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Equivalence of two Series of Spherical Representations of a Free Group (*).

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Summary. – *The spherical principal series of a non-commutative free group may be analytically continued to yield a series of uniformly bounded representations, much as the spherical representations* $\pi_{(1/2)+it}$ *of* $SL(2,\mathbf{R})$ *may be analytically continued in the strip* $0 < \operatorname{Re} z < 1$. *This series of uniformly bounded representations was constructed and studied by A. M. MANTERO and A. ZAPPA. Independently T. PYTLIK and R. SZWARC introduced and studied representations of the free group which contain a series of subrepresentations indexed by spherical functions. Both series consist of irreducible representations and include the spherical complementary series. The aim of this paper is to prove that the non-unitary uniformly bounded representations of the two series are also equivalent.*

Introduction.

The spherical principal series of a non-commutative free group may be analytically continued to yield a series of uniformly bounded representations, much as the spherical representations $\pi_{(1/2)+it}$ of $SL(2,\mathbf{R})$ may be analytically continued in the strip $0 < \operatorname{Re} z < 1$. This series of uniformly bounded representations was constructed and studied by A. M. MANTERO and A. ZAPPA in [4], [5]. Quite independently, T. PYTLIK and R. SZWARC introduced and studied in [7] representations of the free group which contain a series of subrepresentations indexed by spherical functions. Both series consist of irreducible representations and include the spherical complementary series. It is natural therefore to ask if the non-unitary uniformly bounded representations of the two series are equivalent. The purpose of this paper is to prove the equivalence also in the non-unitary case.

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1. – Basic notation and preliminaries.

We refer to [3], [4], [7] and [8] for notations and unexplained results. For any integer $r > 1$, let $\boldsymbol{F}_r$ be the free group on r generators and let Ω be the related Poisson boundary. We denote by ν the probabiliyty measure on Ω defined by

$$\nu(\Omega_x) = (q+1)^{-1} q^{1-|x|}, \qquad x \in \boldsymbol{F}_r,$$

where $q = 2r - 1$, $|x|$ is the length of the word x and $\Omega_x = \{\omega \in \Omega,\ \omega \text{ starts with } x\}$. Moreover for every $x \in \boldsymbol{F}_r$, $\omega \in \Omega$, we denote by $P(x, \omega)$ the related Poisson kernel. Let $\mathcal{F}(\Omega)$ be the linear space generated by the characteristic functions χ_{Ω_x}, for all $x \in \boldsymbol{F}_r$. For any complex number z in the strip $S = \{z \in \boldsymbol{C},\ 0 < \operatorname{Re} z < 1\}$ we consider the representation π_z of $\boldsymbol{F}_r$ acting on $\mathcal{F}(\Omega)$ according to

$$\pi_z(x) f(\omega) = P^z(x, \omega) f(x^{-1}\omega), \qquad x \in \boldsymbol{F}_r,\ \omega \in \Omega,\ f \in \mathcal{F}(\Omega). \tag{1}$$

We recall that the constant function $\mathbf{1}$ on Ω is a cyclic vector for each of these representations and the coefficients

$$\Phi_z(x) = \langle \pi_z(x)\mathbf{1}, \mathbf{1}\rangle, \qquad x \in \boldsymbol{F}_r, \tag{2}$$

are the spherical functions on $\boldsymbol{F}_r$ (see [3]). When $x \in S$, $\operatorname{Re} z = 1/2$, the extension of π_z to $L^2(\Omega)$ gives the spherical principal series of $\boldsymbol{F}_r$. When $z \in S$, $\operatorname{Re} z \neq 1/2$, π_z extends to a uniformly bounded representation acting on a Hilbert space $\mathcal{H}_{\operatorname{Re} z}(\Omega)$, obtained as the completion of $\mathcal{F}(\Omega)$ with respect to a suitable inner product (see [4]). In particular for $\operatorname{Im} z = k\pi/\ln q$, $k \in \boldsymbol{Z}$, we obtain the spherical complementary series of $\boldsymbol{F}_r$.

Let us consider now a family of representations of $\boldsymbol{F}_r$ acting on the linear space $\mathcal{F}(\boldsymbol{F}_r)$ generated by the characteristic functions $\delta_x = \chi_{\{x\}}$, for all $x \in \boldsymbol{F}_r$. Following [7], we define two operators acting on $\mathcal{F}(\boldsymbol{F}_r)$. Let P be the operator on $\mathcal{F}(\boldsymbol{F}_r)$ defined by

$$\begin{aligned} P\delta_e &= 0, \\ P\delta_x &= \delta_{\bar{x}}, \qquad \text{if } x \neq e, \end{aligned}$$

where $\bar{x}$ is the word of $\boldsymbol{F}_r$ obtained from x by deleting the last letter. For any complex number ζ in the open disk $D = \{\zeta \in \boldsymbol{C},\ |\zeta| < 1\}$, we define an invertible operator T_ζ on $\mathcal{F}(\boldsymbol{F}_r)$ by

$$\begin{aligned} T_\zeta\delta_e &= (1-\zeta^2)^{1/2}\delta_e, \\ T_\zeta\delta_x &= \delta_x, \qquad \text{if } x \neq e, \end{aligned}$$

$(1-\zeta^2)^{1/2}$ being the principal branch of the square root of $1-\zeta^2$.

For any $\zeta \in D$ we consider the representation Π_ζ of $\boldsymbol{F}_r$ acting on $\mathcal{F}(\boldsymbol{F}_r)$ obtained by conjugating of the regular representation λ with the invertible operator $(I - \zeta P)T_\zeta$, thus:

$$\Pi_\zeta(x) = T_\zeta^{-1}(I-\zeta P)^{-1}\lambda(x)(I-\zeta P)T_\zeta, \qquad x \in \boldsymbol{F}_r. \tag{3}$$

For any $\zeta \in D$, Π_ζ extends to an uniformly bounded representation on $l^2(\boldsymbol{F}_r)$ and $\{\Pi_\zeta, \zeta \in D\}$ is an analytic family. Moreover, if ζ belongs to $D_0 = \{\zeta \in \boldsymbol{C}, q^{-1/2} < |\zeta| < 1\}$, then Π_ζ decomposes into a direct sum of two representations, one equivalent to the regular representation, and the other, denoted $\tilde{\pi}_\zeta$, irreducible on the space $\operatorname{Ker}(I - \zeta P)T_\zeta$. Let us denote $\mathcal{K}_\zeta(\boldsymbol{F}_r) = \operatorname{Ker}(I - \zeta P)T_\zeta$.

For any $\zeta \in D_0$, the function

$$f_\zeta = \frac{q+1}{q}(1-\zeta^2)^{-1/2}\delta_e + \sum_{n=1}^{\infty}(q\zeta)^{-n}\chi_n, \tag{4}$$

obtained as the projection of δ_e on $\mathcal{K}_\zeta(\boldsymbol{F}_r)$ along the range of $T_\zeta(I - \zeta P^*)$ in $l^2(\boldsymbol{F}_r)$, is a cyclic vector for $\tilde{\pi}_\zeta$ and

$$\Psi_\zeta(x) = \langle f_\zeta, f_{\bar{\zeta}}\rangle^{-1}\langle\tilde{\pi}_\zeta(x) f_\zeta, f_{\bar{\zeta}}\rangle, \qquad x \in \boldsymbol{F}_r, \tag{5}$$

is a spherical function.

There is a bijection between the set $\{\Phi_z, z \in S, 1/2 < \operatorname{Re} z < 1\}$ and the set $\{\Psi_\zeta, \zeta \in D_0\}$. In fact if α denotes the map from the strip $S_0 = \{z \in \boldsymbol{C}, 1/2 < \operatorname{Re} z < 1\}$ onto D_0, defined by $\alpha(z) = q^{z-1}$, then for any $z \in S_0$ and $\zeta = \alpha(z)$ we have

$$\Psi_\zeta = \Phi_z.$$

We note that $\tilde{\pi}_\zeta$ is unitary if and only if $\zeta \in D_0$ is real; moreover $\alpha(z)$ is real if and only if $\operatorname{Im} z = k\pi/\ln q$, $k \in \boldsymbol{Z}$. So the representations $\tilde{\pi}_\zeta$, for ζ real in D_0, coincide (up to unitary equivalence) with the spherical complementary series of $\boldsymbol{F}_r$ defined in [4].

2. - Equivalence between π_z and $\tilde{\pi}_\zeta$.

In order to prove the equivalence between the representations π_z and $\tilde{\pi}_\zeta$, for $\zeta = \alpha(z)$, we define a correspondence between the functions on $\boldsymbol{F}_r$ and those on Ω. Let $\zeta \in D_0$. For any complex function f on $\boldsymbol{F}_r$, we define the following sequence of function on Ω

$$F_0(f;\zeta)(\omega) = (q+1)^{-1}q(1-\zeta^2)^{1/2}f(e),$$
$$F_n(f;\zeta)(\omega) = (q\zeta)^n f(\omega_n), \qquad n \geq 1, \qquad \omega \in \Omega,$$

where ω_n denotes the first n letters of ω.

For any integer n the function $F_n(f;\zeta)$ is measurable (with respect to the finite σ-field $\mathcal{B}_n(\Omega)$ generated by $\{\Omega_x, |x| = n\}$).

LEMMA 1. - (i) *The sequence $(F_n(f;\zeta))$ is a martingale on Ω if and only if $f \in \mathcal{K}_\zeta(\boldsymbol{F}_r)$.*

(ii) *If $F_n(f;\zeta)$ are uniformly bounded with respect to n, then the limit of $F_n(f;\zeta)$ exists a.e. in Ω and defines a bounded measurable function $L_\zeta(f)$.*

PROOF. – For any $n \geq 0$, let E_n denote the n-th conditional expectation with respect to $\mathcal{B}_n$. For any $\omega \in \Omega$ and $x = \omega_n$, we have

$$E_n F_{n+1}(f;\, \zeta)(\omega) = \nu(\Omega_x)^{-1} \int_{\Omega_x} F_{n+1}(\omega')\, d\nu(\omega') =$$

$$= \nu(\Omega_x)^{-1} \sum_{|a| = 1,\, x \perp a} \nu(\Omega_{xa})(q\zeta)^{n+1} f(xa) = (q\zeta)^n \zeta \sum_{|a| = 1,\, x \perp a} f(xa),$$

where $x \perp y$ means that no cancellation is possible in the product xy. Since $f \in \mathcal{H}_\zeta(\boldsymbol{F}_r)$ if and only if $\zeta \sum_{|a| = 1,\, x \perp a} f(xa) = f(x)$, then

$$E_n F_{n+1}(f;\, \zeta)(\omega) = (q\zeta)^n f(\omega_n) = F_n(f;\, \zeta)(\omega)$$

if and only if $f \in \mathcal{H}_\zeta(\boldsymbol{F}_r)$.

Let $\zeta \in D_0$. For any bounded measurable function g on Ω let us define

$$V_\zeta(g) = \int_\Omega Q_\zeta(x, \omega)\, g(\omega)\, d\nu(\omega), \qquad x \in \boldsymbol{F}_r,$$

where

$$Q_\zeta(e, \omega) = (q+1)\, q^{-1} (1 - \zeta^2)^{-1/2},$$
$$Q_\zeta(x, \omega) = (q+1)\, q^{-1} \zeta^{-|x|} \chi_{\Omega_x}(\omega), \qquad \text{if } x \neq e,\ \ \omega \in \Omega.$$

LEMMA 2. – (i) *For every bounded measurable function g on Ω the function $V_\zeta(g)$ belongs to $\mathcal{H}_\zeta(\boldsymbol{F}_r)$ and*

$$L_\zeta(V_\zeta(g)) = g;$$

moreover, for every $f \in \mathcal{H}_\zeta(\boldsymbol{F}_r)$

$$V_\zeta(L_\zeta(f)) = f.$$

(ii) *$V_\zeta(\mathcal{F}(\Omega))$ is the space $\mathcal{H}_\zeta^0(\boldsymbol{F}_r)$ spanned by $\{\tilde{\pi}_\zeta(x)(f_\zeta),\, x \in \boldsymbol{F}_r\}$.*

PROOF. – (i) Let us consider the sequence $(F_n(V_\zeta(g);\, \zeta))$; for a.e. $\omega \in \Omega$

$$\lim_n\, (F_n(V_\zeta(g);\, \zeta)(\omega)) = g(\omega)$$

and $|F_n(V_\zeta(g);\, \zeta)(\omega)| \leq \|g\|_\infty$.

Moreover, if $f \in \mathcal{H}_\zeta(\boldsymbol{F}_r)$, then by definition of the sequence $F_n(f;\, \zeta)$ we have

$$f(x) = \int_\Omega Q_\zeta(x, \omega)\, L_\zeta(f)(\omega)\, d\nu(\omega), \qquad x \in \boldsymbol{F}_r.$$

(ii) By straightforward calculations $F_n(f_\zeta;\, \zeta) = \mathbf{1}$, for any $n \geq 0$, so we have, for any $\zeta \in D_0$,

$$1 = L_\zeta(f_\zeta). \tag{6}$$

Moreover for any $x \in \boldsymbol{F}_r$ we have, for all $n \geqslant 1$

$$F_n(\tilde{\pi}_\zeta(x) f_\zeta;\, \zeta)(\omega) = (q\zeta)^n \tilde{\pi}_\zeta(x) f_\zeta(\omega_n) = (q\zeta)^n \langle f_\zeta, \Pi_\zeta(x^{-1}) \delta_{\omega_n} \rangle .$$

In fact, by Lemmas 1 and 2 of [7], $\Pi_\zeta(x^{-1})$ and $\lambda(x^{-1})$ coincide on the orthogonal complement of the finite dimensional subspace spanned by $\{\delta_x, P\delta_x, \ldots, P^{|x|}\delta_x = \delta_e\}$. Therefore for $n > |x|$,

$$\Pi_\zeta(x^{-1})\delta_{\omega_n} = \lambda(x^{-1})\delta_{\omega_n} = \delta_{x^{-1}\omega_n},$$

hence

$$F_n(\tilde{\pi}_\zeta(x) f_\zeta;\, \zeta)(\omega) = (q\zeta)^n f_\zeta(x^{-1}\omega_n) = (q\zeta)^{n - |x^{-1}\omega_n|} = P^z(x, \omega) .$$

Then by definition (1) we have the following identity:

$$\pi_z(x)\mathbf{1} = L_\zeta(\tilde{\pi}_\zeta(x) f_\zeta), \qquad x \in \boldsymbol{F}_r . \tag{7}$$

THEOREM. – *For any $z \in S_0$ and $\zeta = \alpha(z) = q^{z-1}$, let $\mathcal{R}^0_\zeta$ be the linear map, from the space $\mathcal{K}^0_\zeta(\boldsymbol{F}_r)$ spanned by $\{\tilde{\pi}_\zeta(x)(f_\zeta), x \in \boldsymbol{F}_r\}$, into $\mathcal{F}(\Omega)$, defined by*

$$\mathcal{R}^0_\zeta(f) = c_\zeta L_\zeta(f)$$

where $c_\zeta = \langle f_\zeta, f_{\bar{\zeta}}\rangle^{1/2}$, $\alpha(z) = \zeta$. Then $\mathcal{R}^0_\zeta$ extends to a topological isomorphism of $\mathcal{K}_\zeta(\boldsymbol{F}_r)$ onto $\mathcal{H}_{\mathrm{Re}\, z}(\Omega)$ and

$$\mathcal{R}_\zeta \tilde{\pi}_\zeta(x) = \pi_z(x)\mathcal{R}_\zeta, \qquad x \in \boldsymbol{F}_r .$$

PROOF. – For any $\zeta \in D_0$ we consider the linear map $\mathcal{R}^0_\zeta$; the identities (6) and (7) imply that

$$\mathcal{R}^0_\zeta(\tilde{\pi}_\zeta(x) f_\zeta) = c_\zeta \pi_z(x)\mathbf{1}, \qquad x \in \boldsymbol{F}_r .$$

If ζ is real, then $\mathcal{R}^0_\zeta$ is an isometry. Indeed in this case both the representations $\tilde{\pi}_\zeta$ and π_z, $\zeta = \alpha(z)$, are unitary, so far $x, y \in \boldsymbol{F}_r$, we have

$$\langle \mathcal{R}^0_\zeta \tilde{\pi}_\zeta(x) f_\zeta, \mathcal{R}^0_\zeta \tilde{\pi}_\zeta(y) f_\zeta\rangle_{\mathcal{H}_{\mathrm{Re}\, z}} = c_\zeta^2 \langle \pi_z(x)\mathbf{1}, \pi_z(y)\mathbf{1}\rangle_{\mathcal{H}_{\mathrm{Re}\, z}} =$$

$$= c_\zeta^2 \Phi_z(y^{-1}x) = c_\zeta^2 \Psi_\zeta(y^{-1}x) = \langle \tilde{\pi}_\zeta(y^{-1}x) f_\zeta, f_\zeta\rangle = \langle \tilde{\pi}_\zeta(x) f_\zeta, \tilde{\pi}_\zeta(y) f_\zeta\rangle .$$

Hence in this case $\mathcal{R}^0_\zeta$ extends by continuity to an isometry from $\mathcal{K}_\zeta(\boldsymbol{F}_r)$ onto $\mathcal{H}_{\mathrm{Re}\, z}(\Omega)$.

If $\zeta \in D_0$ is not real, we define an operator A_ζ on $l^2(\boldsymbol{F}_r)$ by

$$A_\zeta \delta_e = (1-\zeta^2)^{1/2}(1-|\zeta|^2)^{-1/2}\delta_e,$$
$$A_\zeta \delta_x = \zeta^{|x|}\,|\zeta|^{-|x|}\,\delta_x, \qquad \text{if } x \neq e.$$

It is easy to prove that A_ζ is invertible on $l^2(\boldsymbol{F}_r)$, maps $\mathcal{H}_\zeta(\boldsymbol{F}_r)$ onto $\mathcal{H}_{|\zeta|}(\boldsymbol{F}_r)$ and

$$\mathcal{R}_\zeta^0 = c_\zeta c_{|\zeta|}^{-1}\,\mathcal{R}_{|\zeta|}^0 \circ A_\zeta.$$

Therefore $\mathcal{R}_\zeta^0$ extends to an invertible operator $\mathcal{R}_\zeta$ from $\mathcal{H}_\zeta(\boldsymbol{F}_r)$ onto $\mathcal{H}_{\mathrm{Re}\,z}(\Omega)$. Finally we note that, by the identities (6) and (7), we have for every $\zeta \in D_0$,

$$\mathcal{R}_\zeta \tilde{\pi}_\zeta(x) f_\zeta = c_\zeta L_\zeta(\tilde{\pi}_\zeta(x) f_\zeta) = c_\zeta \pi_z(x)\,\mathbf{1} = \pi_z(x)(c_\zeta L_\zeta(f_\zeta)) = \pi_z(x)\,\mathcal{R}_\zeta\, f_\zeta, \qquad x \in \boldsymbol{F}_r.$$

So $\mathcal{R}_\zeta$ intertwines the representations $\tilde{\pi}_\zeta$ and π_z.

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