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Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems

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Abstract

In this paper, we generalize the concept of well-posedness to a system of hemivariational inequalities in Banach space. By introducing several concepts of well-posedness for systems of hemivariational inequalities considered, we establish some metric characterizations of well-posedness and prove some equivalence results of strong (generalized) well-posedness between a system of hemivariational inequalities and its derived system of inclusion problems.

Keywords: System of hemivariational inequalities, well-posedness, Clarke's generalization gradient, system of inclusion problems.
2010 MSC: 49J40, 47J22, 49J52.

1. Introduction

Let V_1 and V_2 be two Banach spaces with the dual spaces V_1^* and V_2^*, respectively. For a Banach space V_i, $i = 1, 2$, we denote $\langle \cdot, \cdot \rangle_{V_i^* \times V_i}$ the duality pairing between a Banach space V_i and its dual space V_i^* space and by $\| \cdot \|_{V_i}$, $\| \cdot \|_{V_i^*}$ the norms on the space V_i and its dual space V_i^*, respectively. It is well-known that the product space $V_1 \times V_2$ is also a Banach space with the following norm

$$\|\boldsymbol{u}\|_{V_1 \times V_2} = \|u_1\|_{V_1} + \|u_2\|_{V_2}, \forall\, \boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2.$$

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Suppose that, for $i = 1, 2$, $A_i : V_1 \times V_2 \to V_i^*$ is a mapping from $V_1 \times V_2$ to V_i^*, $J : V_1 \times V_2 \to \mathbb{R}$ is a locally Lipschitz functional on $V_1 \times V_2$ and f_i is a given point in V_i^*.

In this paper, we consider a system of hemivariational inequalities which is specified as follows: Find $(u_1, u_2) \in V_1 \times V_2$ such that

$$\textbf{(SHVI)} \quad \begin{cases} \langle A_1(u_1,u_2) - f_1, v_1 - u_1\rangle_{V_1^*\times V_1} + J_1^\circ(u_1,u_2; v_1 - u_1) \geq 0, & \forall v_1 \in V_1 \\ \langle A_2(u_1,u_2) - f_2, v_2 - u_2\rangle_{V_2^*\times V_2} + J_2^\circ(u_1,u_2; v_2 - u_2) \geq 0, & \forall v_2 \in V_2, \end{cases}$$

where, for $i \neq j = 1, 2$, $J_i^\circ(u_i, u_j; v_i - u_i)$ denotes the generalized directional derivative of the functional $J(\cdot, u_j)$ at u_i in the direction $v_i - u_i$, where $J(\cdot, u_j)$ is a functional on V_i for any given $u_j \in V_j$, that is,

$$J_i^\circ(u_i, u_j; v_i - u_i) = \limsup_{w\to u_i, \lambda\to 0} \frac{J(w + \lambda(v_i - u_i), u_j) - J(w, u_j)}{\lambda}.$$

As a classical concept in optimization theory, the well-posedness has a profound impact on the development of optimization problems and their relate problems such as variational inequalities, inclusion problems, Nash equilibrium problems and others. In 1966, Tykhonov [21] firstly defined well-posedness for a unconstrained global optimization problem, which is called Tykhonov well-posedness and requires the existence and uniqueness of minimizer and the convergence of every minimizing sequence toward the unique minimizer. For constrained optimization problems, another kind of well-posedness, which is called LP well-posedness, was given by Levitin and Polyak in [12]. After that, many kinds of results concerned with well-posedness for various optimization problems were introduced and the well-posedness of optimization problems was studied widely in recent years by a large number of researchers in many fields. For more concept of well-posedness for optimization problems and their detailed studies, refer to [3, 9, 14, 27] and references therein.

It is well known that a minimization problem with differentiability property has a close relationship with a differentiable variational inequality. Therefore, it is a natural idea to study well-posedness for variational inequalities and their related problems. In 1981, Lucchetti and Patrone[15] extended the concept of well-posedness for optimization problems to a variational inequality for the first time. By using Ekeland's theorem, they gave a characterization of Tykhonov's well-posedness for a minimizing problem with a convex lower semi-continuous function on a closed convex set. Since then, many kinds of well-posedness for the optimization problems, such as LP well-posedness and extended well-posedness etc., are introduced to the study of variational inequalities and their related problems, such as equilibrium problems, fixed point problems, and inclusion problems and others. In 2008 and 2010, Fang, Hu and Huang generalized the well-posedness to equilibrium problems and systems of equilibrium problems in [5, 8]. Refer to [2, 11, 13] for more details.

Hemivariational inequality, a class of generalization of variational inequalities, is more recent (hemivariational inequalities are introduced firstly by Panagiotopoulos[19] in 1980s) and concerned with nonsmooth and nonconvex energy functionals. As demonstrated by many researchers in the field of variational inequalities and hemivariational inequalities, hemivariational inequalities and systems of hemivariational inequalities are powerful tools to study many problems in mechanics and engineering such as nonconvex semipermeability problems, unilateral contact problems, masonry structures delamination problems. Therefore, in recent years, many researchers devoted themselves to studying many kinds of hemivariational inequalities arising in mechanical and engineering problems. In terms of literature on hemivariational inequalities and systems of hemivariational inequalities, the studies on solvability (the existence and uniqueness of a solution) and well-posedness for various kinds of hemivariational inequalities are mature. Many famous results have been obtained by many distinguished researchers (refer to [17, 18, 20, 22] for details). The concept of well-posedness for hemivariational inequalities was firstly introduced by Goeleven and Mentagui [7] in 1995 and, further, studied by Xiao, Yang and Huang in [23, 24, 25]. Also, there are a few papers studying the solvability of systems of hemivariational inequalities since, due to the complex structure of systems of hemivariational inequalities, it is much more difficult than the study of hemivariational inequalities. However, as far as the authors knowledge, there is no researcher studying well-posedness for systems of hemivariational inequalities.

Inspired by the research on well-posedness for hemivariational inequalities, in this paper, we generalize the concepts of well-posedness to a system of hemivariational inequalities, establish some metric characterizations of well-posedness and prove the equivalence between well-posedness of a system of hemivariational inequalities and its derived system of inclusion problems. The paper is structured as follows. In Section 2, we briefly recall some preliminaries. In Section 3, we define several concepts of well-posedness for the system of hemivariational inequalities and, with two assumptions on the operators involved, establish some metric characterizations for the system of hemivariational inequalities. In Section 4, we prove two equivalence results of well-posedness between the system of hemivariational inequalities and its derived system of inclusion problems. Finally, in Section 5, we give some concluding remarks on our main results.

2. Preliminaries

In this section, we recall some important notions and useful results on nonlinear analysis, optimization theory and nonsmooth analysis, which can be found in [1, 4, 10, 16, 26].

Definition 2.1. Let V be a Banach space. A sequence $\{u_n\} \subset V$ is said to be convergent if there exists $u \in V$ such that

$$\lim_{n\to\infty} \|u_n - u\|_V = 0.$$

Definition 2.2. Let V be a Banach space and V^* be its dual space. A sequence $\{u_n\} \subset V$ is said to be weakly convergent if there exists $u \in V$ such that

$$\langle u^*, u_n\rangle_{V^*\times V} \to \langle u^*, u\rangle_{V^*\times V}, \quad \forall u^* \in V^*.$$

Definition 2.3. Let V be a Banach space with its dual space V^*. A sequence of functional $\{u_n^*\} \subset V^*$ is said to be *weakly** convergent to a point $u^* \subset V^*$ if

$$\langle u_n^*, u\rangle_{V^*\times V} \to \langle u^*, u\rangle_{V^*\times V}, \quad \forall u \in V.$$

Remark 2.4. If V is not reflexive space, then the *weak** topologies of V^* is weaker than its weak topologies. If V is a reflexive space, then the weak and *weak** topologies on V^* are the same.

Proposition 2.5. *Let V be a Banach space. Then the following statement holds: If $\{u_n\} \subset V$, $\{u_n^*\} \subset V^*$, $u_n \to u$ in V and $u_n^* \to u^*$ weakly* in V^*, then*

$$\langle u_n^*, u_n\rangle_{V^*\times V} \to \langle u^*, u\rangle_{V^*\times V}.$$

Definition 2.6. Let V be a Banach space with its dual space V^* and $T : V \to V^*$ be an single-valued operator on V. The operator T is said to be:

(1) demicontinuous if, for any sequence $\{u_n\} \subset V$ converging to $u \in V$, $T(u_n) \rightharpoonup T(u)$ in V^*;

(2) hemicontinuous if, for all $u, v \in V$, the functional $t \to \langle T(u + t(v - u)), v - u\rangle_{V^*\times V}$ from $[0, 1]$ into $\mathbb{R}$ is continuous at 0_+;

(3) continuous if, for any sequence $\{u_n\} \subset V$ converging to $u \in V$, $T(u_n) \to T(u)$ in V^*.

Remark 2.7. It is easy to see that, if $T : V \to V^*$ is continuous, then it is demicontinuous which, in turn, implies that T is hemicontinuous. If $T : V \to V^*$ is linear and demicontinuous, then it is continuous. It can be shown that for monotone operators $T : V \to V^*$ with $D(T) = V$, the notions of demicontinuity and hemicontinuity coincide.

Definition 2.8. Let V_1 and V_2 be two Banach spaces with their dual spaces V_1^*, V_2^*, respectively. Assume that $T : V_1 \times V_2 \to V_1^*$ is a single-valued operator on $V_1 \times V_2$. The operator T is said to be hemicontinuous with respect to first variable if the operator $T(\cdot, u_2) : V_1 \to V_1^*$ is hemicontinuous on V_1 for any given $u_2 \in V_2$.

Remark 2.9. By similar way, we can define the hemicontinuity of an operator $T : V_1 \times V_2 \to V_2^*$ with respect to second variable.

Definition 2.10. Let V be a real Banach space with its dual space V^* and $T : V \to V^*$ be a single-valued operator on V. The operator T is said to be monotone if

$$\langle Tu - Tv, u - v\rangle_{V^* \times V} \geq 0, \quad \forall u, v \in V.$$

Definition 2.11. Let V_1, V_2 be two Banach spaces and V_1^*, V_2^* be their dual spaces, respectively. Assume that $T : V_1 \times V_2 \to V_1^*$ is a single-valued operator on $V_1 \times V_2$. The operator T is said to be monotone with respect to first variable if the operator $T(\cdot, u_2) : V_1 \to V_1^*$ is monotone on V_1 for any given $u_2 \in V_2$.

Remark 2.12. By similar way, we can define the monotonicity of an operator $T : V_1 \times V_2 \to V_2^*$ with respect to second variable.

Definition 2.13. Let V be a Banach space and $h : V \to \mathbb{R}$ be a functional on V. h is said to be:

(1) Lipschitz continuous on V if there exists a constant $L > 0$ such that

$$|h(u_1) - h(u_2)| \leq L\|u_1 - u_2\|_V, \quad \forall u_1, u_2 \in V;$$

(2) locally Lipschitz continuous on V if, for all $u \in V$, there exists a neighborhood $N(u)$ and a constant $L_u > 0$ such that

$$|h(u_1) - h(u_2)| \leq L_u\|u_1 - u_2\|_V \quad \forall u_1, u_2 \in N(u).$$

Definition 2.14. Let V_1, V_2 be two Banach spaces and $h : V_1 \times V_2 \to \mathbb{R}$ be a functional on $V_1 \times V_2$. The functional h is said to be

(1) Lipschitz continuous with respect to first variable if the operator $h(\cdot, u_2) : V_1 \to V_1^*$ is Lipschitz continuous on V_1 for any given $u_2 \in V_2$;

(2) locally Lipschitz continuous with respect to first variable, if the operator $h(\cdot, u_2) : V_1 \to V_1^*$ is locally Lipschitz continuous on V_1 for any given $u_2 \in V_2$.

Remark 2.15. By similar way, we can define Lipschitz continuity and locally Lipschitz continuity of the operator $h : V_1 \times V_2 \to \mathbb{R}$ with respect to the second variable.

Definition 2.16. Let V be a Banach space, $h : V \to \mathbb{R}$ be a locally Lipschitz functional on V and let $u, v \in V$ be given elements. Clarke's generalized directional derivative of h at the point u in the direction v, denoted by $h^\circ(u; v)$, is defined by

$$h^\circ(u; v) = \lim_{w \to u, \lambda \downarrow 0} \frac{h(w + \lambda v) - h(w)}{\lambda}.$$

Clarke's generalized gradient of h at u, denoted by $\partial h(u)$, is subset of the dual space V^*, which is defined by

$$\partial h(u) = \{\rho \in V^* : h^\circ(u, v) \geq \langle \rho, v\rangle_{V^* \times V},\ \forall v \in V\}.$$

Proposition 2.17. *Let V be a Banach space and V^* be its dual space, $h : V \to R$ be a locally Lipschitz functional on V and let $u, v \in V$ be given elements. Then*

(1) *the function $v \to h^\circ(u, v)$ is finite, positively homogenous and subadditive on V;*

(2) *$h^\circ(u,v)$ is upper semicontinuous on $V \times V$ as a function of (u,v), i.e., for all $u, v \in V$, $u_n \subset V$, $v_n \subset V$ such that $u_n \to u$, $v_n \to v$ in V, we have*

$$\limsup h^\circ(u_n, v_n) \le h^\circ(u,v);$$

(3) *$h^\circ(u,-v) = (-h)^\circ(u,v)$;*

(4) *for all $v \in V$, $\partial h(u)$ is a nonempty convex bounded and weak-compact subset of V^*;*

(5) *for all $v \in V$, one has*

$$h^\circ(u,v) = max\{\langle \xi, v\rangle \ \xi \in \partial h(u)\};$$

(6) *The graph of Clarke's gradient $\partial h(u)$ is closed in $V \times (w^* - V^*)$ topology, where $(w^* - V^*)$ denotes the space V^* equipped with weak* topology, i.e., if $\{u_n\} \subset V$ and $\{u_n^*\} \subset V^*$ are sequences such that $u_n^* \in \partial h(u_n)$, $u_n \to u$ in V and $u_n^* \to u^*$ weakly* in V^*, then $u^* \in \partial h(u)$.*

Definition 2.18. Let $A \subset V$ be a nonempty subset of Banach space V. The measure of noncompactness μ of the set A is defined by:

$$\mu(A) = inf\{\epsilon > 0 : A \subset \cup_{i=1}^n A_i,\ diam|A_i| < \epsilon,\ i = 1, 2, 3, \cdots, n\},$$

where diam $|A_i|$ denotes the diameter of the set A_i.

Definition 2.19. Let A, B be two nonempty subsets of Banach space V. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by:

$$\mathcal{H}(A,B) = max\{e(A,B), e(B,A)\},$$

where $e(A,B) = \sup_{a\in A} d(a,B)$ with $d(a,B) = \inf_{b\in B} \|a-b\|_V$.

Note that, in [10], we can find some more properties of the Hausdorff metric between two sets. At the end of this section, we give a lemma from [6], which is important to our main results.

Lemma 2.20. *Let $C \subset V$ be nonempty closed and convex, $C^* \subset V^*$ be nonempty convex and bounded, $\phi : V \to R$ be proper, convex and lower semi-continuous and $y \in C$ be arbitrary. Assume that, for any $x \in C$, there exist $x^*(x) \in C^*$ such that*

$$\langle x^*(x), x - y\rangle_{V^*\times V} \ge \phi(y) - \phi(x).$$

Then there exists $y^* \in C^*$ such that

$$\langle y^*, x - y\rangle_{V^*\times V} \ge \phi(y) - \phi(x), \quad \forall x \in C.$$

3. Well-Posedness of SHVI with Metric Characterizations

In this section, we introduce the concept of well-posedness for a system of hemivariational inequalities **SHVI** and establish some metric characterization of well-posedness for **SHVI** under some conditions.

Definition 3.1. A sequence $\{\boldsymbol{u}^n\} \subset V_1 \times V_2$ with $\boldsymbol{u}^n = (u_1^n, u_2^n)$ is said to be an approximating sequence for **SHVI** if there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ when $n \to +\infty$ such that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \ge -\epsilon_n\|v_1 - u_1^n\|_{V_1}, & \forall v_1 \in V_1 \\ \langle A_2(u_1^n, u_2^n) - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \ge -\epsilon_n\|v_2 - u_2^n\|_{V_2}, & \forall v_2 \in V_2. \end{cases}$$

Definition 3.2. The system **SHVI** of hemivariational inequalities is said to be strongly (resp., weakly) well-posed if **SHVI** has a unique solution and every approximating sequence for **SHVI** converges strongly (resp., weakly) to the unique solution.

Remark 3.3. It is easy to see that the strong well-posedness of **SHVI** implies the weak well-posedness of **SHVI**. On the contrary, the conclusion is not true in general.

Definition 3.4. The system **SHVI** of hemivariational inequalities is said to be well-posed in generalized sense (or generalized well-posed) if the solution set of **SHVI** is nonempty and, for every approximating sequence, there always exists a subsequence converging to some point of the solution set.

Remark 3.5. Similarly, the strong well-posedness in generalized sense implies the weak well-posedness in generalized sense for **SHVI** while the converse dose not hold in general.

In order to establish the metric characterizations for the well-posedness of **SHVI**, we first define two sets in $V_1 \times V_2$ for any $\epsilon > 0$ as follows:

$$\Omega(\epsilon) = \Big\{(u_1, u_2) \in V_1 \times V_2 : \langle A_1(u_1, u_2) - f_1, v_1 - u_1\rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geqslant -\epsilon\|v_1 - u_1\|_{V_1},$$
$$\langle A_2(u_1, u_2) - f_2, v_2 - u_2\rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geqslant -\epsilon\|v_2 - u_2\|_{V_2}, \forall v_1 \in V_1, v_2 \in V_2\Big\}$$

and

$$\Psi(\epsilon) = \Big\{(u_1, u_2) \in V_1 \times V_2 : \langle A_1(v_1, u_2) - f_1, v_1 - u_1\rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geqslant -\epsilon\|v_1 - u_1\|_{V_1},$$
$$\langle A_2(u_1, v_2) - f_2, v_2 - u_2\rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geqslant -\epsilon\|v_2 - u_2\|_{V_2}, \forall v_1 \in V_1, v_2 \in V_2\Big\}.$$

In order to prove some properties of the sets $\Omega(\epsilon)$ and $\Psi(\epsilon)$, we first give some hypotheses on the operators A_1, A_2 and J in the **SHVI**.

(HA) (1) $A_1 : V_1 \times V_2 \to V_1^*$ is monotone with respect to the first variable.

(2) $A_2 : V_1 \times V_2 \to V_2^*$ is monotone with respect to the second variable.

(3) $A_1 : V_1 \times V_2 \to V_1^*$ is demicontinuous on $V_1 \times V_2$.

(4) $A_2 : V_1 \times V_2 \to V_2^*$ is demicontinuous on $V_1 \times V_2$.

(HJ) (1) J is locally Lipschitz with respective to first variable and second variable on $V_1 \times V_2$.

(2) For any $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$, $J(u_1, u_2) + J(v_1, v_2) = J(u_1, v_2) + J(v_1, u_2)$.

Lemma 3.6. *Let V_1, V_2 be two Banach spaces and V_1^*, V_2^* be their dual spaces, respectively. Suppose that the functional $J : V_1 \times V_2 \to \mathbb{R}$ satisfies the hypothesis* **(HJ)**. *Then, for any sequence $\boldsymbol{u}^n = (u_1^n, u_2^n) \in V_1 \times V_2$ converging to $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ and $v_i^n \in V_i$ converging to $v_i \in V_i$,*

$$\limsup_{n\to\infty} J_i^\circ(u_1^n, u_2^n; v_i^n) \leq J_i^\circ(u_1, u_2; v_i), \tag{3.1}$$

where $i = 1, 2$.

Proof. Let $\boldsymbol{u}^n = (u_1^n, u_2^n) \in V_1 \times V_2$ converge to $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ and, for $i = 1, 2$, $v_i^n \in V_i$ converges to $v_i \in V_i$. By Definition 2.16, Clarke's generalized directional derivative of $J(\cdot, u_2^n)$ at u_1^n in the direction v_1^n is formulated as

$$J_1^\circ(u_1^n, u_2^n; v_1^n) = \limsup_{w_1 \to u_1^n, t\downarrow 0} \frac{J(w_1 + tv_1^n, u_2^n) - J(w_1, u_2^n)}{t}.$$

For all $n \in \mathbb{N}$, by the definition of upper limit, there exist $w_1^n \in V_1$ and $t^n > 0$ such that

$$\|w_1^n - u_1^n\|_{V_1} + t^n < \frac{1}{n}$$

and

$$\frac{J(w_1^n + t^n v_1^n, u_2^n) - J(w_1^n, u_2^n)}{t^n} > J_1^\circ(u_1^n, u_2^n; v_1^n) - \frac{1}{n}. \tag{3.2}$$

In terms of hypothesis **(HJ)**, we have

$$\begin{aligned}
&\frac{J(w_1^n + t^n v_1^n, u_2^n) - J(w_1^n, u_2^n)}{t^n} \\
= \ &\frac{J(w_1^n + t^n v_1, u_2) - J(w_1^n, u_2)}{t^n} + \frac{J(w_1^n + t^n v_1^n, u_2^n) - J(w_1^n + t^n v_1, u_2) + J(w_1^n, u_2) - J(w_1^n, u_2^n)}{t^n} \\
= \ &\frac{J(w_1^n + t^n v_1, u_2) - J(w_1^n, u_2)}{t^n} + \frac{J(w_1^n + t^n v_1^n, u_2) - J(w_1^n + t^n v_1, u_2)}{t^n} \\
&+ \frac{J(w_1^n + t^n v_1^n, u_2^n) - J(w_1^n + t^n v_1^n, u_2) + J(w_1^n, u_2) - J(w_1^n, u_2^n)}{t^n} \\
\leq \ &\frac{J(w_1^n + t^n v_1, u_2) - J(w_1^n, u_2)}{t^n} + L_{u_1}\|v_1^n - v_1\|_{V_1},
\end{aligned} \tag{3.3}$$

where L_{u_1} is the locally Lipschitz constant of functional $J(\cdot, u_2)$ at u_1. It follows from (3.2) and (3.3) that

$$J_1^\circ(u_1^n, u_2^n; v_1^n) - \frac{1}{n} < \frac{J(w_1^n + t^n v_1, u_2) - J(w_1^n, u_2)}{t^n} + L_{u_1}\|v_1^n - v_1\|_{V_1}. \tag{3.4}$$

Taking upper limit $n \to \infty$ at both sides of above inequality (3.4) yields

$$\limsup_{n\to\infty} J_1^\circ(u_1^n, u_2^n; v_1^n) \leq J_1^\circ(u_1, u_2; v_1).$$

Similarly, we can prove that

$$\limsup_{n\to\infty} J_2^\circ(u_1^n, u_2^n; v_2^n) \leq J_2^\circ(u_1, u_2; v_2).$$

This completes the proof. □

Lemma 3.7. *Let V_1, V_2 be two Banach spaces and V_1^*, V_2^* be their dual spaces, respectively. Suppose that $A_1 : V_1 \times V_2 \to V_1^*$ and $A_2 : V_1 \times V_2 \to V_2^*$ satisfy the hypothesis* (**HA**) *and $J : V_1 \times V_2 \to \mathbb{R}$ is a locally Lipschitz functional satisfying* (**HJ**). *Then $\Omega(\epsilon) = \Psi(\epsilon)$ for any $\epsilon > 0$.*

Proof. It is obvious that, with the monotonicity of the operator A_1 with respective to first variable and the operator A_2 with second variable, we can easily prove that $\Omega(\epsilon) \subset \Psi(\epsilon)$ for any $\epsilon > 0$. Thus we only need to prove $\Psi(\epsilon) \subset \Omega(\epsilon)$. To this end, for any $\boldsymbol{u} = (u_1, u_2) \in \Psi(\epsilon)$, we have

$$\begin{cases} \langle A_1(v_1, u_2) - f_1, v_1 - u_1\rangle_{V_1^*\times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geqslant -\epsilon\|v_1 - u_1\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1, v_2) - f_2, v_2 - u_2\rangle_{V_2^*\times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geqslant -\epsilon\|v_2 - u_2\|_{V_2}, & \forall v_2 \in V_2. \end{cases} \tag{3.5}$$

For any $\boldsymbol{w} = (w_1, w_2) \in V_1 \times V_2$ and $t \in [0, 1]$, by letting $v_1 = u_1 + t(w_1 - u_1)$ and $v_2 = u_2 + t(w_2 - u_2)$ in (3.5), we get from (3.5) that

$$\begin{cases} \langle A_1(u_1 + t(w_1 - u_1), u_2) - f_1, t(w_1 - u_1)\rangle_{V_1^*\times V_1} + J_1^\circ(u_1, u_2; t(w_1 - u_1)) \geq -\epsilon t\|w_1 - u_1\|_{V_1}, \\ \langle A_2(u_1, u_2 + t(w_2 - u_2)) - f_2, t(w_2 - u_2)\rangle_{V_2^*\times V_2} + J_2^\circ(u_1, u_2; t(w_2 - u_2)) \geq -\epsilon t\|w_2 - u_2\|_{V_2}. \end{cases}$$

From the property (1) of Proposition 2.17, Clarke's generalized directional derivative is positively homogeneous with respect to its direction. Thus it follows that

$$\begin{cases} \langle A_1(u_1 + t(w_1 - u_1), u_2) - f_1, w_1 - u_1\rangle_{V_1^*\times V_1} + J_1^\circ(u_1, u_2; w_1 - u_1) \geq -\epsilon\|w_1 - u_1\|_{V_1}, \\ \langle A_2(u_1, u_2 + t(w_2 - u_2)) - f_2, w_2 - u_2\rangle_{V_2^*\times V_2} + J_2^\circ(u_1, u_2; w_2 - u_2) \geq -\epsilon\|w_2 - u_2\|_{V_2}. \end{cases} \tag{3.6}$$

From Remark 2.7, the hypothesis **(HA)** implies that operator A_1 and A_2 are hemicontinuous with respective to first variable and second variable respectively. Thus taking limit $t \to 0^+$ at both sides of two inequalities in (3.6) yields

$$\begin{cases} \langle A_1(u_1,u_2) - f_1, w_1 - u_1\rangle_{V_1^*\times V_1} + J_1^\circ(u_1,u_2;w_1-u_1) \geq -\epsilon\|w_1-u_1\|_{V_1}, \\ \langle A_2(u_1,u_2) - f_2, w_2 - u_2\rangle_{V_2^*\times V_2} + J_2^\circ(u_1,u_2;w_2-u_2) \geq -\epsilon\|w_2-u_2\|_{V_2}, \end{cases}$$

which together with the arbitrary of $\boldsymbol{w} = (w_1, w_2) \in V_1 \times V_2$ implies that $\Psi(\epsilon) \subset \Omega(\epsilon)$. This completes the proof. □

Lemma 3.8. *Let V_1 and V_2 be two reflexive Banach spaces and V_1^*, V_2^* be their dual spaces, respectively. Suppose that $A_1 : V_1 \times V_2 \to V_1^*$ satisfies the hypothesis* (3) *in* (**HA**)*, $A_2 : V_1 \times V_2 \to V_2^*$ satisfies the hypothesis* (4) *in* (**HA**)*, and $J : V_1 \times V_2 \to \mathbb{R}$ is a locally Lipschitz functional satisfying* (**HJ**)*. Then, for any $\epsilon > 0$, $\Psi(\epsilon)$ is closed in $V_1 \times V_2$.*

Proof. Let $\boldsymbol{u}^n = (u_1^n, u_2^n) \in \Psi(\epsilon)$ be a sequence converging to $\boldsymbol{u} = (u_1, u_2)$ in $V_1 \times V_2$. Then

$$\begin{cases} \langle A_1(v_1,u_2^n) - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} + J_1^\circ(u_1^n,u_2^n;v_1-u_1^n) \geq -\epsilon\|v_1-u_1^n\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1^n,v_2) - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} + J_2^\circ(u_1^n,u_2^n;v_2-u_2^n) \geq -\epsilon\|v_2-u_2^n\|_{V_2}, & \forall v_2 \in V_2. \end{cases} \tag{3.7}$$

By the hypotheses, A_1 and A_2 are demicontinuous on $V_1 \times V_2$. It follows from Proposition 2.5 that

$$\begin{aligned} &\lim_{n\to\infty} \langle A_1(v_1,u_2^n) - f_1, v_1 - u_1^n\rangle_{V_1^*\times V_1} = \langle A_1(v_1,u_2) - f_1, v_1 - u_1\rangle_{V_1^*\times V_1}, \\ &\lim_{n\to\infty} \langle A_2(u_1^n,v_2) - f_2, v_2 - u_2^n\rangle_{V_2^*\times V_2} = \langle A_2(u_1,v_2) - f_2, v_2 - u_2\rangle_{V_2^*\times V_2}. \end{aligned} \tag{3.8}$$

Moreover, by the hypothesis **(HJ)** on the functional J, Lemma 3.6 implies that

$$\begin{aligned} &\limsup_{n\to\infty} J_1^\circ(u_1^n,u_2^n;v_1-u_1^n) \leq J_1^\circ(u_1,u_2;v_1-u_1), \\ &\limsup_{n\to\infty} J_2^\circ(u_1^n,u_2^n;v_2-u_2^n) \leq J_2^\circ(u_1,u_2;v_2-u_2). \end{aligned} \tag{3.9}$$

Therefore, taking upper limit $n \to \infty$ at both sides of the inequality (3.7), it follows from (3.8) and (3.9) that

$$\begin{cases} \langle A_1(v_1,u_2) - f_1, v_1 - u_1\rangle_{V_1^*\times V_1} + J_1^\circ(u_1,u_2;v_1-u_1) \geq -\epsilon\|v_1-u_1\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1,v_2) - f_2, v_2 - u_2\rangle_{V_2^*\times V_2} + J_2^\circ(u_1,u_2;v_2-u_2) \geq -\epsilon\|v_2-u_2\|_{V_2}, & \forall v_2 \in V_2, \end{cases}$$

which implies that $\boldsymbol{u} = (u_1^n, u_2^n) \in \Psi(\epsilon)$. Thus $\Psi(\epsilon)$ is closed in $V_1 \times V_2$. This completes the proof. □

Remark 3.9. Let V_1 and V_2 be two reflexive Banach spaces with V_1^* and V_2^* being their dual spaces. Suppose that operators $A_1 : V_1 \times V_2 \to V_1^*$, $A_2 : V_1 \times V_2 \to V_2^*$ satisfy the hypothesis **(HA)** and the functional $J : V_1 \times V_2 \to \mathbb{R}$ satisfies the hypothesis **(HJ)**, we can easily get from Lemma 3.7 and Lemma 3.8 that $\Omega(\epsilon) = \Psi(\epsilon)$ is closed in $V_1 \times V_2$.

Theorem 3.10. *Let V_1 and V_2 be two reflexive Banach spaces and V_1^* and V_2^* be their dual spaces, respectively. Suppose that $A_1 : V_1 \times V_2 \to V_1^*$ satisfies the hypothesis* (3) *in* **(HA)***, $A_2 : V_1 \times V_2 \to V_2^*$ satisfies the hypothesis* (4) *in* **(HA)** *and the functional $J : V_1 \times V_2 \to \mathbb{R}$ satisfies the hypothesis* **(HJ)***. Then the system* **SHVI** *is strongly well-posed if and only if*

$$\Omega(\epsilon) \neq \emptyset \;\forall \epsilon > 0 \quad and \quad diam(\Omega(\epsilon)) \to 0 \; as \; \epsilon \to 0.$$

Proof. Necessity: Let **SHVI** be strongly well-posed. Then **SHVI** admits a unique solution $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$, *i.e.*,

$$\begin{cases} \langle A_1(u_1,u_2) - f_1, v_1 - u_1\rangle_{V_1^*\times V_1} + J_1^\circ(u_1,u_2;v_1-u_1) \geq 0, & \forall v_1 \in V_1, \\ \langle A_2(u_1,u_2) - f_2, v_2 - u_2\rangle_{V_2^*\times V_2} + J_2^\circ(u_1,u_2;v_2-u_2) \geq 0, & \forall v_2 \in V_2. \end{cases}$$

This implies that $\boldsymbol{u} \in \Omega(\epsilon)$ for any $\epsilon > 0$, i.e., $\Omega(\epsilon) \neq \emptyset$ for all $\epsilon > 0$. If $diam(\Omega(\epsilon)) \nrightarrow 0$ as $\epsilon \to 0$, then there exist $\boldsymbol{u}^n = (u_1^n, u_2^n), \boldsymbol{p}^n = (p_1^n, p_2^n) \in \Omega(\epsilon_n)$, $d > 0$ and $0 < \epsilon_n \to 0$ such that

$$\|\boldsymbol{u}^n - \boldsymbol{p}^n\|_{V_1 \times V_2} = \|u_1^n - p_1^n\|_{V_1} + \|u_2^n - p_2^n\|_{V_2} > d. \tag{3.10}$$

By the definition of the approximating sequence for **SHVI**, $\boldsymbol{u}^n$ and $\boldsymbol{p}^n$ are two approximating sequences. Thus it follows from the strong well-posedness of **SHVI** that both $\boldsymbol{u}^n$ and $\boldsymbol{p}^n$ converge to the unique solution $\boldsymbol{u}$, which contradicts (3.10).

Sufficiency: Let $\Omega(\epsilon) \neq \varnothing$ and $diam(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$. Then we prove that the system of hemivariational inequalities **SHVI** is strongly well-posed. To this end, suppose that $\{\boldsymbol{u}^n\}$ with $\boldsymbol{u}^n = (u_1^n, u_2^n)$ is an approximating sequence for **SHVI**. Then there exists $0 < \epsilon_n \to 0$ as $n \to \infty$, such that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon_n \|v_1 - u_1^n\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1^n, u_2^n) - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon_n \|v_2 - u_2^n\|_{V_2}, & \forall v_2 \in V_2, \end{cases}$$

which implies $\boldsymbol{u}^n \in \Omega(\epsilon_n)$ for all $n \geq 1$. Since $diam(\Omega(\epsilon_n)) \to 0$ as $n \to +\infty$, $\{\boldsymbol{u}^n\}$ is a Cauchy sequence in $V_1 \times v_2$. Without loss of generality, we suppose that $\{\boldsymbol{u}^n\}$ converges strongly to $\boldsymbol{u} = (u_1, u_2)$ in $V_1 \times V_2$.

Now, we prove that $\boldsymbol{u}$ is a unique solution to the system **SHVI** of hemivariational inequalities. First, since the operators A_1 and A_2 are demicontinuous on $V_1 \times V_2$ and the functional J satisfies the hypothesis **(HJ)**, we can get by similar arguments as (3.8) and (3.9) that

$$\begin{aligned} &\langle A_1(u_1, u_2) - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \\ &\qquad \geq \lim_{n\to\infty} \left\langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n \right\rangle_{V_1^* \times V_1} + \limsup_{n\to\infty} J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \\ &\qquad = \limsup_{n\to\infty} (\langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n)) \\ &\qquad \geq \lim_{n\to\infty} -\epsilon_n \|v_1 - u_1^n\|_{V_1} \\ &\qquad = 0. \end{aligned}$$

By the similar arguments, one has

$$\langle A_2(u_1, u_2) - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq 0.$$

Therefore, $\boldsymbol{u}$ is a solution to the system **SHVI**.

Second, we prove that $\boldsymbol{u}$ is the unique solution to the system **SHVI**. Suppose that $\boldsymbol{u}'$ is another solution to the system **SHVI** of hemivariational inequalities. Since, for any $\epsilon > 0$, $\boldsymbol{u}, \boldsymbol{u}' \in \Omega(\epsilon)$, $\|\boldsymbol{u} - \boldsymbol{u}'\|_{V_1 \times V_2} \leq diam(\Omega(\epsilon))$, which together with the condition $diam(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$ implies that $\boldsymbol{u} = \boldsymbol{u}'$. This completes the proof. □

Theorem 3.11. *Let V_1, V_2 be two Banach spaces and V_1^*, V_2^* be their dual spaces, respectively. Suppose that $A_1 : V_1 \times V_2 \to V_1^*$ and $A_2 : V_1 \times V_2 \to V_2^*$ are two operators on $V_1 \times V_2$ satisfying* **(HA)** *and the functional $J : V_1 \times V_2 \to \mathbb{R}$ satisfies the hypothesis* **(HJ)**. *Then the system* **SHVI** *is generalized well-posed if and only if*

$$\Omega(\epsilon) \neq \varnothing \text{ for any } \epsilon > 0 \text{ and } \mu(\Omega(\epsilon)) \to 0 \text{ as } \epsilon \to 0.$$

Proof. Necessity: Suppose that the system **SHVI** of hemivariational inequalities is generalized well-posed. Then the solution set of the system **SHVI**, $S \neq \emptyset$. This indicates that, for any $\epsilon > 0$, $\Omega(\epsilon) \neq \emptyset$ since $S \subset \Omega(\epsilon)$. Moreover, we claim here that the solution set S of the system **SHVI** is compact. In fact, for any sequence $\{\boldsymbol{u}^n\} \subset S$ with $\boldsymbol{u}^n = (u_1^n, u_2^n)$, $\boldsymbol{u}^n$ is an approximating sequence for **SHVI** and thus there exists a subsequence of $\{\boldsymbol{u}^n\}$ converging to some point of S, which implies that S is compact. To complete the proof of Necessity, we show that $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$. It follows from $S \subset \Omega(\epsilon)$ that

$$\mathcal{H}(\Omega(\epsilon), S) = max\{e(\Omega(\epsilon), S), e(S, \Omega(\epsilon))\} = e(\Omega(\epsilon), S).$$

Since the solution set S is compact, one has

$$\mu(\Omega(\epsilon)) \leq 2\mathcal{H}(\Omega(\epsilon), S) = 2e(\Omega(\epsilon), S).$$

Now, we prove $e(\Omega(\epsilon), S) \to 0$ as $\epsilon \to 0$ to obtian $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$. If not, there exists a constant $l > 0$, a sequence $\{\epsilon_n\} \subset R_+$ with $\epsilon_n \to 0$ and $u_n \in \Omega(\epsilon_n)$ such that

$$\boldsymbol{u}^n \notin S + B(0, l), \tag{3.11}$$

where $B(0, l)$ is an open ball with center 0 and radius l. However, $u_n \in \Omega(\epsilon_n)$ and $\epsilon_n \to 0$ imply that $\{\boldsymbol{u}^n\}$ is an approximating sequence for **SHVI**. It follows the generalized well-posedness of **SHVI** that $\{\boldsymbol{u}^n\}$ has a subsequence converges to some point of $\boldsymbol{u} \in S$, which contradicts (3.11).

Sufficiency: Assume that $\Omega(\epsilon) \neq \varnothing$ for all $\epsilon > 0$ and $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$. We prove the system **SHVI** of hemivariational inequalities is generalized well-posed. First of all, we observe that

$$S = \cap_{\epsilon>0}\Omega(\epsilon).$$

Furthermore, since $\mu(\Omega(\epsilon)) \to 0$ as $\epsilon \to 0$ and, by Remark 3.9, $\Omega(\epsilon)$ is closed for any $\epsilon > 0$, it follows from Theorem on page 412 of [10] that S is nonempty compact and

$$e(\Omega(\epsilon), S) = \mathcal{H}(\Omega(\epsilon), S) \to 0 \ as \ \epsilon \to 0. \tag{3.12}$$

Now, to prove the generalized well-posedness of **SHVI**, let $\boldsymbol{u}^n \in V_1 \times V_2$ with $\boldsymbol{u}^n = (u_1^n, u_2^n)$ be an approximating sequence for **SHVI**. It follows that there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n\rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon_n\|v_1 - u_1^n\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1^n, u_2^n) - f_2, v_2 - u_2^n\rangle_{V_2^* \times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon_n\|v_2 - u_2^n\|_{V_2}, & \forall v_2 \in V_2, \end{cases}$$

which implies $\boldsymbol{u}^n \in \Omega(\epsilon_n)$. This together with (3.12) indicates that

$$d(\boldsymbol{u}^n, S) \leq e(\Omega(\epsilon_n), S) \to 0.$$

Since S is compact, it follows that there exists $\boldsymbol{w}^n \in S$ with $\boldsymbol{w}^n = (w_1^n, w_2^n)$ such that

$$\|\boldsymbol{u}^n - \boldsymbol{w}^n\|_{V_1 \times V_2} = d(\boldsymbol{u}^n, S) \to 0.$$

Again, by the compactness of the solution set S, the sequence $\{\boldsymbol{w}^n\} \subset S$ has a subsequence $\{\boldsymbol{w}^{n_k}\}$ converging to some point $\boldsymbol{w}' \in S$. Thus it follows from

$$\|\boldsymbol{u}^{n_k} - \boldsymbol{w}'\|_{V_1 \times V_2} \leq \|\boldsymbol{u}^{n_k} - \boldsymbol{w}^{n_k}\|_{V_1 \times V_2} + \|\boldsymbol{w}^{n_k} - \boldsymbol{w}'\|_{V_1 \times V_2}$$

that the subsequence $\{\boldsymbol{u}^{n_k}\}$ of $\{\boldsymbol{u}^n\}$ converges to $\boldsymbol{w}'$. Therefore, the system **SHVI** is well-posedness in generalized sense. This completes the proof. □

4. Relations with Well-Posedness of SIP

In this section, we firstly introduce systems of inclusion problems on the product space $V_1 \times V_2$ and define the concept of well-posedness for the system of inclusion problems. Then, we prove the equivalence results between the well-posedness of the system of hemivariational inequalities and the well-posedness of the corresponding system of inclusion problems.

Let V_1 and V_2 be two Banach spaces with V_1^* and V_2^* being their dual spaces, respectively. Suppose that, for $i = 1, 2$, T_i is a set-value mapping from $V_1 \times V_2$ to V_i^*. A system of inclusion problems related to the mappings T_1 and T_2 is defined as follows:

Find $u_1 \in V_1$ and $u_2 \in V_2$ such that

$$\textbf{(SIP)} \quad \begin{cases} 0_1 \in T_1(u_1, u_2), \\ 0_2 \in T_2(u_1, u_2), \end{cases} \tag{4.1}$$

where, for $i = 1, 2$, $0_i \in V_i^*$ represents the zero element in V_i^*. For simplicity, we use the symbols as follows:

$$\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2, \ \ \mathbf{0} = (0_1, 0_2) \in V_1^* \times V_2^*, \ \ \boldsymbol{T}(\boldsymbol{u}) = (T_1(\boldsymbol{u}), T_2(\boldsymbol{u})) \in V_1^* \times V_2^*.$$

This allows us to simplify the system of inclusion problems as follows:

Find $\boldsymbol{u} \in V_1 \times V_2$ such that

$$\mathbf{0} \in \boldsymbol{T}(\boldsymbol{u}).$$

Definition 4.1. A sequence $\{\boldsymbol{u}^n\} \subset V_1 \times V_2$ with $\boldsymbol{u}^n = (u_1^n, u_2^n)$ is called an approximating sequence for the system **SIP** of inclusion problems if $d(\mathbf{0}, \boldsymbol{T}(\boldsymbol{u}^n)) \to 0$ or there exists a sequence $\boldsymbol{p}^n = (p_1^n, p_2^n) \in \boldsymbol{T}(\boldsymbol{u}^n)$ such that $\|\boldsymbol{p}^n\|_{V_1^* \times V_2^*} \to 0$ as $n \to \infty$.

Definition 4.2. The system **SIP** is said to be strongly (resp., weakly) well-posed if it has a unique solution and each approximating sequence converges strongly (resp., weakly) to the unique solution of $\textbf{SIP}(\boldsymbol{T})$.

Definition 4.3. The system **SIP** is said to be strongly (resp., weakly) well-posed in generalized sense (or generalized well-posed) if the solution set S of **SIP** is nonempty and each approximating sequence has a subsequence converging strongly (resp., weakly) to some point of solution set S.

In order to show that the well-posedness of the system of hemivariational inequalities is equivalent to the well-posedness of its corresponding system of inclusion problems, we first give a lemma which establishes the equivalence between the system **SHVI** and its derived system inclusion problems.

Lemma 4.4. *Let V_1, V_2 be two Banach spaces and V_1^* and V_2^* be their dual spaces, respectively. $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ is a solution to the system* **SHVI** *of hemivariational inequalities if and only if it solves the following derived system of inclusion problems:*

Find $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ such that

$$\textbf{(DSIP)} \quad \begin{cases} f_1 \in A_1(u_1, u_2) + \partial_1 J(u_1, u_2) \\ f_2 \in A_2(u_1, u_2) + \partial_2 J(u_1, u_2), \end{cases}$$

where, for $i \neq j = 1, 2$, $\partial_i J(u_1, u_2)$ denotes Clarke's generalized gradient of the functional $J(\cdot, u_j)$ at u_i.

Proof. First of all, we prove the necessity. To this end, assume that $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ is the solution to the system **SHVI** of hemivariational inequalities, i.e.,

$$\begin{cases} \langle A_1(u_1, u_2) - f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \geq 0, & \forall v_1 \in V_1, \\ \langle A_2(u_1, u_2) - f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2) \geq 0, & \forall v_2 \in V_2. \end{cases} \tag{4.2}$$

For any $w_1 \in V_1$, $w_2 \in V_2$, letting $v_1 = u_1 + w_1 \in V_1$, $v_2 = u_2 + w_2 \in V_2$ in (4.2) yields

$$\begin{cases} J_1^\circ(u_1, u_2; w_1) \geq \langle f_1 - A_1(u_1, u_2), w_1 \rangle_{V_1^* \times V_1}, \\ J_2^\circ(u_1, u_2; w_2) \geq \langle f_2 - A_2(u_1, u_2), w_2 \rangle_{V_2^* \times V_2}. \end{cases}$$

It follows from the arbitrary of $w_1 \in V_1$ and $w_2 \in V_2$ that

$$\begin{cases} f_1 \in A_1(u_1, u_2) + \partial_1 J(u_1, u_2), \\ f_2 \in A_2(u_1, u_2) + \partial_2 J(u_1, u_2), \end{cases}$$

which implies that $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ is the solution to the system **DSIP**.

Sufficiency. Suppose that $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ is the solution to the derived system **DSIP** of inclusion problems, i.e.,

$$\begin{cases} f_1 \in A_1(u_1, u_2) + \partial_1 J(u_1, u_2), \\ f_2 \in A_2(u_1, u_2) + \partial_2 J(u_1, u_2). \end{cases}$$

Then there exist $\xi_1 \in \partial_1 J(u_1, u_2)$ and $\xi_2 \in \partial_2 J(u_1, u_2)$ such that

$$f_1 = A_1(u_1, u_2) + \xi_1, \quad f_2 = A_2(u_1, u_2) + \xi_2. \tag{4.3}$$

By multiplying the above two equality (4.3) with $v_1 - u_1 \in V_1$ and $v_2 - u_2 \in V_2$, respectively, we can obtain, by the definition of Clarke's generalized gradient, that

$$\begin{aligned} \langle f_1, v_1 - u_1 \rangle_{V_1^* \times V_1} &= \langle A_1(u_1, u_2) + \xi_1, v_1 - u_1 \rangle_{V_1^* \times V_1} \\ &= \langle A_1(u_1, u_2), v_1 - u_1 \rangle_{V_1^* \times V_1} + \langle \xi_1, v_1 - u_1 \rangle_{V_1^* \times V_1} \\ &\leq \langle A_1(u_1, u_2), v_1 - u_1 \rangle_{V_1^* \times V_1} + J_1^\circ(u_1, u_2; v_1 - u_1) \end{aligned}$$

and

$$\begin{aligned} \langle f_2, v_2 - u_2 \rangle_{V_2^* \times V_2} &= \langle A_2(u_1, u_2) + \xi_2, v_2 - u_2 \rangle_{V_2^* \times V_2} \\ &= \langle A_2(u_1, u_2), v_2 - u_2 \rangle_{V_2^* \times V_2} + \langle \xi_2, v_2 - u_2 \rangle_{V_2^* \times V_2} \\ &\leq \langle A_2(u_1, u_2), v_2 - u_2 \rangle_{V_2^* \times V_2} + J_2^\circ(u_1, u_2; v_2 - u_2). \end{aligned}$$

Therefore, $\boldsymbol{u}$ is the solution of the system **SHVI**. This completes the proof. □

Theorem 4.5. *Let V_1, V_2 be two Banach spaces with V_1^* and V_2^* being their dual spaces, respectively. The system* **SHVI** *of hemivariational inequalities is strongly well-posed if and only if the derived system* **DSIP** *of inclusion problems is strongly well-posed.*

Proof. Necessity: Suppose that the system **SHVI** of hemivariational inequalities is strongly well-posed and thus there exists a unique $\boldsymbol{u} = (u_1, u_2) \in V_1 \times V_2$ solving **SHVI**. It follows from Lemma 4.4 that $\boldsymbol{u}$ is the unique solution to **DSIP**. To prove the well-posedness of **DSIP**, we let $\boldsymbol{u}^n = (u_1^n, u_2^n)$ be an approximating sequence for **DSIP** and prove that $\boldsymbol{u}^n \to \boldsymbol{u}$ as $n \to \infty$. Then there exists a sequence $\boldsymbol{p}^n = (p_1^n, p_2^n)$ such that, for $i = 1, 2$, $p_i^n \in A_i(u_1^n, u_2^n) - f_i + \partial_i J(u_1^n, u_2^n)$ and $\|\boldsymbol{p}_i^n\|_{V_i^*} \to 0$ as $n \to \infty$. It follows that

$$\begin{cases} p_1^n - A_1(u_1^n, u_2^n) + f_1 \in \partial_1 J(u_1^n, u_2^n), \\ p_2^n - A_2(u_1^n, u_2^n) + f_2 \in \partial_2 J(u_1^n, u_2^n). \end{cases}$$

In terms of the definition of Clarke's generalized gradient, one easily obtains

$$\begin{cases} J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq \langle -A_1(u_1^n, u_2^n) + f_1 + p_1^n, v_1 - u_1^n \rangle_{V_1^* \times V_1}, \quad \forall v_1 \in V_1, \\ J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq \langle -A_2(u_1^n, u_2^n) + f_2 + p_2^n, v_2 - u_2^n \rangle_{V_2^* \times V_2}, \quad \forall v_2 \in V_2, \end{cases}$$

which implies that

$$\begin{cases} J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) + \langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} \geq \langle p_1^n, v_1 - u_1^n \rangle_{V_1^* \times V_1} \\ \quad \geq -\|p_1^n\|_{V_1^*} \|v_1 - u_1^n\|_{V_1}, \quad \forall v_1 \in V_1, \\ J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) + \langle A_2(u_1^n, u_2^n) - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} \geq \langle p_2^n, v_2 - u_2^n \rangle_{V_2^* \times V_2} \\ \quad \geq -\|p_2^n\|_{V_2^*} \|v_2 - u_2^n\|_{V_2}, \quad \forall v_2 \in V_2. \end{cases} \tag{4.4}$$

Letting $\epsilon_n = max(\|p_1^n\|_{V_1^*}, \|p_2^n\|_{V_2^*})$, it follows from (4.4) that $\boldsymbol{u}^n$ is an approximating sequence for **SHVI** since $\epsilon^n \to 0$ as $n \to \infty$. By the well-posedness of **SHVI**, $\boldsymbol{u}^n$ strongly converges to the unique solution $\boldsymbol{u}$.

Sufficiency: Let the system **DSIP** of inclusion problems be strongly well-posed. Thus there exists a unique solution $\boldsymbol{u}$ to **DSIP**. In terms of Lemma 4.4, $\boldsymbol{u}$ is the solution to the system **SHVI** of hemivariational inequalities as well. We now suppose that $\boldsymbol{u}^n$ is an approximating sequence for **SHVI**, which implies that there exits $\epsilon_n > 0$ satisfying $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) \geq -\epsilon_n \|v_1 - u_1^n\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1^n, u_2^n) - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) \geq -\epsilon_n \|v_2 - u_2^n\|_{V_2}, & \forall v_2 \in V_2. \end{cases} \tag{4.5}$$

By virtue of Proposition 2.17, one observes that

$$\begin{cases} J_1^\circ(u_1^n, u_2^n; v_1 - u_1^n) = max\{\langle p_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} : p_1 \in \partial_1 J(u_1^n, u_2^n)\}, \\ J_2^\circ(u_1^n, u_2^n; v_2 - u_2^n) = max\{\langle p_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} : p_2 \in \partial_2 J(u_1^n, u_2^n)\}. \end{cases}$$

Thus there exist $p_1(u_1^n, u_2^n, v_1) \in \partial_1 J(u_1^n, u_2^n)$ and $p_2(u_1^n, u_2^n, v_2) \in \partial_2 J(u_1^n, u_2^n)$ such that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + \langle p_1(u_1^n, u_2^n, v_1), v_1 - u_1^n \rangle_{V_1^* \times V_1} \geq -\epsilon_n \|v_1 - u_1^n\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1^n, u_2^n) - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + \langle p_2(u_1^n, u_2^n, v_2), v_2 - u_2^n \rangle_{V_2^* \times V_2} \geq -\epsilon_n \|v_2 - u_2^n\|_{V_2}, & \forall v_2 \in V_2. \end{cases}$$

According to (4) of Proposition (2.17), both $\partial_1 J(u_1^n, u_2^n)$ and $\partial_2 J(u_1^n, u_2^n)$ are nonempty convex and bounded in V_1^*, V_2^* respectively, which indicates that, for $i = 1, 2$, the set $\{A_i(u_1^n, u_2^n) + p_i - f_i : p_i \in \partial_i J(u_1^n, u_2^n)\}$ is also nonempty, convex and bounded in V_i^*. Therefore, for $i = 1, 2$, it follows from Lemma (2.20) with $\phi_i(x) = \epsilon_n \|x - u_i^n\|_{V_i}$ that there exists $p_i^n \in \partial_i J(u_1^n, u_2^n)$, which is independent on v_i, such that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1, v_1 - u_1^n \rangle_{V_1^* \times V_1} + \langle p_1^n, v_1 - u_1^n \rangle_{V_1^* \times V_1} \geq -\epsilon_n \|v_1 - u_1^n\|_{V_1}, & \forall v_1 \in V_1, \\ \langle A_2(u_1^n, u_2^n) - f_2, v_2 - u_2^n \rangle_{V_2^* \times V_2} + \langle p_2^n, v_2 - u_2^n \rangle_{V_2^* \times V_2} \geq -\epsilon_n \|v_2 - u_2^n\|_{V_2}, & \forall v_2 \in V_2. \end{cases} \tag{4.6}$$

In particular, for any $w_1 \in V_1$ and $w_2 \in V_2$, letting $v_1 = u_1^n - w_1$, $v_2 = u_2^n - w_2$ in the above inequality (4.6), one obtains that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1 + p_1^n, -w_1 \rangle_{V_1^* \times V_1} \geq -\epsilon_n \|w_1\|_{V_1}, \\ \langle A_2(u_1^n, u_2^n) - f_2 + p_2^n, -w_2 \rangle_{V_2^* \times V_2} \geq -\epsilon_n \|w_2\|_{V_2}, \end{cases}$$

which together with the arbitrary of $w_1 \in V_1$ and $w_2 \in V_2$ implies that

$$\begin{cases} \langle A_1(u_1^n, u_2^n) - f_1 + p_1^n, w_1 \rangle_{V_1^* \times V_1} \leq \epsilon_n \|w_1\|_{V_1}, & \forall w_1 \in V_1, \\ \langle A_2(u_1^n, u_2^n) - f_2 + p_2^n, w_2 \rangle_{V_2^* \times V_2} \leq \epsilon_n \|w_2\|_{V_2}, & \forall w_2 \in V_2. \end{cases}$$

Thus we have

$$\begin{cases} \|A_1(u_1^n, u_2^n) - f_1 + p_1^n\|_{V_1^*} \leq \epsilon_n \to 0, \\ \|A_2(u_1^n, u_2^n) - f_2 + p_2^n\|_{V_2^*} \leq \epsilon_n \to 0. \end{cases}$$

Moreover, since

$$A_1(u_1^n, u_2^n) - f_1 + p_1^n \in A_1(u_1^n, u_2^n) - f_1 + \partial_1 J(u_1^n, u_2^n)$$

and

$$A_2(u_1^n, u_2^n) - f_2 + p_2^n \in A_2(u_1^n, u_2^n) - f_2 + \partial_2 J(u_1^n, u_2^n),$$

the sequence $\{\boldsymbol{u}^n\}$ with $\boldsymbol{u}^n = (u_1^n, u_2^n)$ is an approximating sequence for **DSIP**. Now, it follows from the well-posedness of **DSIP** that $\{\boldsymbol{u}^n\}$ converges to the unique solution $\boldsymbol{u}$ in $V_1 \times V_2$. Therefore, the system of hemivariational inequalities **SHVI** is strongly well-posed. This completes the proof. □

With the similar arguments in the proof of Theorem 4.5, one can easily prove the following equivalence between the generalized well-posedness of **SHVI** and the generalized well-posedness of the system **DSIP**.

Theorem 4.6. *Let V_1, V_2 be two Banach spaces and V_1^*, V_2^* being their dual spaces, respectively. The system* **SHVI** *of hemivariational inequalities is strongly well-posed in generalized sense if and only if the derived system* **DSIP** *of inclusion problems is strongly well-posed in generalized sense.*

5. Concluding Remarks

The present paper generalizes the concept of well-posedness to a system **SHVI** of hemivariational inequalities in Banach space. Firstly, we give several definitions of well-posedness and, with two assumptions on the operators involved in **SHVI**, establish some metric characterizations of well-posedness for **SHVI** considered. Then, by introducing an equivalence result between the system **SHVI** and a derived system **DSIP** of inclusion problems, we prove that the strong (generalized) well-posedness for **SHVI** is equivalent to the strong (generalized) well-posedness for its **DSIP**.

Several problems related to the well-posedness of systems of hemivariational inequalities remain to be considered in the future study. The first one is to exploit some conditions under which the strong (weak) well-posedness of systems of hemivariational inequalities is equivalent to the existence and uniqueness of their solution. The second one is to generalize the study of well-posedness to systems of hemivariational inequalities involving both nonsmooth functionals and proper, convex and lower semi-continuous functionals, which are referred to as systems of variational-hemivariational inequalities. Finally, there are many other concepts of well-posedness in the literature on optimization problems and variational inequalities, such as α-well-posedness and Levitin-Polyak well-posedness. Extending these concepts of well-posedness to the study of systems of hemivariational inequalities would be interesting in the future.

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