Equivalences of Blocks of Group Algebras

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Let $\mathcal{O}$ be a complete local noetherian ring, whose field of fractions has characteristic zero and residue field has non–zero characteristic. A block algebra over $\mathcal{O}$ is an indecomposable summand of the algebra of a finite group over $\mathcal{O}$.

We introduce briefly (and justify through examples) several types of equivalences. Three levels of equivalences between block algebras seem to be relevant: Morita equivalence, Rickard (derived) equivalence, stable equivalence of Morita type.

We give a classification of various classical "invariants" of block algebras (such as numerical defect, decomposition matrices, defect of irreducible characters, etc.) depending on the type of equivalence we consider between block algebras.

After recalling why, when switching from the algebra point of view to the group point of view, the source algebra is a suitable replacement for the basic algebra, we try to give suitable "group theoretic" refinements of the previous equivalences.

This is an introductory survey : almost no proof is given, the comments are brief and the applications short. We emphasize the "algebra–theoretic approach", which should be viewed as a first approximation to the methods used in block theory, as we try to explain in the last paragraph. In order to simplify the exposition, we restrict ourselves, most of the time, to the case of principal blocks.

1. Basic context and Notation

Let A be a left and right noetherian ring.

We denote by ${}_A\mathbf{mod}$ the abelian category of finitely generated left A–modules, and by ${}_A\mathbf{proj}$ the category of finitely generated projective left A–modules. We denote by $\mathcal{R}(A)$ the Grothendieck group of ${}_A\mathbf{mod}$ and by $\mathcal{R}^{\mathrm{pr}}(A)$ the Grothendieck group of ${}_A\mathbf{proj}$. If X is an object of ${}_A\mathbf{mod}$ (resp. of ${}_A\mathbf{proj}$), we denote by $[X]$ its representative in $\mathcal{R}(A)$ (resp. in $\mathcal{R}^{\mathrm{pr}}(A)$).

We denote by $\mathbf{mod}_A$ the abelian category of finitely generated right A–modules, and by $\mathbf{proj}_A$ the category of finitely generated projective right A–modules. For B another ring, we denote by ${}_A\mathbf{mod}_B$ the category of finitely generated (A, B)–bimodules.

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Let G be a finite group, and let p be a prime number. Let K be a finite extension of the field of p–adic numbers $\mathbb{Q}_p$ which contains the $|G|$-th roots of unity. Thus the group algebra KG is a split semi–simple K–algebra. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_p$. We denote by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, and we set $k := \mathcal{O}/\mathfrak{p}$. If JkG denotes the Jacobson radical of the group algebra kG, the algebra kG/JkG is a split semi–simple k–algebra.

By extension of scalars we get two functors

$$ {}_{\mathcal{O}G}\mathbf{mod} \to {}_{KG}\mathbf{mod} \quad \text{and} \quad {}_{\mathcal{O}G}\mathbf{mod} \to {}_{kG}\mathbf{mod}\,. $$

$$
\begin{tikzcd}
K & & \\
& \mathcal{O} \arrow[ul, hook] \arrow[r, two heads] & k = \mathcal{O}/\mathfrak{p} \\
\mathbb{Q}_p \arrow[uu, hook] & & \\
& \mathbb{Z}_p \arrow[ul, hook] \arrow[uu, hook] \arrow[r, two heads] & \mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p \arrow[uu, hook]
\end{tikzcd}
$$

The Cartan–Decomposition triangle.

For the following classical facts, we refer the reader to [Se], part III.

The set $\mathrm{Irr}(KG)$ of representatives in $\mathcal{R}(KG)$ of the irreducible KG–modules is an orthonormal basis of $\mathcal{R}(KG)$ for the scalar product defined by

$$ \langle [X], [X'] \rangle := \dim \mathrm{Hom}_{KG}(X, X')\,. $$

The set $\mathrm{Irr}(kG)$ of representatives in $\mathcal{R}(kG)$ of the irreducible kG–modules is a $\mathbb{Z}$–basis of $\mathcal{R}(kG)$, while the set $\mathrm{Pim}(kG)$ of representatives in $\mathcal{R}^{\mathrm{pr}}(kG)$ of the indecomposable projective kG–modules is a $\mathbb{Z}$–basis of $\mathcal{R}^{\mathrm{pr}}(kG)$. The pairing

$$ \mathcal{R}^{\mathrm{pr}}(kG) \times \mathcal{R}(kG) \to \mathbb{Z} $$

defined by

$$ \langle [P], [X] \rangle := \dim \mathrm{Hom}_{kG}(P, X) $$

(P an object of ${}_{kG}\mathbf{proj}$, X an object of ${}_{kG}\mathbf{mod}$) defines a duality between $\mathcal{R}^{\mathrm{pr}}(kG)$ and $\mathcal{R}(kG)$.

Let X be a finitely generated KG–module. Let X_0 be a finitely generated $\mathcal{O}$–free $\mathcal{O}G$–module such that $X = K \otimes_{\mathcal{O}} X_0$. Then the corresponding element $[k \otimes X_0]$ in $\mathcal{R}(kG)$ depends only on X, and this defines the decomposition map

$$ \mathrm{dec}^G \colon \mathcal{R}(KG) \to \mathcal{R}(kG)\,. $$

The reduction modulo $\mathfrak{p}$ defines an isomorphism $\mathcal{R}^{\mathrm{pr}}(\mathcal{O}G) \xrightarrow{\sim} \mathcal{R}^{\mathrm{pr}}(kG)$. Identifying $\mathcal{R}^{\mathrm{pr}}(\mathcal{O}G)$ and $\mathcal{R}^{\mathrm{pr}}(kG)$ through this isomorphism, the adjoint of the decomposition map is the linear map ${}^t\mathrm{dec}^G\colon \mathcal{R}^{\mathrm{pr}}(\mathcal{O}G) \to \mathcal{R}(KG)$ which sends the representative of a projective $\mathcal{O}G$–module X onto the representative of the KG–module $K \otimes_{\mathcal{O}} X$.

Finally, the Cartan map $\mathrm{Car}^G\colon \mathcal{R}^{\mathrm{pr}}(kG) \to \mathcal{R}(kG)$ is the linear map which sends the representative in $\mathcal{R}^{\mathrm{pr}}(kG)$ of a projective kG–module X onto its representative in $\mathcal{R}(kG)$.

$$(\mathcal{T}(G)) \qquad \begin{tikzcd} \mathcal{R}(KG) \arrow[rr, "\mathrm{dec}^G"] & & \mathcal{R}(kG) \\ & \mathcal{R}^{\mathrm{pr}}(kG) \arrow[ul, "{}^t\mathrm{dec}^G"] \arrow[ur, "\mathrm{Car}^G"'] & \end{tikzcd}$$

1.1. Theorem.

(1) *The cokernel of* Car^G *is a finite p–group, whose exponent is the order of a Sylow p–subgroup of G.*
(2) *The map* dec^G *is onto, and the image of* ${}^t\mathrm{dec}^G$ *is a pure submodule of* $\mathcal{R}(KG)$.
(3) $\mathrm{Car}^G = \mathrm{dec}^G \cdot {}^t\mathrm{dec}^G$.

2. Blocks

The decomposition of the unity element of $\mathcal{O}G$ into a sum of orthogonal primitive central idempotents $1 = \sum e$ corresponds to the decomposition of the algebra $\mathcal{O}G$ into a direct sum of indecomposable two–sided ideals $\mathcal{O}G = \bigoplus A$ ($A = \mathcal{O}Ge$), called the *blocks* of $\mathcal{O}G$. For A a block of $\mathcal{O}$, we set $KA := K \otimes_{\mathcal{O}} A$ and $kA := k \otimes_{\mathcal{O}} A$.

By reduction modulo $\mathfrak{p}$, a primitive central idempotent remains primitive central, and consequently $kG = \bigoplus kA$ is still a decomposition into a direct sum of indecomposable two–sided ideals, called the blocks of kG.

$$\begin{array}{ccccc} \mathcal{O}G & = & \bigoplus & A \\ \downarrow & & & \downarrow \\ kG & = & \bigoplus & kA \end{array}$$

The augmentation map $\mathcal{O}G \to \mathcal{O}$ factorizes through a unique block of $\mathcal{O}G$ called *the principal block* and denoted by$A_0(\mathcal{O}G)$.

2.A. The invariants of a block.

Let e be a central idempotent of $\mathcal{O}G$ and let $A := \mathcal{O}Ge$ be the corresponding algebra (note that we are not assuming e necessarily primitive, so what follows applies to direct sums of blocks). The idempotent e is the unity element of the algebra A, and A is a symmetric $\mathcal{O}$–algebra for the form

$$t\colon A \to \mathcal{O}\ ,\ \sum_{g\in G} a(g)g \mapsto a(1)\ .$$

Center and projective center. View A as an (A, A)–bimodule. The ring $\mathrm{End}_A(A)_A$ of its endomorphisms is the center $Z(A)$ of A. The set of projective endomorphisms (endomorphisms which factorize through a projective (A, A)–bimodule) is an ideal of $Z(A)$ which is denoted by $Z^{\mathrm{pr}}(A)$ and called *the projective center* of A.

2.1. Proposition. *We have* $Z^{\mathrm{pr}}(A) = \{\sum_{g\in G} gag^{-1} \,|\, (a \in A)\}$.

The set of projective endomorphisms of the (kA, kA)–bimodule kA is denoted by $Z^{\mathrm{pr}}(kA)$. It is equal to the image of $Z^{\mathrm{pr}}(A)$ through the reduction modulo $\mathfrak{p}$ $Z(A) \to Z(kA)$.

c-d-triangle and associated invariants. The Grothendieck groups $\mathcal{R}(KA)$, $\mathcal{R}(kA)$ and $\mathcal{R}^{\mathrm{pr}}(kA)$ are summands of the Grothendieck groups $\mathcal{R}(KG)$, $\mathcal{R}(kG)$ and $\mathcal{R}^{\mathrm{pr}}(kG)$, and the maps dec^G, ${}^t\mathrm{dec}^G$, Car^G restrict to maps which define the "c-d-triangle" of the block A,

$$(\mathcal{T}(A)) \qquad \begin{tikzcd} \mathcal{R}(KA) \arrow[rr, "\mathrm{dec}^A"] & & \mathcal{R}(kA) \\ & \mathcal{R}^{\mathrm{pr}}(kA) \arrow[ul, "{}^t\mathrm{dec}^A"] \arrow[ur, "\mathrm{Car}^A"'] & \end{tikzcd}$$

which we view as endowed with its "metric structure" given by the dualities

$$\mathcal{R}(KA) \times \mathcal{R}(KA) \to \mathbb{Z} \quad \text{and} \quad \mathcal{R}^{\mathrm{pr}}(kA) \times \mathcal{R}(kA) \to \mathbb{Z}\,.$$

We denote by $\mathrm{Irr}(KA)$ (resp. $\mathrm{Irr}(kA)$, $\mathrm{Pim}(kA)$) the set of representatives in the corresponding Grothendieck group of the irreducible KA–modules (resp. of the irreducible kA–modules, of the projective indecomposable kA–modules), called the *canonical basis* of the corresponding $\mathbb{Z}$–modules.

The matrix of Car^A on the canonical basis is called the Cartan matrix of A and denoted by C^A, while the matrix of dec^A on the canonical basis is called the decomposition matrix of A and denoted by D^A.

It is traditional to set

$$\mathrm{k}(A) := |\mathrm{Irr}(KA)| \quad \text{and} \quad \mathrm{l}(A) := |\mathrm{Irr}(kA)| = |\mathrm{Pim}(kA)|\,.$$

The $\mathcal{O}$–rank of $Z(A)$ equals $\mathrm{k}(A)$, while the rank of $Z^{\mathrm{pr}}(kA)$ equals the number of trivial invariant factors of the map Car^A. We set

$$\mathrm{l}^{\mathrm{pr}}(A) := \dim Z^{\mathrm{pr}}(kA)\,.$$

The exponent of the cokernel of Car^A divides the order of a Sylow p–subgroup of G and so has the shape $p^{d(A)}$. The integer $d(A)$ is called the *defect* of the block A.

Defects of irreducible KA–modules.

If X is an irreducible KA–module, we set $p^{d(X)} := \left(\dfrac{|G|}{\dim X}\right)_p$, and we call the integer $d(X)$ the *defect* of X.

2.2. Proposition.

(1) *We have* $d(A) = \sup\{d(X) \,|\, (X \in \mathrm{Irr}(KA))\}$.
(2) *Let* X_0 *be an* $\mathcal{O}$*–free* A*–module such that* $X = K \otimes_{\mathcal{O}} X_0$. *Let* $\mathrm{End}_A^{\mathrm{pr}}(X_0)$ *be the ideal of* $\mathrm{End}_A(X_0)$ *consisting of the projective endomorphisms of* X_0. *Then* $\mathrm{End}_A(X_0)/\mathrm{End}_A^{\mathrm{pr}}(X_0) = p^{d(A)-d(X)}\mathcal{O}$.

Remark.

Let P be a p–group. Then we have $J\mathcal{O}P = \mathfrak{p}\mathcal{O}P + \mathcal{A}\mathcal{O}P$ where $J\mathcal{O}P$ denotes the Jacobson radical of $\mathcal{O}P$ and $\mathcal{A}\mathcal{O}P$ denotes the augmentation ideal of $\mathcal{O}P$. So $\mathcal{O}P$ is itself a block and we have $\mathrm{l}(\mathcal{O}G) = 1$. The c-d-triangle is trivial :

$$\begin{tikzcd}
\mathcal{R}(KP) \arrow[rr, "\dim"] & & \mathcal{R}(kP) \\
& \mathcal{R}^{\mathrm{pr}}(kP) \arrow[ul, "\mathrm{reg}"] \arrow[ur, "|P|"'] &
\end{tikzcd}$$

where the map "reg" maps the generator $|P|$ of $\mathcal{R}^{\mathrm{pr}}(kP)$ onto the representative of the regular representation of KP. Notice that, on the other hand, the category ${}_{\mathcal{O}P}\mathbf{mod}$ is far from being trivial. If P is neither cyclic nor (for $p = 2$) dihedral, semidihedral or generalized quaternion, then the algebra $\mathcal{O}P$ is wild.

2.B. Problems of block theory.

Block theory, as introduced and developed by Richard Brauer, originated mainly in the problem of the classification of finite simple groups. As a first approximation, we may say that the main problem of block theory is *to compare the category* ${}_{\mathcal{O}G}\mathbf{mod}$ *to the "local" categories* ${}_{\mathcal{O}N_G(P)}\mathbf{mod}$*, where* P *runs over the set of non–trivial* p*–subgroups of* G, and $N_G(P)$ denotes the normalizer of P in G.

Remark.

• Let A_0 be the principal block of $\mathcal{O}G$. The structure of ${}_{A_0}\mathbf{mod}$ is closely related to the structure of the group G itself — more precisely, to the structure of $G/O_{p'}(G)$, where $O_{p'}(G)$ denotes the largest normal subgroup of G whose order is relatively prime to p.

For example, let P be a Sylow p–subgroup of G. The following assertions are equivalent:

(i) $\mathrm{l}(A_0) = 1$,
(ii) ${}_{A_0}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}P}\mathbf{mod}$,
(iii) G is p–nilpotent, *i.e.,* isomorphic to the semi–direct product $O_{p'}(G) \rtimes P$.

• The situation is more complicated for non–principal blocks. Indeed, there are blocks A of non–abelian simple groups G which satisfy one of the following equivalent properties ("defect zero") :

(i) There is $X \in \mathrm{Irr}(KA)$ such that $(\dim X)_p$ equals the order of a Sylow p–subgroup of G,
(ii) ${}_{A}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}}\mathbf{mod}$.

For example, the block defined by the Steinberg character of $\mathrm{GL}_n(p^m)$ has the above properties.

More generally, the "nilpotent blocks" A (see [BrPu], [Pu]) are such that ${}_{A}\mathbf{mod} \simeq {}_{\mathcal{O}P}\mathbf{mod}$ for a certain p–group P. We give here an example of such a block in $\mathrm{GL}_n(\ell^m)$ for $\ell \neq p$ (see [Br1] for more details). We view $\mathrm{GL}_n(\ell^m)$ as the group of fixed points of the algebraic group $\mathbf{G} := \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ under the action of the usual Frobenius endomorphism F. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ and $\theta\colon \mathbf{T}^F \to K^\times$ a character of $\mathbf{T}^F$ such that :

- the order of θ is prime to p,

- θ is in general position in $\mathrm{GL}_n(\ell^m)$ (*i.e.*, an element which normalizes $\mathbf{T}$ and fixes θ must centralize $\mathbf{T}$).

We denote by $R_{\mathbf{T}}^{\mathbf{G}}\colon \mathcal{R}(K\mathbf{T}^F) \to \mathcal{R}(K\mathbf{G}^F)$ the linear map defined by Deligne and Lusztig (see [DeLu]). There is a block $A(\mathbf{T},\theta)$ of $\mathcal{O}\mathrm{GL}_n(\ell^m)$ such that

$$\mathrm{Irr}(KA(\mathbf{T},\theta)) = \{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}R_{\mathbf{T}}^{\mathbf{G}}(\theta\eta)\}_\eta$$

where η runs over the set of characters of $\mathbf{T}^F$ whose order is a power of p. The category ${}_{A(\mathbf{T},\theta)}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}\mathbf{T}_p^F}\mathbf{mod}$ where $\mathbf{T}_p^F$ denotes the Sylow p–subgroup of $\mathbf{T}^F$.

3. Morita equivalences between blocks

From now on, we denote by G and H two finite groups, by e and f respectively two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$. We set $A := \mathcal{O}Ge$ and $B := \mathcal{O}Hf$.

3.A. Preliminaries : bimodules and adjunctions.

We first recall in this context well known properties of functors induced by bimodules. Let M be an (A,B)–bimodule. Let X (resp. Y) be an A–module (resp. a B–module).

1. We have $\mathrm{Hom}_A(M \underset{B}{\otimes} Y, X) \simeq \mathrm{Hom}_B(Y, \mathrm{Hom}_A(M,X))$ through the maps

$$\left(\alpha\colon M \underset{B}{\otimes} Y \to X\right) \mapsto (\hat{\alpha}\colon Y \to \mathrm{Hom}_A(M,X)\,,\ y \mapsto (m \mapsto \alpha(m\otimes y)))$$
$$(\beta\colon Y \to \mathrm{Hom}_A(M,X)) \mapsto \left(\hat{\beta}\colon m\otimes y \mapsto \beta(y)(m)\right)$$

2. Let us set $M^\vee := \mathrm{Hom}_A(M,A)$ viewed as an object of ${}_B\mathbf{mod}_A$. We denote by $<,>\colon M \times M^\vee \to A$ the natural A–pairing between M and $M^\vee$. Suppose that $M \in {}_A\mathbf{mod}_B \cap {}_A\mathbf{proj}$. Then the map

$$M^\vee \underset{A}{\otimes} X \to \mathrm{Hom}_A(M,X)$$
$$(m^\vee \underset{A}{\otimes} x) \mapsto (m \mapsto <m,m^\vee>x)$$

is an isomorphism in ${}_B\mathbf{mod}$.

3. Let us set $M^* := \mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})$, viewed as an object of ${}_B\mathbf{mod}_A$. Since the linear form $t_A\colon A \to \mathcal{O}\,,\ \sum_{g\in G} a(g)g \mapsto a(1)$ is a symmetrizing form for A, the maps

$$m^\vee \mapsto t_A\cdot m^\vee \quad \text{and} \quad m^* \mapsto \left(m \mapsto \sum_{g\in G} m^*(g^{-1}m)g\right)$$

are inverse isomorphisms (in ${}_B\mathbf{mod}_A$) between $M^\vee$ and M^*.

Suppose given a (B,A)–bimodule N and a duality $M\times N \to \mathcal{O}$ which is (A,B)–compatible, *i.e.*, $<m,bna> = <amb,n>$ for $a\in A$, $b\in B$, $m\in M$, $n\in N$. From what preceds, we deduce two explicit isomorphisms (respectively in ${}_A\mathbf{mod}_A$ and in ${}_B\mathbf{mod}_B$) :

$$\mathrm{Hom}_B(N,N) \xrightarrow{\sim} \mathrm{Hom}_A(M \underset{B}{\otimes} N, A)$$

$$\mathrm{Hom}_A(M,M) \xrightarrow{\sim} \mathrm{Hom}_B(B, N \underset{A}{\otimes} M).$$

We denote by $\varepsilon_{M,N}$ the image of Id_M through the first isomorphism, and by $\eta_{M,N}$ the image of Id_N through the second isomorphism. The maps $\varepsilon_{M,N}$ and $\eta_{M,N}$ are called the *adjunctions* associated with the pair (M,N).

3.1. Proposition. *With the previous hypothesis and notation, the maps $\varepsilon_{M,N}$ and $\eta_{M,N}$ are computed as follows :*

$$\varepsilon_{M,N} \colon M \underset{B}{\otimes} N \to A\,,\ m \underset{B}{\otimes} n \mapsto \sum_{g \in G} <g^{-1}m, n>g$$

$$\eta_{M,N} \colon B \to N \underset{A}{\otimes} M\,,\ b \mapsto \sum_{i \in I} n_i \underset{A}{\otimes} m_i$$

where $\sum_{i\in I} n_i \underset{A}{\otimes} m_i$ is the element of $N \underset{A}{\otimes} M$ such that, for all $m \in M$,

$$\sum_{g\in G}\sum_{i \in I} <n_i g^{-1}, m> g n_i = bm\,.$$

3.B. Morita theorem and block invariants.

The following statement is a variation on Morita's theorem, applied in the particular case of symmetric algebras.

3.2. Theorem. *The following assertions are equivalent :*

(i) *The categories ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$ are equivalent.*

(ii) *There exist*

- *an (A,B)–bimodule M which is projective both as an A–module and as a module–B,*
- *a (B,A)–bimodule N which is projective both as a B–module and as a module–A,*
- *an (A,B)–compatible $\mathcal{O}$–duality between M and N*

such that $M \underset{B}{\otimes} N \simeq A$ in ${}_A\mathbf{mod}_A$ and $N \underset{A}{\otimes} M \simeq B$ in ${}_B\mathbf{mod}_B$.

Moreover, if the preceding statements are satisfied, then all of the adjunctions $\varepsilon_{M,N}$, $\eta_{M,N}$, $\varepsilon_{N,M}$, $\eta_{N,M}$ are isomorphisms.

Morita equivalence and triangle invariants. If (M,N), as above, defines an equivalence between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$, the pairs $(K \otimes_{\mathcal{O}} M, K \otimes_{\mathcal{O}} N)$ and $(k \otimes_{\mathcal{O}} M, k \otimes_{\mathcal{O}} N)$ define equivalences respectively between ${}_{KA}\mathbf{mod}$ and ${}_{KB}\mathbf{mod}$ and between ${}_{kA}\mathbf{mod}$ and ${}_{kB}\mathbf{mod}$. So the Morita equivalence defined by (M,N) induces bijections

$$\mathrm{Irr}(KA) \simeq \mathrm{Irr}(KB)\,,\ \mathrm{Irr}(kA) \simeq \mathrm{Irr}(kB)\,,\ \mathrm{Pim}(kA) \simeq \mathrm{Pim}(kB)\,.$$

Moreover, by construction of the c-d-triangles (see §1 above), it is clear that the induced isomorphisms $\mathcal{R}(KA) \simeq \mathcal{R}(KB)$, $\mathcal{R}(kA) \simeq \mathcal{R}(kB)$, $\mathcal{R}^{\mathrm{pr}}(kA) \simeq \mathcal{R}^{\mathrm{pr}}(kB)$, commute with the Cartan and the decomposition maps. To summarize :

3.3. Proposition. *A Morita equivalence between A and B induces an isomorphism between the c-d-triangles $\mathcal{T}(A)$ and $\mathcal{T}(B)$, which preserves the canonical basis.*

$$
\begin{tikzcd}
 & & & \mathcal{R}(KB) \arrow[rr, "\mathrm{dec}^B"] & & \mathcal{R}(kB) \\
\mathcal{R}(KA) \arrow[rr, "\mathrm{dec}^A"] \arrow[urrr, leftrightarrow] & & \mathcal{R}(kA) \arrow[urrr, leftrightarrow] & & \mathcal{R}^{\mathrm{pr}}(kB) \arrow[ul] \arrow[ur, "\mathrm{Car}^B"'] & \\
 & \mathcal{R}^{\mathrm{pr}}(kA) \arrow[ul, "{}^t\mathrm{dec}^A"] \arrow[ur, "\mathrm{Car}^A"'] \arrow[urrr, leftrightarrow] & & & &
\end{tikzcd}
$$

As a consequence, a Morita equivalence between A and B preserves all the invariants determined by the c-d-triangles and their canonical basis :

$$\mathrm{k}(A) = \mathrm{k}(B)\,,\ \mathrm{l}(A) = \mathrm{l}(B)\,,\ C^A = C^B\,,\ D^A = D^B\,, d(A) = d(B)\,.$$

Morita equivalence and centers. As it is well known, a Morita equivalence between A and B induces an algebra isomorphism between the centers $Z(A)$ and $Z(B)$, since $Z(A)$ may be viewed as the center of the category ${}_A\mathbf{mod}$. It also results from what follows (see [Ri2]), which also proves the preservation of projective centers.

We denote by A^{op} the opposite algebra of A, and $A^{\mathrm{en}} := A \otimes_{\mathcal{O}} A^{\mathrm{op}}$ the "enveloping algebra". Assume that (M, N) induces a Morita equivalence between A and B. Then $M \otimes_{\mathcal{O}} N$ is endowed with a natural structure of $(A^{\mathrm{en}}, B^{\mathrm{en}})$–bimodule defined by

$$(a \otimes a')(m \otimes n)(b \otimes b') := amb \otimes b'na'$$

and similarly $N \otimes_{\mathcal{O}} M$ is endowed with a natural structure of $(B^{\mathrm{en}}, A^{\mathrm{en}})$–bimodule. The map

$$M \otimes_{\mathcal{O}} N \times N \otimes_{\mathcal{O}} M \to \mathcal{O} \quad , \quad (m \otimes n, n' \otimes m') \mapsto \langle m, n' \rangle \langle m', n \rangle$$

is an $(A^{\mathrm{en}}, B^{\mathrm{en}})$–compatible duality.

3.4. Proposition. *With the previous notation, $(M \otimes_{\mathcal{O}} N, N \otimes_{\mathcal{O}} M)$ defines a Morita equivalence between A^{en} and B^{en} which exchanges A and B.*

3.5. Corollary. *A Morita equivalence between A and B induces an algebra isomorphism between $Z(A)$ and $Z(B)$ which restricts to an isomorphism between $Z^{\mathrm{pr}}(A)$ and $Z^{\mathrm{pr}}(B)$.*

Morita equivalence and defects of irreducible KA–modules. By 3.3, a Morita equivalence induces a bijection $I \colon \mathrm{Irr}(KB) \xrightarrow{\sim} \mathrm{Irr}(KA)$, and it results from 2.2, (2), that I preserves the defects : for all $Y \in \mathrm{Irr}(KB)$, $d(I(Y)) = d(Y)$. This is also a consequence of what follows, which will be generalized later on to the case of a Rickard equivalence.

For $X \in \mathrm{Irr}(KA)$ (resp. $Y \in \mathrm{Irr}(KB)$) we denote by e_X (resp. f_Y) the corresponding primitive idempotent of ZKA (resp. of ZKB). We denote by e (resp. f) the unity element of A (resp. B), so $A = \mathcal{O}Ge$ and $e = \sum_{X\in\mathrm{Irr}(KA)} e_X$ (resp. $B = \mathcal{O}Hf$ and $f = \sum_{Y\in\mathrm{Irr}(KB)} f_Y$). The set $\{e_X\}_{X\in\mathrm{Irr}(KA)}$ is a K–basis of ZKA, and if $\sum_X \lambda_X e_X \in Z(A)$, then $\lambda_X \in \mathcal{O}$ for all $X \in \mathrm{Irr}(KA)$.

Let (M, N) as in 3.2 which induces an equivalence between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$. It induces a bijection $\mathrm{Irr}(KA) \xrightarrow{\sim} \mathrm{Irr}(KB)$ which we denote by $X \mapsto Y_X$. The adjunctions $\eta_{N,M}$ and $\varepsilon_{M,N}$ are isomorphisms in ${}_A\mathbf{mod}_A$, hence $\eta_{N,M} \cdot \varepsilon_{M,N}$ is an automorphism of A in ${}_A\mathbf{mod}_A$ and it restricts to an automorphism of $Z(A)$ viewed as an $\mathcal{O}$–module.

3.6. Proposition. *With the previous notation,*

$$\eta_{N,M} \cdot \varepsilon_{M,N} \colon e \mapsto \sum_{X\in\mathrm{Irr}(KA)} \frac{|G|/\dim X}{|H|/\dim Y_X}\, e_X\,.$$

In particular, $\frac{|G|/\dim X}{|H|/\dim Y_X}$ *is invertible modulo p and the defects are preserved :* $d(X) = d(Y_X)$.

3.7. Corollary. *If e is primitive (i.e., if A is a block), then* $\frac{|G|/\dim X}{|H|/\dim Y_X}$ *is constant modulo p for $X \in \mathrm{Irr}(KA)$.*

3.C. Examples of Morita equivalences.

Clifford Theory. "Clifford theory" is the name of a set of theorems relating representations of a group G with representations of a normal subgroup N of G. It can be viewed as a series of Morita equivalences. We present here the first (and easy) part of Clifford theory : the "reduction to the inertial group".

Let N be a normal subgroup of G. Let f be a central primitive idempotent of $\mathcal{O}N$. We denote by H the stabilizer of f in G (which acts by conjugation on the set of central idempotents of $\mathcal{O}N$). Then f is a central idempotent of $\mathcal{O}H$. We set $B := \mathcal{O}Hf$.

We set $e := \sum_{g\in[G/H]} gfg^{-1}$, where $[G/H]$ denotes a set of representatives of the cosets of G modulo H. Then e is a central idempotent of $\mathcal{O}G$. We set $A := \mathcal{O}Ge$.

Let $M := e\mathcal{O}Gf = \mathcal{O}Gf$, endowed with left multiplication by A and right multiplication by B. Let $N := f\mathcal{O}Ge = f\mathcal{O}G$, viewed similarly as a (B, A)–bimodule. Since $M \underset{B}{\otimes} N \simeq A$ in ${}_A\mathbf{mod}_A$ and $N \underset{A}{\otimes} M \simeq B$ in ${}_B\mathbf{mod}_B$, and since the functors $M \underset{B}{\otimes} \cdot$ and $N \underset{A}{\otimes} \cdot$ are respectively $f \cdot \mathrm{Res}^G_H$ and Ind^G_H, we get :

3.8. Proposition. *The functors $f \cdot \mathrm{Res}^G_H$ and Ind^G_H are inverse equivalences between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$.*

p–nilpotent groups. Let G be a p–nilpotent group, *i.e.*, $G \simeq S \rtimes P$ where P is a p–group and S is a normal p'–subgroup (group with order prime to p) of G.

For every irreducible KS–module X, we denote by f_X the corresponding central primitive idempotent of KS (since S is a p'–group, $f_X \in Z\mathcal{O}S$) and by P_X the stabilizer of f_X in P.

3.9. Proposition. *We have*

$$ {}_{\mathcal{O}G}\mathbf{mod} \simeq \bigoplus_{X\in \mathrm{Irr}(KS) \mod P} {}_{\mathcal{O}P_X}\mathbf{mod}\,. $$

Sketch of proof. Set $e_X := \sum_{[P/P_X]} g f_X g^{-1}$, and $G_X := S \rtimes P_X$. By 3.8 above, we see that ${}_{\mathcal{O}Ge_X}\mathbf{mod}$ is equivalent to ${}_{\mathcal{O}G_X f_X}\mathbf{mod}$. Since $\mathcal{O}G = \bigoplus_{X\in\mathrm{Irr}(KS) \mod P} \mathcal{O}Ge_X$, it suffices to prove that ${}_{\mathcal{O}G_X f_X}\mathbf{mod} \simeq {}_{\mathcal{O}P_X}\mathbf{mod}$.

We may assume $P_X = P$. Let X_0 be an $\mathcal{O}$–free $\mathcal{O}S$–module such that $X = K\otimes_{\mathcal{O}} X_0$ (note that X_0 is unique up to isomorphism). Because $\dim X$ and $|P|$ are relatively prime, there is an action of $S \rtimes P$ on X_0 which extends the action of S. This allows us to define on $M := \mathcal{O}G \underset{\mathcal{O}S}{\otimes} X_0$ a structure of $(\mathcal{O}Ge_X, \mathcal{O}P)$–bimodule as follows :

$$ g.(g_1 \underset{\mathcal{O}S}{\otimes} x).\pi := gg_1 \underset{\mathcal{O}S}{\otimes} \pi^{-1}(x) \quad \text{for } g, g_1 \in G\,,\ x \in X_0\,,\ \pi \in P\,. $$

Similarly, the module $N := X_0^* \underset{\mathcal{O}S}{\otimes} \mathcal{O}G$ is endowed with a structure of $(\mathcal{O}P, \mathcal{O}Ge_X)$–bimodule, and it is not difficult to check that (M, N) induces a Morita equivalence between ${}_{\mathcal{O}Ge_X}\mathbf{mod}$ and ${}_{\mathcal{O}P}\mathbf{mod}$. $\square$

Remark. There are lots of Morita equivalences in the theory of blocks of p–solvable groups, analogous (although sometimes far deeper) to the ones just described for the case of a p–nilpotent group. The classical "Fong reduction theorem" (see [Fo]) may be viewed as the description of a Morita equivalence between two blocks of two p–solvable groups (see for example [Pu1]). The description of blocks of groups of p–length one relies on some highly non trivial Morita equivalences (see [Da]).

On the other hand, Morita equivalences between blocks seem far less frequent for non abelian simple groups. In this case, the equivalence must be weakened to what we call a Rickard equivalence.

4. Rickard equivalences between blocks

4.A. Complexes : Notation and conventions.

As in the previous section, we denote by G and H two finite groups, by e and f respectively two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$, and we set $A := \mathcal{O}Ge$, $B := \mathcal{O}Hf$.

Homomorphisms and tensor product of two complexes. The definitions we use here for the differentials of the homomorphisms and the tensor product of two complexes are slightly different from the usual ones (although they provide complexes isomorphic to the usual ones).

1. Let $X := \left(\cdots \to X^m \overset{d_X^m}{\to} X^{m+1} \to \cdots\right)$ and $Y := \left(\cdots \to Y^m \overset{d_Y^m}{\to} Y^{m+1} \to \cdots\right)$ be complexes in $\mod A$. We set

$$ \mathrm{Hom}_A^{\cdot}(X,Y) := \left(\cdots \to \mathrm{Hom}_A^m(X,Y) \overset{d^m}{\to} \mathrm{Hom}_A^{m+1}(X,Y) \to \cdots\right) $$

where

$$\mathrm{Hom}^m_A(X,Y) := \prod_{i,j,j-i=m} \mathrm{Hom}_A(X^i,Y^j) \quad \text{and}$$

$$d^m\colon \mathrm{Hom}_A(X^i,Y^j) \to \mathrm{Hom}_A(X^i,Y^{j+1}) \times \mathrm{Hom}_A(X^{i-1},Y^j)$$
$$\alpha \mapsto \Big((-1)^m d^j_Y \cdot \alpha, -\alpha \cdot d^{i-1}_X\Big)$$

We set $X^* := \mathrm{Hom}^{\cdot}_{\mathcal{O}}(X,\mathcal{O})$, viewed as a complex in $\mathbf{mod}_A$, and $d^m_{X^*} = -{}^t d_X^{-(m+1)}$.

2. Assume now that Y is a complex in $\mathbf{mod}_A$. We set

$$Y \underset{A}{\otimes} X := \Big(\cdots \to (Y \underset{A}{\otimes} X)^m \xrightarrow{d^m} (Y \underset{A}{\otimes} X)^{m+1} \to \cdots\Big)$$

where

$$(Y \underset{A}{\otimes} X)^m := \bigoplus_{i,j,i+j=m} (Y^j \underset{A}{\otimes} X^i) \quad \text{and}$$

$$d^m\colon (Y^j \underset{A}{\otimes} X^i) \to (Y^{j+1} \underset{A}{\otimes} X^i) \oplus (Y^j \underset{A}{\otimes} X^{i+1})$$
$$y \otimes x \mapsto \Big((-1)^m d^j_Y(y) \otimes x\Big) \oplus \big(y \otimes d^i_X(x)\big)$$

Some classical maps. We denote now by $M := \Big(\cdots \to M^i \xrightarrow{d^i_M} M^{i+1} \to \cdots\Big)$ a *bounded complex of (A,B)–bimodules.*

Let $N := \Big(\cdots \to N^j \xrightarrow{d^j_N} N^{j+1} \to \cdots\Big)$ be a complex in ${}_B\mathbf{mod}_A$. An (A,B)–compatible $\mathcal{O}$–duality between M and N is the following datum :

(1) for all i, an (A,B)–compatible duality $<,>\colon N^i \times M^{-i} \to \mathcal{O}$,

(2) such that the maps d^i_N and $-d_M^{-(i+1)}$ are adjoint for this duality.

- We denote by $\varepsilon_{M^i,N^{-i}}\colon M^i \underset{B}{\otimes} N^{-i} \to A$ the map defined by

$$\varepsilon_{M^i,N^{-i}}(m \otimes n) := \sum_{g \in G} <n, g^{-1}m> g\,,$$

and we denote by $\varepsilon_{M,N}\colon M \underset{B}{\otimes} N \to A$ the chain map defined by

$$\varepsilon_{M,N} := \sum_i (-1)^i \varepsilon_{M^i,N^{-i}}\,.$$

- We denote by $\tau_{N^{-j},M^i}\colon N^{-j} \underset{A}{\otimes} M^i \to \mathrm{Hom}_A(M^j,M^i)$ the map defined by

$$\tau_{N^{-j},M^i}(n \otimes m)\colon m' \mapsto \varepsilon_{M^j,N^{-j}}(m' \otimes n)m\,,$$

and we denote by $\tau_{N,M}\colon N \underset{A}{\otimes} M \to \mathrm{Hom}^{\bullet}_A(M,M)$ the chain map defined by the family $(\tau_{N^{-j},M^i})_{i,j}$.

• Finally, we denote by $\sigma^B_{M^i}\colon B \to \mathrm{Hom}_A(M^i, M^i)$ the morphism which defines the structure of module–B on M, and we denote by $\sigma^B_M\colon B \to \mathrm{Hom}^{\bullet}_A(M,M)$ the chain map defined by

$$b \mapsto \prod_i \sigma^B_{M^i}(b)\,.$$

The adjunctions. Let ${}_A\mathbf{com}$ be the category of complexes in ${}_A\mathbf{mod}$ with chain maps as morphisms, and let ${}_A\mathbf{com}_B$ be the category of complexes in ${}_A\mathbf{mod}_B$ with chain maps as morphisms.

Assume that, as above, M is a bounded complex of (A,B)–bimodules, N is a bounded complex of (B,A)–bimodules with a given (A,B)–compatible duality with M. Assume moreover that each component M^i of M is projective both as an A–module and as a module–B. Then the functors

$$M \underset{B}{\otimes} \bullet\colon {}_B\mathbf{com} \to {}_A\mathbf{com} \quad \text{and} \quad N \underset{A}{\otimes} \bullet\colon {}_A\mathbf{com} \to {}_B\mathbf{com}$$

are adjoint one to the the other on both sides.

The chain map $\tau_{N,M}\colon N \underset{A}{\otimes} M \to \mathrm{Hom}^{\bullet}_A(M,M)$ is an isomorphism in ${}_B\mathbf{com}_B$. We denote by $\eta_{M,N}\colon B \to N \underset{A}{\otimes} M$ the chain map defined by $\eta_{M,N} := \tau^{-1}_{N,M} \cdot \sigma^B_M$.

Definition. *We call adjunctions the two pairs of chain maps of complexes of bimodules*

$$\begin{aligned} \varepsilon_{M,N}\colon M \underset{B}{\otimes} N \to A \quad &\textit{and} \quad \eta_{M,N}\colon B \to N \underset{A}{\otimes} M \\ \varepsilon_{N,M}\colon N \underset{A}{\otimes} M \to B \quad &\textit{and} \quad \eta_{N,M}\colon A \to M \underset{B}{\otimes} N \end{aligned}$$

which define respectively adjunctions for the adjoint pairs $(M \underset{B}{\otimes} \bullet, N \underset{A}{\otimes} \bullet)$ *and* $(N \underset{A}{\otimes} \bullet, M \underset{B}{\otimes} \bullet)$.

4.B. Rickard equivalences and block invariants.

Definition. *We say that A and B are Rickard equivalent if there exists*

- *a bounded complex M in ${}_A\mathbf{com}_B$, each component of which is both projective as an A–module and as a module–B,*
- *a bounded complex N in ${}_B\mathbf{com}_A$, each component of which is both projective as an B–module and as a module–A,*
- *an (A,B)–compatible $\mathcal{O}$–duality between M and N,*

$$\textit{such that} \begin{cases} M \underset{B}{\otimes} N \textit{ is homotopy equivalent to } A \textit{ in } {}_A\mathbf{com}_A \\ N \underset{A}{\otimes} M \textit{ is homotopy equivalent to } B \textit{ in } {}_B\mathbf{com}_B\,. \end{cases}$$

In this case, the complexes M and N are called "Rickard tilting complexes" for A and B.

We denote by $\mathcal{D}^b(A)$ the derived bounded category of ${}_A\mathbf{mod}$.

The following theorem is due to J. Rickard ([Ri1], [Ri3]). Note that it may be viewed as a generalization of Morita theorem 3.2.

4.1. Theorem. *The following assertions are equivalent :*

(i) *The derived bounded categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.*

(ii) *The algebras A and B are Rickard equivalent.*

Moreover, in this case, the adjunctions $\varepsilon_{M,N}$, $\eta_{M,N}$, $\varepsilon_{N,M}$, $\eta_{N,M}$ are all homotopy equivalences between the corresponding complexes of bimodules.

Like in the case of a Morita equivalence (see §3 above), any pair of complexes (M,N) which induces a Rickard equivalence between A and B defines pairs $(K\otimes_{\mathcal{O}} M, K\otimes_{\mathcal{O}} N)$ and $(k\otimes_{\mathcal{O}} M, k\otimes_{\mathcal{O}} N)$ which induce Rickard equivalences between respectively KA and KB, kA and kB.

An object of $\mathcal{D}^b(A)$ is called *perfect* (see [Gro]) if it is isomorphic to a bounded complex of projective A–modules. Let $\mathcal{D}^b_{\mathrm{perf}}(A)$ be the full subcategory of $\mathcal{D}^b(A)$ consisting of perfect complexes. A Rickard equivalence between A and B induces an equivalence of categories between $\mathcal{D}^b_{\mathrm{perf}}(A)$ and $\mathcal{D}^b_{\mathrm{perf}}(B)$.

Rickard equivalences and triangle invariants. The Grothendieck groups of the triangulated categories (see [Gro], and also [Ha]) $\mathcal{D}^b(KA)$, $\mathcal{D}^b(kA)$, $\mathcal{D}^b_{\mathrm{perf}}(kA)$ are respectively the groups $\mathcal{R}(KA)$, $\mathcal{R}(kA)$, $\mathcal{R}^{\mathrm{pr}}(kA)$. Hence a Rickard equivalence induces isomorphisms between these groups. By construction, these isomorphisms commute with the maps of the c-d-triangle (see §1).

The metric structure on the c-d-triangle may be defined in terms of the derived category. For example, the duality between $\mathcal{R}(kA)$ and $\mathcal{R}^{\mathrm{pr}}(kA)$ is defined as follows. For $P := \left(\cdots\to P^i\to P^{i+1}\to\cdots\right)$ a bounded complex of projective A–modules, object of $\mathcal{D}^b_{\mathrm{perf}}(kA)$, and $X := \left(\cdots\to X^i\to X^{i+1}\to\cdots\right)$ an object of $\mathcal{D}^b(kA)$, we have

$$<[P],[X]> := \sum_i (-1)^i \dim \mathrm{Hom}_{\mathcal{D}^b(kA)}(P[i],X)\,.$$

Hence the isomorphisms between triangles defined by a Rickard equivalence preserve the natural $\mathbb{Z}$–dualities.

Since $\mathrm{Irr}(KA)$ is an orthonormal basis of the $\mathbb{Z}$–module $\mathcal{R}(KA)$, it follows that the isomorphism $\mathcal{R}(KA)\xrightarrow{\sim}\mathcal{R}(KB)$ sends an element of $\mathrm{Irr}(KA)$ onto an element of $\{\pm Y \mid (Y\in\mathrm{Irr}(KB))\}$ (while a Morita equivalence induces a bijection between $\mathrm{Irr}(KA)$ and $\mathrm{Irr}(KB)$). There is no analogous property for $\mathrm{Irr}(kA)$ or $\mathrm{Pim}(kA)$.

4.2. Proposition. *A Rickard equivalence between A and B induces an isomorphism between the c-d-triangles $\mathcal{T}(A)$ and $\mathcal{T}(B)$ (viewed as endowed with their natural "metric").*

$$\begin{tikzcd}
 & & & \mathcal{R}(KB) \arrow[rr, "\mathrm{dec}^B"] & & \mathcal{R}(kB) \\
\mathcal{R}(KA) \arrow[rr, "\mathrm{dec}^A"] \arrow[urrr, leftrightarrow] & & \mathcal{R}(kA) \arrow[urrr, leftrightarrow] & & \mathcal{R}^{\mathrm{pr}}(kB) \arrow[ul] \arrow[ur, "\mathrm{Car}^B"'] & \\
 & \mathcal{R}^{\mathrm{pr}}(kA) \arrow[ul, "{}^t\mathrm{dec}^A"] \arrow[ur, "\mathrm{Car}^A"'] \arrow[urrr, leftrightarrow] & & & &
\end{tikzcd}$$

As a consequence, a Rickard equivalence between A and B preserves all the invariants determined by the c-d-triangles and their metrics :

- $\mathrm{k}(A) = \mathrm{k}(B)$, $\mathrm{l}(A) = \mathrm{l}(B)$, $d(A) = d(B)$,
- the Cartan matrices C^A and C^B are equivalent as matrices of quadratic forms over $\mathbb{Z}$ (in particular they have the same invariant factors, and $\mathrm{l}^{\mathrm{pr}}(A) = \mathrm{l}^{\mathrm{pr}}(B)$),
- the decomposition matrices are equivalent as follows : there exists an orthonormal matrix U in $\mathrm{Mat}_{\mathrm{k}(A)}(\mathbb{Z})$ and an invertible matrix V in $\mathrm{Mat}_{\mathrm{l}(A)}(\mathbb{Z})$ such that $D^B = UD^AV$ (and in particular D^A and D^B have the same invariant factors).

Rickard equivalence and centers. A Rickard equivalence between A and B induces an algebra isomorphism between the centers $Z(A)$ and $Z(B)$, since $Z(A)$ may be viewed as the center of the category $\mathcal{D}^b(A)$.

Like in the case of Morita equivalences (see §3), it also results from the following proposition.

4.3. Proposition. *If (M, N) induces a Rickard equivalence between A and B, then $(M \otimes_{\mathcal{O}} N, N \otimes_{\mathcal{O}} M)$ defines a Rickard equivalence between A^{en} and B^{en} which exchanges A and B.*

Since a morphism between two modules factorizes through a projective module if and only if it factorizes through a perfect complex, we get as a consequence :

4.4. Corollary. *A Rickard equivalence between A and B induces an algebra isomorphism between $Z(A)$ and $Z(B)$ which restricts to an isomorphism between $Z^{\mathrm{pr}}(A)$ and $Z^{\mathrm{pr}}(B)$.*

Rickard equivalence and defects of irreducible KA–modules. Like in §3, for $X \in \mathrm{Irr}(KA)$, we denote by e_X the corresponding primitive idempotent of ZKA. We denote by e (resp. f) the unity element of A (resp. B).

Let (M, N) be a pair of complexes which induces a Rickard equivalence between $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$. It sends an element $X \in \mathrm{Irr}(KA)$ onto $\varepsilon_X Y_X$ where $\varepsilon_X = \pm 1$ and $Y_X \in \mathrm{Irr}(KB)$. The adjunctions $\eta_{N,M}$ and $\varepsilon_{M,N}$ are homotopy equivalences between objects of ${}_A\mathbf{com}_A$, hence $\eta_{N,M} \cdot \varepsilon_{M,N}$ is an automorphism of A in ${}_A\mathbf{mod}_A$ and it restricts to an automorphism of $Z(A)$ viewed as an $\mathcal{O}$–module.

4.5. Proposition. *With the previous notation,*

$$\eta_{N,M} \cdot \varepsilon_{M,N} \colon e \mapsto \sum_{X \in \mathrm{Irr}(KA)} \frac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X} \, e_X \, .$$

In particular, $\dfrac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X}$ is invertible modulo p and the defects are preserved : $d(X) = d(Y_X)$.

4.6. Corollary. *If e is primitive in $Z(A)$ (i.e., if A is a block), $\dfrac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X}$ is constant modulo p for $X \in \mathrm{Irr}(KA)$.*

4.C. Examples of Rickard equivalences. Although derived equivalences are conjecturally very frequent in block theory, only a very small number of them is actually proved.

Groups with cyclic Sylow p–subgroups. The following result is a particular case of results proved by Rickard and Linckelmann (see [Ri2] and [Li]) as a consequence of the structure theorem of blocks with cyclic defect groups.

4.7. Theorem. *Assume that G has a cyclic Sylow p–subgroup P. Let us denote by A_0 and B_0 respectively the principal blocks of G and of the normalizer of P in G. Then A_0 and B_0 are Rickard equivalent.*

Remark. At this date (december 1992), no explicit construction of a Rickard tilting complex is known.

The principal 2–block of $\mathfrak{A}_5$ and $\mathfrak{A}_4$. Let us denote by G the alternating group $\mathfrak{A}_5$ on 5 letters, and by H the normalizer of a Sylow 2-subgroup of G, isomorphic to the alternating group $\mathfrak{A}_4$. Let $p = 2$. Let A_0 (resp. B_0) be the principal block of G (resp. H).

It is easy to check that A_0 and B_0 are not Morita equivalent (since, for example, they have different decomposition matrices). Nevertheless, the functors Ind_H^G and Res_H^G induce a stable equivalence between A_0 and B_0. The following unpublished result of Rickard shows that one can "twist" this stable equivalence to get a Rickard equivalence.

Let $\mathcal{A}\mathcal{O}G$ be the augmentation ideal of $\mathcal{O}G$ and let $P(\mathcal{A}\mathcal{O}G) \xrightarrow{d} \mathcal{A}\mathcal{O}G$ be the projective cover of $\mathcal{A}\mathcal{O}G$ viewed as a $(\mathcal{O}G, \mathcal{O}H)$–bimodule. We set

$$M := \left(\cdots \to 0 \to P(\mathcal{A}\mathcal{O}G) \xrightarrow{d} \mathcal{O}G \to 0 \to \cdots \right)$$

and $N := M^*$.

4.8. Theorem. *The pair (M, N) defines a Rickard equivalence between A_0 and B_0.*

A conjecture. The preceding two examples are particular cases of a conjectural general result (see [Br1]).

4.9. Conjecture. *Let G be a finite group whose Sylow p–subgroups are abelian. Let H be the normalizer of one of the Sylow p–subgroups of G. Then the principal blocks of G and H are Rickard equivalent.*

4.D. Perfect isometries.

The preceding conjecture seems hard to prove (or even to check on examples) at the moment. Nevertheless, one of its non–trivial consequences, which should be viewed as the "shadow", at the level of characters, of a Rickard equivalence, has already been checked on a long series of cases.

For the definitions and properties stated in this paragraph, see [Br1].

As in §3, we denote by G and H two finite groups, by e and f respectively two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$, and we set $A := \mathcal{O}Ge$, $B := \mathcal{O}Hf$. From now on, we identify $\mathcal{R}(KG)$ with the group of virtual characters of KG–representations, and $\mathrm{Irr}(KG)$ with the set of irreducible characters.

Let μ be a virtual character of $G \times H$, element of $\mathcal{R}(K[G \times H])$. Then μ corresponds to a linear map $I_\mu \colon \mathcal{R}(KH) \to \mathcal{R}(KG)$ as follows : for $\zeta \in \mathrm{Irr}(KH)$, the function $I_\mu(\zeta)$ is defined by $I_\mu(\zeta)(g) := \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1})\zeta(h)$.

Definition. *We say that a virtual character μ of $G \times H$ is perfect if :*

(pe.1) *for all $g \in G$ and $h \in H$, $|C_G(g)|_p$ and $|C_H(h)|_p$ both divide $\mu(g,h)$,*

(pe.2) *if $\mu(g,h) \neq 0$, then either g and h are both p–regular, or g and h are both p–singular.*

If moreover the map I_μ defined by μ induces an isometric bijection from $\mathcal{R}(KB)$ to $\mathcal{R}(KA)$, we say that I_μ is a perfect isometry between B and A, and that A and B are perfectly isometric.

The connection with Rickard equivalences is made by the following statement.

4.10. Proposition. *Assume that M is a Rickard tilting complex for A and B. Let μ_M be the virtual character of $G \times H$ defined by*

$$\mu_M(g,h) := \sum_i (-1)^i \mathrm{tr}((g, h^{-1}); M^i)\,.$$

Then μ_M defines a perfect isometry between B and A.

The point is that, if A and B are perfectly isometric, their invariants behave "as if" they were Rickard equivalent (see [Br1]) — compare with assertions 4.2, 4.4, 4.5 above.

4.11. Theorem. *Suppose that A and B are perfectly isometric.*

(1) There is an isomorphism between the c-d-triangles $\mathcal{T}(A)$ and $\mathcal{T}(B)$ (viewed as endowed with their natural "metric"). In particular,

- $\mathrm{k}(A) = \mathrm{k}(B)$, $\mathrm{l}(A) = \mathrm{l}(B)$, $d(A) = d(B)$,
- *the Cartan matrices C^A and C^B are equivalent as matrices of quadratic forms over $\mathbb{Z}$,*
- *the decomposition matrices are equivalent as follows : there exists an orthonormal matrix U in $\mathrm{Mat}_{\mathrm{k}(A)}(\mathbb{Z})$ and an invertible matrix V in $\mathrm{Mat}_{\mathrm{l}(A)}(\mathbb{Z})$ such that $D^B = UD^AV$.*

(2) There is an algebra isomorphism between $Z(A)$ and $Z(B)$ which restricts to an isomorphism between $Z^{\mathrm{pr}}(A)$ and $Z^{\mathrm{pr}}(B)$.

(3) There is an automorphism of (A,A)–bimodules of $Z(A)$ such that, if $I_{\mu_M}^{-1}(X) = \varepsilon_X Y_X$, then

$$e \mapsto \sum_{X \in \mathrm{Irr}(KA)} \frac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X}\, e_X\,.$$

In particular, $\dfrac{|G|/\dim X}{|H|/\varepsilon_X \dim Y_X}$ is invertible modulo p and the defects are preserved : $d(X) = d(Y_X)$.

The following conjecture is a weaker form of 4.9.

4.12. Conjecture. *Let G be a finite group whose Sylow p–subgroups are abelian. Let H be the normalizer of one of the Sylow p–subgroups of G. Then the principal blocks of G and H are perfectly isometric.*

The preceding conjecture is known to be true in the following cases :

- for all p, if G is p–solvable ;
- for $p = 2$, in all cases ([FoHa]) ;
- for all p, if G is a sporadic simple group ([Rou]) ;
- for all p, if G is a symmetric group ([Rou]) or an alternating group (Fong, *private communication*) ;
- if G is the group of rational points of a connected reductive algebraic group $\mathbf{G}$ defined over $\mathbb{F}_q$ and p is a prime number which does not divide q and which is good for $\mathbf{G}$ ([BMM], [BrMi]).

4.E. The case of the finite reductive groups.

In the case where G is a "finite reductive group", the conjecture 4.9 can be made more precise and closely linked with the underlying algebraic geometry (for more details, see [BrMa]).

Notation. In this paragraph, we temporarily change our notation to fit with the usual notation of finite reductive groups : our prime p (the characteristic of our field $k := \mathcal{O}/\mathfrak{p}$) is now denoted by ℓ, and q denotes a power of another prime $p \neq \ell$.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, endowed with a Frobenius endomorphism F which defines a rational structure on $\mathbb{F}_q$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, with unipotent radical $\mathbf{U}$, and with F–stable Levi subgroup $\mathbf{L}$. We denote by $\mathrm{Y}(\mathbf{U})$ the associated Deligne–Lusztig variety defined (*cf.* for example [Lu]) by

$$\mathrm{Y}(\mathbf{U}) := \{g(\mathbf{U} \cap F(\mathbf{U})) \in \mathbf{G}/\mathbf{U} \cap F(\mathbf{U})\,;\, g^{-1}F(g) \in F(\mathbf{U})\}\,,$$

and we recall that $\mathbf{G}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by left multiplication while $\mathbf{L}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by right multiplication. It is known (*cf.* [Lu]) that $\mathrm{Y}(\mathbf{U})$ is an $\mathbf{L}^F$–torsor on a variety $\mathrm{X}(\mathbf{U})$, which is smooth of pure dimension equal to $\dim(\mathbf{U}/\mathbf{U} \cap F(\mathbf{U}))$, and which is affine (at least if q is large enough). In particular $\mathrm{X}(\mathbf{U})$ is endowed with a left action of $\mathbf{G}^F$. If $\mathcal{O}$ is a commutative ring, the image of the constant sheaf $\mathcal{O}$ on $\mathrm{Y}(\mathbf{U})$ through the finite morphism $\pi\colon \mathrm{Y}(\mathbf{U}) \to \mathrm{X}(\mathbf{U})$ is a locally constant constant sheaf $\pi_*(\mathcal{O})$ on $\mathrm{X}(\mathbf{U})$. We denote this sheaf by $\mathcal{F}_{\mathcal{O}\mathbf{L}^F}$.

Let ℓ be a prime number which does not divide q and let $\mathcal{O}$ be the ring of integers of a finite extension of the field $\mathbb{Q}_\ell$ of ℓ–adic numbers. For any $\mathbf{G}^F$–equivariant torsion free $\mathcal{O}$–sheaf $\mathcal{F}$ on $\mathrm{X}(\mathbf{U})$, we denote by $\mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F})$ the algebra of endomorphisms of the "ℓ–adic cohomology" complex $\mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}), \mathcal{F})$ viewed as an element of the derived bounded category $\mathcal{D}^b(\mathcal{O}\mathbf{G}^F)$ of the category of finitely generated $\mathcal{O}\mathbf{G}^F$–modules.

We set $\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U})) := \mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F})$ and $\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U})) := \mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F})$.

Note that the algebra $\mathcal{H}_{\mathcal{O}}(\mathrm{Y}(\mathbf{U}))$ contains the group algebra $\mathcal{O}\mathbf{L}^F$ as a subalgebra.

For K an extension of $\mathcal{O}$, we set $\mathcal{H}_K(\mathrm{X}(\mathbf{U}), \mathcal{F}) := K \otimes_{\mathcal{O}} \mathcal{H}_{\mathcal{O}}(\mathrm{X}(\mathbf{U}), \mathcal{F})$.

The data.

- Let ℓ be a prime number, $\ell \neq p$, which does not divide $|Z(\mathbf{G})/Z^o(\mathbf{G})|$ nor $|Z(\mathbf{G}^*)/Z^o(\mathbf{G}^*)|$, and which is good for $\mathbf{G}$. We assume that the Sylow ℓ–subgroups of $\mathbf{G}^F$ are *abelian*.

- Let $\mathcal{O}$ be the ring of integers of a finite unramified extension k of the field of ℓ–adic numbers $\mathbb{Q}_\ell$, with residue field k, such that the finite group algebra $k\mathbf{G}^F$ is split.
- Let $A = \mathcal{O}\mathbf{G}^F e$ be the principal block of $\mathcal{O}\mathbf{G}^F$. Let S be a Sylow ℓ–subgroup of $\mathbf{G}^F$, let $\mathbf{L} := C_{\mathbf{G}}(S)$, and let f be the principal block idempotent of $\mathcal{O}\mathbf{L}^F$.

The group $\mathbf{L}$ is a rational Levi subgroup of $\mathbf{G}$. We have $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{L})$, and we set $W_{\mathbf{G}^F}(\mathbf{L}) := N_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$. The group S is a Sylow ℓ–subgroup of $Z(\mathbf{L})^F$, and ℓ does not divide $|W_{\mathbf{G}^F}(\mathbf{L})|$.

Conjectures. There exist

• a parabolic subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and Levi complement $\mathbf{L}$,

• a finite complex $\mathbf{\Upsilon} = (\cdots \to \mathbf{\Upsilon}^{n-1} \to \mathbf{\Upsilon}^n \to \mathbf{\Upsilon}^{n+1} \to \cdots)$ of $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodules, which are finitely generated projective as $\mathcal{O}\mathbf{G}^F$–modules as well as $\mathcal{O}\mathbf{L}^F$–modules,

with the following properties.

(C1) Viewed as an object of the category $\mathcal{D}^b({}_{\mathcal{O}\mathbf{G}^F}\mathbf{mod}_{\mathcal{O}\mathbf{L}^F})$, the complex $\mathbf{\Upsilon}$ is isomorphic to $\mathrm{R}\Gamma_c(\mathrm{Y}(\mathbf{U}))$. In particular, for each n, the n-th homology group of $\mathbf{\Upsilon}$ is isomorphic, as an $(\mathcal{O}\mathbf{G}^F, \mathcal{O}\mathbf{L}^F)$–bimodule, to $\mathcal{O} \otimes_{\mathbb{Z}_\ell} \mathrm{H}^n_c(\mathrm{Y}(\mathbf{U}), \mathbb{Z}_\ell)$.

(C2) The idempotent e acts as the identity on the complex $\mathbf{\Upsilon}.f$,

(C3) • the structure of complex of $(A, \mathcal{O}\mathbf{L}^F f)$–bimodules of $\mathbf{\Upsilon}.f$ extends to a structure of complex of $(A, f\mathcal{H}_\mathcal{O}(\mathrm{Y}(\mathbf{U}))f)$–bimodules, all of which are projective as right $f\mathcal{H}_\mathcal{O}(\mathrm{Y}(\mathbf{U}))f$–modules,
• the complexes $(\mathbf{\Upsilon}.f \otimes_{f\mathcal{H}_\mathcal{O}(\mathrm{Y}(\mathbf{U}))f} f.\mathbf{\Upsilon}^*)$ and A are homotopy equivalent as complexes of (A, A)–bimodules,
• the complexes $(f.\mathbf{\Upsilon}^* \otimes_{\mathcal{O}\mathbf{G}^F e} \mathbf{\Upsilon}.f)$ and $f\mathcal{H}_\mathcal{O}(\mathrm{Y}(\mathbf{U}))f$ are homotopy equivalent as complexes of $(f\mathcal{H}_\mathcal{O}(\mathrm{Y}(\mathbf{U}))f, f\mathcal{H}_\mathcal{O}(\mathrm{Y}(\mathbf{U}))f)$–bimodules.

(C4) The algebra $f\mathcal{H}_\mathcal{O}(\mathrm{Y}(\mathbf{U}))f$ is isomorphic to the principal block $\mathcal{O}N_{\mathbf{G}^F}(S)f$.

5. Stable equivalences of Morita type

5.A. Definition and first remarks.

An example. Some blocks may look very similar without being Morita equivalent nor even Rickard equivalent. This is often the case in the following situation :

(p–t.i.) We assume that the Sylow p–subgroups of G are t.i., *i.e.,* for S a Sylow subgroup, for all $g \in G$, one has $S \cap gSg^{-1} = \{1\}$ or S. We set $H := N_G(S)$ and we denote respectively by A and B the principal blocks of G and H.

It is easy to see that the functors Ind_H^G and Res_H^G induce inverse stable equivalences between ${}_A\mathbf{mod}$ and ${}_B\mathbf{mod}$. Such a stable equivalence has certain properties we shall formalize below : it is a "stable equivalence of Morita type".

The known examples show that, under hypothesis (p–t.i.), if S is non abelian and G is a non abelian simple group, the algebras A and B are not necessarily Morita nor Rickard equivalent, although they have the same numbers of irreducible characters : $\mathrm{k}(A) = \mathrm{k}(B)$ and $\mathrm{l}(A) = \mathrm{l}(B)$, according to Alperin's conjecture ([Al]).

For example, if $G = \mathrm{Sz}(8)$ and $p = 2$, the algebras A and B have non isomorphic centers (G. Cliff, *private communication*) and their Cartan matrices are not quadratically equivalent ([Br]).

Stable equivalences, Morita and Rickard equivalences. As before, we denote by G and H two finite groups, by e and f two central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively, and we set $A := \mathcal{O}Ge$, $B := \mathcal{O}Hf$. We denote by ${}_A\mathbf{stab}$ and ${}_B\mathbf{stab}$ the stable categories of A and B respectively. The stable categories have a natural structure of triangulated categories (see for example [Ha]).

Definition. *We say that A and B are "stably equivalent à la Morita" (or that there is a stable equivalence of Morita type between A and B) if there exist*

- *an (A, B)–bimodule M which is projective as an A–module and as a module–B,*
- *a (B, A)–bimodule N which is projective as a B–module and as a module–A,*
- *an (A, B)–compatible $\mathcal{O}$–duality between M and N,*

$$\text{such that } \begin{cases} M \underset{B}{\otimes} N \text{ is stably equivalent to } A \text{ in } {}_A\mathbf{mod}_A \\ N \underset{A}{\otimes} M \text{ is stably equivalent to } B \text{ in } {}_B\mathbf{mod}_B\,. \end{cases}$$

The following statement is trivial.

5.1. Proposition. *A stable equivalence of Morita type between A and B induces an equivalence of triangulated categories between the stable categories ${}_A\mathbf{stab}$ and ${}_B\mathbf{stab}$*

It is obvious that if A and B are Morita equivalent, then they are stably equivalent à la Morita. Since the stable category ${}_A\mathbf{stab}$ is equivalent to the quotient category $\mathcal{D}^b(A)/\mathcal{D}^b_{\mathrm{perf}}(A)$ (see [Ri2]), we see that if A and B are Rickard equivalent, then they are stably equivalent. In fact, Rickard proved a more precise result ([Ri3], 5.5)[1]:

5.2. Proposition. *Assume that A and B are Rickard equivalent. Then A and B are stably equivalent à la Morita.*

5.B. Stable equivalences of Morita type and block invariants.

Stable triangle invariants. Let us set

$$\mathcal{R}^{\mathrm{st}}(KA) := \mathcal{R}(KA)/\mathrm{im}({}^t\mathrm{dec}^A) = \text{“}\mathcal{R}(KA)/\mathcal{R}^{\mathrm{pr}}(kA)\text{”}$$
$$\mathcal{R}^{\mathrm{st}}(kA) := \mathcal{R}(kA)/\mathrm{im}(\mathrm{Car}^A) = \text{“}\mathcal{R}(kA)/\mathcal{R}^{\mathrm{pr}}(kA)\text{”}\,.$$

The triangle $\mathcal{T}(A)$ defines by quotient a morphism of abelian groups :

$$(\mathcal{T}^{\mathrm{st}}(A)) \qquad \mathcal{R}^{\mathrm{st}}(KA) \xrightarrow{\mathrm{dec}^A_{\mathrm{st}}} \mathcal{R}^{\mathrm{st}}(kA)$$

Since $\mathcal{R}^{\mathrm{st}}(kA)$ is the Grothendieck group of the stable category of kA (viewed as triangulated category), a stable equivalence of Morita type induces an isomorphism between $\mathcal{R}^{\mathrm{st}}(kA)$ and $\mathcal{R}^{\mathrm{st}}(kB)$. More precisely :

5.3. Proposition. *A stable equivalence of Morita type between A and B induces*

(1) *an isomorphism between $\mathcal{T}^{\mathrm{st}}(A)$ and $\mathcal{T}^{\mathrm{st}}(B)$,*
(2) *an isometry between $\ker \mathrm{dec}^A$ and $\ker \mathrm{dec}^B$.*

[1] Strictly speaking, this result concerns group algebras over a *field* ; but it can easily be extended to our context.

$$\begin{tikzcd}
 & \mathcal{R}^{\mathrm{st}}(KB) \arrow[rr, "\mathrm{dec}^{B}_{\mathrm{st}}"] & & \mathcal{R}^{\mathrm{st}}(kB) \\
\mathcal{R}^{\mathrm{st}}(KA) \arrow[ur, leftrightarrow] \arrow[rr, "\mathrm{dec}^{A}_{\mathrm{st}}"] & & \mathcal{R}^{\mathrm{st}}(kA) \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

As a consequence, a stable equivalence of Morita type preserves the numerical invariants attached to the "stable triangles" $\mathcal{T}^{\mathrm{st}}$. For example :

$$\mathrm{k}(A) - \mathrm{l}(A) = \mathrm{k}(B) - \mathrm{l}(B)\,,\ \mathrm{l}(A) - \mathrm{l}^{\mathrm{pr}}(A) = \mathrm{l}(B) - \mathrm{l}^{\mathrm{pr}}(B)\,,\ d(A) = d(B)\,,$$

and the Cartan matrices C^A and C^B have the same non trivial invariant factors.

Stable equivalences of Morita type and centers. We set $Z^{\mathrm{st}}(A) := Z(A)/Z^{\mathrm{pr}}(A)$ and we call this algebra the *stable center* of A.

It is not true in general that the stable center of an algebra is the center of its stable category. Nevertheless :

5.4. Proposition. *A stable equivalence of Morita type between A and B induces an algebra isomorphism between $Z^{\mathrm{st}}(A)$ and $Z^{\mathrm{st}}(B)$.*

Proof. Let ${}_A\mathbf{stab}_A$ denote the stable category of A^{en}, and let ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ denote the full subcategory of ${}_A\mathbf{stab}_A$ whose objects are the (A,A)–bimodules which are projective as A–module and as module-A. Assume that (M,N) induces a stable equivalence of Morita type between A and B. Then the pair $(M \underset{\mathcal{O}}{\otimes} N\,,\ N \underset{\mathcal{O}}{\otimes} M)$ (where $M \underset{\mathcal{O}}{\otimes} N$ is viewed as an $(A^{\mathrm{en}}, B^{\mathrm{en}})$–bimodule and $N \underset{\mathcal{O}}{\otimes} M$ is viewed as a $(B^{\mathrm{en}}, A^{\mathrm{en}})$–bimodule, as in §3 above) induce inverse equivalences between ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ and ${}_B\mathbf{stab}_B^{\mathrm{pr}}$ which exchange A and B. The assertion follows from the fact that $Z^{\mathrm{st}}(A)$ is the algebra of endomorphisms of A in ${}_A\mathbf{stab}_A^{\mathrm{pr}}$. $\square$

Example. Let ΩA denote the kernel of the multiplication map

$$A \otimes_{\mathcal{O}} A \to A\ ,\ a \otimes a' \mapsto aa'\,.$$

Then the pair of (A,A)–bimodules $(\Omega A, (\Omega A)^*)$ induces a self stable equivalence of Morita type of A. Let X be an A–module and let $\pi\colon P \to X$ be a surjective morphism, where P is a projective A–module. Then there is a unique isomorphism $\ker \pi \xrightarrow{\sim} \Omega A \underset{A}{\otimes} X$ in ${}_A\mathbf{stab}$.

Remark. It is not known at the moment whether the existence of a stable equivalence of Morita type between A and B implies that $\mathrm{k}(A) = \mathrm{k}(B)$, and, if so, if there is a bijection between $\mathrm{Irr}(KA)$ and $\mathrm{Irr}(KB)$ which preserves the defects.

6. Inputting the group action

In all what has been stated so far, A and B might as well have been symmetric algebras over $\mathcal{O}$ — the groups themselves did not play an essential role. In what follows, we give brief indications on the actual methods of *group* representation theory.

6.A. Defect groups and source algebras.

Let $\Delta\colon G \to G\times G$ be the diagonal morphism. As $G\times G$–module, $\mathcal{O}G$ is isomorphic to $\mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{O}$.

6.1. Theorem–Definition. *(*[Gre], [Al2], [Pu2]*) Let A be a block of $\mathcal{O}G$.*

(1) The vertices of the $\mathcal{O}[G\times G]$–module A are the $G\times G$–conjugates of ΔD, where D is a p–subgroup of G. The G–conjugates of D are called the defect groups of A.

(2) Let S be an indecomposable summand of $\mathrm{Res}_{G\times D}^{G\times G} A$ with vertex ΔD. Such an S, viewed as an A–module, is unique up to isomorphism, and is a progenerator of ${}_A\mathbf{mod}$. We call source algebra of A the algebra $\mathrm{Sce}(A) := \mathrm{End}_{\mathcal{O}G}(S)$, viewed as endowed with the natural morphism $D \to \mathrm{Sce}(A)^\times$.

Thus in particular a source algebra of A is Morita equivalent to A. But the source algebra contains much more information than the Morita type of A. One can prove[2] that it contains all the "local information" of the block A, such as the category of subpairs ([AlBr]), the vertices and sources of indecomposable A–modules, and the generalized c-d-triangles (see below). The source algebra may be seen as the "*group* representation version" of the basic algebra.

6.B. Generalized c-d-triangles.

Definition. For x an element of finite order of a group, we let $\zeta_x \in \overline{\mathbb{Q}}$ be a root of unity of the same order as x. If A is any ring, we set $\mathcal{R}_x(A) := \mathbb{Z}[\zeta_x] \otimes_{\mathbb{Z}} \mathcal{R}(A)$.

Let G be a finite group. As in [Se], chap. 18, we identify now $\mathcal{R}(KG)$, $\mathcal{R}(kG)$ and $\mathcal{R}^{\mathrm{pr}}(kG)$ with various subgroups of the group of $\mathcal{O}$–valued class functions on G.

Let x be a p–element of G. We denote by $C_G(x)$ its centralizer in G. The generalized decomposition map $\mathrm{dec}^{G,x}\colon \mathcal{R}_x(KG) \to \mathcal{R}_x(kC_G(x))$ is defined as follows :

For $\chi \in \mathcal{R}_x(KG)$, $\mathrm{dec}^{G,x}(\chi)$ is the class function on $C_G(x)$ defined by

$$\mathrm{dec}^{G,x}(\chi)(y) := \begin{cases} \chi(xy) & \text{if } y \text{ is a } p'\text{–element}\,, \\ 0 & \text{if not}\,. \end{cases}$$

The generalized c-d-triangle associated with x is

$$(\mathcal{T}(G,x)) \qquad \begin{tikzcd} \mathcal{R}_x(KG) \arrow[rr, "\mathrm{dec}^{G,x}"] & & \mathcal{R}_x(kC_G(x)) \\ & \mathcal{R}_x^{\mathrm{pr}}(kC_G(x)) \arrow[ul, "{}^t\mathrm{dec}^{G,x}"] \arrow[ur, "\mathrm{Car}^{C_G(x)}"'] & \end{tikzcd}$$

Notice that $\mathcal{T}(G,1) = \mathcal{T}(G)$.

The generalized decomposition matrix is the matrix of the map $\mathrm{dec}^{G,x}$ on the natural basis $\mathrm{Irr}(KG)$ and $\mathrm{Irr}(kC_G(x))$.

[2] see for example [Br4] for a brief account and some bibliographical references of Puig's work along these lines.

The triangle of a block. To simplify the exposition, we assume from now on that *A is the principal block of $\mathcal{O}G$*[3]. For x a p–element of G, we denote by A_x the principal block of $\mathcal{O}C_G(x)$. Then the combination of Brauer's Second and Third Main Theorems (see for example [Fe]) implies that the image of $\mathcal{R}_x(KA)$ through $\mathrm{dec}^{G,x}$ is contained in $\mathcal{R}_x(kA_x)$, from which one defines the corresponding triangle :

$$(\mathcal{T}(A,x)) \qquad \begin{tikzcd} \mathcal{R}_x(KA) \arrow[rr, "\mathrm{dec}^{A,x}"] & & \mathcal{R}_x(kA_x) \\ & \mathcal{R}_x^{\mathrm{pr}}(kA_x) \arrow[ul, "{}^t\mathrm{dec}^{A,x}"] \arrow[ur, "\mathrm{Car}^{A_x}"'] & \end{tikzcd}$$

Of course one has $\mathcal{T}(A,1) = \mathcal{T}(A)$.

6.C. Equivalences "with groups".

Puig equivalences.

Definition. *We say that two blocks A and B of two finite groups G and H are Puig equivalent*[4] *if, denoting by D (resp. E) a defect group of A (resp. B), there exist a group isomorphism $D \xrightarrow{\sim} E$ and an algebra isomorphism $\mathrm{Sce}(A) \xrightarrow{\sim} \mathrm{Sce}(B)$ such that the following diagram is commmutative*

$$\begin{tikzcd} \mathcal{O}D \arrow[r, "\sim"] \arrow[d] & \mathcal{O}E \arrow[d] \\ \mathrm{Sce}(A) \arrow[r, "\sim"] & \mathrm{Sce}(B) \end{tikzcd}$$

Example. Let us use again the notation introduced in §4.E above : our prime p (the characteristic of our field $k := \mathcal{O}/\mathfrak{p}$) is now denoted by ℓ, and q denotes a power of another prime $p \neq \ell$.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, endowed with a Frobenius endomorphism F which defines a rational structure on $\mathbb{F}_q$. We assume for simplicity that $(\mathbf{G}, F)$ is split. Let W be the Weyl group of $\mathbf{G}$.

Assume that ℓ does not divide $|W|$ and divides $(q-1)$. Then the Sylow ℓ–subgroups of $\mathbf{G}^F$ are abelian, and the centralizer in $\mathbf{G}$ of a Sylow ℓ–subgroup is a Levi subgroup of an F–stable parabolic subgroup of $\mathbf{G}$. Let $\mathbf{H}$ be the normalizer in $\mathbf{G}$ of a Sylow ℓ–subgroup of $\mathbf{G}^F$.

Then ([Pu4]) the principal ℓ–blocks of $\mathbf{G}^F$ and of $\mathbf{H}^F$ are Puig equivalent.

6.2. Puig Conjecture. [5] *Given a finite p–group D, there exists only a finite number of interior D–algebras over $\mathcal{O}$ which are the source algebras of some block of some finite group.*

The validity of this conjecture would imply in particular that there is only a finite number of Morita types for blocks with a given defect group.

[3] otherwise we would have to introduce the subpairs and the Brauer elements as in [AlBr].
[4] Puig says "isomorphic".
[5] Stated in the conference on representation of finite group, Oberwolfach, 1982.

Puig equivalences as "equivalences with groups".

Let us first recall the definition of the "Brauer functor" (see [Br3]). For V an $\mathcal{O}G$–module and P a p–subgroup of G, we set

$$\mathrm{Br}_P(V) := V^P/(\sum_{Q<P} \mathrm{Tr}_Q^P(V^Q) + \mathfrak{p}V^P)\,,$$

where V^P denotes the set of fixed points of V under P, and where $\mathrm{Tr}_Q^P(v) := \sum_{x\in[P/Q]} x(v)$ for Q a subgroup of P and $v \in V^Q$. It defines a functor

$$\mathrm{Br}_P : {}_{\mathcal{O}G}\mathbf{mod} \to {}_{k[N_G(P)/P]}\mathbf{mod}\,.$$

In particular, if V is a permutation P–module, $\mathrm{Br}_P(V)$ is a permutation $(N_G(P)/P)$–module.

From now on, to simplify the exposition, the following hypothesis will be in force :

(A1) G and H are two finite groups with a common Sylow p–subgroup D, and D is abelian,

(A2) $N_G(D)/C_G(D) \simeq N_H(D)/C_H(D)$ — note that this implies that the Frobenius categories $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(H)$ (see for example [Br4]) are equivalent.

We denote by A and B the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$. Whenever P is a subgroup of D, we denote by A_P and B_P the principal blocks of $\mathcal{O}C_G(P)$ and $\mathcal{O}C_H(P)$.

Assume that A and B are Puig equivalent. Then there exists a family (M_P, N_P) (P runs over the set of subgroups of D) where, for each P, M_P is an (A_P, B_P)–bimodule and N_P is a (B_P, A_P)–bimodule such that :

(pu1) (M_P, N_P) induces a Morita equivalence between A_P and B_P.

(pu2) As an $\mathcal{O}[C_G(P) \times C_H(P)]$–module, M_P is a summand of $\mathrm{Ind}_{\Delta(D)}^{C_G(P)\times C_H(P)}\mathcal{O}$, where $\mathcal{O}$ is the trivial D–module.

(pu3) $k \otimes M_P \simeq \mathrm{Res}_{C_G(P)\times C_H(P)}^{N_{G\times H}(\Delta(P))} \mathrm{Br}_{\Delta(P)}(M_{\{1\}})\,.$

Such a family (M_P, N_P) induces in particular an isomorphism between all generalized "local" c-d-triangles

$$\mathcal{T}(A_P, x) \stackrel{\sim}{\longleftrightarrow} \mathcal{T}(B_P, x) \quad (\text{for all } x \in D)$$

which preserves the canonical basis.

Rickard equivalences with groups. As just seen, a Puig equivalence may be seen as a "Morita equivalence with groups" The preceding formulation of a Puig equivalence allows us to define (still under the hypothesis (A1) and (A2)) what is a "Rickard equivalence with groups".

We still denote by A and B the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ and, for P a subgroup of D, by A_P and B_P the principal blocks of $\mathcal{O}C_G(P)$ and $\mathcal{O}C_H(P)$.

We say that A and B are "Rickard equivalent with groups" if there exists a family (M_P, N_P) (P runs over the set of subgroups of D) where, for each P, M_P is a bounded complex of (A_P, B_P)–bimodules and N_P is a bounded complex of (B_P, A_P)–bimodules such that :

(ri1) (M_P, N_P) induces a Rickard equivalence between A_P and B_P.

(ri2) As $\mathcal{O}[C_G(P) \times C_H(P)]$–modules, M_P^n is a summand of $\mathrm{Ind}_{\Delta(D)}^{C_G(P)\times C_H(P)} X_P^n$, where X_P^n is a permutation D–module.

(ri3) $k \otimes M_P \simeq \mathrm{Res}_{C_G(P)\times C_H(P)}^{N_{G\times H}(\Delta(P))} \mathrm{Br}_{\Delta(P)}(M_{\{1\}})$.

Such a family (M_P, N_P) induces in particular an isometry between all generalized "local" c-d-triangles

$$\mathcal{T}(A_P, x) \overset{\sim}{\longleftrightarrow} \mathcal{T}(B_P, x) \quad (\text{for } x \in D)$$

corresponding to what is called an "isotypie" in [Br1].

Some unpublished work of J. Rickard shows the relevance of the preceding definition. In particular, complexes with properties (ri2) and (ri3) above occur naturally for finite reductive groups in the context of étale cohomology (see [Ri5]).

On stable equivalences. Let us end with a result which has been often used in applications to structure of finite groups. Consider a slighly more general situation than (A1) and (A2). Now G and H are two finite groups with a common Sylow p–subgroup D. The group D is not necessarily abelian, but we still assume that G and H have "the same fusion" on p–subgroups, *i.e.*, the embedding of D in both G and H defines an equivalence between the Frobenius categories $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(H)$.

Let e and f be central idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. We set $A := \mathcal{O}Ge$ and $B := \mathcal{O}Hf$. For P a subgroup of D, we set $\overline{e}_P := \mathrm{Br}_P(e)$, $\overline{f}_P := \mathrm{Br}_P(f)$, and $\overline{A}_P := kC_G(P)\overline{e}_P$, $\overline{B}_P := kC_H(P)\overline{f}_P$.

Let M be an (A, B)–bimodule and N be a (B, A)–bimodule. For each subgroup P of D, we set $\overline{M}_P := \mathrm{Br}_{\Delta(P)}(M)$ and $\overline{N}_P := \mathrm{Br}_{\Delta(P)}(N)$.

6.3. Theorem. *Assume that*

(st1) *M is a summand of $\mathrm{Ind}_{\Delta(D)}^{G\times H} X$, where X is a permutation D–module.*

(st2) *For each non trivial subgroup P of D, $(\overline{M}_P, \overline{N}_P)$ induces a Morita equivalence between $\overline{A}_P$ and $\overline{B}_P$.*

Then (M, N) induces a stable equivalence of Morita type between A and B.

Example. The following situation is a direct generalization of the (p–t.i.)–case mentioned in §5.

6.4. *Assume that H is a subgroup of G with index prime to p, and with the following property :*
(p–s.c.) whenever P is a p–subgroup of H, we have $N_G(P) = N_H(P)O_{p'}C_G(P)$.
Let A and B be the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$ respectively, with unity elements e and f. Then the functors $e.\mathrm{Ind}_H^G$ and $f.\mathrm{Res}_H^G$ induce inverse stable equivalences of Morita type between A and B.

The preceding statement has several applications to some "non–simplicity criteria" for finite groups. In this spirit, an important open question is to find a direct and "representation theoretic" proof to the Z_p^*–theorem for p odd, which would provide a significant simplification in the classification of finite simple groups.

6.5. Theorem. *Let H be a subgroup of G which controls the fusion of p–subgroups in G (i.e., the inclusion of H in G induces an equivalence between the Frobenius* *categories $\mathfrak{Fr}_p(G)$ and $\mathfrak{Fr}_p(H)$). Assume that H is the centralizer in G of a p–subgroup of G. Then $G = HO_{p'}(G)$.*

For $p = 2$, the preceding theorem is due to Glauberman ([Gl]). For p odd, it is a consequence of the classification of finite simple groups. An important work of G. Robinson ([Ro1], [Ro2]) makes plausible to find a direct proof using representation theory.

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