

EQUIVALENCE OF MARKOV PROCESSES⁽¹⁾

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1. Introduction. The question of the measure theoretic equivalence of one stochastic process with respect to another has been studied in several contexts. For example, A. V. Skorokhod [16], [17] has considered the equivalence of Markov processes defined by means of stochastic differential equations, L. A. Shepp [15] has obtained criteria for the equivalence of Gaussian processes and the Wiener process, and E. B. Dynkin [7] has obtained a global criterion for the absolute continuity of one Markov process with respect to another in terms of multiplicative functionals.

In this paper we investigate the equivalence of Hunt processes from a local point of view. After introducing the basic notation and definitions in §2 we discuss three special cases in §3 which motivate and illustrate the general theory which is to be developed. In §4 a class of martingales is associated with a Markov process; this leads to the proof of an extended Markov property in §5. The extended Markov property is central in proving our results on the equivalence of Hunt processes. §§6 and 7 are concerned with certain technical results which lead to the statement and proof of the main results in §§8 and 9. In §8 it is shown that if two Hunt processes arising from Feller semigroups are not equivalent then at least one of four basic types of singularity must occur. The four basic types of singularity are: singularity on the germ field, singularity on the tail field, local singularity at a stopping time and jump singularity at a stopping time. §9 contains a deeper study of local singularities. In particular, if the processes have the additional property which we call local smoothness then the existence of local singularities at a stopping time is equivalent to the existence of singularities on the germ field. Finally, in §10 the property of local smoothness is characterized in potential theoretic terms.

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2. Notation and definitions. In this section we introduce the basic notation and review the important definitions. For a more complete discussion of the standard material refer to E. B. Dynkin [7] and P. A. Meyer [13]. A summary of the basic definitions in the theory of Markov processes is also found in R. K. Gettoor [8]. Basic results of measure theory are often used without explicit reference; such material is found in the basic texts of J. Neveu [14] and P. R. Halmos [9].

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2.1. *The state space.* Let (E, \mathcal{E}) be a measurable space such that E is a locally compact separable metric space with metric $\rho(\cdot, \cdot)$ and \mathcal{E} is the σ -field of Borel subsets of E . Note that each open subset and each closed subset in such a space is a countable union of compact sets. Hence every set in the algebra generated by the open (or compact) subsets is the countable union of compacts. Moreover all finite Borel measures are inner regular. Let Δ be the point at infinity if E is not compact or an isolated point if E is compact. Let $E_\Delta = E \cup \{\Delta\}$ and \mathcal{E}_Δ be the σ -field of Borel subsets of E_Δ .

2.2. *The basic measure space.* Let Ω be the set of all functions, $\omega(\cdot)$, from $[0, \infty]$ to E_Δ which are right continuous, have left-hand limits on $[0, \infty)$, have the property that if $\omega(t) = \Delta$, then $\omega(s) = \Delta$ for all $s \geq t$, and satisfy $\omega(\infty) = \Delta$. ω_Δ represents the element of Ω defined by $\omega_\Delta(t) = \Delta$ for all $t \in [0, \infty]$. If $\omega \in \Omega$, the mapping $t \rightsquigarrow X_t(\omega) = X(t, \omega) \equiv \omega(t)$ is called the *path* of ω . For $t \in [0, \infty)$, $X_t^-(\omega) = \lim_{s \nearrow t} X_s(\omega)$. For $t \in [0, \infty]$ let \mathcal{F}_t be the σ -field of subsets of Ω generated by sets of the form $\{\omega : X_s(\omega) \in \Gamma\}$ with $s \in [0, t]$ and $\Gamma \in \mathcal{E}_\Delta$ and for $t \in [0, \infty)$ let $\mathcal{M}_t \equiv \bigcap_{s > t} \mathcal{F}_s$, $\mathcal{M}_\infty \equiv \mathcal{F}_\infty$. \mathcal{M}_0 is known as the *germ field*.

PROPOSITION 2.2.1. *For any $s \in [0, \infty]$, \mathcal{F}_s is generated by a countable algebra of sets.*

Proof. Let \mathcal{A}_s be the algebra of subsets of E generated by sets of the form $\{\omega : X_r(\omega) \in \Gamma_j\}$ with r a rational in $[0, s]$ and Γ_j an element of the countable base $\{\Gamma_j : j = 1, 2, 3, \dots\}$ of open sets for E . It suffices to show that a set of the form $\{\omega : X_r(\omega) \in \Gamma\}$ with r an irrational number in $[0, s]$ and Γ an open subset of E is contained in $\sigma(\mathcal{A}_s)$, the σ -field generated by \mathcal{A}_s . But the right continuity of the paths implies that

$$(2.2.1) \quad \{\omega : X_r(\omega) \in \Gamma\} = \bigcup_{j \in \mathcal{J}} \bigcap_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\omega : X_{r_n}(\omega) \in \Gamma_j\}$$

where $\{r_n\}$ is a sequence of rationals which decrease to r and $\mathcal{J} \equiv \{j : \Gamma_j \subset \Gamma\}$. Since $\bigcup_{j \in \mathcal{J}} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\omega : X_{r_n}(\omega) \in \Gamma_j\} \in \sigma(\mathcal{A}_s)$, the proof is complete.

A mapping $T: \Omega \rightarrow [0, \infty]$ is a *stopping time* with respect to the σ -fields $(\mathcal{M}_t)_{t \in [0, \infty]}$ if for every $t \in [0, \infty]$, $\{T < t\} \in \mathcal{M}_t$. If T is a stopping time, then \mathcal{M}_T denotes the sub- σ -field of sets $A \in \mathcal{F}_\infty$ such that $A \cap \{T \leq t\} \in \mathcal{M}_t$ for every $t \in [0, \infty]$.

If $\zeta(\omega) \equiv \inf\{t : X_t(\omega) = \Delta\}$, then ζ is a stopping time and is called the *lifetime* of the path.

If the mapping $Y(\cdot, \cdot): ([0, t] \times \Omega, \mathcal{B}_{[0,t]} \times \mathcal{M}_t) \rightarrow (F, \mathcal{F})$ where (F, \mathcal{F}) is a measurable space and $\mathcal{B}_{[0,t]}$ is the σ -field of Borel subsets of $[0, t]$ is measurable for every $t \in [0, \infty)$, then the stochastic process $Y(\cdot, \cdot)$ is said to be *progressively measurable*.

PROPOSITION 2.2.2. (i) $X_s(\omega)$ is a progressively measurable stochastic process.

(ii) If T is a stopping time, then X_T is \mathcal{M}_T -measurable.

Proof. See P. A. Meyer [13, Chapter 4, §3].

2.3. *Definition of a Markov process.* The system $\pi = (\Omega, \mathcal{F}_\infty, (P_x)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ is a *strong Markov process* with respect to the σ -fields $(\mathcal{M}_t)_{t \in [0, \infty]}$ if:

(2.3.1) $P_x(X_0 = x) = 1$ for each $x \in E_\Delta$,

(2.3.2) the mapping $x \rightsquigarrow P_x(A)$ is \mathcal{E}_Δ -measurable for $A \in \mathcal{F}_\infty$, and

(2.3.3) for any stopping time T , $\Gamma \in \mathcal{E}_\Delta$ and $s \in [0, \infty)$,

$$P_x(X_{T+s} \in \Gamma | \mathcal{M}_T) = P_{X(T)}(X_s \in \Gamma),$$

P_x -almost surely for each $x \in E_\Delta$.

E_x denotes the expectation with respect to the measure P_x . The Markov process is said to be *continuous* if the paths are continuous on $[0, \zeta)$, P_x -almost surely for each $x \in E$. It is said to be *uninterrupted* if for every $x \in E$, $P_x(\zeta = \infty) = 1$.

2.4. *The semigroup.* Given an \mathcal{E}_Δ -measurable bounded function, f , let

$$P_t f(x) \equiv \int f(y) P_x(X_t \in dy).$$

$C_0(E_\Delta)$ designates the set of continuous functions, f , on E_Δ such that $\lim_{y \rightarrow \Delta} f(y) = 0$ and $f(\Delta) = 0$. The semigroup $\{P_t : t \geq 0\}$ is a *Feller semigroup* if:

(2.4.1) for every function $f \in C_0(E_\Delta)$ and $t > 0$, $P_t f \in C_0(E_\Delta)$, and

(2.4.2) for every function $f \in C_0(E_\Delta)$, $\|P_t f - f\| \rightarrow 0$ as $t \rightarrow 0$ where

$$\|g\| \equiv \sup \{|g(x)| : x \in E\}.$$

The following result is well known.

PROPOSITION 2.4.1. *If $\{P_t : t \geq 0\}$ is a Feller semigroup, then*

(i) *if $f \in C_0(E_\Delta)$, $P_t f(x)$ is jointly continuous in t and x on $[0, \infty) \times E_\Delta$, and*

(ii) *$P_x(\zeta = 0) = 0$ for each $x \in E$.*

2.5. *Definition of a Hunt process.* The strong Markov process π is said to be *quasi-left continuous* if for any increasing sequence of stopping times $\{T_n\}$ with limit T , $X(T_n) \rightarrow X(T)$, P_x -almost surely on $\{T < \infty\}$ for each $x \in E$. A strong Markov process as described in 2.3 which is quasi-left continuous is known as a *Hunt process*. In the remainder of this paper the term Hunt process refers to a quasi-left continuous strong Markov process in the canonical form described in 2.2 and 2.3 with a Feller semigroup.

2.6. *The translation operators.* For $t \in [0, \infty]$ the *translation operator* $\theta_t: \Omega \rightarrow \Omega$ is defined by

$$(\theta_t \omega)(s) = \omega(s+t), \quad t < \infty,$$

$$\theta_\infty \omega = \omega_\Delta \quad \text{for every } \omega.$$

If $T: \Omega \rightarrow [0, \infty]$ is a stopping time, $\theta_T: \Omega \rightarrow \Omega$ is defined by

$$(\theta_T \omega) \equiv \theta_{T(\omega)}(\omega).$$

It is easy to verify the following result (cf. P. Courrège and P. Priouret [5] and E. B. Dynkin [7]).

PROPOSITION 2.6.1. (i) $\theta_t: (\Omega, \mathcal{F}_{t+s}) \rightarrow (\Omega, \mathcal{F}_s)$ is measurable, that is, $\theta_t^{-1}\mathcal{F}_s \subset \mathcal{F}_{t+s}$.
 (ii) $\theta_t: (\Omega, \mathcal{M}_{t+s}) \rightarrow (\Omega, \mathcal{M}_s)$ is measurable, that is, $\theta_t^{-1}\mathcal{M}_s \subset \mathcal{M}_{t+s}$.
 (iii) If T is a stopping time, $\theta_T: (\Omega, \mathcal{M}_{T+s}) \rightarrow (\Omega, \mathcal{F}_s)$ is measurable, that is, $\theta_T^{-1}\mathcal{F}_s \subset \mathcal{M}_{T+s}$. Also $\theta_T^{-1}\mathcal{M}_s \subset \mathcal{M}_{T+s}$.

2.7. *The zero-one law.* The following zero-one law is well known.

PROPOSITION 2.7.1. If $(\Omega, \mathcal{F}_\infty, (P_x)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ is a strong Markov process with respect to the σ -fields $(\mathcal{M}_t)_{t \in [0, \infty)}$ and $A \in \mathcal{M}_0$, then for each $x \in E_\Delta$, $P_x(A)$ is either zero or one.

Proof. Refer to E. B. Dynkin [7, Theorem 3.1]. It should also be noted that two zero-one measures on a σ -field which are equivalent are in fact identical.

2.8. *The completed σ -fields.* For any finite measure μ on $(E_\Delta, \mathcal{E}_s)$, P_μ is a measure on $(\Omega, \mathcal{F}_\infty)$ defined by $P_\mu(A) = \int P_x(A)\mu(dx)$. \mathcal{F}_∞^π is defined to be the intersection of all the P_μ -completions of \mathcal{F}_∞ . Each measure P_μ can be extended to \mathcal{F}_∞^π . If \mathcal{G} is a sub- σ -field of \mathcal{F}_∞ , $\tilde{\mathcal{G}}^\pi$ is defined to be such that $A \in \tilde{\mathcal{G}}^\pi$ if for each μ there exists a set $A_\mu \in \mathcal{G}$ such that $A - A_\mu$ and $A_\mu - A$ are in \mathcal{F}_∞^π and $P_\mu(A - A_\mu) = P_\mu(A_\mu - A) = 0$. It can be shown that π is also a strong Markov process with respect to the σ -fields $(\tilde{\mathcal{M}}_t^\pi)_{t \in [0, \infty]}$ (E. B. Dynkin [7, Theorem 3.12]). In the literature it is customary to replace the σ -fields $(\mathcal{M}_t)_{t \in [0, \infty)}$ by $(\tilde{\mathcal{M}}_t^\pi)_{t \in [0, \infty)}$. However, since if π_1 and π_2 are not equivalent $\tilde{\mathcal{M}}_t^{\pi_1}$ and $\tilde{\mathcal{M}}_t^{\pi_2}$ need not be identical, we avoid this convention unless it is explicitly stated to the contrary. However the following result is very useful.

PROPOSITION 2.8.1. Let T be an $(\tilde{\mathcal{M}}_t^\pi)_{t \in [0, \infty)}$ -stopping time. Then for each μ there is a $(\mathcal{M}_t)_{t \in [0, \infty)}$ -stopping time T^μ such that $P_\mu(T \neq T^\mu) = 0$.

Proof. Refer to E. B. Dynkin [7, Lemma 3.4].

If Γ is an \mathcal{E} -measurable set and

$$T_\Gamma \equiv \inf \{t : X_t(\omega) \in \Gamma\} \text{ if } \{t : X_t(\omega) \in \Gamma\} \neq \emptyset, \\ \equiv +\infty \text{ otherwise,}$$

then T_Γ is an $(\tilde{\mathcal{M}}_t^\pi)_{t \in [0, \infty)}$ -stopping time (P. A. Meyer [13, Chapter 4, T 52]). Note that if Γ is compact, then $X_{T_\Gamma} \in \Gamma$ on $\{T_\Gamma < \infty\}$ and if T_Γ^π is an $(\mathcal{M}_t)_{t \in [0, \infty)}$ -stopping time such that $P_\mu(T_\Gamma \neq T_\Gamma^\pi) = 0$, then $X_{T_\Gamma^\pi} \in \Gamma$, P_μ -almost surely on $\{T_\Gamma^\pi < \infty\}$.

2.9. *Classification of stopping times.* Let us briefly review P. A. Meyer's classification of stopping times [12], [13]. Let π be a Hunt process. A stopping time T is said to be *totally inaccessible in the weak sense* at x_0 on a set A , $x_0 \in E$, $A \in \mathcal{F}_\infty$, if for every increasing sequence $\{T_n\}$ of stopping times which converge to T ,

$$(2.9.1) \quad P_{x_0}(\{T_n < T \text{ for all } n, T < \infty\} \cap A) = 0.$$

A stopping time T is said to be *accessible* at x_0 on a set A , $x_0 \in E$, $A \in \mathcal{F}_\infty$, if there is an increasing sequence of stopping times $\{T_n\}$, with $T_n \leq T$ for all n , and such that

$$(2.9.2) \quad \lim_{n \rightarrow \infty} T_n = T$$

P_{x_0} -almost surely on A , and

$$(2.9.3) \quad T_n < T \quad \text{for all } n,$$

P_{x_0} -almost surely on $A \cap \{T > 0\}$. Note that every stopping time T is accessible on $\{T = \infty\}$. The definitions are usually stated in terms of $(\tilde{\mathcal{M}}_t^x)_{t \in [0, \infty]}$ -stopping times. However in view of Proposition 2.8.1 this difference is inconsequential. The following result is of great importance.

PROPOSITION 2.9.1. *The stopping time T is accessible at x_0 on A if and only if $t \rightsquigarrow X_t$ is continuous at T , P_{x_0} -almost surely on $A \cap \{T < \infty\}$.*

Proof. Refer to P. A. Meyer [12].

2.10. *The definitions of equivalence.* In the sequel $\pi_1 = (\Omega, \mathcal{F}_\infty, (P_x^1)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ and $\pi_2 = (\Omega, \mathcal{F}_\infty, (P_x^2)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ designate a pair of Hunt processes. π_1 and π_2 are said to be *equivalent* if for each $x \in E_\Delta$ and $A \in \mathcal{F}_\infty$, $P_x^1(A) = 0$ if and only if $P_x^2(A) = 0$. π_1 and π_2 are *equivalent in finite time* if for each $x \in E_\Delta$, $t < \infty$ and $A \in \mathcal{M}_t$, $A \subset \{\zeta > t\}$, $P_x^1(A) = 0$ if and only if $P_x^2(A) = 0$; if this is true even if $A \not\subset \{\zeta > t\}$, π_1 and π_2 are *strongly equivalent in finite time*. A set $A \in \mathcal{F}_\infty$ such that $P_x^1(A) = 0$, $P_x^2(A) > 0$ is said to be a (π_1, π_2, x) -singularity.

3. Three examples.

EXAMPLE 1. Let $\pi = (\Omega, \mathcal{F}_\infty, (P_x)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ be the standard one-dimensional Brownian motion. Let $\pi_1 = (\Omega, \mathcal{F}_\infty, (P_x^1)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ and $\pi_2 = (\Omega, \mathcal{F}_\infty, (P_x^2)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ be a pair of one-dimensional continuous strong Markov processes represented in the canonical form of 2.3 which are induced by the solutions of the stochastic integral equations:

$$(3.1) \quad X_1(t) = X_1(0) + \int_0^t a_1(X_1(s)) ds + \int_0^t \sigma_1(X_1(s)) dX(s),$$

and

$$(3.2) \quad X_2(t) = X_2(0) + \int_0^t a_2(X_2(s)) ds + \int_0^t \sigma_2(X_2(s)) dX(s)$$

where the stochastic integrals are evaluated with respect to the Brownian motion. In order to insure existence and uniqueness we also assume that

$$(3.3) \quad a_i(\cdot) \text{ and } \sigma_i(\cdot), \quad i = 1, 2, \text{ are continuous and } \sigma_i(\cdot) > 0, \quad i = 1, 2,$$

and

$$(3.4) \quad \text{there exists a constant } K > 0 \text{ such that} \\ (a_1(x))^2 + (a_2(x))^2 + (\sigma_1(x))^2 + (\sigma_2(x))^2 \leq K(1 + x^2).$$

In the terminology of K. Itô and H. P. McKean [11], the solutions $X_1(t)$ and $X_2(t)$ are nonstandard descriptions of π_1 and π_2 .

A. V. Skorokhod [16], [17] has shown that π_1 and π_2 are strongly equivalent in

finite time to the continuous strong Markov processes π_3 and π_4 which are induced by the unique solutions of the stochastic integral equations

$$(3.5) \quad X_3(t) = X_3(0) + \int_0^t \sigma_1(X_3(s)) dX(s),$$

and

$$(3.6) \quad X_4(t) = X_4(0) + \int_0^t \sigma_2(X_4(s)) dX(s)$$

respectively. A result originally due to V. A. Volkonskii (refer to K. Itô and H. P. McKean [11]) states that nonstandard descriptions of π_3 and π_4 are

$$X_i(s_i) = X(f_i^{-1}(s_i)), \quad i = 3, 4,$$

where $s_i = f_i(t, \omega)$, $i = 3, 4$, are the additive functionals defined by

$$f_3(t, \omega) = \int_0^t \sigma_1^{-2}(X(s, \omega)) ds \quad \text{and} \quad f_4(t, \omega) = \int_0^t \sigma_2^{-2}(X(s, \omega)) ds$$

where $X(s)$ represents the standard Brownian motion. In particular this implies that for any real number a

$$P_{x_0}^t \left(\limsup_{t \downarrow 0} \frac{X(t) - x_0}{(2t \log \log 1/t)^{1/2}} = a \right) = 1$$

if and only if

$$P_{x_0}^i \left(\limsup_{t \downarrow 0} \frac{X(f_i^{-1}(t)) - x_0}{(2t \log \log 1/t)^{1/2}} = a \right) = 1, \quad i = 3, 4.$$

PROPOSITION 3.1.

$$P_{x_0}^i \left(\limsup_{t \downarrow 0} \frac{X(t) - x_0}{(2t \log \log 1/t)^{1/2}} = \sigma_i(x_0) \right) = 1, \quad i = 3, 4.$$

Proof. Recall that the classical law of the iterated logarithm for Brownian motion states that

$$P_{x_0} \left(\limsup_{s \downarrow 0} \frac{X(s) - x_0}{(2s \log \log 1/s)^{1/2}} = 1 \right) = 1.$$

Hence it suffices to show that if ω_0 is such that

$$(3.7) \quad \limsup_{s \downarrow 0} \frac{X(s, \omega_0) - x_0}{(2s \log \log 1/s)^{1/2}} = 1,$$

then

$$(3.8) \quad \limsup_{t \downarrow 0} \frac{X(f_i^{-1}(t, \omega_0)) - x_0}{(2t \log \log 1/t)^{1/2}} = \sigma_i(x_0), \quad i = 3, 4.$$

Because of the right continuity of paths, (3.8) is equivalent to

$$(3.9) \quad \limsup_{t \downarrow 0} \frac{X(t, \omega_0) - x_0}{(2s_i(t, \omega_0) \log \log 1/s_i(t, \omega_0))^{1/2}} = \sigma_i(x_0), \quad i = 3, 4.$$

Hence we must show that for $1 > \varepsilon > 0$,

$$(3.10) \quad X(t, \omega_0) - x_0 > \sigma_i(x_0)(1 - \varepsilon)(2s_i(t, \omega_0) \log \log 1/s_i(t, \omega_0))^{1/2}$$

for arbitrarily small t , and

$$(3.11) \quad X(t, \omega_0) - x_0 < \sigma_i(x_0)(1 + \varepsilon)(2s_i(t, \omega_0) \log \log 1/s_i(t, \omega_0))^{1/2}$$

for all sufficiently small t . We prove (3.10), (3.11) is proved in an analogous manner.

Because of (3.7) we can assume that for arbitrarily small values of t ,

$$(3.12) \quad X(t, \omega_0) - x_0 > (1 - \varepsilon/2)(2t \log \log 1/t)^{1/2}.$$

Hence it suffices to show that for any ρ satisfying $1 > \rho > 0$,

$$\lim_{t \downarrow 0} \frac{(2t \log \log 1/t)^{1/2}}{(2s_i(t, \omega_0) \log \log 1/s_i(t, \omega_0))^{1/2}} \geq \sigma_i(x_0)(1 - \rho), \quad i = 3, 4.$$

For $0 < \eta_i < 1/\sigma_i^2(x_0)$ let t_0 be chosen so that for $t \leq t_0$,

$$|\sigma_i^{-2}(X(s_i(t, \omega_0))) - \sigma_i^{-2}(x_0)| < \eta_i, \quad i = 3, 4.$$

But then if $t \leq t_0$, $0 < t/\sigma_i^2(x_0) - \eta_i t \leq s_i(t, \omega_0) \leq t/\sigma_i^2(x_0) + \eta_i t$. But then

$$\begin{aligned} & \lim_{t \downarrow 0} \frac{(2t \log \log 1/t)^{1/2}}{(2s_i(t, \omega_0) \log \log 1/s_i(t, \omega_0))^{1/2}} \\ & \geq \lim_{t \downarrow 0} \frac{(2t \log \log 1/t)^{1/2}}{(2(t/\sigma_i^2(x_0) + \eta_i t) \log \log (1/(t/\sigma_i^2(x_0) - \eta_i t)))^{1/2}} \\ & = \frac{\sigma_i(x_0)}{(1 + \eta_i \sigma_i^2(x_0))^{1/2}} \lim_{t \downarrow 0} \left(\frac{\log \log 1/t}{\log \log 1/t(1/\sigma_i^2(x_0) - \eta_i)} \right)^{1/2} \\ & = \frac{\sigma_i(x_0)}{(1 + \eta_i \sigma_i^2(x_0))^{1/2}}, \quad i = 3, 4, \\ & \geq (1 - \rho)\sigma_i(x_0) \quad \text{if } \eta_i, (i = 3, 4), \text{ are chosen to be sufficiently small.} \end{aligned}$$

The last limit indicated above is found by a double application of De l'Hôpital's rule. Hence (3.10) follows and the proof is complete.

COROLLARY. π_1 and π_2 are equivalent in finite time if and only if they are instantaneously equivalent.

Proof. Since

$$\left\{ \limsup_{t \downarrow 0} \frac{X(t) - X(0)}{(2t \log \log 1/t)^{1/2}} = a \right\} \in \mathcal{M}_0,$$

the proposition implies that if π_1 and π_2 are instantaneously equivalent, then $\sigma_1(x) = \sigma_2(x)$ for all x . On the other hand if $\sigma_1(x) = \sigma_2(x)$ for all $x \in E$, then the results of A. V. Skorokhod imply that π_1 and π_2 are strongly equivalent in finite time.

REMARKS. 3.1. It can be shown that if $a_1(x) \geq a_2(x) + c$ for all $x \in E$ where c is a

constant, then π_1 and π_2 are not equivalent on \mathcal{F}_∞ even when $\sigma_1(x) = \sigma_2(x)$ for all $x \in E$.

3.2. In the above proposition it is shown that for a one-dimensional diffusion the measures restricted to the germ fields actually determine the diffusive part of the infinitesimal generator and thus determine the equivalence class (in finite time) of the process. However the extension of this result to higher dimensions has not yet been established. Nevertheless one would expect the extension to be valid and our results on equivalence lend support to this supposition.

The following two examples show that instantaneous equivalence does not always imply equivalence in finite time and illustrate two of the types of singularities which are discussed in §8.

EXAMPLE 2. An example of a pair of strong Markov processes which are instantaneously equivalent but not equivalent in finite time is furnished by the pair of deterministic processes:

$$(3.13) \quad X_1(t) \equiv X_1(0) + t,$$

and

$$(3.14) \quad X_2(t) \equiv X_2(0) + t - n(t + X_2(0)) + n(X_2(0)),$$

where $n(t)$ designates the greatest integer less than or equal to t . Note that the second process defined by (3.14) is not quasi-left continuous.

EXAMPLE 3. Let $\pi = (\Omega, \mathcal{F}_\infty, (P_x)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ be a Hunt process such that for some $x_0 \in E$ and $t_0 < \infty$, $P_{x_0}(\zeta < t_0) > 0$. Let $E_{\Delta, \Delta'} = E \cup \{\Delta\} \cup \{\Delta'\}$ where Δ' is an isolated point and $\Delta' \notin E \cup \{\Delta\}$. Let Ω' be the set of right continuous functions, $\omega(\cdot)$, from $[0, \infty]$ to $E_{\Delta, \Delta'}$ such that:

$$(3.15) \quad \omega(\cdot) \text{ has left-hand limits on } [0, \infty),$$

$$(3.16) \quad \text{if } \omega(t) = \Delta, \text{ then } \omega(s) = \Delta \quad \text{for all } s \geq t,$$

$$(3.17) \quad \text{if } \omega(t) = \Delta', \text{ then } \omega(s) = \Delta' \quad \text{for all } s \geq t, \text{ and}$$

$$(3.18) \quad \omega(\infty) \in \{\Delta\} \cup \{\Delta'\}.$$

Let \mathcal{F}'_∞ be the σ -field generated by the random variables $(X_t)_{t \in [0, \infty)}$. The measures $(P_x)_{x \in E}$ can be extended to $(\Omega', \mathcal{F}'_\infty)$ by defining $P_x^1(A) \equiv P_x(A \cap \Omega)$ for $A \in \mathcal{F}'_\infty$. Let $\pi_1 \equiv (\Omega', \mathcal{F}'_\infty, (P_x^1)_{x \in E_{\Delta, \Delta'}}, (X_t)_{t \in [0, \infty)})$. Let $\Phi: \Omega' \rightarrow \Omega'$ be defined by:

$$\Phi(\omega)(s) = \omega(s) \quad \text{if } \omega(s) \in E,$$

$$\Phi(\omega)(s) = \Delta' \quad \text{if } \omega(s) = \Delta,$$

and

$$\Phi(\omega)(s) = \Delta \quad \text{if } \omega(s) = \Delta'.$$

For $A \in \mathcal{F}'_\infty$, let $P_x^2(A) \equiv P_x^1(\Phi^{-1}(A))$ and let $\pi_2 = (\Omega', \mathcal{F}'_\infty, (P_x^2)_{x \in E_{\Delta, \Delta'}}, (X_t)_{t \in [0, \infty)})$. Then π_1 and π_2 are instantaneously equivalent and even equivalent in finite time. However they are not strongly equivalent in finite time, in fact $P_{x_0}^1(X_{t_0} = \Delta) > 0$, $P_{x_0}^2(X_{t_0} = \Delta) = 0$.

4. **The stochastic processes** $Y_A(t, \cdot)$. In this section we assign to each Hunt process $\pi = (\Omega, \mathcal{F}_\infty, (P_x)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ and set $A \in \mathcal{F}_\infty$ a stochastic process $Y_A(t, \cdot)$. These processes are introduced in order to formulate an extended Markov property in §5. The extended Markov property implies that $Y_A(t, \cdot)$ is for each $x \in E$, a version of the martingale $P_x(A | \mathcal{M}_t)$.

Before introducing the processes $Y_A(t, \cdot)$ it is necessary to review some useful notation and results due to P. Courrège and P. Priouret [5]. If $H: \Omega \rightarrow [0, \infty]$ is measurable, an equivalence relation R_H on Ω is defined by:

$$\begin{aligned} \omega \sim \omega' (R_H) \text{ if and only if } & H(\omega) = H(\omega') \text{ and} \\ & X_s(\omega) = X_s(\omega') \text{ for all } s \leq H(\omega). \end{aligned}$$

A second equivalence relation R_{H+} is defined by:

$\omega \sim \omega' (R_{H+})$ if and only if $H(\omega) = H(\omega')$ and there exists an $\varepsilon > 0$ such that $X_s(\omega) = X_s(\omega')$ for all $s \leq H(\omega) + \varepsilon$.

- PROPOSITION 4.1.** (i) \mathcal{F}_t is the sub- σ -field of sets of \mathcal{F}_∞ which are saturated for R_t .
(ii) \mathcal{M}_t is the sub- σ -field of sets of \mathcal{F}_∞ saturated for R_{t+} .
(iii) If T is an (\mathcal{M}_t) -stopping time, then \mathcal{M}_T is the sub- σ -field of sets of \mathcal{F}_∞ saturated for R_{T+} .

The following version of Galmarino's lemma will also be required.

PROPOSITION 4.2. In order that $T: \Omega \rightarrow [0, \infty]$ be an (\mathcal{M}_t) -stopping time it is necessary and sufficient that T be \mathcal{F}_∞ -measurable and that for each $t \in [0, \infty]$, $\omega \in \Omega$, $\omega' \in \Omega$, $T(\omega) < t$ and $\omega' \sim \omega(R_t)$ implies that $T(\omega') = T(\omega)$.

Proof. Refer to P. Courrège and P. Priouret [5].

Let $A \in \mathcal{F}_\infty$ and $\omega \in \Omega$. Then for $t \in [0, \infty)$,

$$(4.1) \quad A^*(t, \omega) \equiv \{\omega' : \omega' \in A, \omega' \sim \omega(R_t)\},$$

and

$$(4.2) \quad A(t, \omega) \equiv \theta_t A^*(t, \omega) = \{\theta_t \omega' : \omega' \in A^*(t, \omega)\}$$

and for $t = \infty$,

$$(4.3) \quad A(\infty, \omega) \equiv \emptyset \quad \text{if } \omega \notin A,$$

and

$$A(\infty, \omega) \equiv \{\omega_\Delta\} \quad \text{if } \omega \in A.$$

The fact that $A(t, \omega) \in \mathcal{F}_\infty$ is verified in the following proposition.

- PROPOSITION 4.3.** (i) If $A \in \mathcal{F}_s$ and $t < s$, then for each $\omega \in \Omega$, $A(t, \omega) \in \mathcal{F}_{s-t}$.
(ii) $(\bigcup_{i=1}^\infty A_i)(t, \omega) = \bigcup_{i=1}^\infty A_i(t, \omega)$ for each $\omega \in \Omega$.
(iii) $(\bigcap_{i=1}^\infty A_i)(t, \omega) = \bigcap_{i=1}^\infty A_i(t, \omega)$ for each $\omega \in \Omega$.
(iv) $A^c(t, \omega) = (A(t, \omega))^c \cap \{\omega' : X_0(\omega') = X_t(\omega)\}$ for each $\omega \in \Omega$.

Proof.

$$(ii) (\bigcup_{i=1}^{\infty} A_i)(t, \omega) = \theta_t[(\bigcup_{i=1}^{\infty} A_i)^*(t, \omega)] = \theta_t[\bigcup_{i=1}^{\infty} A_i^*(t, \omega)] = \bigcup_{i=1}^{\infty} \theta_t A_i^*(t, \omega) = \bigcup_{i=1}^{\infty} A_i(t, \omega).$$

$$(iii) (\bigcap_{i=1}^{\infty} A_i)(t, \omega) = \theta_t[(\bigcap_{i=1}^{\infty} A_i)^*(t, \omega)] = \theta_t[\bigcap_{i=1}^{\infty} A_i^*(t, \omega)] = \bigcap_{i=1}^{\infty} \theta_t A_i^*(t, \omega) = \bigcap_{i=1}^{\infty} A_i(t, \omega).$$

$$(iv) A^c(t, \omega) = \theta_t(A^{c*}(t, \omega)) = \theta_t[(A^*(t, \omega))^c \cap \{\omega' : X_t(\omega') = X_t(\omega)\}] = (A(t, \omega))^c \cap \{\omega' : X_0(\omega') = X_t(\omega)\}.$$

To prove (i) it thus suffices to show that $A(t, \omega) \in \mathcal{F}_{s-t}$ when

$$A = \{\omega : X_v(\omega) \in \Gamma_1, X_t(\omega) \in \Gamma_2, X_u(\omega) \in \Gamma_3\}$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ are in \mathcal{E} and $v < t < u \leq s$. But then

$$A(t, \omega) = \{\omega' : X_0(\omega') = X_t(\omega), X_{u-t}(\omega') \in \Gamma_3\} \quad \text{if } X_v(\omega) \in \Gamma_1, X_t(\omega) \in \Gamma_2, \\ = \emptyset \quad \text{if } X_v(\omega) \notin \Gamma_1 \text{ or } X_t(\omega) \notin \Gamma_2.$$

Hence $A(t, \omega) \in \mathcal{F}_{s-t}$ and the proof is complete.

For $A \in \mathcal{F}_{\infty}$ and Hunt process $\pi = (\Omega, \mathcal{F}_{\infty}, (P_x)_{x \in E_A}, (X_t)_{t \in [0, \infty)})$ the real-valued stochastic process $(\Omega, \mathcal{F}_{\infty}, (P_x)_{x \in E_A}, (Y_A(t))_{t \in [0, \infty)})$ is defined as follows:

$$(4.4) \quad Y_A(t, \omega) \equiv P_{X(t, \omega)}(A(t, \omega)) \quad \text{for } t \in [0, \infty], \omega \in \Omega.$$

Note that $Y_{\Omega}(t, \omega) \equiv 1$.

PROPOSITION 4.4. (i) If $A \cap B = \emptyset$, then $Y_{A \cup B}(t, \omega) = Y_A(t, \omega) + Y_B(t, \omega)$ for each $\omega \in \Omega$ and $t \in [0, \infty]$.

(ii) If $A_i \uparrow A$ or $A_i \downarrow A$, then $Y_A(t, \omega) = \lim_{i \rightarrow \infty} Y_{A_i}(t, \omega)$ for each $\omega \in \Omega$ and $t \in [0, \infty]$.

(iii) If $A \in \mathcal{F}_{\infty}$, then $Y_A(t, \cdot)$ is measurable with respect to \mathcal{F}_t .

Proof. (i) By Proposition 4.3, (ii) and (iv), if $A \cap B = \emptyset$, then

$$Y_{A \cup B}(t, \omega) = P_{X(t, \omega)}((A \cup B)(t, \omega)) \\ = P_{X(t, \omega)}(A(t, \omega)) + P_{X(t, \omega)}(B(t, \omega)) = Y_A(t, \omega) + Y_B(t, \omega).$$

(ii) We prove (ii) for $A_i \uparrow A$, the proof for $A_i \downarrow A$ is similar. If $A_i \uparrow A$, then by Proposition 4.3. (i), $A_i(t, \omega) \uparrow A(t, \omega)$. But then

$$Y_{A_i}(t, \omega) = P_{X(t, \omega)}(A_i(t, \omega)) \uparrow P_{X(t, \omega)}(A(t, \omega)) = Y_A(t, \omega).$$

In view of (i) and (ii) it suffices to prove (iii) for a set of the form

$$A = \{X(t_1) \in \Gamma_1, \dots, X(t_i) \in \Gamma_i, X(t) \in \Gamma_0, X(t_{i+1}) \in \Gamma_{i+1}, \dots, X(t_n) \in \Gamma_n\}$$

where $\Gamma_i \in \mathcal{E}, i=0, \dots, n$ and $t_1 < t_2 < \dots < t_i < t < t_{i+1} < \dots < t_n$. But then

$$A(t, \omega) = \{X(t_{i+1}-t) \in \Gamma_{i+1}, \dots, X(t_n-t) \in \Gamma_n\} \cap \{X(0) = X_t(\omega)\} \\ \text{if } \omega \in \{X(t_1) \in \Gamma_1, \dots, X(t_i) \in \Gamma_i, X(t) \in \Gamma_0\}, \\ = \emptyset \quad \text{if } \omega \notin \{X(t_1) \in \Gamma_1, \dots, X(t_i) \in \Gamma_i, X(t) \in \Gamma_0\}.$$

Hence

$$P_{X(t, \omega)}(A(t, \omega)) = \chi_{\{X(t_1) \in \Gamma_1, \dots, X(t_i) \in \Gamma_i, X(t) \in \Gamma_0\}} \cdot P_{X(t, \omega)}(X(t_{i+1}-t) \in \Gamma_{i+1}, \dots, X(t_n-t) \in \Gamma_n),$$

where $\chi_K(\cdot)$ designates the indicator function of the set K . The result then follows since

$$P_{X(t, \omega)}(X(t_{i+1}-t) \in \Gamma_{i+1}, \dots, X(t_n-t) \in \Gamma_n)$$

is a composition of the \mathcal{F}_t -measurable mapping $\omega \rightsquigarrow X_t(\omega)$ and the measurable mapping $y \rightsquigarrow P_y(X(t_{i+1}-t) \in \Gamma_{i+1}, \dots, X(t_n-t) \in \Gamma_n)$.

PROPOSITION 4.5. *If $A \in \mathcal{F}_\infty$, then $Y_A(\cdot, \cdot)$ is progressively measurable.*

Proof. Proposition 4.4 implies that the class of sets A , for which $Y_A(\cdot, \cdot)$ is progressively measurable is closed under the operations of taking complements, disjoint unions and monotone limits. Since E is a locally compact separable metric space, it suffices to show that $Y_A(\cdot, \cdot)$ is progressively measurable for a set of the form

$$(4.4) \quad A = \{X(t_1) \in \Gamma_1, \dots, X(t_m) \in \Gamma_m\}$$

with $\Gamma_1, \dots, \Gamma_m$ compact and $t_1 < t_2 < \dots < t_m$. For $t_{i-1} \leq s < t$

$$Y_A(s, \omega) = \chi_{\{X(t_1) \in \Gamma_1\}} \cdots \chi_{\{X(t_{i-1}) \in \Gamma_{i-1}\}} \cdot P_{X(s, \omega)}(X(t_i-s) \in \Gamma_i, \dots, X(t_m-s) \in \Gamma_m).$$

Since the Γ_i , $i=1, \dots, n$ are compact, for each i there is a sequence $\{f_i^n\}$ of $C_0(E)$ functions such that $f_i^n(x) \downarrow \chi_{\Gamma_i}(x)$ as $n \rightarrow \infty$ (P. R. Halmos [9, Chapter 10]). But then

$$\begin{aligned} P_{X(s, \omega)}(X(t_i-s) \in \Gamma_i, \dots, X(t_m-s) \in \Gamma_m) &= \lim_{n \rightarrow \infty} E_{X(s, \omega)}(f_i^n(X(t_i-s)) \cdot f_{i+1}^n(X(t_{i+1}-s)) \cdots f_m^n(X(t_m-s))) \\ &= \lim_{n \rightarrow \infty} P_{t_i-s}(f_i^n(y_i) \cdots (P_{t_m-t_{m-1}} f_m^n(y_m)))(X(s, \omega)). \end{aligned}$$

But Proposition 2.4.1 implies that $P_{t_i-s}(f_i^n(y_i) \cdots (P_{t_m-t_{m-1}} f_m^n(y_m)))(y)$ is jointly continuous with respect to s and y for $s \in [t_{i-1}, t_i]$ and $y \in E$. Since $X(s, \omega)$ is a right continuous function of s for each ω , $P_{t_i-s}(f_i^n(y_i) \cdots (P_{t_m-t_{m-1}} f_m^n(y_m)))(X(s, \omega))$ is therefore right continuous for $s \in [t_{i-1}, t_i]$ for each ω . Hence $Y_A(\cdot, \cdot)$ is progressively measurable (cf. P. A. Meyer [13, Chapter 4, T 47]).

We now introduce the concept of a relative stopping time. Let T be a stopping time and let $t \leq T(\omega_0)$, $\omega_0 \in \Omega$. Note that if $\omega \in \Omega$ and $X_0(\omega) = X_t(\omega_0)$ then there is a path $\omega' \in \Omega$ such that $\omega' \sim \omega_0(R_t)$ and such that $\omega = \theta_t \omega'$. In this case define $T'(\omega) \equiv T(\omega') - t$. If $X_0(\omega) \neq X_t(\omega_0)$ let $T'(\omega) \equiv +\infty$. T' is called the (T, ω_0, t) -relative stopping time.

PROPOSITION 4.6. *The (T, ω_0, t) -relative stopping time is a stopping time.*

Proof. It is easy to verify that T' is a measurable mapping from Ω to $[0, \infty]$. In particular, for $s < \infty$, $\{T' < s\} = \{X(0) = X(t, \omega_0)\} \cap \{T < s + t\}(t, \omega)$. The result then follows from Proposition 4.3 which implies that $\{T < s + t\}(t, \omega) \in \mathcal{M}_s$.

5. The extended Markov property. In this section the extended Markov property is established. However, before doing this it is necessary to introduce a new class of σ -fields.

Given a stopping time T , \mathcal{M}_T^* is the σ -field $\theta_T^{-1}\mathcal{F}_\infty$ which is a sub- σ -field of \mathcal{F}_∞ (Proposition 2.6.1). It is easy to verify that \mathcal{M}_T^* is the σ -field generated by the random variables $(X_{T+s})_{s \geq 0}$.

PROPOSITION 5.1. *For any (\mathcal{M}_t) -stopping time T , $\mathcal{M}_T \vee \mathcal{M}_T^*$, the σ -field generated by \mathcal{M}_T and \mathcal{M}_T^* is identical to \mathcal{F}_∞ .*

Proof. Clearly $\mathcal{M}_T \vee \mathcal{M}_T^* \subset \mathcal{F}_\infty$. Hence it suffices to show that $\{X_t \in \Gamma\}$ for $t \geq 0$ and Γ an open subset of E is measurable with respect to $\mathcal{M}_T \vee \mathcal{M}_T^*$.

For $\Gamma \in \mathcal{E}$,

$$\{X_t \in \Gamma\} = (\{T \geq t\} \cap \{X_t \in \Gamma\}) \cup (\{T < t\} \cap \{X_t \in \Gamma\}).$$

But $\{T \geq t\} \cap \{X_t \in \Gamma\} \in \mathcal{M}_T$. Hence it suffices to show that $\{T < t\} \cap \{X_t \in \Gamma\} \in \mathcal{M}_T^*$. Note that $T_n \equiv \sum_{i=0}^\infty \chi_{\{i/n \leq T < (i+1)/n\}}((i+1)/n)$, $n = 1, 2, 3, \dots$, is a decreasing sequence of stopping times with limit T . Then if $\eta_n \equiv (T + t - T_n + 1/n) \vee T$, $\eta_n \rightarrow t \vee T$, and the η_n , $n = 1, 2, 3, \dots$, are measurable with respect to \mathcal{M}_T . Hence the η_n , $n = 1, 2, 3, \dots$, are stopping times (P. A. Meyer [1, Chapter 4, T 38]). The right continuity of the paths and the fact that Γ is open imply that

$$\{X_t \in \Gamma\} \cap \{T < t\} = \bigcup_{j \in \mathcal{J}} \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty (\{T < t\} \cap \{X_{\eta_n} \in \Gamma_j\})$$

where the Γ_j , $j = 1, 2, 3, \dots$, and \mathcal{J} are chosen exactly as in the proof of Proposition 2.2.1. But for each n and j ,

$$\begin{aligned} \{T < t\} \cap \{X_{\eta_n} \in \Gamma_j\} &= \bigcup_{i=0}^\infty (\{T < t\} \cap \{X_{\eta_n} \in \Gamma_j\} \cap \{T_n = i/n\}) \\ &= \bigcup_{i=0}^\infty (\{T < t\} \cap \{X(T+t-(i-1)/n) \in \Gamma_j\} \cap \{T_n = 1/n\}). \end{aligned}$$

Since $t \geq (i-1)/n$ on $\{T < t\} \cap \{T_n = i/n\}$, therefore

$$\{T < t\} \cap \{X(T+t-(i-1)/n) \in \Gamma_j\} \cap \{T_n = i/n\} \in \mathcal{M}_T^*.$$

Hence $\{T < t\} \cap \{X_t \in \Gamma\} \in \mathcal{M}_T^*$ and the proof is complete. The following corollary has also been proved.

COROLLARY. *For any stopping time T the events of the form $\{X_{T+t} \in \Gamma\}$ with $\Gamma \in \mathcal{E}$ and $t \geq 0$ and $\{X_t \in \Gamma\} \cap \{T \geq t\}$ with $\Gamma \in \mathcal{E}$ and $t \geq 0$ generate the σ -field \mathcal{F}_∞ .*

THEOREM 5.1 (THE EXTENDED MARKOV PROPERTY). *For any stopping time T , $A \in \mathcal{F}_\infty$, $B \in \mathcal{M}_T$ and any $x \in E$,*

$$\begin{aligned}
 P_x(A \cap B) &= \int_B P_{X(T)}(A(T(\omega), \omega)) dP_x(\omega) \\
 (5.1) \qquad &= \int_B Y_A(T(\omega), \omega) dP_x(\omega) \\
 &= E_x(\chi_B(\omega) \cdot Y_A(T(\omega), \omega)).
 \end{aligned}$$

In other words for every $x \in E$, $Y_A(t, \cdot)$ is a version of the martingale $E_x(\chi_A | \mathcal{M}_t)$ for which the optional sampling theorem is valid.

Proof. $P_x(A \cap B) = P_x(A \cap B \cap \{T < \infty\}) + P_x(A \cap B \cap \{T = \infty\})$. But

$$\begin{aligned}
 P_x(A \cap B \cap \{T = \infty\}) &= \int_{B \cap \{T = \infty\}} \chi_A \cdot P_\Delta(\omega_\Delta) dP_x(\omega) \\
 &= E_x(\chi_{B \cap \{T = \infty\}} \cdot Y_A(T(\omega), \omega)).
 \end{aligned}$$

Hence it suffices to prove (5.1) in the case in which $B \subset \{T < \infty\}$. Note that since $Y_A(\cdot, \cdot)$ is progressively measurable, $Y_A(T(\omega), \omega)$ is measurable with respect to \mathcal{M}_T and hence the integral on the right-hand side of (5.1) is well defined.

We first show that the class of sets A for which (5.1) is satisfied is closed under disjoint unions and monotone limits. If $A \cap C = \emptyset$ and both A and C satisfy (5.1), then

$$\begin{aligned}
 P_x(B \cap (A \cup C)) &= P_x(A \cap B) + P_x(C \cap B) \\
 &= \int_B Y_A(T(\omega), \omega) dP_x(\omega) + \int_B Y_C(T(\omega), \omega) dP_x(\omega) \\
 &= \int_B Y_{A \cup C}(T(\omega), \omega) dP_x(\omega) \quad \text{by Proposition 4.4,}
 \end{aligned}$$

and therefore $A \cup C$ satisfies (5.1). If $A_i \downarrow A$ as $i \rightarrow \infty$ and $A_i, i = 1, 2, 3, \dots$, satisfy (5.1), then

$$\begin{aligned}
 P_x(A \cap B) &= \lim_{i \rightarrow \infty} P_x(A_i \cap B) = \lim_{i \rightarrow \infty} \int_B Y_{A_i}(T(\omega), \omega) dP_x(\omega) \\
 &= \int_B \lim_{i \rightarrow \infty} Y_{A_i}(T(\omega), \omega) dP_x(\omega) \quad \text{by the monotone convergence theorem} \\
 &= \int_B Y_A(T(\omega), \omega) dP_x(\omega) \quad \text{by Proposition 4.4.}
 \end{aligned}$$

Therefore A satisfies (5.1).

Hence in view of Proposition 5.1 it suffices to prove (5.1) for sets of the form

$$(5.2) \qquad A_0 = C \cap \{X(T+t_1) \in \Gamma_1\} \cap \dots \cap \{X(T+t_n) \in \Gamma_n\},$$

where $0 \leq t_1 < t_2 < \dots < t_n$, $\Gamma_i \in \mathcal{E}$, $i = 1, \dots, n$, and $C \in \mathcal{M}_T$. In fact in view of the Corollary to Proposition 5.1, it suffices to consider sets C of the form

$$C = \{X(s_1) \in \Lambda_1, \dots, X(s_m) \in \Lambda_m\} \cap \{s_m \leq T < s_{m+1}\},$$

where $0 \leq s_1 < \dots < s_{m+1}$ and $\Lambda_i \in \mathcal{E}$, $i = 1, \dots, m$. In this case it is easy to verify that the strong Markov property implies that

$$(5.3) \quad P_x(A_0 \cap B) = \int_{B \cap C} P_{X(T(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) dP_x(\omega),$$

(cf. E. B. Dynkin [7, Theorem 3.11]). Hence it suffices to show that if A_0 is defined by (5.2) then

$$(5.4) \quad \int_{B \cap C} P_{X(T(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) dP_x(\omega) = \int_B Y_{A_0}(T(\omega), \omega) dP_x(\omega).$$

Let $H = \{X_T^- = X_T\} \cap \{T < \infty\}$ and $F \equiv \{X_T^- \neq X_T\} \cap \{T < \infty\}$. For $k = 1, 2, 3, \dots$, define

$$\begin{aligned} J_0^k(\cdot) &= 0, \\ J_{n+1}^k(\omega) &= \inf \{s : s > J_n^k(\omega), \rho(X_s(\omega), X_s^-(\omega)) > 1/k\}, \\ &= +\infty \quad \text{if the above set is empty.} \end{aligned}$$

Since the paths possess limits from the left on $[0, \infty)$ and are right continuous,

$$F = \left[\left(\bigcup_{k=1}^{\infty} \bigcup_{n=0}^{\infty} \{T = J_n^k\} \right) \cup (F \cap \{T = \zeta\}) \right] \cap \{T < \infty\}.$$

Note that $\{T = J_n^k\} \in \mathcal{M}_{J_n^k} \cap \mathcal{M}_T$ for any k and n . If $\omega \in \Omega$ and if $\omega \sim \omega'(R_{J_n^k})$, then $J_n^k(\omega') = J_n^k(\omega)$.

We now prove (5.1) for the case $T \equiv J_n^k$ for some k and n . If A_0 is defined by (5.2) (with $T \equiv J_n^k$), then

$$\begin{aligned} A_0(J_n^k(\omega), \omega) &= \{X(t_1) \in \Gamma_1\} \cap \dots \cap \{X(t_n) \in \Gamma_n\} \cap \{X(0) = X(J_n^k(\omega))\} \quad \text{if } \omega \in C \\ &= \emptyset \quad \text{if } \omega \notin C, \end{aligned}$$

and therefore

$$Y_{A_0}(J_n^k(\omega), \omega) = \chi_C \cdot P_{X(J_n^k(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n).$$

Therefore

$$\int_{B \cap C} P_{X(J_n^k(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) dP_x(\omega) = \int_B Y_{A_0}(J_n^k(\omega), \omega) dP_x(\omega).$$

Since we have shown that (5.4) implies (5.1), therefore for any $A \in \mathcal{F}_\infty$, $B \in \mathcal{M}_{J_n^k}$ and $x \in E$,

$$P_x(A \cap B) = \int_B Y_A(J_n^k(\omega), \omega) dP_x(\omega).$$

In particular for an arbitrary stopping time T ,

$$(5.5) \quad P_x(T = J_n^k) = \int_{\{T = J_n^k\}} Y_{(T = J_n^k)}(J_n^k(\omega), \omega) dP_x(\omega)$$

and therefore

$$(5.6) \quad Y_{\{T=J_n^k\}}(J_n^k(\omega), \omega) = 1, \quad P_x\text{-almost surely on } \{T = J_n^k\}.$$

We now proceed to prove (5.4) for an arbitrary stopping time T when $B \subset F$. If $\omega \in \{T=J_n^k\}$, then

$$(5.7) \quad A_0(J_n^k(\omega), \omega) = (A_0 \cap \{T = J_n^k\})(J_n^k(\omega), \omega) + (A_0 \cap \{T > J_n^k\})(J_n^k(\omega), \omega).$$

Then (5.6) implies that

$$\begin{aligned} Y_{A_0}(T(\omega), \omega) &= P_{X(T(\omega))}(A_0 \cap \{T = J_n^k\})(J_n^k(\omega), \omega) \\ &= P_{X(T(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) \cdot \chi_C, \end{aligned}$$

P_x -almost surely on $\{T=J_n^k\}$. A similar result holds for $\{T=\zeta\} \cap F$. Hence it is easy to verify that

$$\int_{B \cap C \cap F} P_{X(T(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) dP_x(\omega) = \int_{B \cap F} Y_{A_0}(T(\omega), \omega) dP_x(\omega).$$

Proposition 2.9.1 implies that there is an increasing sequence of stopping times $\{S_n\}$, $S_n \leq T$ for all n , such that:

$$(5.8) \quad \text{if } Q \equiv \{S_n < T \text{ for all } n, \lim_{n \rightarrow \infty} S_n = T\}, \text{ then } P_x(Q \cap H) = P_x(H).$$

Let $S_\infty \equiv \lim_{n \rightarrow \infty} S_n$ and $Q' \equiv \{S_n < S_\infty \text{ for all } n\}$.

We next prove (5.1) for the case $T \equiv S_\infty$. If $\omega \in Q'$ and $\omega' \sim \omega(R_{S_\infty})$, then $S_n(\omega') = S_n(\omega)$ for all n (cf. Proposition 4.2) and consequently $S_\infty(\omega') = S_\infty(\omega)$. If $\omega \in Q'$ and if A_0 is defined by (5.2) (with $T \equiv S_\infty$), then

$$\begin{aligned} A_0(S_\infty(\omega), \omega) &= \{X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n\} \cap \{X(0) = X(S_\infty(\omega))\} \quad \text{if } \omega \in C \\ &= \emptyset \quad \text{if } \omega \notin C, \end{aligned}$$

and therefore

$$Y_{A_0}(S_\infty(\omega), \omega) = P_{X(S_\infty(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) \chi_C.$$

Hence it is easy to verify that for any set $A \in \mathcal{F}_\infty$ and any set $B \in \mathcal{M}_{S_\infty}$ and any x such that $P_x(B \cap Q') = P_x(B)$, then

$$(5.9) \quad P_x(B \cap A) = \int_B Y_A(S_\infty(\omega), \omega) dP_x(\omega).$$

In particular for an arbitrary stopping time T , if $P_x(\{T=S_\infty\} \cap Q') = P_x(\{T=S_\infty\})$, then

$$P_x(\{T = S_\infty\}) = \int_{\{T=S_\infty\}} Y_{\{T=S_\infty\}}(S_\infty(\omega), \omega) dP_x(\omega)$$

and therefore

$$(5.10) \quad P_{X(S_\infty(\omega))}(\{T = S_\infty\}(S_\infty(\omega), \omega)) = 1, \quad P_x\text{-almost surely on } \{T = S_\infty\}.$$

We are now ready to prove (5.4) for an arbitrary stopping time T when $B \subset H$. Let $\{S_n\}$ be chosen to satisfy (5.8). For $\omega \in \Omega$ let $T'_\omega(\cdot)$ be the $(T, \omega, S_\infty(\omega))$ -relative stopping time. Then (5.10) implies that $P_{X(S_\infty(\omega))}(T'_\omega(\cdot) = 0) = 1$, P_x -almost surely on H . For $\omega \in \Omega$ let $T''_\omega(\cdot)$ be the $(T, \omega, T(\omega))$ -relative stopping time. If $\omega \in Q$, then $S_\infty(\omega) = T(\omega)$ and therefore

$$(5.11) \quad P_{X(T(\omega))}(T''_\omega(\cdot) = 0) = 1, \quad P_x\text{-almost surely on } H.$$

But if A_0 is defined by (5.2) and if $\omega \in Q$, then

$$(5.12) \quad \begin{aligned} A_0(T(\omega), \omega) &= \{X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n\} \\ &\quad \cap \{T''_\omega(\cdot) = 0\} \cap \{X(0) = X(T(\omega))\} \\ &\quad \cup A' \cap \{T''_\omega(\cdot) > 0\} \quad \text{if } \omega \in C \\ &= A'' \cap \{T''_\omega(\cdot) > 0\} \quad \text{if } \omega \notin C \end{aligned}$$

for some $A' \in \mathcal{F}_\infty$ and $A'' \in \mathcal{F}_\infty$. Thus (5.11) implies that

$$Y_{A_0}(T(\omega), \omega) = \chi_C P_{X(T(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n),$$

P_x -almost surely on H . Hence for A_0 defined by (5.2),

$$\int_{B \cap H} Y_{A_0}(T(\omega), \omega) dP_x(\omega) = \int_{C \cap B \cap H} P_{X(T(\omega))}(X(t_1) \in \Gamma_1, \dots, X(t_n) \in \Gamma_n) dP_x(\omega)$$

and the proof is complete.

COROLLARY 1. *If $A \in \mathcal{M}_{T+t}$, then except for a set of ω of P_x -probability zero $A(T(\omega), \omega) \supset B(\omega)$ such that $B(\omega) \in \mathcal{M}_t$ and such that $P_{X(T(\omega))}(A(T(\omega), \omega) - B(\omega)) = 0$.*

Proof. The proof follows immediately from (5.6), (5.7), (5.11), and (5.12).

COROLLARY 2. *If $t < T(\omega_0)$, then $Y_A(t, \omega_0) = \int Y_A(T(\omega'), \omega') dP_{X(t, \omega_0)}^*(\omega')$, where $P_{X(t, \omega_0)}^*$ is the measure on \mathcal{F}_∞ defined by*

$$P_{X(t, \omega_0)}^*(A) \equiv P_{X(t, \omega_0)}(A(t, \omega_0)), \quad A \in \mathcal{F}_\infty.$$

Proof. Let T' be the (T, ω_0, t) -relative stopping time. Then Theorem 5.1 implies that

$$\begin{aligned} P_{X(t, \omega_0)}(A(t, \omega_0)) &= \int Y_{A(t, \omega_0)}(T'(\omega'), \omega') dP_{X(t, \omega_0)}(\omega') \\ &= \int Y_A(T(\omega'), \omega') dP_{X(t, \omega_0)}^*(\omega'). \end{aligned}$$

The last equality is an immediate consequence of the definitions of T' and $P_{X(t, \omega_0)}^*$.

REMARKS 5.1. Since the σ -fields (\mathcal{M}_t) are right continuous, the process $Y_{A+}(t, \omega) \equiv \lim_{s \downarrow t} Y_A(s, \omega)$ is a right continuous modification of the martingale $Y_A(t, \cdot)$ (refer to P. A. Meyer [13, Chapter 6, T 4]). In fact Professor P. A. Meyer has proved in a private communication that $Y_A(\cdot, \cdot)$ is actually well measurable [13,

Chapter 8] and therefore [13, Chapter 8, T 20] it is easy to verify that $Y_A(\cdot, \omega) = Y_{A^+}(\cdot, \omega)$ P_x -almost surely for each x . However we make no use of this property in this paper.

5.2. From another point of view $Y.(t, \cdot)$ can be interpreted as a regular conditional probability with respect to the σ -field \mathcal{M}_t . The fact that $Y.(t, \omega)$ is a *regular conditional probability*, that is, for each $\omega \in \Omega$, $Y.(t, \omega)$ is a probability measure, follows from Proposition 4.4.

5.3. If A is a set of the form $\{X_{T+s} \in \Gamma\}$, $\Gamma \in \mathcal{E}$, $s \geq 0$, then Theorem 5.1 reduces to the usual statement of the strong Markov property. If $A \in \mathcal{M}_T^*$, then Theorem 5.1 reduces to a result of E. B. Dynkin [7, Theorem 3.11]. Hence Theorem 5.1 can be thought of as an extension of the usual strong Markov property to sets which need not belong to \mathcal{M}_T^* . Theorem 5.1 is also analogous to the Fubini theorem for product measures (applied to indicator functions of sets) and thus could also be interpreted as a Fubini-Markov theorem.

6. Nonequivalent pairs of Markov processes. Let

$$\pi_1 = (\Omega, \mathcal{F}_\infty, (P_x^1)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)}) \quad \text{and} \quad \pi_2 = (\Omega, \mathcal{F}_\infty, (P_x^2)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$$

be a pair of Hunt processes which are instantaneously equivalent. It is assumed that for some $x_0 \in E$ and $A_0 \in \mathcal{F}_\infty$, $P_{x_0}^1(A_0) = 0$, $P_{x_0}^2(A_0) = a_0 > 0$, that is, A_0 is a (π_1, π_2, X_0) singularity. $Y_A^1(\cdot, \cdot)$ and $Y_A^2(\cdot, \cdot)$ denote the martingales defined in §5 for π_1 and π_2 respectively. For any real number s , let $\mathcal{D}_s \equiv \{r - s : r \geq s, r \text{ rational}\}$.

PROPOSITION 6.1. For each $s \in [0, \infty)$ and $A \in \mathcal{F}_\infty$,

$$T_A^s(\omega) \equiv \inf \{t : t \in \mathcal{D}_s, Y_A^1(t, \omega) > 0\} \\ \equiv +\infty \quad \text{if } \{t : t \in \mathcal{D}_s, Y_A^1(t, \omega) > 0\} = \emptyset.$$

Then for each $s \in [0, \infty)$, $T_A^s(\cdot)$ is a stopping time.

Proof. $\{\omega : T_A^s(\omega) < t\} = \bigcup_{r \in [0, t] \cap \mathcal{D}_s} \{\omega : Y_A^1(r, \omega) > 0\}$. But Proposition 4.4 implies that $\{\omega : Y_A^1(r, \omega) > 0\} \in \mathcal{M}_t$ if $r < t$ and therefore T_A^s is a stopping time with respect to the σ -fields $(\mathcal{M}_t)_{t \in [0, \infty)}$.

PROPOSITION 6.2. For any $s \in [0, \infty)$ and $A \in \mathcal{F}_\infty$,

- (i) if $P_x^1(A) = 0$, then $P_x^1(T_A^s < \infty) = 0$, and
- (ii) if $P_x^1(A) > 0$, then $P_x^1(T_A^s = 0) = 1$.

Proof. (i) Since $\{T_A^s < \infty\} = \bigcup_{r \in \mathcal{D}_s} \{Y_A^1(r, \omega) > 0\}$, $P_x^1(T_A^s < \infty) > 0$ implies that for some $r \in \mathcal{D}_s$, $P_x(Y_A^1(r, \omega) > 0) > 0$. But then Theorem 5.1 implies that $P_x^1(A) > 0$.

(ii) If $P_x^1(T_A^s = 0) = 0$, then for a sequence $\{r_i\}$ of elements of \mathcal{D}_s such that $r_i \downarrow 0$ as $i \rightarrow \infty$, Theorem 5.1 implies that for each i

$$P_x^1(A) = E_x^1(Y_A^1(r_i, \cdot)) \leq P_x^1(T_A^s \leq r_i).$$

Since $P_x^1(T_A^s \leq r_i) \rightarrow 0$ as $i \rightarrow \infty$, therefore $P_x^1(A) = 0$. Since $\{T_A^s = 0\} \in \mathcal{M}_0$, the result then follows from the zero-one law (Proposition 2.7.1).

REMARKS. 6.1. For $s=0$ we let $T_A^0=T_A$. Since π_1 and π_2 are assumed to be instantaneously equivalent and $P_{x_0}^1(A_0)=0$, then Proposition 6.2 implies that $P_{x_0}^1(T_{A_0}=0)=P_{x_0}^2(T_{A_0}=0)=0$.

6.2. Let S be a stopping time and let $A \in \mathcal{M}_S$. It is easy to verify that for each $\omega \in \Omega$ either $T_A(\omega) \leq S(\omega)$ or $T_A(\omega) = \infty$. In fact if $\infty > t > S(\omega)$, then $A(t, \omega)$ is either Ω or \emptyset depending on whether $\omega \in A$ or not (cf. Proposition 4.1).

We next state and prove three technical results which are used in §8.

PROPOSITION 6.3. *Let S be an (\mathcal{M}_t) -stopping time such that $S \leq T_{A_0}$ and such that $P_{x_0}^2(A_0 \cap \{S < T_{A_0}\}) = P_{x_0}^2(A_0)$. Then $P_{x_0}^2(A_0 \cap \{Y_{A_0}^1(S(\omega), \omega) > 0\}) = 0$.*

Proof. If $Y_{A_0}^1(S(\omega), \omega) > 0$, then $P_{X(S(\omega))}^1(A_0(S(\omega), \omega)) > 0$ and Proposition 6.2 implies that $P_{X(S(\omega))}^1(T_{A_0(S(\omega), \omega)}^{S(\omega)} = 0) = 1$. Since π_1 and π_2 are instantaneously equivalent, this implies that $P_{X(S(\omega))}^2(T_{A_0(S(\omega), \omega)}^{S(\omega)} = 0) = 1$. If $Y_{A_0}^1(S(\omega), \omega) = 0$, then $P_{X(S(\omega))}^1(A_0(S(\omega), \omega)) = 0$ and Proposition 6.2 implies that

$$(6.1) \quad P_{X(S(\omega))}^1(T_{A_0(S(\omega), \omega)}^{S(\omega)} > 0) = P_{X(S(\omega))}^2(T_{A_0(S(\omega), \omega)}^{S(\omega)} > 0) = 1.$$

If $\omega' \sim \omega(R_{S(\omega)})$, then Proposition 4.2 implies that $S(\omega') \geq S(\omega)$. Therefore $\{T_{A_0} > S\}(S(\omega), \omega) \subset \{\omega' : T_{A_0(S(\omega), \omega)}^{S(\omega)}(\omega') > 0\}$. But Theorem 5.1 implies that

$$\begin{aligned} P_{x_0}^2(A_0) &= P_{x_0}^2(A_0 \cap \{T_{A_0} > S\}) \\ &= \int P_{X(S(\omega))}^2[\{T_{A_0} > S\}(S(\omega), \omega)] \cdot P_{X(S(\omega))}^2(A_0(S(\omega), \omega)) dP_{x_0}^2(\omega) \\ &\leq \int P_{X(S(\omega))}^2[T_{A_0(S(\omega), \omega)}^{S(\omega)}(\omega') > 0] \cdot P_{X(S(\omega))}^2(A_0(S(\omega), \omega)) dP_{x_0}^2(\omega) \\ &= P_{x_0}^2(A_0 \cap \{Y_{A_0}^1(S(\cdot), \cdot) = 0\}) \quad \text{by (6.1).} \end{aligned}$$

Hence $P_{x_0}^2(A_0 \cap \{Y_{A_0}^1(S(\cdot), \cdot) > 0\}) = 0$ and the proof is complete.

PROPOSITION 6.4. $P_{x_0}^2(A_0 \cap B_0) = P_{x_0}^2(A_0)$ where $B_0 \equiv \{Y_{A_0}^1(T_{A_0}(\cdot), \cdot) > 0\}$.

Proof. As in the proof of Theorem 5.1 let

$$H \equiv \{X_{T_{A_0}}^- = X_{T_{A_0}}\} \cap \{T_{A_0} < \infty\} \quad \text{and} \quad F = \{X_{T_{A_0}}^- \neq X_{T_{A_0}}\} \cap \{T_{A_0} < \infty\}.$$

It is easy to verify that $P_{x_0}^2(B_0 \cap A_0 \cap \{T_{A_0} = \infty\}) = P_{x_0}^2(A_0 \cap \{T_{A_0} = \infty\})$. To prove that $P_{x_0}^2(B_0 \cap A_0 \cap F) = P_{x_0}^2(A_0 \cap F)$ it suffices to show that

$$P_{x_0}^2(B_0 \cap A_0 \cap F \cap \{T_{A_0} = \zeta\}) = P_{x_0}^2(A_0 \cap F \cap \{T_{A_0} = \zeta\})$$

and that $P_{x_0}^2(B_0 \cap A_0 \cap \{T_{A_0} = J_n^k\}) = P_{x_0}^2(A_0 \cap \{T_{A_0} = J_n^k\})$ for $k, n = 1, 2, 3, \dots$, where the $J_n^k; k, n = 1, 2, 3, \dots$, are defined as in the proof of Theorem 5.1. Noting that $\{T_{A_0} > J_n^k\}(J_n^k(\omega), \omega) \in \mathcal{M}_0$, we can easily verify that Theorem 5.1 implies

$$\begin{aligned} P_{x_0}^2(A_0 \cap \{T_{A_0} = J_n^k\}) &= \int_{\{T_{A_0} = J_n^k\}} P_{X(J_n^k(\omega))}^2(A_0(J_n^k(\omega), \omega)) \\ &\quad \cdot P_{X(J_n^k(\omega))}^2(\{T_{A_0} = J_n^k\}(J_n^k(\omega), \omega)) \cdot dP_{x_0}^2(\omega). \end{aligned}$$

But Proposition 6.2 and the instantaneous equivalence of π_1 and π_2 imply that for $\omega \in \{T_{A_0} = J_n^k\}$,

$$P_{X(J_n^k(\omega))}^2(\{T_{A_0} = J_n^k\}(J_n^k(\omega), \omega)) = P_{X(J_n^k(\omega))}^1(\{T_{A_0} = J_n^k\}(J_n^k(\omega), \omega)) = \chi_{B_0}.$$

Hence $P_{x_0}^2(A_0 \cap \{T_{A_0} = J_n^k\}) = P_{x_0}^2(A_0 \cap B_0 \cap \{T_{A_0} = J_n^k\})$. A similar result may be obtained for $(T = \zeta) \cap F$.

To prove that $P_{x_0}^2(A_0 \cap B_0 \cap H) = P_{x_0}^2(A \cap H)$, note that the proof of Theorem 5.1 (in particular (5.11)) implies that there is a set $Q \in \mathcal{F}_\infty$ such that

$$P_{x_0}^2(H \cap Q) = P_{x_0}^2(H)$$

and such that $P_{X(T_{A_0}(\omega))}^2(T_{A_0(T_{A_0}(\omega), \omega)}^{T_{A_0}(\omega)} = 0) = 1$ for each $\omega \in Q$. But Proposition 6.2 and the instantaneous equivalence of π_1 and π_2 imply that $P_{X(T_{A_0}(\omega))}^2(T_{A_0(T_{A_0}(\omega), \omega)}^{T_{A_0}(\omega)} = 0) = 1$ if and only if $Y_{A_0}^1(T_{A_0}(\omega), \omega) > 0$. Hence it follows that $P_{x_0}^2(A_0 \cap B_0 \cap H) = P_{x_0}^2(A_0 \cap H)$.

REMARK 6.3. It is easy to verify that $P_{x_0}^1(B_0) = 0$, $P_{x_0}^2(B_0) \geq a_0$ and that $B_0 \in \mathcal{M}_{T_{A_0}}$. Furthermore, $P_{x_0}^2(A_0 - B_0) = P_{x_0}^1(A_0 - B_0) = 0$. Finally, Theorem 5.1, Corollary 2 implies that $T_{B_0} \geq T_{A_0}$. Remark 6.2 implies that either $T_{B_0} \leq T_{A_0}$ or $T_{B_0} = \infty$, and therefore

$$\begin{aligned} T_{B_0}(\omega) &= T_{A_0}(\omega) && \text{if } Y_{A_0}^1(T_{A_0}(\omega), \omega) > 0 \\ &= \infty && \text{if } Y_{A_0}^1(T_{A_0}(\omega), \omega) = 0. \end{aligned}$$

PROPOSITION 6.5. *There is a set $B'_0 \supset A_0$, $B'_0 \in \mathcal{M}_{T_{A_0}}$, such that $T_{A_0} = T_{B'_0}$.*

Proof. For any rational $r \in [0, 1/n]$, let T_r^n , $n = 1, 2, 3, \dots$, be the stopping time defined by

$$T_r^n = \sum_{i=-1}^{\infty} \chi_{(r+i/n \leq T_{A_0} < r+(i+1)/n)} \cdot (r+(i+1)/n) + \chi_{(T_{A_0} = \infty)} \cdot (\infty).$$

Note that $T_{A_0} + 1/n \geq T_r^n \geq T_{A_0}$ and that the definition of T_{A_0} implies that

$$A_0 \subset \bigcup_{r \in [0, 1/n] \cap \mathcal{R}_0} \{Y_{A_0}^1(T_r^n(\omega), \omega) > 0\} \equiv B_n.$$

Hence $A_0 \subset B'_0 \equiv \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n$. But since $B_n \in \mathcal{M}_{T_{A_0} + 1/n}$, $B'_0 \in \mathcal{M}_{T_{A_0}}$. Finally, Remark 6.2 implies that $T_{B'_0} \leq T_{A_0}$ or $= +\infty$ and Theorem 5.1, Corollary 2 implies that $T_{B'_0} \geq T_{A_0}$; in fact for $t < T_{A_0}(\omega)$, $Y_{B'_0}^1(t, \omega) > 0$ if and only if $Y_{A_0}^1(t, \omega) > 0$. But the definition of B'_0 implies that if $T_{A_0}(\omega) < \infty$ then $T_{B'_0(\omega)} < \infty$ so that $T_{A_0} = T_{B'_0}$. Note that this implies that $P_{x_0}^1(B'_0) = 0$.

7. The canonical singularities. Let π_1 and π_2 be a pair of Hunt processes. For $t > 0$ and $0 < \alpha \leq 1$ let

$$K_t^\alpha \equiv \{x \in E: \text{for some } C \in \mathcal{F}_t, P_x^1(C) = 0, P_x^2(C) \geq \alpha\}.$$

For a fixed value of α , K_t^α decreases as t decreases.

PROPOSITION 7.1. *For each $0 < \alpha \leq 1$ and $t > 0$, K_t^α is an \mathcal{E} -measurable set.*

Proof. Proposition 2.2.1 implies that there exists a countable algebra \mathcal{C}_t of sets which generates \mathcal{F}_t , say $\mathcal{C}_t = \{B_m : m = 1, 2, 3, \dots\}$. If

$$K_t^{\alpha, n} \equiv \bigcup_{m=1}^{\infty} \{x : P_x^1(B_m) < 1/2^n, P_x^2(B_m) \geq \alpha - 1/2^n\}, \text{ then } K_t^{\alpha, n} \in \mathcal{C}.$$

It is easy to verify that $K_t^\alpha \subset \bigcap_{n=1}^{\infty} K_t^{\alpha, n}$ by the approximation theorem for measures. On the other hand if $x \in \bigcap_{n=1}^{\infty} K_t^{\alpha, n}$, then for each n there is a set $A_n \in \mathcal{F}_t$ such that $P_x^1(A_n) < 1/2^n$ and $P_x^2(A_n) \geq \alpha - 1/2^n$. But then $P_x^1(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$, and

$$P_x^2\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) \geq \alpha$$

so that $x \in K_t^\alpha$. Hence $K_t^\alpha = \bigcap_{n=1}^{\infty} K_t^{\alpha, n}$ and therefore $K_t^\alpha \in \mathcal{C}$.

PROPOSITION 7.2. π_1 and π_2 are instantaneously equivalent if and only if $\bigcap_{t>0} K_t^\alpha = \emptyset$ for each $\alpha > 0$.

Proof. If $x \in \bigcap_{t>0} K_t^\alpha$, then for any positive integer n there is a set $A_n \in \mathcal{F}_{1/n}$ such that $P_x^1(A_n) = 0$ and $P_x^2(A_n) \geq \alpha$. But then $A \equiv \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \in \mathcal{M}_0$ and $P_x^1(A) = 0$ and $P_x^2(A) \geq \alpha$. Hence π_1 and π_2 are not instantaneously equivalent. On the other hand if π_1 and π_2 are not instantaneously equivalent then there is a set $A \in \mathcal{M}_0$ and $x \in E$ such that $P_x^1(A) = 0$, $P_x^2(A) = 1$ and therefore $x \in K_t^\alpha$ for all $t > 0$ and $\alpha \leq 1$.

To each set K_t^α , $t > 0$, $0 < \alpha \leq 1$, we now construct a canonical singular event \hat{K}_t^α . For $B_m \in \mathcal{C}_t$, $m = 1, 2, 3, \dots$, and each positive integer n , let

$$K_m^n \equiv \{x : P_x^1(B_m) < 1/2^n, P_x^2(B_m) \geq \alpha - 1/2^n\} \cap K_t^\alpha,$$

and

$$\begin{aligned} \hat{K}^n \equiv & (\{X(0) \in K_1^n\} \cap B_1) \cup (\{X(0) \in K_2^n - K_1^n\} \cap B_2) \cup \dots \\ & \cup \left(\left\{ X(0) \in \left(K_r^n - \bigcup_{s=1}^{r-1} K_s^n \right) \right\} \cap B_r \right) \cup \dots, \end{aligned}$$

and let $\hat{K}_t^\alpha \equiv \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \hat{K}^n$. Since for each m , $P_x(B_m)$ is measurable (cf. 2.3.2), it is easy to verify that $\hat{K}_t^\alpha \in \mathcal{F}_t$. Moreover $P_x^1(\hat{K}^n) \leq 1/2^n$ for each $x \in E$ and $P_x^2(\hat{K}^n) \geq \alpha - 1/2^n$ for each $x \in K_t^\alpha$. Hence $P_x^1(\hat{K}_t^\alpha) = 0$ for each $x \in E$ and $P_x^2(\hat{K}_t^\alpha) \geq \alpha$ for each $x \in K_t^\alpha$.

8. Classification of singularities. In this section we classify the types of singularities which can occur for a pair (π_1, π_2) of Hunt processes.

8.1. Types of singularities. An event $A \in \mathcal{F}_\infty$ is a (π_1, π_2, X_0) germ field singularity if $A \in \mathcal{M}_0$, $P_{x_0}^1(A) = 0$ and $P_{x_0}^2(A) > 0$.

$A \in \mathcal{F}_\infty$ is a (π_1, π_2, x_0) singularity of tail type if $P_{x_0}^1(A) = 0$, $P_{x_0}^2(A) > 0$ and $Y_A^1(t) = 0$, $P_{x_0}^2$ -almost surely for all $t \in [0, \infty)$.

$A \in \mathcal{F}_\infty$ is a (π_1, π_2, x_0) tail field singularity if A belongs to the tail field $\mathcal{F}_\infty^* \equiv \bigcap_{t>0} \mathcal{F}_t^*$, and A is a (π_1, π_2, x_0) tail type singularity.

$A \in \mathcal{F}_\infty$ is a (π_1, π_2, x_0) jump singularity at T if:

(i) T is a stopping time which is totally inaccessible in the weak sense on A with respect to $P_{x_0}^2$ and $A \in \mathcal{M}_T$,

(ii) for any stopping time S such that $P_{x_0}^2(A \cap \{S < T\}) = P_{x_0}^2(A)$, $Y_A^1(S) = 0$, $P_{x_0}^2$ -almost surely on A , and

(iii) $P_{x_0}^1(A) = 0$, $P_{x_0}^2(A) > 0$.

$A \in \mathcal{F}_\infty$ is a (π_1, π_2, x_0) pseudo local singularity at T if T is a stopping time and there is an increasing sequence of stopping times $\{T_n\}$ such that:

(i) $T_n < T < \infty$ for every n , $P_{x_0}^2$ -almost surely on A ,

(ii) $\lim_{n \rightarrow \infty} T_n = T$, $P_{x_0}^2$ -almost surely on A ,

(iii) $A \in \mathcal{M}_T$, $P_{x_0}^1(A) = 0$, $P_{x_0}^2(A) > 0$ and

(iv) for every n , $Y_A^1(T_n) = 0$, $P_{x_0}^2$ -almost surely on A . In such case it is easy to verify that $T_A(\omega)$ is either $T(\omega)$ or ∞ , $P_{x_0}^2$ -almost surely on A .

Let $\{T_n\}$ be an increasing sequence of stopping times, such that $T_n \leq T$ for each n . The σ -field $\mathcal{M}_{(T_n), T}$ is defined to be $\bigcap_{n=1}^\infty \mathcal{M}_{T_n}^* \cap \mathcal{M}_T$.

$A \in \mathcal{F}_\infty$ is said to be local at a stopping time T with respect to $P_{x_0}^2$ if there is an increasing sequence of stopping times $\{T_n\}$ such that $T_n \leq T$ for each n and such that:

(i) $T_n < T$ for each n and $\lim_{n \rightarrow \infty} T_n = T < \infty$, both $P_{x_0}^2$ -almost surely on a set B which contains A ,

(ii) T_{n+1} is measurable with respect to $\mathcal{M}_{T_n}^*$, and

(iii) A belongs to the local σ -field $\mathcal{M}_{(T_n), T}$.

REMARK 8.1.1. It can be shown that the $P_{x_0}^2$ -completion of the σ -field $\mathcal{M}_{(T_n), T}$ restricted to the set B is independent of the choice of the sequence $\{T_n\}$ as long as $\{T_n\}$ satisfies (i) and (ii). $\mathcal{M}_{(T_n), T}$ is known as the local σ -field at T with respect to $P_{x_0}^2$. Loosely speaking, events in $\mathcal{M}_{(T_n), T}$ depend only on the behavior of the paths near T .

$A \in \mathcal{F}_\infty$ is a (π_1, π_2, x_0) local singularity at T if A is a local event at the stopping time T with respect to $P_{x_0}^2$ and A is a (π_1, π_2, x_0) pseudo local singularity at T .

8.2. *A property of pseudo local singularities.* Let G be a (π_1, π_2, x_0) pseudo local singularity such that $P_{x_0}^2(G) = a > 0$. Then there is an increasing sequence of stopping times $\{T_n\}$ such that $T_n < T_G < \infty$ for all n and $\lim_{n \rightarrow \infty} T_n = T_G$, both $P_{x_0}^2$ -almost surely on G . Without loss of generality the sequence of stopping times $\{T_n\}$ can be chosen so that for a given ϵ , $0 < \epsilon < a/2$, $P_{x_0}^2(G \cap \{T_G \geq T_n + 1/n\}) < \epsilon/2^n$. Let $G_n \equiv \{T_G < T_n + 1/n\} \cap G$. Then $G_n \in \mathcal{M}_{T_n + 1/n}$ (refer to P. A. Meyer [13, Chapter 4, T 39]). Corollary 1 of Theorem 5.1 implies that there is a set $Q \in \mathcal{F}_\infty$ such that $P_{x_0}^2(G \cap Q) = P_{x_0}^2(G)$ and such that for every n and $\omega \in Q$,

$$(8.2.1) \quad G_n(T_n(\omega), \omega) \supset G_n^+(\omega) \quad \text{with} \quad G_n^+(\omega) \in \mathcal{M}_{1/n},$$

and such that $P_{X(T_n(\omega))}^2(G_n(T_n(\omega), \omega) - G_n^+(\omega)) = 0$. Then we have the following result.

PROPOSITION 8.2.1. $P_{x_0}^2(G \cap (\bigcap_{n=1}^\infty \{X(T_n) \in K_{2/n}^b\})) \geq b$ where $b \equiv (a - \epsilon)/2$.

Proof. If $G_\infty \equiv \bigcap_{n=1}^\infty G_n$, then $P_{x_0}^2(G_\infty) \geq a - \varepsilon$. Consider

$$\begin{aligned} N(\omega) &\equiv \min \{n : Y_{G_\infty}^2(T_n(\omega), \omega) \leq b\} \\ &\equiv +\infty \quad \text{if } \{n : Y_{G_\infty}^2(T_n(\omega), \omega) \leq b\} = \emptyset. \end{aligned}$$

Let $\tilde{T}(\omega) \equiv T_{N(\omega)}$, where $T_\infty(\omega) \equiv T_G(\omega)$. Since

$$\{\tilde{T} < t\} = \bigcup_{n=1}^\infty \{T_n < t\} \cap \{Y_{G_\infty}^2(T_n) \leq b, Y_{G_\infty}^2(T_{n-1}) > b, \dots, Y_{G_\infty}^2(T_1) > b\},$$

it follows that $\{\tilde{T} < t\} \in \mathcal{M}_t$ and therefore \tilde{T} is a stopping time. But Theorem 5.1 implies that

$$\begin{aligned} 2b &\leq P_{x_0}^2(G_\infty) = \int P_{X(\tilde{T}(\omega))}^2(G_\infty(\tilde{T}(\omega), \omega)) dP_{x_0}^2(\omega) \\ &= \int_{\{\tilde{T}=T_G\}} P_{X(\tilde{T}(\omega))}^2(G_\infty(\tilde{T}(\omega), \omega)) dP_{x_0}^2(\omega) \\ &\quad + \int_{\{\tilde{T}<T_G\}} P_{X(\tilde{T}(\omega))}^2(G_\infty(\tilde{T}(\omega), \omega)) dP_{x_0}^2(\omega) \\ &= \int_{\{Y_{G_\infty}^2(\tilde{T}(\omega), \omega) < b\}} Y_{G_\infty}^2(\tilde{T}(\omega), \omega) dP_{x_0}^2(\omega) \\ &\quad + \int_{\{Y_{G_\infty}^2(\tilde{T}(\omega), \omega) \geq b\}} Y_{G_\infty}^2(\tilde{T}(\omega), \omega) dP_{x_0}^2(\omega). \end{aligned}$$

But then $\int_{\{Y_{G_\infty}^2(\tilde{T}(\omega), \omega) \geq b\}} Y_{G_\infty}^2(\tilde{T}(\omega), \omega) dP_{x_0}^2(\omega) \geq b$ and therefore

$$P_{x_0}^2\left(G_\infty \cap \left\{\bigcap_{n=1}^\infty Y_{G_\infty}^2(T_n(\omega), \omega) \geq b\right\}\right) \geq b.$$

Since for each n

$$P_{x_0}^2(G_\infty \cap \{Y_{G_\infty}^1(T_n(\omega), \omega) \geq b\}) = 0,$$

then in view of (8.2.1)

$$P_{x_0}^2\left(G_\infty \cap \left(\bigcap_{n=1}^\infty \{X(T_n) \in K_{2/n}^b\}\right)\right) \geq b$$

and the proof is complete.

COROLLARY. *There exist compact sets $\tilde{K}_{2/n}^b \subset K_{2/n}^b$ such that*

$$P_{x_0}^2\left(G_\infty \cap \left(\bigcap_{n=1}^\infty \{X(T_n) \in \tilde{K}_{2/n}^b\}\right)\right) \geq b/2.$$

Proof. This follows immediately since the measures $P_{x_0}^2(X(T_n) \in \cdot)$ are inner regular on \mathcal{E}_Δ .

In the case in which C is a (π_1, π_2, x_0) tail type singularity, a similar type of argument yields the following result.

PROPOSITION 8.2.2. *If C is a (π_1, π_2, x_0) tail type singularity such that $P_{x_0}^2(C) = a$ and $s \in [0, \infty)$, then*

$$P_{x_0}^2(C \cap \{X(s) \in K_{\infty}^{a/2}\}) \geq a/2.$$

In the case in which D is a (π_1, π_2, x_0) jump singularity, a similar type of argument yields the following result.

PROPOSITION 8.2.3. *If D is a (π_1, π_2, x_0) jump singularity at T_D with*

$$P_{x_0}^2(D \cap \{S < T_D < S + t\}) = a' > 0$$

for some $t \in [0, \infty)$, then

$$P_{x_0}^2(X(S) \in K_{2t}^{a'/2}) \geq a'/2.$$

8.3. A result on tail type singularities.

THEOREM 8.3.1. *If (π_1, π_2) have a tail type singularity then they have a tail field singularity.*

Proof. Let C be a (π_1, π_2, x_0) tail type singularity with $P_{x_0}^2(C) = a$. Then Proposition 8.2.2 implies that $P_{x_0}^2(C \cap \{X(s) \in K_{\infty}^{a/2}\}) \geq a/2$ for every $s \in [0, \infty)$. But then $P_{x_0}^2(\theta_s^{-1} \hat{K}_{\infty}^{a/2}) \geq a^2/4$ and $P_{x_0}^1(\theta_s^{-1} \hat{K}_{\infty}^{a/2}) = 0$. Moreover $\theta_s^{-1} \hat{K}_{\infty}^{a/2} \in \mathcal{M}_s^*$. Hence if $C_0 \equiv \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \theta_k^{-1} \hat{K}_{\infty}^{a/2}$, then $C_0 \in \mathcal{F}_{\infty}^*$, $P_{x_0}^2(C_0) > 0$ and $P_{x_0}^1(C_0) = 0$. Hence C_0 is a tail field singularity.

8.4. A result on local singularities.

THEOREM 8.4.1. *If (π_1, π_2) have a pseudo local singularity, then (π_1, π_2) have a local singularity. Moreover for some $\alpha > 0$, $K_t^{\alpha} \neq \emptyset$ for all $t > 0$.*

Proof. Let G be a (π_1, π_2, x_0) pseudo local singularity at a stopping time T with $P_{x_0}^2(G) = a$. But then Proposition 8.2.1 implies that for $0 < \epsilon < a/2$,

$$(8.4.1) \quad P_{x_0}^2\left(\bigcap_{n=1}^{\infty} (X(T_n) \in \tilde{K}_{2/n}^b) \cap G\right) \geq b/2$$

where $b = (a - \epsilon)/2$, $\tilde{K}_{2/n}^b$ is a compact subset of $K_{2/n}^b$ and $\{T_n\}$ is a sequence of stopping times such that $\lim_{n \rightarrow \infty} T_n = T$, $P_{x_0}^2$ -almost surely on G . This implies that $K_{2/n}^b \neq \emptyset$ for every positive integer n .

Consider the sequence of stopping times defined by:

$$\begin{aligned} S_1^b &\equiv 0, \\ S_n^b &\equiv S_{n-1}^b + S_n \circ \theta_{S_{n-1}^b} \quad \text{for } n = 2, 3, \dots, \end{aligned}$$

where the S_n , $n = 2, 3, \dots$, are defined as follows. If

$$\begin{aligned} \tilde{S}_n &\equiv \inf \{s : X_s(\omega) \in \tilde{K}_{2/n}^b\}, \quad n = 2, 3, \dots \\ &\equiv +\infty \quad \text{if the above set is empty,} \end{aligned}$$

then \tilde{S}_n is an $(\mathcal{M}_{\tilde{t}^{\pi_2}})$ -stopping time. We then let S_n be an (\mathcal{M}_t) -stopping time chosen

so that $P_\mu^2(\tilde{S}_n \neq S_n) = 0$ where $\mu(\cdot) \equiv P_{x_0}^2(X(S_{n-1}^b) \in \cdot)$. Let $S_0^b \equiv \lim_{n \rightarrow \infty} S_n^b$, $\tilde{S}_0^b \equiv \theta_{S_0^b}^{-1} \tilde{K}_{2/n}^b$ and $\hat{S}_0^b \equiv \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty \tilde{S}_n^b$. The event $\hat{S}_0^b \cap (S_0^b < \infty)$ is a local event and $P_{x_0}^1(\hat{S}_0^b) = 0$. Moreover (8.4.1) implies that $P_{x_0}^2(S_0^b < \infty) > 0$ and therefore $P_{x_0}^2(\hat{S}_0^b \cap \{S_0^b < \infty\}) > 0$. It is then easy to verify that $\hat{S}_0^b \cap \{S_0^b < \infty\}$ is a (π_1, π_2, x_0) local singularity at S_0^b .

8.5. *A remark on jump singularities.* Let J_n^k , $n, k = 1, 2, 3, \dots$, be the stopping times defined in the proof of Theorem 5.1. Let $\mathcal{M}_{J_n^k}^j$ be the σ -field generated by the triple of random variables $(J_n^k, X_{J_n^k}, X_{J_n^k}^-)$ and \mathcal{M}_ζ^j be the σ -field generated by the random variables ζ and $\chi_{\{\zeta < \infty\}} \cdot X_\zeta^-$. Then $\mathcal{M}_{J_n^k}^j \subset \mathcal{M}_{J_n^k}$ and $\mathcal{M}_\zeta^j \subset \mathcal{M}_\zeta$. We call the σ -fields $\mathcal{M}_{J_n^k}^j$, $k, n = 1, 2, 3, \dots$, and \mathcal{M}_ζ^j the *jump σ -fields*. An event A is a (π_1, π_2, x_0) *jump field singularity* if A belongs to one of the jump σ -fields and is a (π_1, π_2, x_0) jump singularity at the corresponding jump time. If the pair (π_1, π_2) is either free of jump singularities or has jump field singularities then it is said to have the *J-property*. The pair (π_1, π_2) of Example 3 of §3 has the *J-property*. However, the question of the characterization of pairs of processes which have the *J-property* is not considered in this paper.

A. V. Skorokhod [17] has considered the question of jump singularities for Markov processes which are defined by stochastic integral equations.

8.6. *The decomposition theorem.* Let π_1 and π_2 be a pair of Hunt processes which are instantaneously equivalent.

THEOREM 8.6.1. *If A_0 is a (π_1, π_2, x_0) singularity, then $A_0 \subset C \cup D \cup G$ where C is either empty or a tail type singularity, D is either empty or a jump singularity and G is either empty or a pseudo local singularity.*

Proof. Given A_0 let B'_0 be defined as in Proposition 6.5. Let $C \equiv B'_0 \cap \{T_{A_0} = \infty\}$. If $P_{x_0}^2(C) > 0$, then for each $t \in [0, \infty)$, $Y_C^1(t) = 0$, $P_{x_0}^2$ -almost surely (Proposition 6.3). Hence C is a tail type singularity. Let $D \equiv B'_0 \cap \{T_{A_0} < \infty\} \cap \{X_{T_{A_0}} \neq X_{T_{A_0}}^-\}$ and let S be a stopping time such that $P_{x_0}^2(D \cap \{S < T_{A_0}\}) = P_{x_0}^2(D)$. Then Propositions 6.3 and 6.5 imply that $Y_D^1(S) = 0$, $P_{x_0}^2$ -almost surely on D . In addition, T_{A_0} is π_2 -totally inaccessible in the weak sense at x_0 on D (Proposition 2.9.1). Hence D is a (π_1, π_2, x_0) jump singularity. Let $G \equiv B'_0 \cap \{T_{A_0} < \infty\} \cap \{X_{T_{A_0}} = X_{T_{A_0}}^-\}$. Then Proposition 2.9.1 implies the existence of an increasing sequence of stopping times $\{T_n\}$ such that $\lim_{n \rightarrow \infty} T_n = T_{A_0}$ and $T_n < T_{A_0} < \infty$ for every n , both $P_{x_0}^2$ -almost surely on G . Propositions 6.3 and 6.5 imply that for every n , $Y_G^1(T_n) = 0$, $P_{x_0}^2$ -almost surely on G . Hence G is a pseudo local singularity and the proof is complete.

8.6. *Continuous Hunt processes.* Let π_1 and π_2 be a pair of continuous Hunt processes which are instantaneously equivalent. Then we have the following result.

THEOREM 8.7.1. (i) *If π_1 and π_2 are free of local singularities then π_1 and π_2 are equivalent in finite time.*

(ii) *If in addition π_1 and π_2 are not strongly equivalent in finite time, then π_1 and π_2 must have a jump singularity at ζ .*

(iii) If π_1 and π_2 are uninterrupted, then π_1 and π_2 are strongly equivalent in finite time.

Proof. (i) If A_0 is a (π_1, π_2, x_0) or a (π_2, π_1, x_0) singularity and $A_0 \in \mathcal{F}_t$ for some $t \in [0, \infty)$ and $A_0 \subset (\zeta > t)$, then $T_{A_0} < \zeta$ on $\{T_{A_0} < \infty\}$. But then T_{A_0} is accessible on A_0 which in view of Theorem 8.4.1 implies the existence of a local singularity thus yielding a contradiction.

(ii) If A_0 is a (π_1, π_2, x_0) or a (π_2, π_1, x_0) singularity which belongs to \mathcal{F}_t for some $t \in [0, \infty)$, then since π_1 and π_2 are free of local singularities A must be a jump singularity. But if π_1 and π_2 are continuous, $J_n^k = +\infty$, P_x^k -almost surely, $i = 1, 2$, for each $x \in E$, every positive integer k and every $n = 1, 2, 3, \dots$. Hence A must be a jump singularity at ζ .

(iii) If π_1 and π_2 are uninterrupted and continuous, then there can be no jump singularities. Hence if π_1 and π_2 are free of local singularities, then π_1 and π_2 are strongly equivalent in finite time.

8.8. *Remarks on the examples of §3.* Example 1 of §3 is an example of a pair of Markov processes which can have only tail field singularities. In Example 2 the pair (π_1, π_2) have local singularities at the stopping times $T_n = n$, where n is a positive integer. Note that in this case π_2 is not quasi left continuous and that the jumps occur at accessible times. Example 3 illustrates the phenomena of jump singularities. Although the processes in this example are not in the canonical form we have considered, the results that we have obtained are nevertheless applicable. As noted above the pair of processes in Example 3 has the J -property.

9. **Locally smooth Markov processes.** In this section we show that for a large class of Hunt processes the existence of local singularities implies the existence of germ field singularities.

Let $\pi = (\Omega, \mathcal{F}_\infty, (P_x)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ be a strong Markov process with respect to the σ -fields $(\mathcal{M}_t^i)_{t \in [0, \infty)}$. The Markov process π is *locally smooth* if the following is satisfied. Let x be any point in E ; let $\{T_n\}$ be an increasing sequence of stopping times and T be a stopping time such that $T_n < T$ for all n and $\lim_{n \rightarrow \infty} T_n = T$, both P_x -almost surely on $\{T < \zeta\}$, and let $\{\Gamma_n\}$ be a sequence of \mathcal{E}_Δ -measurable sets. Then for any positive integer N ,

$$(9.1) \quad P_x \left(X(T) \in C_N, \bigcap_{n=1}^{\infty} \{X(T_n) \in \Gamma_n\} \right) = 0,$$

where $C_N \equiv \{y : y \in E, P_y(T \cup_{n=N}^\infty \Gamma_n = 0) = 0\}$, and for $\Gamma \in \mathcal{E}_\Delta$,

$$\begin{aligned} T_\Gamma &\equiv \inf \{t : X_t \in \Gamma\} \\ &\equiv +\infty \quad \text{if the above set is empty.} \end{aligned}$$

(Note that it can be shown that C_N is measurable with respect to the σ -field of universally measurable subsets of E_Δ (R. K. Gettoor [8]).)

In §10 we introduce the fine topology connected with the Markov process. The

property of local smoothness can be thought of as quasi left continuity in the fine topology. We begin by obtaining some preliminary results.

PROPOSITION 9.1. *Let $A_n \in \mathcal{M}_{T_n}$, $n=1, 2, 3, \dots$, where $\{T_n\}$ is a decreasing sequence of stopping times such that $\lim_{n \rightarrow \infty} T_n = 0$, P_x -almost surely for some $x \in E_\Delta$. If $A \equiv \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n$, then $A \in \mathcal{M}_0^x$, the completion of \mathcal{M}_0 with respect to the measure P_x .*

Proof. It suffices to show that $A \in \mathcal{M}_t$ for each $t > 0$. Since $A \in \mathcal{M}_{T_n}$ for each n , it follows that $A \cap \{T_n \leq t\} \in \mathcal{M}_t$ for each $t > 0$. Since

$$A = (A \cap \{T_n \leq t\}) \cup (A \cap \{T_n > t\}),$$

it suffices to show that $A \cap \lim_{n \rightarrow \infty} \{T_n > t\} \in \mathcal{M}_0^x$. But

$$A \cap \lim_{n \rightarrow \infty} \{T_n > t\} \subset \{\lim_{n \rightarrow \infty} T_n > 0\}.$$

Since $\{\lim_{n \rightarrow \infty} T_n > 0\} \in \mathcal{M}_0$ and $P_x(\{\lim_{n \rightarrow \infty} T_n > 0\}) = 0$, therefore

$$A \cap \left(\lim_{n \rightarrow \infty} \{T_n > t\}\right) \in \mathcal{M}_0^x.$$

PROPOSITION 9.2. *Let $\{\Gamma_n\}$ be a sequence of compact subsets of E such that for every positive integer N and some $y \in E - \bigcup_{n=1}^\infty \Gamma_n$, $P_y(T_{\bigcup_{n=N}^\infty \Gamma_n} = 0) = 1$. Then for any $\varepsilon > 0$ there exists a decreasing sequence of stopping times $\{T_n^*\}$ and an integral valued function $f(n)$, $n \leq f(n) < \infty$, such that*

- (i) $T_n^* > 0$ for each n and $\lim_{n \rightarrow \infty} T_n^* = 0$, P_y -almost surely, and
- (ii) $P_y(\bigcap_{n=1}^\infty \{X(T_n^*) \in \bigcup_{m=n}^{f(n)} \Gamma_m\}) \geq 1 - \varepsilon$.

Proof. For any integral valued function $f(n)$, $n \leq f(n) < \infty$, let

$$\begin{aligned} T_0^* &\equiv \infty \\ T_n^* &\equiv T_{\bigcup_{m=n}^{f(n)} \Gamma_m} \wedge 1/n \wedge T_{n-1}^*, \quad n = 1, 2, 3, 4, \dots \end{aligned}$$

Since $T_{\bigcup_{m=n}^{f(n)} \Gamma_m} \downarrow 0$, P_y -almost surely as $r \rightarrow \infty$, then $P_y(X(T_n^*) \in \bigcup_{m=n}^{f(n)} \Gamma_m) \uparrow 1$ as $f(n) \rightarrow \infty$. Hence we can choose $f(n)$ so that $P_y(X(T_n^*) \in \bigcup_{m=n}^{f(n)} \Gamma_m) > 1 - \varepsilon/2^n$. But then $P_y(\bigcap_{n=1}^\infty \{X(T_n^*) \in \bigcup_{m=n}^{f(n)} \Gamma_m\}) \geq 1 - \varepsilon$. The fact that $T_n^* > 0$ for each n , P_y -almost surely follows from the right continuity of the paths.

COROLLARY. *There is a sequence of (\mathcal{M}_t) -stopping times $\{S_n\}$ satisfying (i) and (ii).*

Proof. This follows immediately from Propositions 9.2 and 2.8.1.

Let π_1 and π_2 be a pair of instantaneously equivalent locally smooth Hunt processes. Then we have the following result.

THEOREM 9.1. *The pair (π_1, π_2) is free of local singularities except possibly at the stopping time ζ .*

Proof. Let us assume the contrary; in particular let A_0 be a (π_1, π_2, x_0) local singularity at a stopping time T with $T < \zeta$, $P_{x_0}^2$ -almost surely on A_0 . Then by the

corollary to Proposition 8.2.1, $P_{x_0}^2(\{X(\lim_{n \rightarrow \infty} T_n) \in E\} \cap (\bigcap_{n=1}^{\infty} \{X(T_n) \in \tilde{K}_{2/n}^b\})) > 0$ for some $b > 0$ and some increasing sequence of stopping times $\{T_n\}$, where the $\tilde{K}_{2/n}^b, n = 1, 2, 3, \dots$, are compact subsets of the sets $K_{2/n}^b$. Since π_2 is locally smooth, there is a point $y \in E$ such that for each $N = 1, 2, 3, \dots$,

$$P_y^2(T_{\bigcup_{m=N}^{\infty} \tilde{K}_{2/m}^b} = 0) = 1.$$

Note that y is a fine boundary point of each of the sets $K_{2/m}^b$. We proceed to show that $y \in \bigcap_{t>0} K_t^b$ which contradicts the assumption that the processes are instantaneously equivalent (Proposition 7.2).

Since $\bigcap_{m=1}^{\infty} K_{2/m}^b = \emptyset$ we can assume without loss of generality that $y \notin \bigcup_{m=1}^{\infty} \tilde{K}_{2/m}^b$. Let $\{S_n\}$ be a sequence of (\mathcal{M}_t) -stopping times constructed as in the corollary to Proposition 9.2 such that for some integral-valued function $f(n)$,

$$P_y^2\left(\bigcap_{n=1}^{\infty} \{X(S_n) \in \bigcup_{m=n}^{f(n)} \tilde{K}_{2/m}^b\}\right) \geq 1 - \varepsilon$$

for some $\varepsilon < b$. Then

$$\theta_{S_n}^{-1}\left[\bigcup_{m=n}^{f(n)} \tilde{K}_{2/m}^b\right] \in \mathcal{M}_{S_n + 2/n} \quad (\text{Proposition 2.6.1}),$$

$$P_y^1\left[\theta_{S_n}^{-1}\left(\bigcup_{m=n}^{f(n)} \tilde{K}_{2/m}^b\right)\right] = 0,$$

and

$$P_y^2\left(\theta_{S_n}^{-1}\left(\bigcup_{m=n}^{f(n)} \tilde{K}_{2/m}^b\right)\right) \geq b - \varepsilon.$$

If $B \equiv \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \theta_{S_n}^{-1}(\bigcup_{m=n}^{f(n)} \tilde{K}_{2/m}^b)$ then $P_y^1(B) = 0$ and $P_y^2(B) \geq b - \varepsilon$. Furthermore, Proposition 9.1 implies that $B \in \mathcal{M}_0^{P_y^2} = \mathcal{M}_0^{P_y^1}$. Hence $y \in \bigcap_{t>0} K_t^b$, yielding a contradiction and the proof is complete.

COROLLARY. *If (π_1, π_2) are a pair of locally smooth and instantaneously equivalent continuous Hunt processes, then π_1 and π_2 are equivalent in finite time.*

Proof. This follows immediately from Theorem 8.7.1(i) and Theorem 9.1.

REMARK 9.1. In view of Proposition 9.3 the equivalence class (for equivalence in finite time) of a locally smooth continuous Hunt process is described by the mapping

$$(9.1) \quad \Phi: E \rightarrow \Gamma_0 \text{ where } \Gamma_0 \text{ is the Stone space of } \mathcal{M}_0, \text{ and } \Phi(x) \equiv \{A : A \in \mathcal{M}_0, P_x(A) = 0\}.$$

It would be interesting if one could formulate some reasonable set of sufficient conditions in order that a mapping of the form (9.1) be associated with a continuous locally smooth Hunt process. If we call a continuous locally smooth Hunt process whose germ field probabilities are described by (9.1) a *solution* to the correspondence (9.1), then this is the problem of finding sufficient conditions for

the existence of a solution. Since it is shown in Example 1 of §3 that Φ uniquely determines and is uniquely determined by the diffusive term of the infinitesimal generator (or stochastic differential equation) for a one-dimensional diffusion one might think of (9.1) as a generalized stochastic differential equation.

We next formulate a necessary condition for (9.1) to have a solution. Let A_i , $i=1, 2, 3, \dots$, be sets of \mathcal{M}_0 such that for every $x \in E$, $P_x(A_i)=1$. Then if $\{r_i : i=1, 2, 3, \dots\}$ is any countable dense subset of $[0, \varepsilon)$ for some $\varepsilon > 0$, let $\hat{A}_i \equiv \theta_{r_i}^{-1} A_i$ and $\hat{A} \equiv \bigcup_{n=1}^{\infty} \bigcap_{\{i: r_i \leq 1/n\}} A_i$. Then $\hat{A} \in \mathcal{M}_0$ and a necessary condition for the existence of a solution to (9.1) is that for every $x \in E$, $P_x(\hat{A})=1$.

In general such a solution, if it exists, is not unique. In fact E. B. Dynkin [7] and P. Courrège [4] have systematically studied the construction of processes which are equivalent in finite time by means of multiplicative functionals. Another interesting problem is to formulate side conditions which would pick out a unique solution to (9.1). In the case of stochastic differential equations for uninterrupted diffusions on the real line (cf. §3, Example 1) this is done by prescribing the drift term of the stochastic differential equation, that is, the limit:

$$\lim_{t \downarrow 0} E_x((X(t) - x)/t) \equiv a(x).$$

10. Characterization of locally smooth processes. In §9 it was shown that equivalence in finite time is equivalent to instantaneous equivalence for the class of continuous locally smooth Hunt processes. It turns out that the definition of local smoothness is intimately connected with certain concepts in probabilistic potential theory. In this section we discuss these connections. We begin by reviewing a few ideas from probabilistic potential theory. The reader is referred to G. A. Hunt [10], E. B. Dynkin [7] or R. K. Gettoor [8] for more details. In this section $\pi = (\Omega, \mathcal{F}_\infty, (P_x)_{x \in E_\Delta}, (X_t)_{t \in [0, \infty)})$ is a Hunt process which is considered as a strong Markov process with respect to the σ -fields $(\mathcal{M}_t^x)_{t \in [0, \infty)}$.

If $\Gamma \subset E_\Delta$, then x is *irregular* for Γ if there exists a Borel set $\Gamma' \supset \Gamma$ such that $P_x(T_{\Gamma'} > 0) = 1$. If x is not irregular for Γ , then it is said to be *regular* for Γ . Let Γ^r denote the set of regular points of Γ . A set Γ is *open in the fine topology* if each $x \in \Gamma$ is irregular for Γ^c . A set $\Gamma \subset E_\Delta$ is said to be *nearly analytic* if for each μ there are analytic sets Γ_1 and Γ_2 such that $\Gamma_1 \subset \Gamma \subset \Gamma_2$ and $P_\mu(X_t \in \Gamma_2 - \Gamma_1 \text{ for some } t \geq 0) = 0$. A set Γ is *thin* if it is contained in a nearly analytic set Γ' which has no regular points. A set is *semipolar* if it can be covered by a countable union of thin sets. Note that if $\Gamma \in \mathcal{E}$ then it can be shown that $\Gamma - \Gamma^r$ is semipolar. A set Γ is *polar* if it is contained in a nearly analytic set Γ' such that $P_x(T_{\Gamma'} < \infty) = 0$ for each $x \in E$. Note that every polar set is semipolar.

A nonnegative function f on E_Δ with $f(\Delta) = 0$ which is measurable with respect to the σ -field of universally measurable sets is said to be *excessive* if $P_t f \leq f$ for each $t \geq 0$ and $P_t f \rightarrow f$ as $t \rightarrow 0$ pointwise. In particular the constant functions are excessive for any Hunt process having a Feller semigroup.

PROPOSITION 10.1. (i) If f is excessive, the function $t \rightsquigarrow f(X_t)$ is right continuous and finite for all $t \geq s$ if $f(X_s)$ is finite, P_x -almost surely for each $x \in E$.

- (ii) For each $x \in E$, $f(X_s)$ is a P_x -supermartingale.
- (iii) An excessive function is continuous in the fine topology.

Proof. Refer to E. B. Dynkin [7].

An excessive function f is regular if $t \rightsquigarrow f(X_t)$ is continuous whenever $t \rightsquigarrow X_t$ is continuous, P_x -almost surely for each $x \in E$. A function f is said to be a harmonic function if both f and $-f$ are excessive.

PROPOSITION 10.2. If there exists an excessive function f which is not regular, then the Markov process is not locally smooth.

Proof. For $\epsilon > 0$, let $T^\epsilon \equiv \inf \{t : |f(X_t) - f^-(X_t)| > \epsilon, X_t = X_t^-\}$, where

$$f^-(X_t) \equiv \lim_{s \uparrow t} f(X_s).$$

(Recall that $f(X_t)$ is free of oscillatory discontinuities, P_{x_0} -almost surely for each $x_0 \in E$ (P. A. Meyer [13, Chapter 6, T 3]).) If f is a nonregular excessive function, then for some $\epsilon_0 > 0$ and $x_0 \in E$, $P_{x_0}(T^{\epsilon_0} < \infty) > 0$. But then for some positive integer k ,

$$P_{x_0}(|f(X_{T^{\epsilon_0}}) - f^-(X_{T^{\epsilon_0}})| > \epsilon_0, k\epsilon_0/4 \leq f(X_{T^{\epsilon_0}}) < (k+1)\epsilon_0/4) > 0.$$

Let

$$A_{\epsilon_0} \equiv \{|f(X_{T^{\epsilon_0}}) - f^-(X_{T^{\epsilon_0}})| > \epsilon_0, k\epsilon_0/4 \leq f(X_{T^{\epsilon_0}}) < (k+1)\epsilon_0/4\}.$$

Let $\{T_n\}$ be an increasing sequence of stopping times such that $T_n < T^{\epsilon_0}$ for all n and $\lim_{n \rightarrow \infty} T_n = T^{\epsilon_0}$, both P_{x_0} -almost surely on A_{ϵ_0} , and let

$$\Gamma \equiv \{x : f(x) \notin [(k-1)\epsilon_0/4, (k+2)\epsilon_0/4]\}$$

and

$$\Gamma' \equiv \{x : f(x) \in [k\epsilon_0/4, (k+1)\epsilon_0/4]\}.$$

Then $\Gamma' \subset \{x : P_x(T_\Gamma = 0) = 0\}$ because of the fine continuity of f . Since $P_{x_0}(A_{\epsilon_0}) > 0$, there is a positive integer N such that $P_{x_0}(\bigcap_{n=N}^\infty X(T_n) \in \Gamma, X(T) \in \Gamma') > 0$. Hence the process is not locally smooth.

REMARK 10.1. It can also be proved that if there is a semipolar set which is not polar, then the process is not locally smooth. However we omit the proof.

Let $\Gamma \subset E$ and let f be an excessive function. If $R_\Gamma^\Gamma \equiv \inf \{u : u \text{ excessive, } u \geq f \text{ on } \Gamma\}$, then R_Γ^Γ is called the *réduite* of f on Γ . We make the following assumptions concerning the *réduite*.

ASSUMPTION B. (i) There is a unique excessive function, \hat{R}_Γ^Γ , which differs from R_Γ^Γ on a semipolar set. \hat{R}_Γ^Γ is called the *balayage* of f on Γ .

(ii) If $y \notin \Gamma$ is irregular for Γ , then there is a neighborhood N_y of y such that $R_\Gamma^\Gamma \cap N_y(y) < 1$.

(iii) The set $\{y : y \in \Gamma, \hat{R}_\Gamma^\Gamma \neq 1\}$ is semipolar.

REMARK 10.2. M. Brelot [2] has shown that Assumption B is valid if the harmonic functions satisfy a certain set of axioms. In particular these axioms are satisfied if the infinitesimal generator is an elliptic differential operator with smooth coefficients. In the general probabilistic context, the balayage and réduite are discussed in the forthcoming book of R. M. Blumenthal and R. K. Gettoor [3].

THEOREM 10.1. *If Assumption B is satisfied and*

(10.1) *every semipolar set is polar, and*

(10.2) *every finite excessive function is regular, then the Markov process is locally smooth.*

Proof. Let us show that if π is not locally smooth and (10.1) is satisfied, then (10.2) is not satisfied. If π is not locally smooth then there is a sequence of stopping times $\{T_n\}$, a stopping time T , a sequence $\{\Gamma_n\}$ of \mathcal{E}_Δ -measurable sets and a point $x \in E$ such that:

- (i) $T_n < T$ for all n and $\lim_{n \rightarrow \infty} T_n = T$, both P_x -almost surely on $\{T < \zeta\}$, and
- (ii) for some positive integer N ,

$$P_x \left(\bigcap_{n=N}^{\infty} \{X(T_n) \in \Gamma_n\}, X(T) \in C_N \right) > 0,$$

where $C_N \equiv \{y : y \in E, P_y(T_{K_N} = 0) = 0\}$ and $K_N \equiv \bigcup_{n=N}^{\infty} \Gamma_n$, $N = 1, 2, 3, \dots$. As noted above $(C_N \cap K_N)$ is semipolar and hence by (10.1) is polar. If $y \in C_N - K_N$, then y is an irregular point for K_N . Assumption B implies that there is a neighborhood Λ_i of y (Λ_i is assumed to belong to a countable base for the open sets of E , say $\{\Lambda_i\}$) for which $R_{\Gamma_N}^{K_N \cap \Lambda_i}(y) < 1$. In other words

$$y \in \Sigma_i \equiv \{y : R_{\Gamma_N}^{K_N \cap \Lambda_i}(y) < 1\} \cap \Lambda_i.$$

But then $C_N - K_N \subset \bigcup_{i=1}^{\infty} \Sigma_i$ where the union runs over the countable base $\{\Lambda_i\}$. Let $\hat{\Sigma}_i \equiv \{y : \hat{R}_{\Gamma_N}^{K_N \cap \Lambda_i}(y) < 1\} \cap \Lambda_i$. Since $\hat{R}_{\Gamma_N}^{K_N \cap \Lambda_i} = R_{\Gamma_N}^{K_N \cap \Lambda_i}$ except on a semipolar and hence by (10.1) a polar set, there is some i_0 such that

$$P_x \left(\bigcap_{n=N}^{\infty} \{X(T_n) \in \Gamma_n\}, X(T) \in \hat{\Sigma}_{i_0} \right) > 0.$$

However $\hat{R}_{\Gamma_N}^{K_N \cap \Lambda_{i_0}}$ is excessive and $\{y : \hat{R}_{\Gamma_N}^{K_N \cap \Lambda_{i_0}}(y) < 1, y \in K_N \cap \Lambda_{i_0}\}$ is a polar set. Therefore since $X_T = X_T^-$, P_x -almost surely on $\{T < \zeta\}$ (cf. Proposition 2.9.1), $\hat{R}_{\Gamma_N}^{K_N \cap \Lambda_{i_0}}(X(t))$ is discontinuous at T , P_x -almost surely on the set $(\bigcap_{n=N}^{\infty} \{X(T_n) \in \Gamma_n\}, X(T) \in \hat{\Sigma}_{i_0})$ which has positive P_x -probability. But then (10.2) is not satisfied and the proof is complete.

REMARK 10.3. A discussion of (10.1) and (10.2) and their interrelation is given in G. A. Hunt [10] and R. K. Gettoor [8].

J. L. Doob [6] has shown that the harmonic functions and excessive functions of the Wiener process in a Euclidean space are the harmonic and superharmonic

functions, respectively, of classical potential theory. However the classical superharmonic and harmonic functions are such that Assumption B as well as (10.1) and (10.2) are satisfied (refer to M. Brelot [1] and E. B. Dynkin [7]). Hence the Wiener process is locally smooth.

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