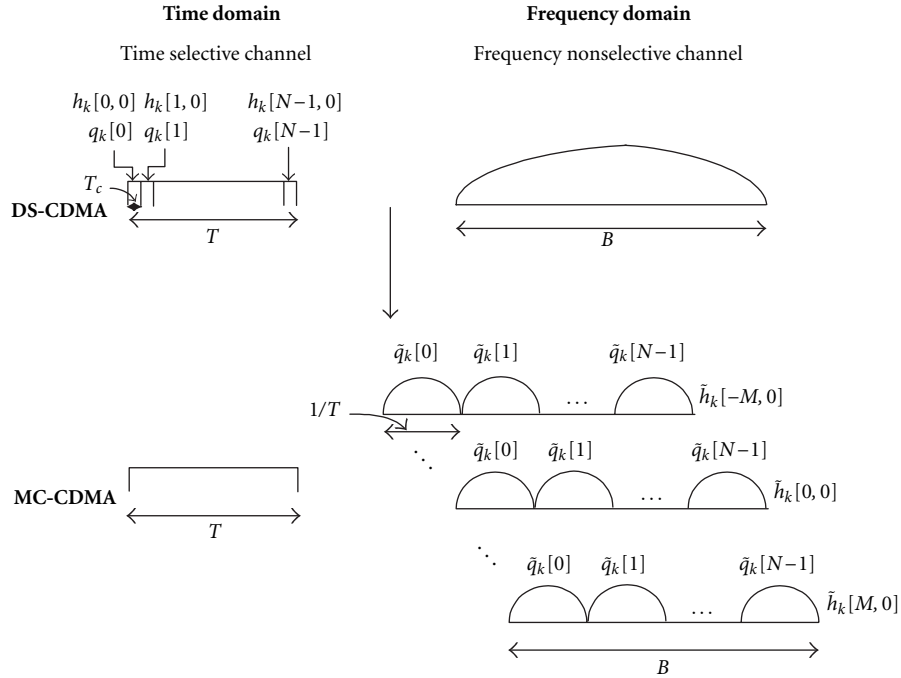


FIGURE 4: DS-CDMA and MC-CDMA systems in a TS channel. The MC-CDMA basis waveforms experience multiple Doppler dispersion in frequency due to time selectivity. The DS-CDMA system is diagonal since each basis waveform (chip) encounters nonselective fading in the time domain.

are narrowband subcarriers with $B_o \ll \Delta f_c$. Hence, each basis function encounters nonselective fading and the resultant channel matrix is diagonal. The diversity due to frequency selectivity is exploited via $L+1$ multipath in DS-CDMA system and via $L+1$ subcarriers separated by Δf_c in the MC-CDMA system. The diversity order in both systems is $L+1$, which is equal to the rank of the channel matrix for both systems, that is, $\text{rank}(\mathbf{R}_{h_k, h_k}) = \text{rank}(\mathbf{R}_{\tilde{h}_k, \tilde{h}_k}) = L+1$.

The behavior of the two systems in TS channels is completely analogous to that discussed above for FS channels except for interchanging time and frequency domains and DS-CDMA and MC-CDMA systems. Specifically, in a DS-CDMA system in a TS channel, the channel matrix is diagonal as long as $T_o \ll \Delta t_c$ —each basis function (chip) encounters nonselective fading. On the other hand, each MC-CDMA system basis function suffers TS fading since $T_o = T > \Delta t_c$. This is manifested as multiple Doppler components, as illustrated in Figure 4, analogous to multipath components in a DS-CDMA system in an FS channel. Consequently, the MC-CDMA system over a TS channel requires a frequency domain RAKE receiver to do multiple Doppler combining. The order of diversity in both systems is the same and is equal to $2M+1$ —the channel covariance matrix for both systems has a rank of $2M+1$. This duality between MC-CDMA and DS-CDMA as well as between TS and FS channels, will be used in Section 5 to derive equivalences between different systems.

Using the previous discussion, we can deduce intuitive insights about diagonalizing the TFS channel in Section 3.3.3. We define the notion of block fading (BF) for

an MC-CDMA system in an FS channel and a DS-CDMA system in a TS channel. For an MC-CDMA system, if the whole bandwidth is divided into N_f subbands, where N_f is chosen sufficiently larger than $L+1$, then under BF assumption, the N_f subcarriers in the same subband (corresponding to Δf_c) encounter identical fading coefficients.⁶ More precisely, if the set of indices of subcarriers in the i th coherence subband is $\Omega_i^{(f)}$, $i = 0, 1, \dots, N_f - 1$, then, under BF assumption (recall (16)), $\tilde{h}_k[m, n_1] \approx \tilde{h}_k[m, n_2]$ for all $n_1, n_2 \in \Omega_i^{(f)}$. A similar definition holds for a DS-CDMA system in a TS channel. If the whole symbol duration is divided into N_t time slots, where N_t is chosen sufficiently larger than $2M+1$, then, under BF assumption, the N_f chips in the same time slot (corresponding to Δt_c) encounter identical fading coefficients.⁷ That is, if the set of indices of chips in the i th coherence time slot is $\Omega_i^{(t)}$, $i = 0, 1, \dots, N_t - 1$, then, under BF assumption (recall (8)), $h_k[p_1, l] \approx h_k[p_2, l]$ for all $p_1, p_2 \in \Omega_i^{(t)}$.

Now, consider an MC-CDMA system in a TFS channel. Divide the bandwidth into N_f coherence subbands such that BF holds for the N_f subcarriers per subband. Now, per subband, the channel is frequency nonselective but may be TS. That is, each subband resembles a TS channel and multiple Doppler components are generated. Hence, to diagonalize

⁶Note that N_f sufficiently larger than $L+1$ is equivalent to B_o sufficiently smaller than Δf_c .

⁷Note that N_t sufficiently larger than $2M+1$ is equivalent to T_o sufficiently smaller than Δt_c .

the system (analogous to DS-CDMA system over a TS channel), we need to choose basis waveforms that are narrower in time so that each encounters nonselective fading. This is precisely the case with the basis waveforms of the TF-CDMA system in Section 3.3.3. Compared to an MC-CDMA system, the symbol duration is decreased by a factor of N_t , that is, $T_o = T/N_t$ and, consequently, the bandwidth increases N_t times to $B_o = BN_t/N = B/N_f$. Since B_o equals the subband bandwidth, flat fading in the frequency domain is still preserved. A dual way of looking at the TF-CDMA system in Section 3.3.3 is via a DS-CDMA systems in a TFS channel. In this case, T is divided into well-chosen N_t slots such that BF holds in time. Per time slot, the N_f time samples encounter FS but time nonselective fading. To diagonalize the system, we need to decrease the bandwidth of each basis waveform by a factor of N_f and increase the time duration by the same factor to achieve nonselective fading. This idea is illustrated in Figures 5 and 6.

5. EQUIVALENCES

In this section, we derive the sufficient conditions for the CDMA systems in Section 3 to be equivalent. We say that two systems are equivalent if both attain the same SINR for any given set of channel realizations for all users. We can easily deduce that, if two systems are equivalent, then both also attain the same P_e . Before introducing the equivalence results, we derive the linear MMSE detector, the expression for SINR, and P_e .

We can note from (12), (20), and (28) that the sampled received vector for a given system x is

$$\begin{aligned} \mathbf{r}_x &= b_1 \mathbf{g}_1^{(x)} + \sum_{k=2}^K b_k \mathbf{g}_k^{(x)} + \mathbf{v}_x \\ &= b_1 \mathbf{g}_1^{(x)} + \mathbf{i}_x + \mathbf{v}_x = \mathbf{g}_1^{(x)} + \epsilon_x, \end{aligned} \quad (55)$$

where $\mathbf{g}_k^{(x)} = \mathbf{Q}_k^{(x)} \mathbf{h}_k^{(x)}$, $\mathbf{Q}^{(x)}$ is a matrix that is a function of the k th user signature code for system x , $\mathbf{h}^{(x)}$ is the corresponding channel coefficient vector, \mathbf{i}_x is the MAI vector affecting the desired signal component, \mathbf{v}_x is the noise vector, and ϵ_x is the noise plus interference vector. The linear MMSE detector for user 1 takes the form $\hat{b}_1 = \text{sign}\{\text{real}[\mathbf{w}_x^{\text{MMSE}H} \mathbf{r}_x]\}$ and $\mathbf{w}_x^{\text{MMSE}}$ is chosen to solve

$$\mathbf{w}_x^{\text{MMSE}} = \min_{\mathbf{w}} E[|\mathbf{w}^H \mathbf{r}_x - b_1|^2]. \quad (56)$$

The solution is the well-known Wiener filter

$$\mathbf{w}_x^{\text{MMSE}} = \mathbf{R}_{\mathbf{r}_x \mathbf{r}_x}^{-1} \mathbf{g}_1^{(x)}. \quad (57)$$

The SINR can be written as [14]

$$\text{SINR}^{(x)} = \mathbf{g}_1^{(x)H} \mathbf{R}_{\epsilon_x \epsilon_x}^{-1} \mathbf{g}_1^{(x)}, \quad (58)$$

where $\mathbf{R}_{\epsilon_x \epsilon_x} = \sum_{k=2}^K \mathbf{g}_k^{(x)} \mathbf{g}_k^{(x)H} + \sigma^2 \mathbf{I}_N$. There is no closed form expression for the receiver P_e in the presence of MAI.

However, parallel to the discussion in [15], we note that, for sufficiently large N , the Gaussian approximation of the interference at the output of the MMSE is fairly accurate. In this case, the conditional P_e given $\mathbf{h}_1^{(x)}, \dots, \mathbf{h}_K^{(x)}$ can be approximated as

$$P_e^{(x)}(\mathbf{h}_1^{(x)}, \dots, \mathbf{h}_K^{(x)}) = Q\left(\sqrt{2 \mathbf{g}_1^{(x)H} \mathbf{R}_{\epsilon_x \epsilon_x}^{-1} \mathbf{g}_1^{(x)}}\right), \quad (59)$$

where $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-x^2/2} dx$. To find the average P_e , we need to average (59) over the distribution of $\mathbf{h}_1^{(x)}, \dots, \mathbf{h}_K^{(x)}$. However, under the assumption of treating the MAI term at the output of the MMSE detector as Gaussian noise, we only need to average (59) over $\mathbf{h}_1^{(x)}$ (or equivalently over $\mathbf{g}_1^{(x)}$). Recalling that $\mathbf{g}_1^{(x)}$ is a complex Gaussian vector, the average $P_e^{(x)}$ over the distribution of $\mathbf{g}_1^{(x)}$ is [1]

$$\begin{aligned} P_e^{(x)} &= E[P_e(\mathbf{h}_1^{(x)})] \\ &= \sum_{l=1}^D \frac{\pi_l}{2} \left[1 - \sqrt{\frac{\mu_l}{1 + \mu_l}} \right], \quad \pi_l = \prod_{i=1, i \neq l}^D \frac{\mu_l}{\mu_l - \mu_i}, \end{aligned} \quad (60)$$

where μ_l , $l = 1, 2, \dots, D$, are the nonzero eigenvalues of $\mathbf{R}_{\epsilon_x \epsilon_x}^{-1} \mathbf{R}_{\mathbf{g}_1^{(x)} \mathbf{g}_1^{(x)}}$. For the numerical results in Section 6, we adopt this Gaussian approximation and use the SINR and P_e expressions in (58) and (60), respectively.

Recalling that two systems are equivalent if both attain the same SINR, from the expression in (58), we note that a sufficient condition for systems x and y to be equivalent is to have

$$\mathbf{g}_k^{(y)} = \mathbf{F} \mathbf{g}_k^{(x)}, \quad k = 1, 2, \dots, K, \quad (61)$$

where \mathbf{F} is an $N \times N$ unitary transformation, that is, $\mathbf{F} \mathbf{F}^H = \mathbf{F}^H \mathbf{F} = \mathbf{I}_N$. From (12), (20), and (28) or (40), we note that we can always write $\mathbf{g}_k^{(x)}$ as

$$\mathbf{g}_k^{(x)} = \mathbf{Q}_k^{(x)} \mathbf{h}_k^{(x)} = \mathbf{H}_k^{(x)} \mathbf{q}_k^{(x)}, \quad (62)$$

where the multiplication order in the first equality is flipped in the second one using the appropriate $N \times N$ channel matrix $\mathbf{H}_k^{(x)}$. Now, (61) becomes

$$\mathbf{H}_k^{(y)} \mathbf{q}_k^{(y)} = \mathbf{F} \mathbf{H}_k^{(x)} \mathbf{q}_k^{(x)}. \quad (63)$$

If $\mathbf{q}_k^{(y)} = \mathbf{F} \mathbf{q}_k^{(x)}$, then condition (63) for equivalence becomes

$$\mathbf{H}_k^{(x)} = \mathbf{F}^H \mathbf{H}_k^{(y)} \mathbf{F}. \quad (64)$$

Note that $\mathbf{H}^{(x)}$ and $\mathbf{H}^{(y)}$ have identical statistics since \mathbf{F} is unitary.

In Sections 5.1, 5.2, and 5.3, we implicitly solve the following problem. Given $\mathbf{q}_k^{(x)}$, $\mathbf{H}_k^{(x)}$, and $\mathbf{H}_k^{(y)}$, find an \mathbf{F} such that $\mathbf{q}_k^{(y)} = \mathbf{F} \mathbf{q}_k^{(x)}$ and (64) is satisfied. In other words, for the same channel, realization systems x and y have identical performance when $\mathbf{q}_k^{(x)}$ and $\mathbf{q}_k^{(y)}$ are chosen as the signature

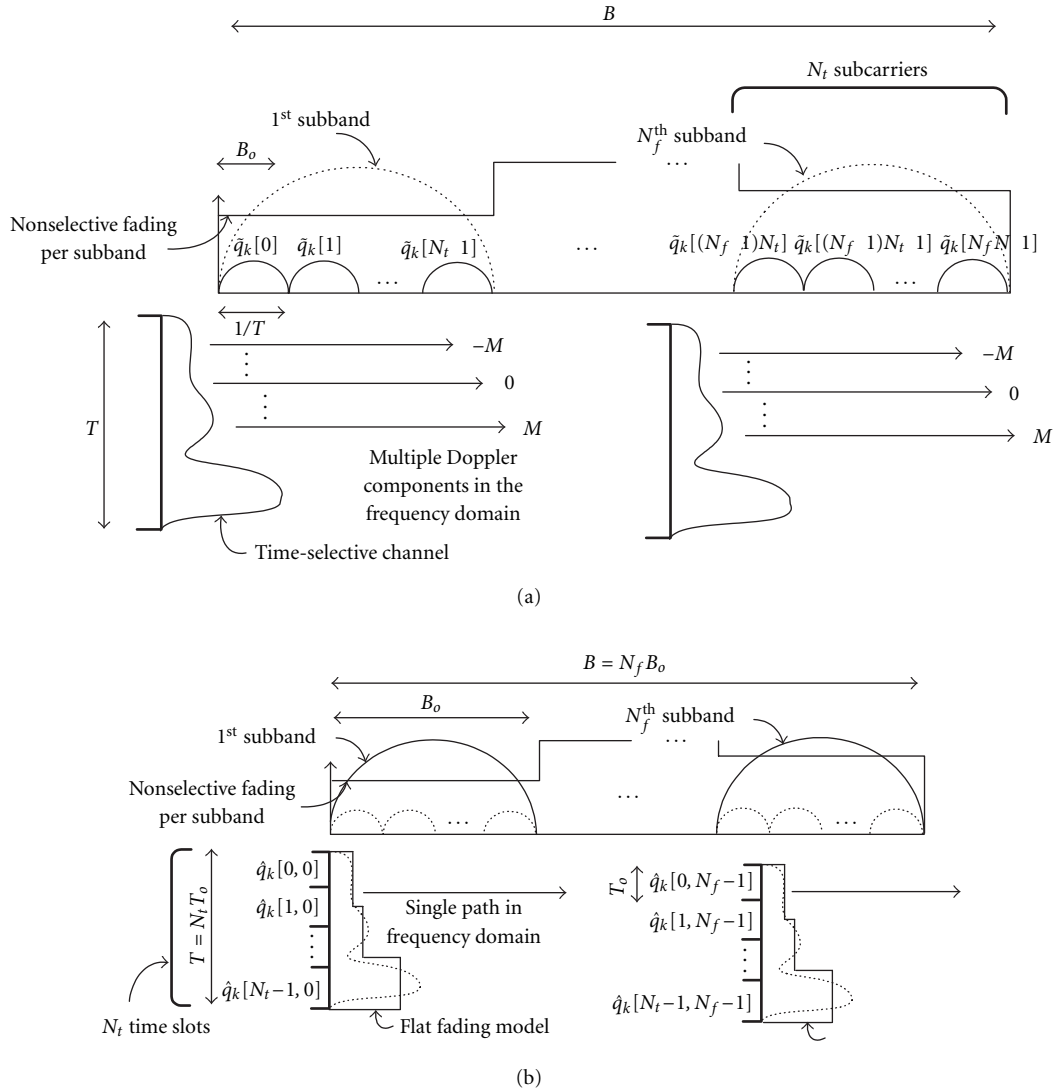


FIGURE 5: Diagonalizing a TFS channel via a TF-CDMA system starting from an MC-CDMA system. The bandwidth is divided into N_f subbands, each containing N_t subcarriers that fade similarly. To diagonalize TFS channel, each basis waveform duration is reduced by N_t and its bandwidth is increased by N_t so that the new basis waveforms encounter nonselective fading in time and frequency. (a) MC-CDMA system. (b) TF-CDMA system.

codes. We start by finding sufficient conditions for equivalences in an FS channel. Using the duality relations described in Section 4, in conjunction with necessary additional proofs, we show the equivalence conditions for the TS and TFS channels in Sections 5.2 and 5.3, respectively.

5.1. Equivalences in an FS channel

We first recall from Section 3.3.5 that, in an FS channel, the MC-DS-CDMA system only exhibits Cases 1 and 3. In Proposition 1, we state the sufficient conditions for a DS-CDMA system to be equivalent to an MC-CDMA system. In Proposition 2, we state the equivalence conditions for an MC-CDMA system and an MC-DS-CDMA system. The equivalence conditions for a DS-CDMA system and an MC-DS-CDMA system will then follow.

Proposition 1. *DS-CDMA system is equivalent to an MC-CDMA system in an FS channel if the following condition is satisfied:*

$$\tilde{\mathbf{q}}_k = \mathbf{D}_N \mathbf{q}_k, \quad (65)$$

where \mathbf{D}_N is the $N \times N$ discrete Fourier transform (DFT) matrix with $D_{n,m} = (1/\sqrt{N})e^{-j(2\pi nm/N)}$.

Proof. Recall that \mathbf{Q}_k and \mathbf{h}_k are the signature code matrix and channel vector for the DS-CDMA system and $\tilde{\mathbf{Q}}_k$ and $\tilde{\mathbf{h}}_k$ are the corresponding quantities in the MC-CDMA system. As discussed in Section 3.1, \mathbf{Q}_k is a Toeplitz matrix in an FS channel. In Appendix A, we show that

$$\tilde{\mathbf{h}}_k = \mathbf{D}_N(1:L+1)\mathbf{h}_k \quad (66)$$

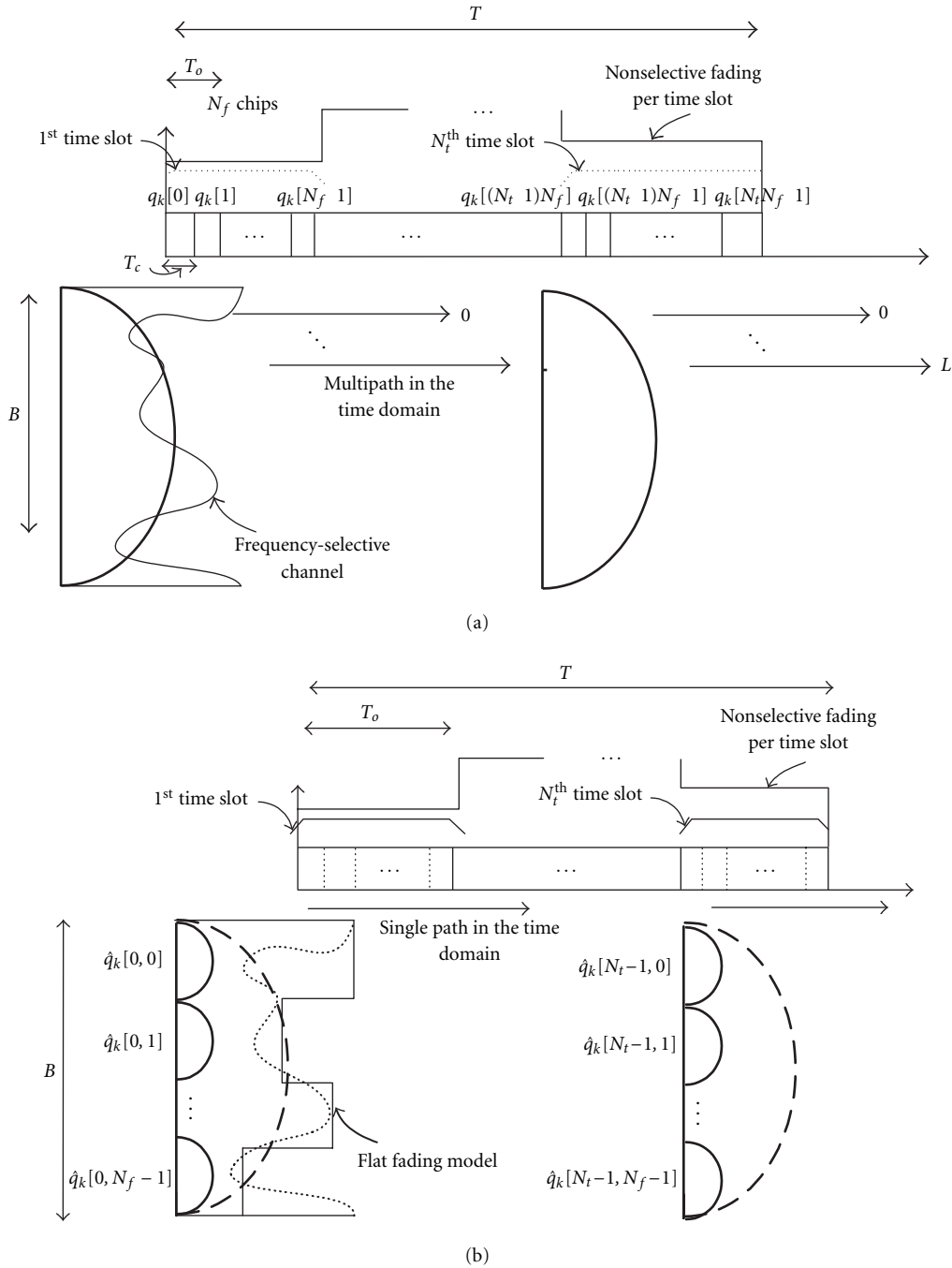


FIGURE 6: Diagonalizing TFS channel via TF-CDMA system starting from a DS-CDMA system. The symbol duration is divided into N_t time slots, each containing N_f time samples that fade similarly. To diagonalize the TFS channel, each basis waveform bandwidth is reduced by N_f and its duration is increased by N_f so that the new basis waveforms encounter flat fading in time and frequency. (a) DS-CDMA system. (b) TF-CDMA system.

so that $\mathbf{D}_N(L+2:N)^H \tilde{\mathbf{h}}_k = \mathbf{0}$. Hence, using (12),

$$\mathbf{g}_k = \mathbf{Q}_k \mathbf{D}_N(1:L+1)^H \tilde{\mathbf{h}}_k = \tilde{\mathbf{Q}}_k \mathbf{D}_N^H \tilde{\mathbf{h}}_k, \quad (67)$$

where $\tilde{\mathbf{Q}}_k = [\mathbf{Q}_k \ \mathbf{X}]$ for any appropriately sized arbitrary matrix \mathbf{X} . We note that, in general, $L \ll N$ so that \mathbf{Q}_k can be approximated with a circulant matrix. We can then choose \mathbf{X}

so that $\tilde{\mathbf{Q}}_k$ is circulant and multiply \mathbf{g}_k by \mathbf{D}_N to get

$$\mathbf{D}_N \mathbf{g}_k \approx \mathbf{D}_N \tilde{\mathbf{Q}}_k \mathbf{D}_N^H \tilde{\mathbf{h}}_k = \tilde{\tilde{\mathbf{Q}}}_k \tilde{\mathbf{h}}_k = \tilde{\tilde{\mathbf{g}}}_k, \quad (68)$$

where $\tilde{\tilde{\mathbf{Q}}}_k = \text{diag}\{\mathbf{D}_N \mathbf{q}_k\}$ since the DFT matrix diagonalizes circulant matrices. We can see that $\tilde{\mathbf{g}}_k = \tilde{\mathbf{Q}}_k \tilde{\mathbf{h}}_k$ in (20) for an FS channel is equal to $\tilde{\tilde{\mathbf{g}}}_k = \tilde{\tilde{\mathbf{Q}}}_k \tilde{\mathbf{h}}_k$ in (68) provided that

$\tilde{\mathbf{q}}_k = \mathbf{D}_N \mathbf{q}_k$. The proof then follows from our definition of equivalences in (61). \square

Proposition 2. *The MC-DS-CDMA system in Cases 1 and 3 is equivalent to an MC-CDMA system provided that the following condition is satisfied:*

$$\tilde{\mathbf{q}}_k = \bar{\mathbf{D}}_{N_t} \hat{\mathbf{q}}_k^{(f)}, \quad (69)$$

where

$$\bar{\mathbf{D}}_{N_t} = \begin{bmatrix} \mathbf{D}_{N_t} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{N_t} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{D}_{N_t} \end{bmatrix} \quad (70)$$

and $\hat{\mathbf{q}}_k^{(f)} = [\hat{\mathbf{q}}_{k,0}^{(f)T} \cdots \hat{\mathbf{q}}_{k,N_f-1}^{(f)T}]^T$ of dimension $N \times 1$.

Proof. We first prove Proposition 2 for the MC-DS-CDMA system in Case 1. We recall that, in an FS channel, each subband n in the MC-DS-CDMA system in Case 1 encounters multipath dispersion. The proof of this proposition is then obtained by applying the proof of Proposition 1 per subband n . We start with writing $\hat{\mathbf{g}}_k^{(f)}$ in (28) as

$$\hat{\mathbf{g}}_k^{(f)} = [\hat{\mathbf{g}}_{k,0}^{(f)T} \cdots \hat{\mathbf{g}}_{k,N_f-1}^{(f)T}]^T, \quad (71)$$

where $\hat{\mathbf{g}}_{k,n}^{(f)} = \hat{\mathbf{Q}}_k^{(f)}(n) \hat{\mathbf{h}}_{k,n}^{(f)}$. Following the proof in Appendix A, we can show that

$$\tilde{\mathbf{h}}_{k,n} = \mathbf{D}_{N_t} (1 : L_o + 1) \hat{\mathbf{h}}_{k,n}^{(f)}, \quad (72)$$

where $\tilde{\mathbf{h}}_{k,n} = [\tilde{h}_{k,n}[0, nN_t] \cdots \tilde{h}_{k,n}[0, (n+1)N_t - 1]]$ is the n th segment of $\tilde{\mathbf{h}}_k$ of length N_t . Recall from Section 3.3.5 that, in an FS channel, $\hat{\mathbf{Q}}_k^{(f)}(n)$ is Toeplitz constituted from $\hat{\mathbf{q}}_{k,n}^{(f)}$. Hence, using (72) and following the proof of Proposition 1, we can show that

$$\begin{aligned} \mathbf{D}_{N_t} \hat{\mathbf{g}}_{k,n}^{(f)} &\approx \mathbf{D}_{N_t} \hat{\mathbf{Q}}_k^{(f)}(n) \mathbf{D}_{N_t} (1 : L_o + 1)^H \tilde{\mathbf{h}}_{k,n} \\ &= \ddot{\mathbf{Q}}_{k,n} \tilde{\mathbf{h}}_{k,n}, \end{aligned} \quad (73)$$

where $\ddot{\mathbf{Q}}_{k,n} = \text{diag}\{\mathbf{D}_{N_t} \hat{\mathbf{q}}_{k,n}^{(f)}\}$. That is,

$$\bar{\mathbf{D}}_{N_t} \hat{\mathbf{g}}_k^{(f)} \approx \ddot{\mathbf{Q}}_k \tilde{\mathbf{h}}_k = \check{\mathbf{g}}_k, \quad (74)$$

where $\check{\mathbf{Q}}_k = \text{diag}\{\bar{\mathbf{D}}_{N_t} \hat{\mathbf{q}}_k^{(f)}\}$. We can see that $\check{\mathbf{g}}_k$ in (20), for the FS channel, is equal to $\check{\mathbf{g}}_k$ in (74) provided that $\tilde{\mathbf{q}}_k = \bar{\mathbf{D}}_{N_t} \hat{\mathbf{q}}_k^{(f)}$. The proof for Case 1 MC-DS-CDMA system then follows.

The proof of Proposition 2 for an MC-DS-CDMA system in Case 3 is a special case of the proof above when $L_o = 0$ (so that each subband encounters frequency nonselective fading). To see this, we first recall that, in this case, the BF assumption is valid since $B_o \ll \Delta f_c$. From Section 4, under BF assumption, $\tilde{\mathbf{h}}_k$ can be written as

$$\tilde{\mathbf{h}}_k \approx \begin{bmatrix} \tilde{h}_k[0, 0] \mathbf{1}_{N_t}^T & \cdots & \tilde{h}_k[0, N_f - 1] \mathbf{1}_{N_t}^T \end{bmatrix}^T \quad (75)$$

$$\approx \begin{bmatrix} \hat{h}_k[0, 0] \mathbf{1}_{N_t}^T & \cdots & \hat{h}_k[0, N_f - 1] \mathbf{1}_{N_t}^T \end{bmatrix}^T, \quad (76)$$

that is, $\tilde{\mathbf{h}}_{k,n} = \tilde{h}_k[0, n] \mathbf{1}_{N_t}$. In (76), we used the fact that $\tilde{h}_k[0, n] = \hat{h}_k[0, n]$ for FS channels. To recognize this fact, substitute $\tilde{c}_k(t, f) = \tilde{c}_k(0, f)$ and $\tilde{C}_k(\theta, f) = \tilde{c}_k(0, f) \delta(\theta)$ in (16) and (49), respectively, where $\delta(\cdot)$ is the Dirac delta function. Now, starting from (53), using (75) and (76), and parallel to the first part of this proof, we can easily prove (74) for a Case 3 MC-DS-CDMA system. \square

We note that Case 3 MC-DS-CDMA system is equivalent to an MC-CDMA system over an FS channel for any

$$\bar{\mathbf{F}}_{N_t} = \begin{bmatrix} \mathbf{F}_{N_t} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{N_t} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{F}_{N_t} \end{bmatrix}, \quad (77)$$

where \mathbf{F}_{N_t} is any $N_t \times N_t$ unitary transformation. An easy way to recognize this fact is to consider (64). In this case, the channel matrices for Case 3 MC-DS-CDMA system and MC-CDMA system are

$$\begin{aligned} \hat{\mathbf{H}}_k &= \begin{bmatrix} \hat{h}_k[0, 0] \mathbf{I}_{N_t} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \hat{h}_k[0, 1] \mathbf{I}_{N_t} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \hat{h}_k[0, N_f - 1] \mathbf{I}_{N_t} \end{bmatrix}, \\ \tilde{\mathbf{H}}_k &= \begin{bmatrix} \tilde{h}_k[0, 0] \mathbf{I}_{N_t} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \tilde{h}_k[0, 1] \mathbf{I}_{N_t} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \tilde{h}_k[0, N_f - 1] \mathbf{I}_{N_t} \end{bmatrix}, \end{aligned} \quad (78)$$

respectively. It is then clear that $\hat{\mathbf{H}}_k = \bar{\mathbf{F}}_{N_t}^H \tilde{\mathbf{H}}_k \bar{\mathbf{F}}_{N_t}$ for any $\bar{\mathbf{F}}_{N_t}$. An interesting special case of $\bar{\mathbf{F}}_{N_t}$ is \mathbf{I}_N , that is, the MC-DS-CDMA and the MC-CDMA systems are also equivalent if $\tilde{\mathbf{q}}_k = \hat{\mathbf{q}}_k^{(f)}$.

The results obtained in this section are summarized in Table 2. Note that the third column in Table 2 is obtained by combining the results in the first two columns.

5.2. Equivalences in a TS channel

Building on the intuition in Section 4, a similar set of results analogous to those in Section 5.1 can be obtained for a TS channel. We recall from Section 3.3.5 that, in a TS channel, the MC-DS-CDMA system exhibits only Cases 2 and 3. In this case, Proposition 1 in Section 5.1 still holds for a TS channel. Parallel to Proposition 2, Proposition 3 shows the equivalence conditions for a DS-CDMA system and an MC-DS-CDMA system over a TS channel. The conditions for an MC-CDMA system to be equivalent to an MC-DS-CDMA system then follow.

TABLE 2: Summary of equivalence results for an FS channel. B_o refers to the basis waveform bandwidth in an MC-DS-CDMA system.

MC-DS-CDMA	MC-CDMA	DS-CDMA
⇕	⇕	⇕
MC-CDMA	DS-CDMA	MC-DS-CDMA
$\tilde{\mathbf{q}}_k = \begin{cases} \tilde{\mathbf{D}}_{N_t} \hat{\mathbf{q}}_k^{(f)}, & B_o > \Delta f_c, \\ \tilde{\mathbf{F}}_{N_t} \hat{\mathbf{q}}_k^{(f)}, & B_o \ll \Delta f_c \end{cases}$	$\mathbf{q}_k = \mathbf{D}_N^H \tilde{\mathbf{q}}_k$	$\hat{\mathbf{q}}_k^{(f)} = \begin{cases} \tilde{\mathbf{D}}_{N_t}^H \mathbf{D}_N \mathbf{q}_k, & B_o > \Delta f_c, \\ \tilde{\mathbf{F}}_{N_t}^H \mathbf{D}_N \mathbf{q}_k, & B_o \ll \Delta f_c \end{cases}$

Proposition 3. *The MC-DS-CDMA system in Cases 2 and 3 is equivalent to a DS-CDMA system provided that the following condition is satisfied:*

$$\mathbf{q}_k = \tilde{\mathbf{D}}_{N_f}^H \hat{\mathbf{q}}_k^{(t)}, \quad (79)$$

where $\tilde{\mathbf{D}}_{N_f}$ is defined as $\tilde{\mathbf{D}}_{N_t}$ in (69) with N_t replaced by N_f , and $\hat{\mathbf{q}}_k^{(t)} = [\hat{\mathbf{q}}_{k,0}^{(t)T} \cdots \hat{\mathbf{q}}_{k,N_t-1}^{(t)T}]^T$ of dimension $N \times 1$.

The proofs of Proposition 1 for a TS channel and Proposition 3 are analogous to the proofs of Proposition 1 for an FS channel and Proposition 2 in Section 5.1, respectively, under the following substitutions: MC-CDMA \rightarrow DS-CDMA, MC-DS-CDMA Case 1 \rightarrow MC-DS-CDMA Case 2, $\mathbf{q}_k \rightarrow \tilde{\mathbf{q}}_k$, $\tilde{\mathbf{q}}_k \rightarrow \mathbf{q}_k$, $\hat{\mathbf{q}}_k^{(f)} \rightarrow \hat{\mathbf{q}}_k^{(t)}$, $\tilde{\mathbf{h}}_{k,n} \rightarrow \mathbf{h}_{k,p}$, where $\mathbf{h}_{k,p} = [\tilde{h}_k[pN_f, 0] \cdots \tilde{h}_k[(p+1)N_f - 1, 0]]$ is the p th segment of $\tilde{\mathbf{h}}_k$ of length N_f , $N_t \rightarrow N_f$, $N_f \rightarrow N_t$, $l \rightarrow m$, $L \rightarrow 2M + 1$, $L_o \rightarrow 2M_o + 1$, $x^{(f)} \rightarrow x^{(t)}$ for any quantity x , $\mathbf{D}_N \rightarrow \mathbf{D}_N^H$, $\mathbf{D}_{N_t} \rightarrow \mathbf{D}_{N_f}^H$, $\tilde{\mathbf{D}}_{N_t} \rightarrow \tilde{\mathbf{D}}_{N_f}^H$, $\tilde{\mathbf{F}}_{N_t} \rightarrow \tilde{\mathbf{F}}_{N_f}^H$.

Table 3 summarizes the equivalence results in a TS channel. Again, the third column in the table is obtained by combining the results in columns one and two.

There is a subtle point that needs to be explained when proofs analogous to those in Section 5.1 are derived in the case of TS channel. In an FS channel, l only takes positive values and the one-sided DFT operation was used in the obtained results. In a TS channel, as stated above, l maps to m which takes symmetric values around 0 as seen in (16) and (37). Now, using a single-sided DFT operation, as done in the proofs in Section 5.1, will imply a phase shift. We can easily show that the equations analogous to (66) and (72) are

$$\mathbf{D}_N^H(1 : 2M + 1) \tilde{\mathbf{h}}_k = \mathbf{h}_k \odot \mathbf{e}_{M,N}, \quad (80)$$

$$\mathbf{D}_{N_f}^H(1 : 2M_o + 1) \hat{\mathbf{h}}_{k,p}^{(t)} = \tilde{\mathbf{h}}_{k,n} \odot \mathbf{e}_{M_o, N_f}, \quad (81)$$

respectively, where $\mathbf{e}_{M,N} = [1 \ e^{j(2\pi M/N)} \cdots e^{j(2\pi(N-1)M/N)}]^T$. We note that the presence of phase shift does not change the equivalence conditions since it does not affect the SINR value and the eigenvalues of $\mathbf{g}_k^{(x)}$ that governs P_e . Starting from this point, the phase shift will thus be ignored in the analysis.

5.3. Equivalences in a TFS channel

In this section, we prove the equivalence conditions for the general case of a TFS channel. In TFS channel, the MC-DS-CDMA system has the three cases described in Section 3.3. In

this case, Propositions 1, 2, and 3 still hold as shown in the sequel.

5.3.1 Proof of Proposition 1

Before proving Proposition 1 for TFS channel, we first introduce the following lemma.

Lemma 1. *The channel coefficient defined in (8) is given by*

$$h_k[p, l] = \sum_{m=-M}^M H_k[m, l] e^{j(2\pi m(p+1/2)/N)}, \quad (82)$$

where

$$H_k[m, l] = \int_0^{T_m} \int_{-B_d}^{B_d} C_k(\theta, \tau) \text{sinc}(m - \theta T) \text{sinc}(l - B\tau) \times e^{-j\pi(m-\theta T)} e^{-j\pi(l-B\tau)} d\theta d\tau, \quad (83)$$

where $C_k(\theta, \tau) = \int c_k(t, \tau) e^{-j2\pi\theta t} dt$.

Proof. See Appendix B. \square

Now, recalling (11) and (12), we write \mathbf{g}_k as

$$\mathbf{g}_k = \sum_{l=0}^L \mathbf{Q}_k(l) \mathbf{h}_{k,l} \quad (84)$$

$$= \sum_{l=0}^L \mathbf{Q}_k(l) \sum_{m=-M}^M H_k[m, l] \mathbf{e}_{m,N}(l) \quad (85)$$

$$= \sum_{m=-M}^M \sum_{l=0}^L H_k[m, l] (\mathbf{q}_k(l) \odot \mathbf{e}_{m,N}(l)) \quad (86)$$

$$= \sum_{m=-M}^M \hat{\mathbf{Q}}_{k,m} \hat{\mathbf{h}}_{k,m}, \quad (87)$$

where in (85) we used Lemma 1, in (86) (recall that $\mathbf{Q}_k(l)$ in (11) is diagonal) we replaced $\mathbf{Q}_k(l) \mathbf{e}_{m,N}(l)$ by $\mathbf{q}_k(l) \odot \mathbf{e}_{m,N}(l)$, and in (87)

$$\hat{\mathbf{Q}}_{k,m} = [\mathbf{q}_k(0) \odot \mathbf{e}_{m,N}(0) \cdots \mathbf{q}_k(L) \odot \mathbf{e}_{m,N}(L)] \quad (88)$$

(recall that $\hat{\mathbf{Q}}_{k,m}$ is Toeplitz) and

$$\hat{\mathbf{h}}_{k,m} = [H_k[m, 0] \cdots H_k[m, L]]^T. \quad (89)$$

TABLE 3: Summary of equivalence results for a TS channel. T_o refers to the duration of the basis waveform in an MC-DS-CDMA system.

MC-DS-CDMA	DS-CDMA	MC-CDMA
⇕	⇕	⇕
DS-CDMA	MC-CDMA	MC-DS-CDMA
$\mathbf{q}_k = \begin{cases} \mathbf{D}_{N_f}^H \hat{\mathbf{q}}_k^{(t)}, & T_o > \Delta t_c, \\ \bar{\mathbf{F}}_{N_f}^H \hat{\mathbf{q}}_k^{(t)}, & T_o \ll \Delta t_c \end{cases}$	$\tilde{\mathbf{q}}_k = \mathbf{D}_N \mathbf{q}_k$	$\hat{\mathbf{q}}_k^{(t)} = \begin{cases} \bar{\mathbf{D}}_{N_f} \mathbf{D}_N^H \tilde{\mathbf{q}}_k, & T_o > \Delta t_c, \\ \bar{\mathbf{F}}_{N_f} \mathbf{D}_N^H \tilde{\mathbf{q}}_k, & T_o \ll \Delta t_c \end{cases}$

TABLE 4: Summary of equivalence results for a TFS channel—Part 1. T_o and B_o refer to the duration and bandwidth of MC-DS-CDMA basis waveforms.

MC-DS-CDMA	MC-CDMA	DS-CDMA
⇕	⇕	⇕
MC-CDMA	DS-CDMA	MC-DS-CDMA
$\tilde{\mathbf{q}}_k = \begin{cases} \bar{\mathbf{D}}_{N_t} \hat{\mathbf{q}}_k^{(f)}, & B_o > \Delta f_c, T_o \ll \Delta t_c, \\ \bar{\mathbf{D}}_{N_t} \hat{\mathbf{q}}_k^{(f)}, & B_o \ll \Delta f_c, T_o \ll \Delta t_c \end{cases}$	$\mathbf{q}_k = \mathbf{D}_N^H \tilde{\mathbf{q}}_k$	$\hat{\mathbf{q}}_k^{(f)} = \begin{cases} \bar{\mathbf{D}}_{N_t}^H \mathbf{D}_N \mathbf{q}_k, & B_o > \Delta f_c, T_o \ll \Delta t_c, \\ \bar{\mathbf{D}}_{N_t}^H \mathbf{D}_N \mathbf{q}_k, & B_o \ll \Delta f_c, T_o \ll \Delta t_c \end{cases}$

Using the same approach as in Appendix A, we can show that

$$\hat{\mathbf{h}}_{k,m} = \mathbf{D}_N^H \tilde{\mathbf{h}}_{k,m}, \quad (90)$$

where $\tilde{\mathbf{h}}_{k,m}$ is given in (19). Using (90) in (87), then

$$\mathbf{D}_N \mathbf{g}_k = \sum_{m=-M}^M \mathbf{D}_N \hat{\mathbf{Q}}_{k,m} \mathbf{D}_N^H \tilde{\mathbf{h}}_{k,m} \quad (91)$$

$$= \sum_{m=-M}^M \tilde{\mathbf{Q}}_k(m) \tilde{\mathbf{h}}_{k,m} = \tilde{\mathbf{g}}_k, \quad (92)$$

where $\tilde{\mathbf{Q}}_k(m) = \text{diag}\{\mathbf{D}_N[\mathbf{q}_k \odot \mathbf{e}_m]\} = \text{diag}\{[\mathbf{D}_N \mathbf{q}_k](m)\}$, and the second equality is obtained from the properties of DFT [16]. We can see that $\tilde{\mathbf{g}}_k$ in (92) is $\tilde{\mathbf{g}}_k$ in (20) for a TFS channel provided that $\tilde{\mathbf{q}}_k = \mathbf{D}_N \mathbf{q}_k$. Hence, Proposition 1 is proved for a TFS channel.

5.3.2 Proof of Proposition 2

Starting with an MC-DS-CDMA system in Case 1, we note from Section 3.3.1 that, per subband, there exists L_o multipath components due to frequency selectivity, and, per path, different chips fade differently due to time selectivity. This scenario per subband is similar to that in Section 5.3.1. The proof is obtained by mimicking the proof in Section 5.3.1 per subband. This is similar to what was done in Section 5.1 when Proposition 2 was proved by mimicking the proof of Proposition 1 per subband.

To prove the equivalence for the MC-DS-CDMA system in Case 3, we start with an MC-CDMA system in a TFS channel and note that, since $B_o \ll \Delta f_c$, the BF assumption holds. Under this assumption, the noise-free received signal in (15)

can be written as

$$\begin{aligned} \tilde{x}_k(t) \approx & b_k \sum_{i=0}^{N_f-1} \sum_{m=-M}^M \tilde{h}_k[m, i] \\ & \times \sum_{n \in \Omega_i} \tilde{q}_k[n] v_T(t) e^{j2\pi n t / T} e^{j(2\pi(n+1/2)t/T)}. \end{aligned} \quad (93)$$

In view of (93), the MC-CDMA system in a TFS channel encounters multiple Doppler components per subband. Recalling the proof of Proposition 2 in Section 5.1, going from MC-DS-CDMA to MC-CDMA, the increase in T_o does not cause frequency dispersion due to absence of time selectivity. However, it causes $B_o < \Delta f_c$ eliminating the time dispersion. Similar scenario is experienced in this case per subband except for interchanging time and frequency domains. In particular (as discussed in Section 4), to diagonalize the system, we increase B_o , and that does not cause time dispersion (due to the BF assumption per subband) and decrease T_o such that $T_o < \Delta t_c$ to eliminate frequency dispersion so that the resulting TF-CDMA basis waveforms encounter nonselective fading. After drawing this analogy, we can see that the proof of Proposition 2 for MC-DS-CDMA system in Case 3 over a TFS channel is analogous to that of Proposition 2 for MC-DS-CDMA system in Case 1 over an FS channel except for interchanging time and frequency domains and the same condition $\tilde{\mathbf{q}}_k = \mathbf{D}_N \hat{\mathbf{q}}_k^{(f)}$ for equivalence holds.

5.3.3 Proof of Proposition 3

The proof in this case is analogous to that in Section 5.3.2 by using the substitutions in Section 5.2.

The equivalence results for a TFS channel are summarized in Tables 4 and 5. In both tables, the last column is obtained by combining the results in the first two columns.

TABLE 5: Summary of equivalence results for a TFS channel—Part 2. T_o and B_o refer to the duration and bandwidth of MC-DS-CDMA basis waveforms.

MC-DS-CDMA	DS-CDMA	MC-CDMA
⇕	⇕	⇕
DS-CDMA	MC-CDMA	MC-DS-CDMA
$\mathbf{q}_k = \begin{cases} \bar{\mathbf{D}}_{N_f}^H \hat{\mathbf{q}}_k^{(t)}, T_o > \Delta t_c, B_o \ll \Delta f_c, \\ \bar{\mathbf{D}}_{N_f}^H \hat{\mathbf{q}}_k^{(t)}, T_o \ll \Delta t_c, B_o \ll \Delta f_c \end{cases}$	$\hat{\mathbf{q}}_k = \mathbf{D}_N \mathbf{q}_k$	$\hat{\mathbf{q}}_k^{(t)} = \begin{cases} \bar{\mathbf{D}}_{N_f} \mathbf{D}_N^H \hat{\mathbf{q}}_k, T_o > \Delta t_c, B_o \ll \Delta f_c, \\ \bar{\mathbf{D}}_{N_f} \mathbf{D}_N^H \hat{\mathbf{q}}_k, T_o \ll \Delta t_c, B_o \ll \Delta f_c \end{cases}$

5.3.4 Important remarks

For the sake of completeness, it is necessary to make the following remarks.

Remark 1. In a TFS channel as opposed to an FS (TS) channel, not any $\bar{\mathbf{F}}_{N_f}$ ($\bar{\mathbf{F}}_{N_f}$) unitary transformation satisfies equivalence. The reason is that, in a TFS channel, the channel matrix in MC-CDMA (DS-CDMA) system is no longer diagonal but rather Toeplitz due to time (frequency) selectivity. Hence, there will always be a DFT operation between the channel coefficients of MC-DS-CDMA system in Case 3 and those of an MC-CDMA (DS-CDMA) system.

Remark 2. From Tables 4 and 5, given \mathbf{q}_k , we have two possible conditions for MC-DS-CDMA system in Case 3 to be equivalent to DS-CDMA system

- (1) $\hat{\mathbf{q}}_k^{(f)} = \bar{\mathbf{D}}_{N_f}^H \mathbf{D}_N \mathbf{q}_k$;
- (2) $\hat{\mathbf{q}}_k^{(t)} = \bar{\mathbf{D}}_{N_f} \mathbf{q}_k$.

We can show that both unitary transformations $\mathbf{F}_1 = \bar{\mathbf{D}}_{N_f} \mathbf{D}_N$ and $\mathbf{F}_2 = \bar{\mathbf{D}}_{N_f}^H$ satisfy (64) when systems x and y are MC-DS-CDMA in Case 3 and DS-CDMA, respectively. A similar argument holds for an MC-DS-CDMA in Case 3 and an MC-CDMA systems.

Remark 3. Even though prior literature touched on the connections between different CDMA systems in FS channels (and, to a lesser extent, TS channel), we note that such connections in TFS were not well studied and are not as clear as the simple case of FS channels. Moreover, most of the results in the prior works address single user scenarios. Propositions 1, 2, and 3 extend the connections between the different CDMA systems to TFS channels building on the time-frequency duality as explained in Section 4. In addition, all the results address multiuser scenarios where MMSE detection is used at the receiver.

We also note that, from an implementational point of view, the available hardware may favor one system over the other. For instance, the problem of peak-to-average ratio in MC-CDMA system may make implementing DS-CDMA system easier. Similarly, the difficulty in implementing RAKE receivers in high data rate systems may make implementing MC-CDMA system easier. We showed in Propositions 1, 2, and 3 that all the systems can perform as well provided that the sufficient conditions are met. We again emphasize that

the equivalent conditions were not clear in prior literature for TFS channels.

6. NUMERICAL RESULTS

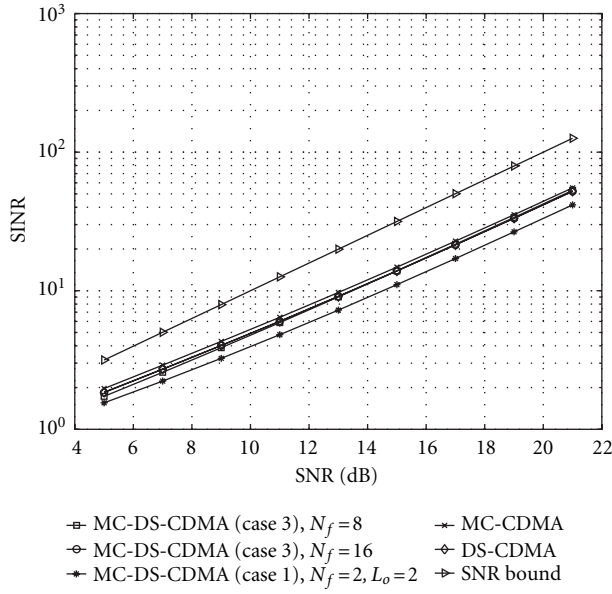
In this section, we illustrate the results obtained in Section 5. We show the near identical performance of the different CDMA systems when using the sufficient conditions in Propositions 1, 2, and 3 even after relaxing the assumptions used in their proofs.⁸ The goal is to emphasize the validity of these assumptions and hence the equivalent conditions. We consider a system of $K = 20$ users and a WSSUS channel model with $L + 1 = 4$ resolvable paths. We use the separability assumption in (3). We also choose (recall Section 2.2) $\Phi_\theta(\theta) = 1/4\pi B_d \sqrt{1 - (\theta/B_d)^2}$, $|\theta| < B_d$ and 0 elsewhere [17] and set $\Psi(\tau)$ to be constant over τ . Each user employs a distinct Gold code of length $N = 32^9$ for signal spreading and the receiver employs an MMSE detector. We use SINR and P_e , defined in Section 5, as measures for performance and adopt the equivalence conditions stated in Tables 1, 2, 3, and 4.

Figures 7a and 7b show the performance of the different CDMA systems in an FS channel. We show the performance of MC-DS-CDMA system for its two cases. For Case 1, we choose $N_f = 2$ so that $L_o + 1 = 2$. For Case 3, two different values of N_f are considered $N_f = 8, 16$. Figure 7a shows the SINR for the different systems as well as the signal-to-noise ratio (SNR) in the absence of interference which is an upper bound on the SINR. Figure 7b shows the P_e for the different systems. As expected from Propositions 1 and 2, the performance of the different systems is virtually indistinguishable.

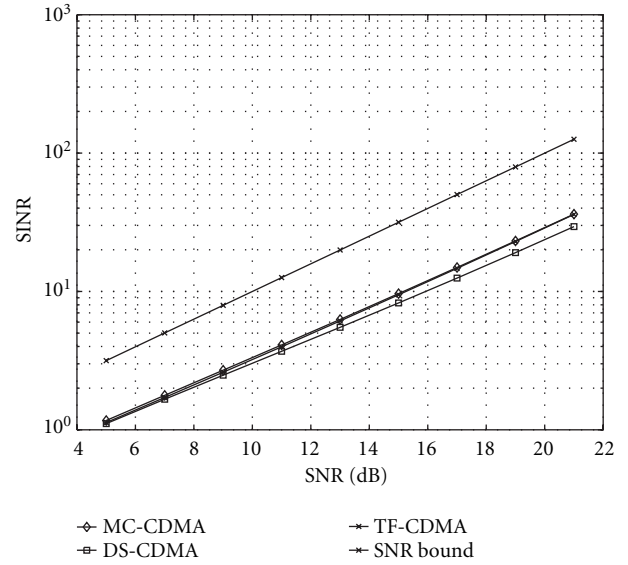
Figures 8a and 8b show the SINR and P_e , respectively, for the MC-CDMA, DS-CDMA, and MC-DS-CDMA Case 3 systems in TFS channel with $B_d T = 0.2$. For the MC-DS-CDMA Case 3 system, we take $N_f = 8$ and ideally assume that, under this choice, every time-frequency basis waveform encounters nonselective fading in time and frequency—that is, we ignore the relatively negligible dispersion effects that

⁸Basically, any assumption that was made on the signaling matrices $\mathbf{Q}_k^{(x)}$ (such as approximating Toeplitz matrices by circulant ones) is relaxed in the expressions of SINR and P_e in (58) and (60), respectively.

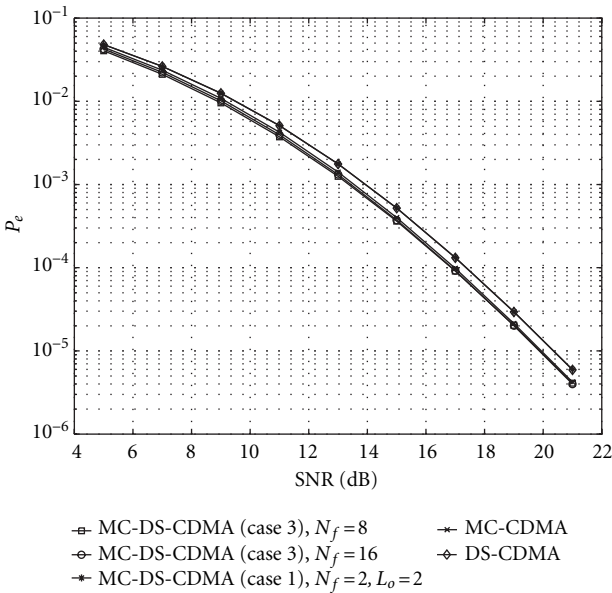
⁹The Gold code length is 31, and an additional common bit is added to every signature code to have an overall length of 32.



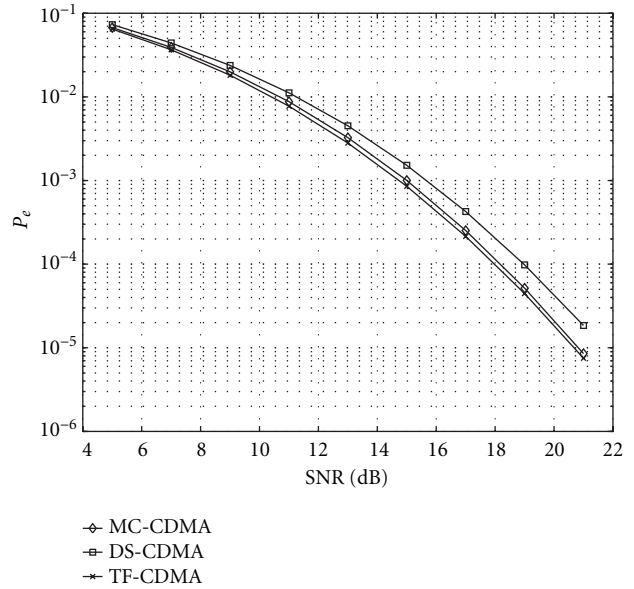
(a)



(a)



(b)



(b)

FIGURE 7: Performance of various equivalent systems over an FS channel for $K = 20$ users as a function of SNR. (a) SINR. (b) P_e .

FIGURE 8: The performance of MC-CDMA, DS-CDMA, and TF-CDMA (MC-DS-CDMA) systems over a TFS channel ($B_d T = 0.2$) for $K = 20$ users as a function of SNR. (a) SINR. (b) P_e .

still remain in an MC-DS-CDMA system. Again, the performance of the three systems is virtually identical.

7. CONCLUSION

In this paper, we have studied linear MMSE multiuser detection for a class of CDMA systems in FS, TS, and TFS channels. We discussed a time-frequency duality between various systems operating over TFS channels. The time-frequency

duality exposes the channel conditions under which different systems are ideal (diagonalize the channel matrix) DS-CDMA systems are ideal in TS channels, MC-CDMA in FS channels, and MC-DS-CDMA (TF-CDMA) in TFS channels. We use this duality to develop equivalences between systems operating over a TFS channel and systems operating either over FS or TS channels. Building on the time-frequency duality and the equivalences, we derive sufficient conditions relating the signature codes of the different systems for them to

perform in a near-identical fashion in FS, TS, and TFS channels. Numerical results are provided to support the analysis. The results of this paper relating the various systems in different channel conditions enable cross-leveraging of design insights for the widely used DS-CDMA, MC-CDMA, and MC-DS-CDMA systems.

APPENDICES

A. PROOF OF (66)

Recall from (16) that, in FS channel, (i.e., $M = 0$ in (15)), $\tilde{h}_k(0, n) = \tilde{C}_k(0, (n + 1/2)/T)$. For (66) to hold, it suffices to prove that $\tilde{h}_k(0, n)$ is the n th component of the DFT of $\{h_k(0, l), l = 0, 1, \dots, L\}$, which we show as follows:

$$\begin{aligned}
& \sum_{l=0}^L h_k[0, l] e^{-j(2\pi l n/N)} \\
&= \sum_{l=0}^L e^{-j(2\pi l n/N)} \int_0^{T_m} c_k(0, \tau) \text{sinc}(l - B\tau) e^{j\pi(l - B\tau)} dt \\
&= \int_0^{T_m} c_k(0, \tau) e^{-j\pi B\tau} \sum_{l=0}^L \text{sinc}(l - B\tau) e^{-j2\pi l(n/N + 1/2)} dt \\
&= \int_0^{T_m} c_k(0, \tau) e^{-j\pi B\tau} e^{-j2\pi(n/N + 1/2)B\tau} dt \\
&= \int_0^{T_m} c_k(0, \tau) e^{-j(2\pi n\tau/T)} dt \\
&= \tilde{C}_k\left(0, \frac{n}{T}\right) \\
&= \tilde{h}_k(0, n)
\end{aligned} \tag{A.1}$$

B. PROOF OF LEMMA 1

Starting with the right-hand side of (82)

$$\begin{aligned}
& \sum_{m=-M}^M H_k[m, l] e^{j(2\pi m(p+1/2)/N)} \\
&= \int_0^{T_m} \int_{-B_d}^{B_d} C_k(\theta, \tau) \text{sinc}(l - B\tau) e^{-j\pi(l - B\tau)} e^{j\pi(\theta T)} \\
&\quad \times \sum_{m=-M}^M \text{sinc}(m - \theta T) e^{-j\pi m} e^{j(2\pi m(p+1/2)/N)} d\theta d\tau
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
&= \int_0^{T_m} \int_{-B_d}^{B_d} C_k(\theta, \tau) \text{sinc}(l - B\tau) e^{-j\pi(l - B\tau)} e^{j\pi(\theta T)} e^{j\pi M} \\
&\quad \times e^{-j(2\pi M(p+1/2)/N)} \sum_{v=0}^{2M} \text{sinc}(v - M - \theta T) \\
&\quad \times e^{-j\pi v} e^{j(2\pi v(p+1/2)/N)} d\theta d\tau
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&= \int_0^{T_m} \int_{-B_d}^{B_d} C_k(\theta, \tau) \text{sinc}(l - B\tau) e^{-j\pi(l - B\tau)} e^{j\pi(\theta T)} e^{j\pi M} \\
&\quad \times e^{-j(2\pi M(p+1/2)/N)} e^{j2\pi((p+1/2)/N - 1/2)(M + \theta T)} d\theta d\tau \\
&= \int_0^{T_m} \int_{-B_d}^{B_d} C_k(\theta, \tau) \text{sinc}(l - B\tau) e^{-j\pi(l - B\tau)} \\
&\quad \times e^{j2\pi\theta T((p+1/2)/N)} d\theta d\tau \\
&= \int_0^{T_m} c_k((p + 1/2)T_c, \tau) \text{sinc}(l - B\tau) e^{-j\pi(l - B\tau)} d\tau \\
&= h_k[p, l],
\end{aligned} \tag{B.3}$$

where in (B.2), we did a change of variables $v = m + M$.

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