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Equivalence Relations Induced by Interval Valued (S, T)-fuzzy h-ideals (k-ideals) of Semirings

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Abstract: We introduce the notion of interval valued intutionistic (S,T)-fuzzy left h-ideal (k-ideal) of a hemiring (semiring). We describe some characteristic properties and connections. Finally, we construct some equivalence relations induced by interval valued intutionistic (S,T)-fuzzy left h-ideals (k-ideals).

Key words: Semiring . hemiring . interval valued intuitionistic (S,T)-fuzzy h-ideal (k-ideal) . fuzzy characteristic

INTRODUCTION

Semirings and hemirings (semirings with zero and commutative addition) appear in a natural manner in some applications to the theory of automata and formal languages [1-3]. It is a well known result that regular languages form so-called star semirings. According to the well known theorem of Kleene, the languages, or sets of words, recognized by finite-state automata are precisely those that are obtained from letters of input alphabets by application of the operations: sum (union), concatenation (product) and Kleene star (Kleene closure). If a language is represented as a formal series with the coefficients in a Boolean hemiring, then the Kleene theorem can be well described by the Kleen-Schutzenberger theorem. Moreover, if the coefficient hemiring (semiring) is a field, then a series is rational if and only if its syntactic algebra [2-4] for details) has a finite rank. The class of K-fuzzy semirings ($K\cup\{+\infty\}$, min, max), where K denotes the subset of the power set of R which is closed under operations min, +, or max, has many interesting applications. Min-max-plus computation (and suitable semirings) are used in several areas, e.g., in differential equations. Many other applications with references can be found in [2] and in a guide to the literature on semirings and their applications [5]. Ideals of hemirings and semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogous in hemirings and semirings using only ideals. Henriksen [6] defined in a more restricted class of ideals in semiring, which is called the class of k-ideals, with the property that if the semiring R is a ring then a complex in R is a k-ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals in hemirings, called now h-ideals, has been given and investigated by Izuka [7] and La Torre [8]. It is interesting that the regularity of hemirings can be characterized by fuzzy h-ideals [9]. General properties of fuzzy k-ideals are described in [10-12]. Other important results connected with fuzzy h-ideals in hemirings were obtained in [13, 14].

After the introduction of fuzzy sets by Zadeh [15], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [16] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [16-18]. In 1975, Zadeh [19] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are intervals of numbers instead of the numbers. Such fuzzy sets have some applications in the technological scheme of the functioning of a silo-farm, with pneumatic transportation, in a plastic products company and in medicine (for more details see the book [18]. It is interesting to observe that the fuzzy calculus, used for artificial intelligence purposes, indeed involves essentially (min, max)-semirings which (in some sense) gives the impulse to the study of intuitionistic fuzzy sets in semirings. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, real analysis, measure theory etc. Also the notion of fuzzy ideals in rings and semirings have seriously studied by many mathematicians [13, 20, 22]. Recently, some researchers are trying to present new views of fuzzy algebraic structures as intuitionistic fuzzy algebraic structures [13, 23-31].

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In this paper on the basis of the concept of the interval valued intuitionistic fuzzy sets introduced by K. Atanassov [17, 18] we introduce the notion of interval valued intuitionistic fuzzy h-ideal (k-ideal) of a hemiring (semiring) with respect to t-norm T and s-norm S. Then we characterize all of them based on special kind of levels $U(\tilde{M}_A;[t,s])$ and $L(\tilde{N}_A;[t,s])$, which is a generalization of classic level subsets. Then we investigate their properties and connections with left h-ideals (k-ideals) of the corresponding hemirings (semirings). At the following the behavior of these structures under homomorphisms is investigated. Also we describe the relationship between some natural equivalence relations and induced partitions on the set of all interval valued intutionistic (S,T)-fuzzy h-ideals and k-ideals. In particular, we see that there is a bijective map between partitions and the set of all interval valued intuitionistic (S,T)-fuzzy h-ideals and k-ideals of hemiring (semiring) R.

PRELIMINARIES AND NOTATIONS

By a semiring we mean an algebraic system (R, +,.) consisting of a nonempty set R together with binary operations on R called add and multiplication such that (R, +) and (R,.) are semigroups and for all $x,y,z\in R$, we have $x(y+z) = x.y+x.z$ and $(y+z).x = y.x+z.x$ which are called distributive. By a zero we mean an element $0\in R$ such that $0.x = 0 = x.0$ and $0+x = x = x+0$ for all $x\in R$. A semiring with zero and a commutative semigroup (R, +) is called a hemiring. By a subsemring of a semiring R we mean a non-empty subset S of R such that for all $x,y\in S$, we have $x,y\in S$ and $x+y\in S$. By a left (right) ideal of a semiring we mean a subsemiring I of R such that for all $y\in R$ and $x\in I$ we have $x.y\in I(x.y\in I)$. By an ideal, we mean a subsemiring of R which both a left and right ideal of R. A left (right) ideal I of a semiring of R is called left (right) k-ideal if $y\in I(x.y\in I)$. By an ideal, we mean a subsemiring of R which both a left and a right ideal of R. A left (right) ideal I of a semiring R is called left (right) k-ideal if $y\in I$ and $x+y\in I$ imply that $x\in I$. Also a left (right) ideal I of a hemiring R is called left (right) h-ideal if for any $x,z\in R$ and $a,b\in I$ from $x+a+z = b+z$ it follows $x\in I$ [2, 8].

A fuzzy set $\mu:R\to[0,1]$ is called a fuzzy left ideal of semiring R if for all $x,y\in R$ we have $\mu(x+y)\geq\min\{\mu(x),\mu(y)\}$ and $\mu(xy)\geq\mu(y)$. A fuzzy left ideal μ is called a fuzzy left k-ideal of a semiring R if for all $x,y\in R$ we have $\mu(x)\geq\min\{\mu(y),\mu(x+y)\}$. Also a fuzzy left ideal μ is called a fuzzy left h-ideal of a hemiring R if for all $a,b,x,z\in R$ from $x+a+z = b+z$ it follows [10-12]:

$$\mu(x)\geq\min\{\mu(a),\mu(b)\}$$

By an interval number $\tilde{a}$ we mean [19] an interval $[a^-, a^+]$, where $0\leq a^-\leq a^+\leq 1$. The set of all interval numbers is denoted by D[0,1]. The interval [a,a] is identified with the number $a\in[0,1]$. For interval numbers

$$\tilde{a}_i=[a_i^-,a_i^+]\in D[0,1], i\in I,$$

we define

$$\inf\tilde{a}_i=[\wedge_{i\in I}a_i^-,\wedge_{i\in I}a_i^+],\ \sup\tilde{a}_i=[\vee_{i\in I}a_i^-,\vee_{i\in I}a_i^+]$$

and put

(1) $\tilde{a}_1\leq\tilde{a}_2\Leftrightarrow a_1^-\leq a_2^-$ and $a_1^+\leq a_2^+$,

(2) $\tilde{a}_1=\tilde{a}_2\Leftrightarrow a_1^-=a_2^-$ and $a_1^+=a_2^+$,

(3) $\tilde{a}_1<\tilde{a}_2\Leftrightarrow\tilde{a}_1\leq\tilde{a}_2$ and $\tilde{a}_1\neq\tilde{a}_2$,

(4) $k\tilde{a}=[ka^-,ka^+]$, whenever $0\leq k\leq 1$.

It is clear that $(D[0,1],\leq,\vee,\wedge)$ is a complete lattice with 0 = [0,0] as the least element and 1 = [1,1] as the greatest element.

By an interval valued fuzzy set F on X we mean [11] the set

$$F=\{(x,[\mu_F^-(x),\mu_F^+(x)])\mid x\in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subsets of X such that $\mu_F^-(x)\leq\mu_F^+(x)$ for all $x\in X$. Putting

$$\mu_F(x)=[\mu_F^-(x),\mu_F^+(x)],$$

we see that $F=\{(x,\mu_F(x))\mid x\in X\}$, where $\mu_F:X\to D[0,1]$. As it is well-known, the function

$$\delta:[0,1]\times[0,1]\to[0,1]$$

is called a t-norm (resp. s-norm) if δ satisfies the conditions:

(i) $\delta(x,1)=x$ (resp. $\delta(x,0)=x$)

(ii) $\delta(x,y)=\delta(y,x)$

(iii) $\delta(\delta(x,y),z)=\delta(x,\delta(y,z))$

(iv) $\delta(x,u)\leq\delta(x,w)$

for all $x,y,z,u,w\in[0,1]$, where $u\leq w$. A t-norm (resp. s-norm) is called idempotent if $\delta(x,x) = x$, for all $x\in[0,1]$ [19].

If δ is an idempotent t-norm (s-norm), then the mapping Δ: $D[0,1]\times D[0,1]\to D[0,1]$ defined by $\Delta(\tilde{a}_1, \tilde{a}_2) = [\delta(a_1^-, a_2^-), \delta(a_1^+, a_2^+)]$ is, as it is not difficult to verify, an idempotent t-norm (s-norm, respectively) and is called an idempotent interval t-norm (s-norm, respectively).

According to Atanassov [17, 18] an interval valued intuitionistic fuzzy set on X is defined as an object of the from

$$A = \{x, \tilde{M}_A(x), \tilde{N}_A(x)) \mid x \in X\}$$

where $\tilde{M}_A$ and $\tilde{N}_A$ are interval valued fuzzy sets on X such that

$$[0,0] \le \tilde{M}_A(x) + \tilde{N}_A(x) \le [1,1]$$

for all $x\in X$. For the sake of simplicity, in the following such interval intuitionistic fuzzy sets will be denoted by $A = (\tilde{M}_A, \tilde{N}_A)$.

INTERVAL VALUED INTUITIONISTIC (S,T)-FUZZY h-IDEALS (k-Ideals) OF Semirings

In the sequel, T and S denote idempotent interval t-norm and s-norm respectively, unless otherwise specified.

Definition 3.1: An interval valued intuitionistic fuzzy set $A = (\tilde{M}_A, \tilde{N}_A)$ of a hemiring R is called an interval valued intuitionistic (S,T)-fuzzy left h-ideal if for all $x,y,z,a,b \in R$:

(1) $\tilde{M}_A(x+y) \ge T(\tilde{M}_A(x), \tilde{M}_A(y))$,

(2) $\tilde{N}_A(x+y) \le S(\tilde{N}_A(x), \tilde{N}_A(y))$,

(3) $\tilde{M}_A(xy) \ge \tilde{M}_A(y)$,

(4) $\tilde{N}_A(xy) \le \tilde{N}_A(y)$,

(5) $x + a + z = b + z$ implies

$$\tilde{M}_A(x) \ge T(\tilde{M}_A(a), \tilde{M}_A(b)),$$

(6) $x + a + z = b + z$ implies

$$\tilde{N}_A(x) \le S(\tilde{N}_A(a), \tilde{N}_A(b)).$$

An interval valued intuitionistic fuzzy set $A = (\tilde{M}_A, \tilde{N}_A)$ satisfying the first four conditions is called an interval valued intuitionistic fuzzy left ideal.

Definiton 3.2: An interval valued intuitionistic fuzzy left ideal $A = (\tilde{M}_A, \tilde{N}_A)$ of a semiring R is called an interval valued intuitionistic (S,T)-fuzzy left k-ideal if for all $x,y \in R$:

(1) $\tilde{M}_A(x) \ge T(\tilde{M}_A(y), \tilde{M}_A(x+y))$,

(2) $\tilde{N}_A(x) \le S(\tilde{N}_A(y), \tilde{N}_A(x+y))$.

Clearly every interval valued intuitionistic (S,T)-fuzzy left h-ideal is an interval valued intuitionistic (S,T)-fuzzy left k-ideal, but following example shows that in general case the converse is not true.

Example: On a four element semiring (R,+,.) defined by the following two tables:

+	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Consider the interval valued fuzzy set $A = (\tilde{M}_A, \tilde{N}_A)$ as follows

$$\tilde{M}_A(x) = \begin{cases} [0.4, 0.5], & \text{if } x = 0 \\ [0.2, 0.25], & \text{if } x \ne 0 \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [0.2, 0.3], & \text{if } x = 0 \\ [0.7, 0.75], & \text{if } x \ne 0 \end{cases}$$

It is easy to verify that $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy k-ideal. But $A = (\tilde{M}_A, \tilde{N}_A)$ is not an interval valued intuitionistic (S,T)-fuzzy h-ideal. Because a+0+b = 0+b and

$$\tilde{M}_A(a) = [0.2, 0.25] \not\ge [0.4, 0.5] = T([0.4,0.5],[0.4,0.5]) = T(\tilde{M}_A(0), \tilde{M}_A(0))$$

Example: Let N^* be the set of non-negative integers and

$$\tilde{M}_A(x) = \begin{cases} [0.9, 0.95] & \text{if } x \in <4> \\ [0.4, 0.5] & \text{if } x \in <2> - <4> \\ [0, 0.05] & \text{otherwise} \end{cases}$$

$$\tilde{N}_A(x)=\begin{cases}[0,0.05] & \text{if } x\in<4>\\ [0.4,0.5] & \text{if } x\in<2>-<4>\\ [0.9,0.95] & \text{otherwise}\end{cases}$$

where <n> denotes the set of all integers divided by n. It is routine to calculate that $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy h-ideal of hemiring $(N^*,+,0)$. Therefore $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy k-ideal.

With any interval valued intuitionistic fuzzy set $A=(\tilde{M}_A,\tilde{N}_A)$ of a hemiring (semiring) R there are connected two levels:

$$U(\tilde{M}_A;[t,s])=\{x\in R \mid \tilde{M}_A(x)\geq[t,s]\}$$

$$L(\tilde{N}_A;[t,s])=\{x\in R \mid \tilde{N}_A(x)\leq[t,s]\}$$

Theorem 3.3: $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of hemiring (semiring) R if and only if for all $t,s\in[0,1], t\leq s, U(\tilde{M}_A;[t,s])$ and $L(\tilde{N}_A;[t,s])$ are left h-ideals (k-ideals) of hemiring (semiring) R.

Proof: Let $A=(\tilde{M}_A,\tilde{N}_A)$ be an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R. Then for every $x,y\in U(\tilde{M}_A;[t,a])$ and r∈R we have:

$$\tilde{M}_A(x)\geq[t,s] \text{ and } \tilde{M}_A(x)\geq[t,s].$$

Hence

$$T(\tilde{M}_A(x),\tilde{M}_A(y))\geq T([t,s],[t,s])=[t,s],$$

and so

$$\tilde{M}_A(x+y)\geq T(\tilde{M}_A(x),\tilde{M}_A(y))\geq[t,s].$$

This means $x+y\in U(\tilde{M}_A;[t,s])$. Also

$$\tilde{M}_A(rx)\geq \tilde{M}_A(x)\geq[t,s].$$

This means $rx\in U(\tilde{M}_A;[t,s])$. Therefore $U(\tilde{M}_A;[t,s])$ is a left ideal of R.

Suppose that $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal, $x,z\in R, a,b\in U(\tilde{M}_A;[t,s])$ and $x+a+z=b+z$. Then $\tilde{M}_A(a)\geq[t,s]$ and so $\tilde{M}_A(b)\geq[t,s]$. Also

$$\tilde{M}_A(x)\geq T(\tilde{M}_A(a),\tilde{M}_A(b))\geq[t,s].$$

Thus $x\in U(\tilde{M}_A;[t,s])$ and so $U(\tilde{M}_A;[t,s])$ is a left h-ideal of hemiring R. Similarly we can show that $U(\tilde{M}_A;[t,s])$ is a left h-ideal of hemiring R. Similarly we can show that $L(\tilde{M}_A;[t,s])$ is a left h-ideal of hemiring R.

Now suppose that $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left k-ideal,

$$x\in R, a\in U(\tilde{M}_A;[t,s]),\ x+a\in U(\tilde{M}_A;[t,s]).$$

Then $\tilde{M}_A(a)\geq[t,s]$ and $\tilde{M}_A(x+a)\geq[t,s]$. Also

$$\tilde{M}_A(x)\geq T(\tilde{M}_A(a),\tilde{M}_A(x+a))\geq[t,s].$$

Thus $x\in U(\tilde{M}_A;[t,s])$ and so $U(\tilde{M}_A;[t,s])$ is a left k-ideal of semiring R. Similarly we can show that $L(\tilde{M}_A;[t,s])$ is a left k-ideal of semiring R.

Conversely, assume that for every [t,s]∈D[0,1] any non-empty $U(\tilde{M}_A;[t,s])$ is a left h-ideal (k-ideal) of R. If $[t_0,s_0]=T(\tilde{M}_A(x),\tilde{M}_A(y))$ for some x,y∈R, then $x,y\in U(\tilde{M}_A;[t_0,s_0])$ and so

$$x+y\in U(\tilde{M}_A;[t_0,s_0]).$$

Therefore

$$\tilde{M}_A(x+y)\geq[t_0,s_0]=T(\tilde{M}_A(x),\tilde{M}_A(y)).$$

Also if

$$[t'_0,s'_0]=\tilde{M}_A(x),$$

then

$$x\in U(\tilde{M}_A;[t'_0,s'_0]),$$

and so $rx\in U(\tilde{M}_A;[t'_0,s'_0])$ for every r∈R. Therefore $\tilde{M}_A(rx)\geq[t'_0,s'_0]=\tilde{M}_A(x)$. This proves that $\tilde{M}_A$ is an interval valued T-fuzzy left ideal of R. Similarly, we can show that $\tilde{N}_A$ is an interval valued S-fuzzy left ideal of R. Therefore $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left ideal of R.

Suppose that for every [t,s]∈D[0,1] any nonempty $U(\tilde{M}_A;[t,s])$ is a left h-ideal, $x,z\in R$, $x+a+z=b+z$ and

$$T(\tilde{M}_A(a),\tilde{M}_A(b))=[t_0,s_0].$$

Thus $a,b\in U(\tilde{M}_A;[t_0,s_0])$ and so

$$x\in U(\tilde{M}_A;[t_0,s_0])$$

This means

$$\tilde{M}_A(x) \geq [t_0, s_0] = T(\tilde{M}_A(a), \tilde{M}_A(b)).$$

Therefore $\tilde{M}_A$ is an interval valued T-fuzzy left h-ideal of hemiring R. Similarly we can show that $\tilde{N}_A$ is an interval valued S-fuzzy left h-ideal of hemiring R. Therefore $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal of hemiring R.

Now suppose that for every $[t,s] \in D[0,1]$ any nonempty $U(\tilde{M}_A;[t,s])$ is a left k-ideal, $a,x \in R$ and

$$T(\tilde{M}_A(a), \tilde{M}_A(x+a)) = [t_0, s_0],$$

then $a, a+x \in U(\tilde{M}_A;[t_0, s_0])$ and so $x \in U(\tilde{M}_A;[t_0, s_0])$. This means $\tilde{M}_A$ is an interval valued T-fuzzy left k-ideal of semiring R. Similarly, we can show that $\tilde{N}_A$ is an internal valued S-fuzzy left k-ideal of R. Therefore $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left k-ideal of semiring R.

Let $A = (\tilde{M}_A, \tilde{N}_A)$ be an interval valued intuitionistic fuzzy set of semiring R and let t, s, $t', s' \in [0,1]$ such that $t \leq s$ and $t' \leq s'$. Put

$$R^{[t,s]}_{[t',s']} = \{x \in R \mid \tilde{M}_A(x) \geq [t,s], \tilde{N}_A(x) \leq [t',s']\}.$$

Clearly

$$R^{[t,s]}_{[t',s']} = U(\tilde{M}_A;[t,s]) \cap L(\tilde{N}_A;[t',s']).$$

Corollary 3.4: $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of hemiring (semiring) R if and only if for all $t, s, t', s' \in [0,1], t \leq s, t' \leq s', R^{[t,s]}_{[t',s']}$ is a left h-ideal (k-ideal) of hemiring (semiring) R.

Proof: It is immediately by Theorem 3.3.

Definiton 3.5: Let f: X→Y be a mapping and $A = (\tilde{M}_A, \tilde{N}_A)$ and $B = (\tilde{M}_B, \tilde{N}_B)$ interval valued intuitionistic sets of X and Y, respectively. Then the image

$$f[A] = (f(\tilde{M}_A), f(\tilde{N}_A))$$

of A is an interval valued intuitionistic fuzzy set of Y defined by

$$f(\tilde{M}_A)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} \tilde{M}_A(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ [0,0] & \text{otherwise} \end{cases}$$

$$f(\tilde{N}_A)(y) = \begin{cases} \inf_{z \in f^{-1}(y)} \tilde{N}_A(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ [1,1] & \text{otherwise} \end{cases}$$

for all $y \in Y$. The inverse image

$$f^{-1}(B) = (f^{-1}(\tilde{M}_B), f^{-1}(\tilde{N}_B))$$

of B is an interval valued intuitionistic fuzzy set of X defined by

$$f^{-1}(\tilde{M}_B)(x) = \tilde{M}_{f^{-1}(B)}(x) = \tilde{M}_B(f(x)),$$
$$f^{-1}(\tilde{N}_B)(x) = \tilde{N}_{f^{-1}(B)}(x) = \tilde{N}_B(f(x))$$

for all $x \in X$.

Definition 3.6: Let R_1 and R_2 be two hemirings (semirings). A mapping f: $R_1 \to R_2$ is called a homomorphism if for all $x,y \in R_1$ we have f(x+y) = f (x) + f (y) and f(x.y) = f (x).f (y).

Lemma 3.7: Let R_1 and R_2 be two hemirings (semirings) and f: $R_1 \to R_2$ an epimorphism.

(i) If I is a left h-ideal (k-ideal) of R_1, then f(I) is a left h-ideal (k-ideal) of R_2.
(ii) If J is a left h-ideal (k-ideal) of R_2, then f^{-1} (J) is a left h-ideal (k-ideal) of R_1.

Proof: Straightforward.

Theorem 3.8: Let R_1 and R_2 be two hemirings (semirings) and f: $R_1 \to R_2$ an epimorphism.

(i) If $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_1, then the image $f[A] = (f(\tilde{M}_A), f(\tilde{N}_A))$ of A is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_2.
(ii) If $B = (\tilde{M}_B, \tilde{N}_B)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_2, then the inverse image

$$x \in U(\tilde{M}_A;[\gamma, \gamma']$$

is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_1.

Proof: (i) Let $A=(\tilde{M}_A,\tilde{N}_A)$ be an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_1. By Theorem 3.3, $U(\tilde{M}_A;[t,s])$ and $L(\tilde{N}_A;[t,s])$ are left h-ideals (k-ideals) of R_1 for every $[t,s]\in D[0,1]$. Therefore, by Lemma 3.7, $f(U(\tilde{M}_A;[t,s]))$ and $f(L(\tilde{N}_A;[t,s]))$ are left h-ideals (k-ideals) of R_2. But

$$U(f(\tilde{M}_A);[t,s])=f(U(\tilde{M}_A;[t,s]))$$

and

$$L(f(\tilde{N}_A);[t,s])=f(L(\tilde{N}_A;[t,s])),$$

so the levels

$$U(f(\tilde{M}_A);[t,s]) \text{ and } L(f(\tilde{N}_A);[t,s])$$

are left h-ideals (k-ideals) of R_2. Therefore f[A] is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_2.

(ii) For any $x,y\in R_1$, we have

$$\tilde{M}_{f^{-1}(B)}(x+y)=\tilde{M}_B(f(x+y))\geq T(\tilde{M}_B(f(x)),$$
$$\tilde{M}_B(f(y)))=T(\tilde{M}_{f^{-1}(B)}(x),\tilde{M}_{f^{-1}(B)}(y)).$$

Also we have

$$\tilde{M}_{f^{-1}(B)}(xy)=\tilde{M}_B(f(xy))=\tilde{M}_B(f(x)f(y))\geq$$
$$\tilde{M}_B(f(y))=\tilde{M}_{f^{-1}(B)}(y).$$

This proves that $\tilde{M}_{f^{-1}(B)}$ is an interval valued T-fuzzy left ideal of R_1. Similarly we can show that $\tilde{N}_{f^{-1}(B)}$ is an interval valued S-fuzzy left ideal of R_1. Therefore

$$f^{-1}(B)=(f^{-1}(\tilde{M}_B),f^{-1}(\tilde{N}_B))$$

is an interval valued intuitionistic (S,T)-fuzzy left ideal of R_1.

Let $B=(\tilde{M}_B,\tilde{N}_B)$ be an interval valued intuitionistic (S,T)-fuzzy left h-ideal and $x,z,a,b\in R_1$. If $x+a+z=b+z$ then $f(x)+f(a)+f(z)=f(b)+f(z)$ and we have

$$\tilde{M}_{f^{-1}(B)}(x)=\tilde{M}_B(f(x))\geq T(\tilde{M}_B(f(a)),$$
$$\tilde{M}_B(f(b)))=T(\tilde{M}_{f^{-1}(B)}(a),\tilde{M}_{f^{-1}(B)}(b)).$$

So $\tilde{M}_{f^{-1}(B)}$ is an interval valued T-fuzzy left h-ideal of R_1. Similarly we can prove that $\tilde{N}_{f^{-1}(B)}$ is an interval valued S-fuzzy left h-ideal of R_1. Therefore

$$f^{-1}(B)=(f^{-1}(\tilde{M}_B),f^{-1}(\tilde{N}_B))$$

is an interval valued intuitionistic (S,T)-fuzzy left h-ideal of R_1.

Also if $B=(\tilde{M}_B,\tilde{N}_B)$ is an interval valued intuitionistic (S,T)-fuzzy left k-ideal, for any $x,a\in R_1$ we have

$$\tilde{M}_{f^{-1}(B)}(x)=\tilde{M}_B(f(x))\geq T(\tilde{M}_B(f(a)),$$

$$\begin{aligned}\tilde{M}_B(f(a)+f(x)))&=T(\tilde{M}_B(f(a)),\tilde{M}_B(f(a+x)))\\&=T(\tilde{M}_{f^{-1}(B)}(a),\tilde{M}_{f^{-1}(B)}(a+x))\end{aligned}$$

So $\tilde{M}_{f^{-1}(B)}$ is an interval valued T-fuzzy left k-ideal of R_1. Similarly we can prove that $\tilde{N}_{f^{-1}(B)}$ is an interval valued S-fuzzy left k-ideal of R_1. Therefore

$$f^{-1}(B)=(f^{-1}(\tilde{M}_B),f^{-1}(\tilde{N}_B))$$

is an interval valued intuitionistic (S,T)-fuzzy left k-ideal of R_1. This completes the proof.

Theorem 3.9: Let f: $R_1\to R_2$ be an epimorphism of hemirings (semirings). If $f^{-1}(A)=(f^{-1}(\tilde{M}_A),f^{-1}(\tilde{N}_A))$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_1, then $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_2.

Proof: It is a easy to verify that $A=(\tilde{M}_A,\tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left ideal of R_2. Suppose that

$$f^{-1}(A)=(f^{-1}(\tilde{M}_A),f^{-1}(\tilde{N}_A))$$

is an interval valued intuitionistic (S,T)-fuzzy left h-ideal. Since f is a surjective mapping, for $x,a,b,z\in R_2$ there are $x_1,a_1,b_1,z_1\in R_1$ such that

$$x=f(x_1),a=f(a_1),b=f(b_1),z=f(z_1).$$

Now if x+a+z = b+z then

$$f(x_1)+f(a_1)+f(z_1)=f(b_1)+f(z_1)$$

and so

$$\tilde{M}_A(x)=\tilde{M}_A(f(x_1))\geq T(\tilde{M}_A(f(a_1)),$$

$$\tilde{M}_A(f(b_1)))=T(\tilde{M}_A(a),\tilde{M}_A(b)).$$

Similarly we can prove that $\tilde{N}_A(x) \le S(\tilde{N}_A(a), \tilde{N}_A(b))$, which means that $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal of R_2.
Also suppose that

$$f^{-1}(A) = (f^{-1}(\tilde{M}_A), f^{-1}(\tilde{N}_A))$$

is an interval valued intuitionistic (S-T)-fuzzy left k-ideal. For $x,a \in R_2$ there are $x_1, a_1 \in R_1$ such that $x = f(x_1), a = f(a_1)$. We have

$$\tilde{M}_A(x) = \tilde{M}_A(f(x_1)) \ge T(\tilde{M}_A(f(a_1))$$

$$\tilde{M}_A(f(x_1) + f(a))) = T(\tilde{M}_A(a), \tilde{M}_A(x+a))$$

Similarly we can show that $\tilde{N}_A(x) \le S(\tilde{N}_A(a),$ $\tilde{N}_A(x+a))$. Therefore $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S-T)-fuzzy left k-ideal of R_2.

Theorem 3.10: Let f: $R_1 \to R_2$ be an epimorphism of hemirings (semirings). Then $f^{-1}(A) = (f^{-1}(\tilde{M}_A), f^{-1}(\tilde{N}_A))$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of R_1 if and only if $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S-T)-fuzzy left k-ideal (k-ideal) of R_2.

Proof: The result is a consequence of the Theorems 3.8 and 3.9.

Theorem 3.11: Let I be a nonempty subset of a hemiring (semiring) R. Then interval valued fuzzy set $A = (\tilde{M}_A, \tilde{N}_A)$ defined by

$$\tilde{M}_A(x) = \begin{cases} [a_2, a_2'], & \text{if } x \in I \\ [a_1, a_1'], & \text{if } x \notin I \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [\beta_2, \beta_2'], & \text{if } x \in I \\ [\beta_1, \beta_1'], & \text{if } x \notin I \end{cases}$$

where

$$[0,0] \le [a_1, a_1'] < [a_2, a_2'] \le [1,1],$$

$$[0,0] \le [\beta_2, \beta_2'] < [\beta_1, \beta_1'] \le [1,1]$$

and $a_i' + \beta_i' \le 1$ for i=1,2, is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) if and only if I is a left h-ideal (k-ideal) of hemiring (semiring) R.

Proof: We prove that I is a left h-ideal (k-ideal). The proof of I is a left ideal is straightforward. Let $A = (\tilde{M}_A, \tilde{N}_A)$ be an interval valued intuitionistic (S,T)-fuzzy left h-ideal, x+a+z=b+z and $a,b \in I$. Then

$$\tilde{M}_A(a) = \tilde{M}_A(b) = [a_2, a_2'],$$

thus

$$\tilde{M}_A(x) \ge T(\tilde{M}_A(a), \tilde{M}_A(b)) = [a_2, a_2']$$

and so $x \in I$. Therefore I is a left h-ideal.

Also if $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left k-ideal, $x+a \in I$ and $a \in I$, then

$$\tilde{M}_A(x) \ge T(\tilde{M}_A(a), \tilde{M}_A(x+a)) = [a_2, a_2'],$$

and so $x \in I$. Therefore I is a left k-ideal.

Conversely, suppose that I is a left h-ideal (k-ideal). The proof of $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left ideal of R is straightforward. If I is a left h-ideal and x+a+z = b+z, we have the following cases:

(1) $a,b \in I$, which implies that $x \in I$ and so we have

$$\tilde{M}_A(x) = [a_2, a_2'] = T(\tilde{M}_A(a), \tilde{M}_A(b))$$

(2) $a,b \notin I$, so we have

$$\tilde{M}_A(x) \ge [a_1, a_1'] = T(\tilde{M}_A(a), \tilde{M}_A(b))$$

(3) $a \in I$, $b \notin I$, so we have

$$\tilde{M}_A(x) \ge [a_1, a_1'] \ge T(\tilde{M}_A(a), \tilde{M}_A(b))$$

(4) $a \notin I$, $b \in I$, is similar to the case (3).

Thus $\tilde{M}_A$ is an interval valued T-fuzzy left h-ideal. Similarly we can prove that $\tilde{N}_A$ is an interval valued S-fuzzy left h-ideal. Therefore $(\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal.

Also if I is a k-ideal we have the following cases:

(1) $x, x+y \in I$, which implies that $y \in I$ and so we have

$$\tilde{M}_A(y) = [a_2, a_2'] = T(\tilde{M}_A(x), \tilde{M}_A(x+y))$$

(2) $x, x+y \notin I$, so we have

$$\tilde{M}_A(y) \ge [a_2, a_2'] = T(\tilde{M}_A(x), \tilde{M}_A(x+y))$$

(3) $x \in I, x+y \notin I$ so we have

$$\tilde{M}_A(y) \geq [a_2, a_2'] = T(\tilde{M}_A(x), \tilde{M}_A(x+y))$$

(4) $x \notin I, x+y \in I$ is similar to the case (3).

Thus $\tilde{M}_A$ is an interval valued T-fuzzy left k-ideal. Similarly we can prove that $\tilde{N}_A$ is an interval valued S-fuzzy left k-ideal. Therefore $(\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left k-ideal.

Cleary every interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) is an interval valued intuitionistic (S,T)-fuzzy left ideal, but the converse is not true in general as can be seen in the following example.

Example: The set N* of all non-negative integers is a hemiring with respect to usual addition and multiplication. Define an interval valued intuitionistic set $A = (\tilde{M}_A, \tilde{N}_A)$ of N* by

$$\tilde{M}_A(x) = \begin{cases} [0, 0.05], & \text{if } 0 \leq x < 5 \\ [0.5, 0.5], & \text{if } 5 \leq x < 7 \\ [0.7, 0.8], & \text{if } x \geq 7 \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [0.9, 0.95], & \text{if } 0 \leq x < 5 \\ [0.5, 0.5], & \text{if } 5 \leq x < 7 \\ [0, 0.1], & \text{if } x \geq 7 \end{cases}$$

It is easy to show that $A = (\tilde{M}_A, \tilde{N}_A)$ is an interval valued intuitionistic (S,T)-fuzzy left ideal of N*, but it is neither an interval valued intuitionistic (S,T)-fuzzy left h-ideal nor an interval valued intuitionistic (S,T)-fuzzy left k-ideal of N*. To see this we have

$$\tilde{M}_A(3) = [0, 0.1] \ngeq T(\tilde{M}_A(6), \tilde{M}_A(3+6)) = T([0.5, 0.5], [0.7, 0.8])$$

Also if x = 1, a = 7, z = 2 and b = 8, then x+a+z = b+z and

$$\tilde{N}_A(1) = [0.9, 1] \nleq S(\tilde{N}_A(7), \tilde{N}_A(8)) = [0, 0.1]$$

Lemma 3.12: Let $A = (\tilde{M}_A, \tilde{N}_A)$ be an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of a hemiring (semiring) R and $x \in R$. Then $\tilde{M}_A(x) = [a, a'], \tilde{N}_A(x) = [\beta, \beta']$ if and only if

$$x \in U(\tilde{M}_A; [a, a']), x \notin U(\tilde{M}_A; [\gamma, \gamma'])$$

and

$$x \in L(\tilde{N}_A; [\beta, \beta']), x \notin L(\tilde{N}_A; [\delta, \delta'])$$

for all $[\gamma, \gamma'] > [a, a']$ and $[\delta, \delta'] < [\beta, \beta']$.

Proof: Let

$$\tilde{M}_A(x) = [a, a'] \text{ and } \tilde{N}_A(x) = [\beta, \beta'].$$

Clearly

$$x \in U(\tilde{M}_A; [a, a'])$$

and

$$x \in L(\tilde{N}A; [\beta, \beta']).$$

Now if $[\gamma, \gamma'] > [a, a']$ and $x \in U(\tilde{M}_A; [\gamma, \gamma'])$, then

$$\tilde{M}_A(x) \geq [\gamma, \gamma'] > [a, a']$$

which is a contradiction. Thus

$$x \notin U(\tilde{M}_A; [\gamma, \gamma']).$$

Similarly we can prove that for

$$[\delta, \delta'] < [\beta, \beta'], x \notin L(\tilde{N}_A; [\delta, \delta']).$$

Conversely, let

$$\tilde{M}_A(x) = [a_0, a_0'] \text{ and } \tilde{N}_A(x) = [\beta_0, \beta_0'].$$

Since

$$x \in U(\tilde{M}_A; [a, a']) \text{ and } x \in L(\tilde{N}_A; [\beta, \beta']),$$

so

$$[a_0, a_0'] = \tilde{M}_A(x) \geq [a, a']$$

and

$$[\beta_0, \beta_0'] = \tilde{N}_A(x) \leq [\beta, \beta']$$

If $[a_0, a_0'] > [a, a']$ then by hypothesis

$$x \notin U(\tilde{M}_A; [a_0, a_0'])$$

which is a contradiction. Thus $[a_0, a_0'] = [a, a']$. Similarly we can show that $[\beta_0, \beta_0'] = [\beta, \beta']$.

Definition 3.13: A left h-ideal (k-ideal) I of a hemiring (semiring) R is said to be characteristic if $\wp(I) = I$ for all $\wp \in Aut(R)$, where Aut (R) is the set of all automorphisms of R.

Definition 3.14: An interval valued intuitionistic fuzzy set $A = (\tilde{M}_A, \tilde{N}_A)$ is called characteristic if

$$\tilde{M}_A(\psi(x)) = \tilde{M}_A(x) \text{ and } \tilde{N}_A(\psi(x)) = \tilde{N}_A(x)$$

for all x∈R and $\psi \in Aut(R)$.

Theorem 3.15: An interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) $A = (\tilde{M}_A, \tilde{N}_A)$ is characteristic if and only if $U(\tilde{M}_A;[s,t])$ and $L(\tilde{N}_A;[s,t])$ are characteristic left h-ideal (k-ideal) of hemiring (semiring) R, for all [s,t]∈D[0,1].

Proof: By Theorem 3.3, it is enough we prove $A = (\tilde{M}_A, \tilde{N}_A)$ is interval valued fuzzy characteristic if and only if $U(\tilde{M}_A;[s,t])$ and $L(\tilde{N}_A;[s,t])$ are characteristic for all $[s,t] \in D[0,1]$. If $A = (\tilde{M}_A, \tilde{N}_A)$ is interval valued fuzzy characteristic, $[a,a'] \in Im(\tilde{M}_A)$, $\psi \in Aut(R)$ and $x \in U(\tilde{M}_A;[a,a'])$, then

$$\psi^{-1}(\tilde{M}_A)(x) = \tilde{M}_A(\psi(x)) = \tilde{M}_A(x) \geq [a,a'],$$

which means that $\psi(x) \in U(\tilde{M}_A;[a,a'])$. Then

$$\psi(U(\tilde{M}_A;[a,a'])) \subseteq U(\tilde{M}_A;[a,a']).$$

Also for each $x \in U(\tilde{M}_A,[a,a'])$ there exists y∈R such that $\psi(y) = x$, thus we have

$$\tilde{M}_A(y) = \psi^{-1}(\tilde{M}_A)(y) = \tilde{M}_A(\psi(y))$$
$$= \tilde{M}_A(x) \geq [a,a']$$

So $y \in U(\tilde{M}_A;[a,a']))$ and

$$x \in \psi(y) \in \psi(U(\tilde{M}_A;[a,a']))$$

Hence it is concluded that

$$\psi(U(\tilde{M}_A;[a,a'])) = U(\tilde{M}_A;[a,a']).$$

Similarly we can show

$$\psi(L(\tilde{N}_A;[\beta,\beta'])) = L(\tilde{N}_A,[\beta,\beta'])$$

This means $U(\tilde{M}_A;[a,a']$ and $L(\tilde{N}_A;[\beta,\beta'])$ are characteristic. Conversely, for

$$x \in R, \psi \in Aut(R), \tilde{M}_A(x) = [a,a']$$

and

$$\tilde{N}_A(x) = [\beta,\beta']$$

by Lemma 3.12, we have

$$x \in U(\tilde{M}_A;[a,a']), x \notin U(\tilde{M}_A;[\gamma,\gamma'])$$

and

$$x \in L(\tilde{N}_A;[\beta,\beta']), x \notin L(\tilde{N}_A;[\delta,\delta'])$$

for all $[\gamma,\gamma'] > [a,a']$ and $[\delta,\delta'] < [\beta,\beta']$. Thus we can conclude that

$$f(x) \in f(U(\tilde{M}_A;[a,a'])) = U(\tilde{M}_A;[a,a'])$$

and

$$f(x) \in f(L(\tilde{N}_A;[\beta,\beta']=))L(\tilde{N}_A;[\beta,\beta'])$$

which means $\tilde{M}_A(f(x)) \geq [a,a']$ and $\tilde{N}_A(f(x)) \leq [\beta,\beta']$. Now if

$$\tilde{M}_A(f(x)) = [\gamma,\gamma'] > [a,a']$$

and

$$\tilde{N}_A(f(x)) = [\delta,\delta'] < [\beta,\beta'].$$

we have

$$f(x) \in U(\tilde{M}_A;[\gamma,\gamma']) = f(U(\tilde{M}_A;[\gamma,\gamma']))$$

and

$$f(x) \in L(\tilde{N}_A;[\delta,\delta']) = f(L(\tilde{N}_A;[\delta,\delta'])).$$

This implies that $x \in U(\tilde{M}_A;[\gamma,\gamma'])$ and

$x \in L(\tilde{N}_A;[\delta,\delta'])$, which is a contradiction. Thus

$$\tilde{M}_A(f(x)) = \tilde{M}_A(x) \text{ and } \tilde{N}_A(f(x)) = \tilde{N}_A(x).$$

Therefore $A = (\tilde{M}_A, \tilde{N}_A)$ is interval valued fuzzy characteristic.

EQUIVALENCE RELATIONS ON INTERVAL VALUED INTUITIONISTIC FUZZY H-IDEALS (K-IDEALS)

Let $[a,a'] \in D[0,1]$ be fixed $(a \neq 0, a' \neq 1)$ and Ih(R) (Ik(R)) be the family of all interval valued intuitionistic (S,T)-fuzzy left h-ideals (k-ideals) of a hemiring (semiring) R. For any $A = (\tilde{M}_A, \tilde{N}_A)$ and $B = (\tilde{M}_B, \tilde{N}_B)$ from Ih(R) (Ik(R)), we define two binary relations $U^{[a,a']}$ $U^{[a,a']}$ and $L^{[a,a']}$ on Ih (R) (Ik (R)) as follows:

$$AU^{[a,a']}B \Leftrightarrow U(\tilde{M}_A;[a,a']) = U(\tilde{M}_B;[a,a'])$$

$$AL^{[a,a']}B \Leftrightarrow L(\tilde{N}_A;[a,a']) = L(\tilde{N}_B;[a,a'])$$

Clearly these two relations $U^{[a,a']}$ and $L^{[a,a']}$ are equivalence relations. Hence Ih (R) (Ik (R)) can be divided in to the equivalence classes of $U^{[a,a']}$ and $L^{[a,a']}$, denoted by $[A]_{U^{[a,a']}}$ and $[A]_{L^{[a,a']}}$, respectively, for any $A = (\tilde{M}_A, \tilde{N}_A)$ in Ih(R) (Ik(R)). The corresponding quotient stes will be denoted as $Ih(R)/U^{[a,a']}$ ($Ik(R)/U^{[a,a']}$ and $Ih(R)/L^{[a,a']}$ ($Ik(R)/L^{[a,a']}$, respectively. If h (R) (k (R)) is the set of all left h-ideals (k-ideals) of R, we define two maps $U_{[a,a']}$ and $L_{[a,a']}$ for each $A = (\tilde{M}_A, \tilde{N}_A) \in Ih(R)$ as follows:

$$U_{[a,a']}: Ih(R) \to h(R) \cup \{\phi\} \text{ by}$$

$$U_{[a,a']}(A) = U(\tilde{M}_A;[a,a']),$$

$$L_{[a,a']}: Ih(R) \to h(R) \cup \{\phi\} \text{ by}$$

$$L_{[a,a']}(A) = L(\tilde{N}_A;[a,a']),$$

Exactly such these maps can be defined on Ik(R). Clearly if A = B, then

$$U(\tilde{M}_A;[a,a']) = U(\tilde{M}_B;[a,a'])$$

and

$$L(\tilde{N}_A;[a,a']) = L(\tilde{N}_B;[a,a']). .$$

Therefore $U_{[a,a']}$ and $L_{[a,a']}$ are well-defined.

Lemma 4.1: Let $0 \le \beta' \le 1$ and $I \in h(R)(I \in k(R))$. Then $A = (\tilde{M}_A, \tilde{N}_A)$, defined as follows, is interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of hemiring (semiring) R:

$$\tilde{M}_A(x) = \begin{cases} [\beta',1], & \text{if } x \in I \\ [0,0], & \text{if } x \notin I \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [0,0], & \text{if } x \in I \\ [\beta',1], & \text{if } x \notin I \end{cases}$$

Proof: It is a consequence of Theorem 3.11.

Theorem 4.2: For any $[a,a'] \in D[0,1]$, the maps $U_{[a,a']}$ and $L_{[a,a']}$ are surjective.

Proof: Let $\tilde{0}$ and $\tilde{1}$ be interval valued fuzzy sets on hemiring (semiring) R defined by $\tilde{0}(x) = [0,0]$ and $\tilde{1}(x) = [\beta',1]$ such that $\beta' > a'$ for all $x \in R$. Then $A_0 = (\tilde{0}, \tilde{1})$ is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of hemiring (semiring) R and

$$U_{[a,a']}(A_0) = \phi = L_{[a,a']}(A_0)$$

for any $[a,a'] \in D[0,1]$ with $a \neq 0$ and $a' \neq 1$. Moreover, for any $I \in h(R)(I \in k(R))$, we define $A = (\tilde{M}_A, \tilde{N}_A)$ as follows:

$$\tilde{M}_A(x) = \begin{cases} [\beta',1], & \text{if } x \in I \\ [0,0], & \text{if } x \notin I \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [0,0], & \text{if } x \in I \\ [\beta',1], & \text{if } x \notin I \end{cases}$$

By Lemma 4.1,

$$A = (\tilde{M}_A, \tilde{N}_A) \in Ih(R)(A = (\tilde{M}_A, \tilde{N}_A) \in Ik(R))$$

Also

$$U_{[a,a']}(A) = U(\tilde{M}_A;[a,a']) = I$$

and

$$L_{[a,a']}(A) = L(\tilde{M}_A;[a,a']) = I.$$

This means $U_{[a,a']}$ and $L_{[a,a']}$ are surjective.

Theorem 4.3: For any $[a,a'] \in D[0,1]$, there are bijective maps from

$$Ih(R)/U^{[a,a']} \; (ik(R)/U^{[a,a']}$$

and

$$Ih(R)/L^{[a,a']}(Ik(R)/L^{[a,a']})$$

to $\quad h(R) \cup \{\phi(k(R) \cup \{\phi\}).$

Proof: Let $[a,a'] \in D[0,1]$. Define $U^*_{[a,a']}$ and $L^*_{[a,a']}$ as follows:

$$U^*_{[a,a']}: Ih(R)/U^{[a,a']} \to h(R) \cup \{\phi\} \text{ by}$$

$$U^*_{[a,a']}([A]_{U^{[a,a']}}) = U_{[a,a']}(A),$$

$$L^*_{[a,a']}: Ih(R)/L^{[a,a']} \to h(R) \cup \{\phi\} \text{ by}$$

$$L^*_{[a,a']}([A]_{L^{[a,a']}}) = L_{[a,a']}(A).$$

Exactly such these maps can be defined on $Ik(R)/U^{[a,a']}$ and $Ik(R)/L^{[a,a']}$. Since $U_{[a,a']}$ and $L_{[a,a']}$ are well-defined, so $U^*_{[a,a']}$ and $L^*_{[a,a']}$. If

$$U(\tilde{M}_A;[a,a']) = U(\tilde{M}_B;[a,a'])$$

and

$$L(\tilde{N}_A;[a,a']) = L(\tilde{N}_B;[a,a'])$$

for some

$$A = (\tilde{M}_A, \tilde{N}_A)$$

and

$$B = (\tilde{M}_A, \tilde{N}_A)$$

from Ih(R)(Ik(R)) then $AU^{[a,a']}B$ and $AL^{[a,a']}B$, so $[A]_{U^{[a,a']}} = [B]_{U^{[a,a']}}$ and $[A]_{L^{[a,a']}} = [B]_{L^{[a,a']}}$, which means $U^*_{[a,a']}$ and $L^*_{[a,a']}$ are injective. Let I∈h(R)(I∈k(R)) then for $A = (\tilde{M}_A, \tilde{N}_A)$ defined by:

$$\tilde{M}_A(x) = \begin{cases} [\beta',1], & \text{if } x \in I \\ [0,0], & \text{if } x \notin I \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [0,0], & \text{if } x \in I \\ [\beta',1], & \text{if } x \notin I \end{cases}$$

where β'>a', is an interval valued intuitionistic (S,T)-fuzzy left h-ideal (k-ideal) of hemiring (semiring) R. We have

$$U^*_{[a,a']}([A]_{U^{[a,a']}}) = U(\tilde{M}_A;[a,a']) = I$$

and

$$L^*_{[a,a']}([A]_{L^{[a,a']}}) = L(\tilde{N}_A;[a,a']) = I.$$

Moreover

$$U^*_{[a,a']}([A_0]_{U^{[a,a']}}) = U(\tilde{0};[a,a']) = \phi$$

and

$$L^*_{[a,a']}([A_0]_{L^{[a,a']}} = L(\tilde{1};[a,a']) = \phi,$$

where $A_0 = (\tilde{0},\tilde{1})$ was defined in the proof of Theorem 4.2. Therefore $U^*_{[a,a']}$ and $L^*_{[a,a']}$ are surjective.

Now for any $[a,a'] \in D[0,1]$ with $a,a' \leq 0.5$, we have a new relation $W^{[a,a']}$ on Ih (R) (Ik(R)) as follows:

$$AW^{[a,a']}B \Leftrightarrow U(\tilde{M}_A;[a,a']) \cap L(\tilde{N}_A;[a,a']) = U(\tilde{M}_B;[a,a']) \cap L(\tilde{N}_B;[a,a']),$$

where

$$A = (\tilde{M}_A, \tilde{N}_A) \text{ and } B = (\tilde{M}_B, \tilde{N}_B)$$

Clearly $W^{[a,a']}$ is an equivalence relation.

Theorem 4.4: The map

$$I_{[a,a']} : Ih(R) \to h(R) \cup \{\phi\}$$

defined by

$$I_{[a,a']}(A) = U(\tilde{M}_A;[a,a']) \cap L(\tilde{N}_A;[a,a']),$$

where $A = (\tilde{M}_A, \tilde{N}_A)$ is surjective for any $[a,a'] \in D[0,1]$ with a,a'≤0.5 (such this map can be defined on Ik(R)).

Proof: If $[a,a'] \in D[0,1]$ is fixed (a,a'≤0.5) then for $A_0 = (\tilde{0},\tilde{1}) \in Ih(R)$, defined in the proof of Theorem 4.2, we have

$$I_{[a,a']}(A_0) = U(\tilde{0};[a,a']) \cap L(\tilde{1};[a,a']) = \phi.$$

Also for any $I \in h(R)(I \in k(R))$, by Lemma 4.1, there exists

$$A = (\tilde{M}_A, \tilde{N}_A) \in Ih(R)$$

$$(A = (\tilde{M}_A, \tilde{N}_A) \in Ik(R))$$

defined by:

$$\tilde{M}_A(x) = \begin{cases} [\beta',1], & \text{if } x \in I \\ [0,0], & \text{if } x \notin I \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [0,0], & \text{if } x \in I \\ [\beta',1], & \text{if } x \notin I \end{cases}$$

where β'>a'. Such that

$$I_{[a,a']}(A) = U(\tilde{M}_A;[a,a']) \cap L(\tilde{N}_A;[a,a'] = I.$$

Theorem 4.5: For any $[a,a'] \in D[0,1]$ with a,a'≤0.5, there is a bijective map from $Ih(R)/W^{[a,a']}$ $(Ik(R)/W^{[a,a']})$ to

$$h(R) \cup \{\phi\} \; (k(R) \cup \{\phi\}).$$

Proof: Let $I^*_{[a,a']} : Ih(R)/W^{[a,a']} \to h(R) \cup \{\phi\}$,

where $[a,a'] \in D[0,1]$ and $a,a' \leq 0.5$ be defined by $I^*_{[a,a']}([A]_{W^{[a,a']}}) = I_{[a,a']}(A)$ for each $[A]_{W^{[a,a']}} \in Ih(R)/W^{[a,a']}$ (such this map can be defined on $Ik(R)/W^{[a,a']}$). If

$$I^*_{[a,a']}([A]_{W^{[a,a']}}) = I^*_{[a,a']}([B]_{W^{[a,a']}})$$

for some

$$[A]_{W^{[a,a']}}, [B]_{W^{[a,a']}} \in Ih(R)/W^{[a,a']},$$

then

$$U(\tilde{M}_A;[a,a']) \cap L(\tilde{N}_A;[a,a']) = U(\tilde{M}_B;[a,a']) \cap L(\tilde{N}_B;[a,a']),$$

which implies $AW^{[a,a']}B$ and so $[A]_{W^{[a,a']}} = [B]_{W^{[a,a']}}$. This means $I^*_{[a,a']}$ is injective. $I^*_{[a,a']}$ is surjective because

$$I^*_{[a,a']}([A_0]_{W^{[a,a']}}) = I_{[a,a']}(A_0) = \phi$$

for $A_0 = (\tilde{0}, \tilde{1}) \in Ih(R)$ defined in the proof of Theorem 4.2. Also if I∈h (R) (I∈k (R), then $I^*_{[a,a']}([A]_{W^{[a,a']}}) = I_{[a,a']}(A) = I$, where $A = (\tilde{M}_A, \tilde{N}_A)$ is defined as follows:

$$\tilde{M}_A(x) = \begin{cases} [\beta',1], & \text{if } x \in I \\ [0,0], & \text{if } x \notin I \end{cases}$$

$$\tilde{N}_A(x) = \begin{cases} [0,0], & \text{if } x \in I \\ [\beta',1], & \text{if } x \notin I \end{cases}$$

where $\beta' > a'$. Therefore $I^*_{[a,a']}$ is a bijective map.

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