Equivalences of pushdown systems are hard

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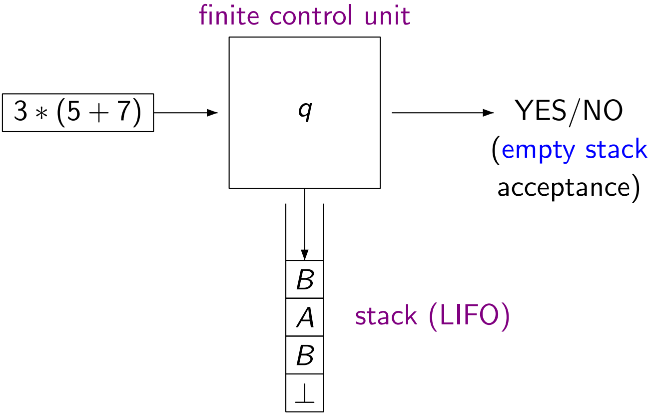
Deterministic pushdown automata; language equivalence

$M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$



Decidability of $L(M_1) \stackrel{?}{=} L(M_2)$ was open since 1960s (Ginsburg, Greibach).
First-order schemes (1970s, 1980s, ..., B. Courcelle, ...).

Solution

- Sénizergues G.:
  L(A)=L(B)? Decidability results from complete formal systems.
  Theoretical Computer Science 251(1-2): 1-166 (2001)
  (a preliminary version appeared at ICALP'97; Gödel prize 2002)
- Stirling C.: Decidability of DPDA equivalence.
  Theoretical Computer Science 255, 1-31, 2001
- Sénizergues G.: L(A)=L(B)? A simplified decidability proof.
  Theoretical Computer Science 281(1-2): 555-608 (2002)
- Stirling C.: Deciding DPDA equivalence is primitive recursive.
  ICALP 2002, Lecture Notes in Computer Science 2380, 821-832, Springer 2002 (longer draft paper on the author's web page)
- Sénizergues G.: The Bisimulation Problem for Equational Graphs of Finite Out-Degree.
  SIAM J.Comput., 34(5), 1025–1106 (2005)
  (a preliminary version appeared at FOCS'98)

Outline

Part 1

- **Deterministic case is in TOWER.**
  Equivalence of first-order schemes (or det-FO-grammars, or deterministic pushdown automata (DPDA)) is in TOWER, i.e. "close" to elementary. (The known lower bound is P-hardness.)

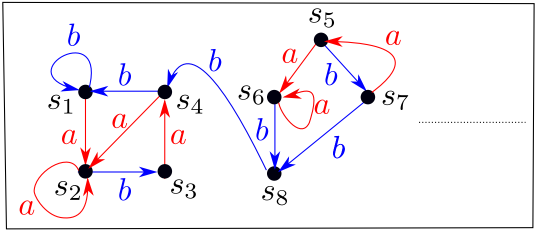
Part 2

- **Nondeterministic case is Ackermann-hard.**
  Bisimulation equivalence of first-order grammars (or PDA with deterministic popping ε-moves) is Ackermann-hard, and thus not primitive recursive (but decidable).

Part 1

Equivalence of det-FO-grammars (or of DPDA) is in TOWER.

(Det-)labelled transition systems (LTSs); trace equivalence

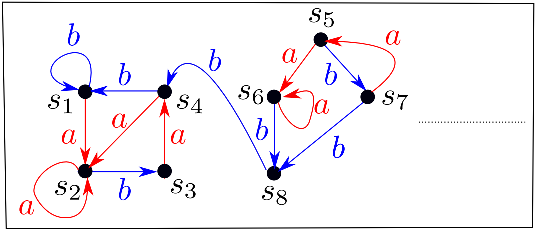


$\mathcal{L} = (\mathcal{S}, \mathcal{A}, (\xrightarrow{a})_{a \in \mathcal{A}})$

$\mathcal{S} = \{s_1, s_2, s_3, \ldots\}$

$\mathcal{A} = \{a, b\}$ $\quad \xrightarrow{a} \subseteq \mathcal{S} \times \mathcal{S} \quad \xrightarrow{b} \subseteq \mathcal{S} \times \mathcal{S}$

(Det-)labelled transition systems (LTSs); trace equivalence



$s_1 \xrightarrow{ab} s_3 \xrightarrow{a}$

$s_5 \xrightarrow{ab} s_8 \stackrel{a}{\not\rightarrow}$

$s_1 \underset{\sim_2}{\not\sim_3} s_5$

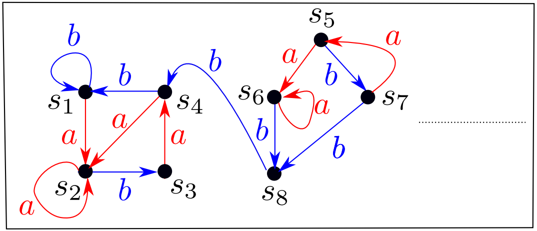
$EL(s_1, s_5) = 2$

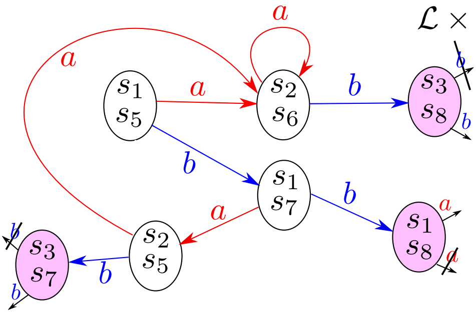
$s \sim_k t$ ... $\forall w \in \mathcal{A}^{\leq k} : s \xrightarrow{w} \Leftrightarrow t \xrightarrow{w}$

$s \sim_\omega t$ ... $\forall k : s \sim_k t$

$EL(s,t) = \max\{k \mid s \sim_k t\}$

(Det-)labelled transition systems (LTSs); trace equivalence





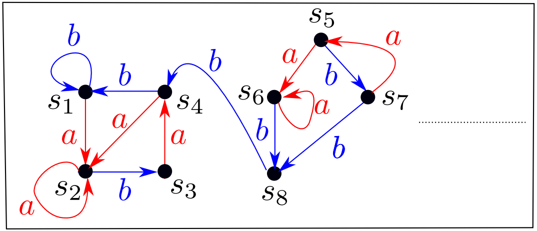
$$\mathcal{L} \times \mathcal{L} = (\mathcal{S} \times \mathcal{S}, \mathcal{A}, (\xrightarrow{a})_{a \in \mathcal{A}})$$

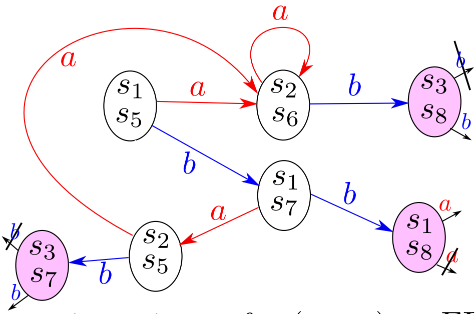
$$(r_1, r_2) \xrightarrow{a} (r_1', r_2')$$

$$\Updownarrow$$

$$r_1 \xrightarrow{a} r_1' \;\wedge\; r_2 \xrightarrow{a} r_2'$$

(Det-)labelled transition systems (LTSs); trace equivalence





$a\,b$ is a witness for (s_1, s_5) ... EL drops by 1 in each step

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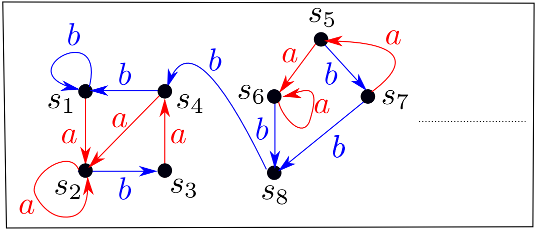
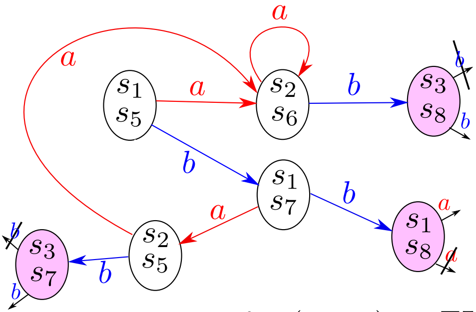




Observation:

$r \sim_{k+1}$ $s \begin{array}{c}\not\sim_{k+1}\\ \sim_k\end{array} t$

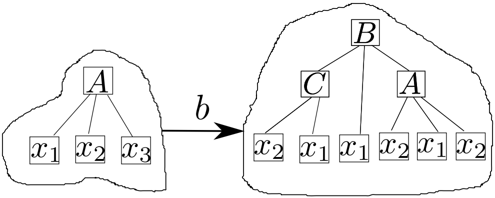
$\Downarrow$

$r \begin{array}{c}\not\sim_{k+1}\\ \sim_k\end{array} t$

$a\,b$ is a witness for (s_1, s_5) ... EL drops by 1 in each step

FO-grammar $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ ... rules $A(x_1, \ldots, x_m) \xrightarrow{a} E$

$$A(x_1, x_2, x_3) \xrightarrow{b} B(C(x_2, x_1), x_1, A(x_2, x_1, x_2))$$



$$B(x_1, x_2, x_3) \xrightarrow{a} x_2$$

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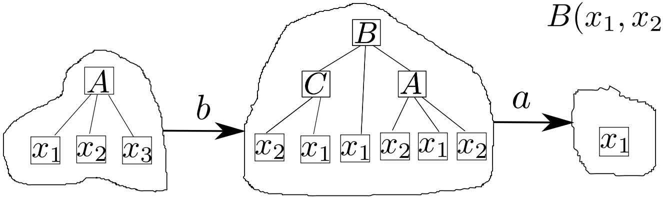

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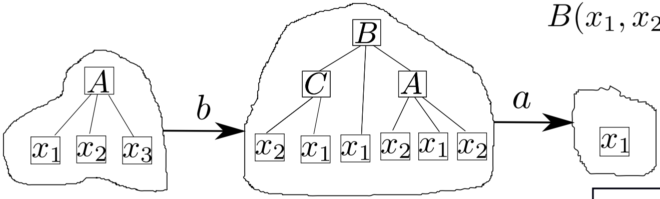
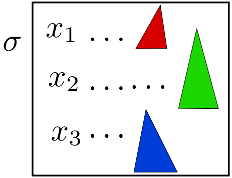


$$F \xrightarrow{a} G \text{ implies } F\sigma \xrightarrow{a} G\sigma$$



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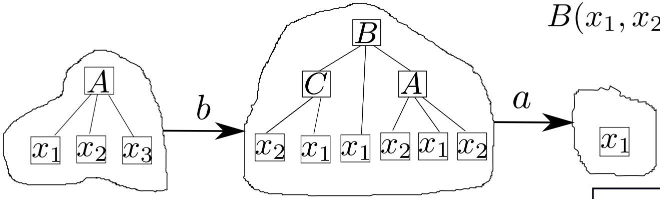

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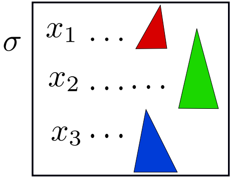
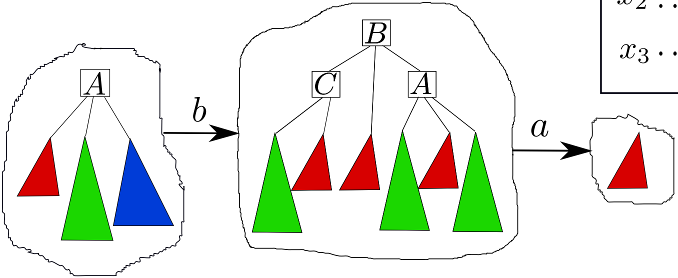
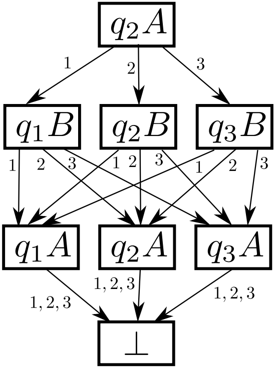


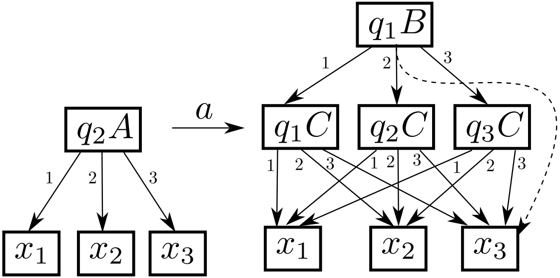


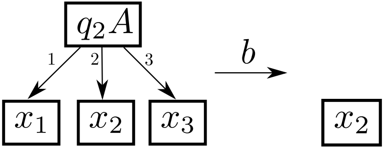
(D)pda from a first-order term perspective

$Q = \{q_1, q_2, q_3\}$
configuration q_2ABA

(pushing) rule $q_2A \xrightarrow{a} q_1BC$

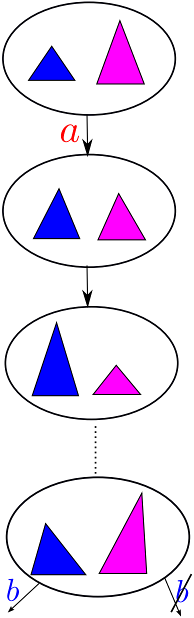






(popping) rule $q_2A \xrightarrow{b} q_2$ $\qquad$ $q_2C \xrightarrow{\varepsilon} q_3$

Bounding lengths of witnesses (where EL keeps dropping)



Theorem.

There is an elementary function g such that for any
det-FO grammar $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ and $T \not\sim U$ of size n we have

$$EL(T, U) \leq tower(g(n)).$$

$tower(0) = 1$
$tower(n+1) = 2^{tower(n)}$

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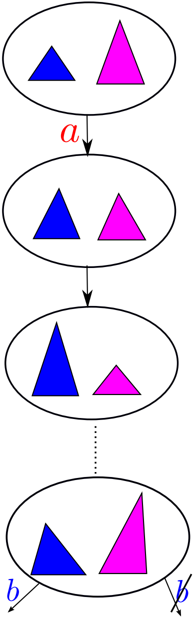

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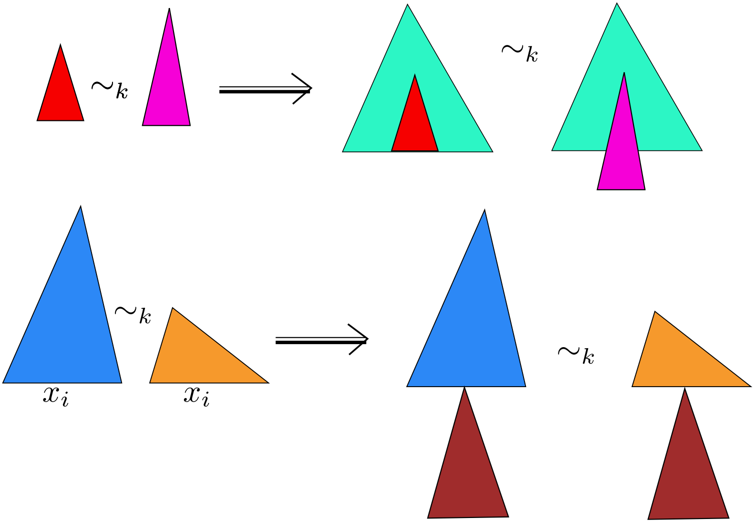
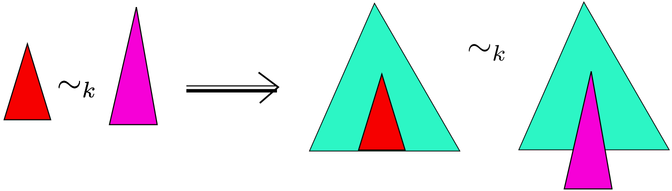
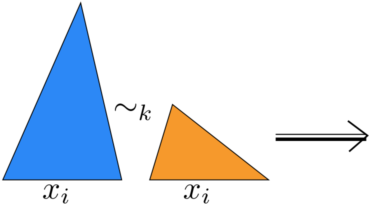
Proof is based on two ideas:

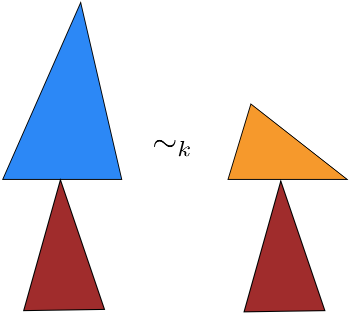
1. "Synchronize" the growth of lhs-terms and rhs-terms *while not changing the respective eq-levels*. (Hence no repeat.)
2. Derive a tower-bound on the size of terms in the (modified) sequence.

Congruence properties of $\sim_k$ and $\sim$



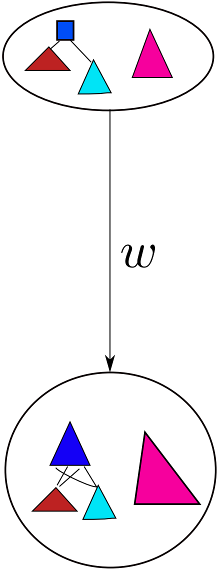






$x_i \stackrel{\not\sim_1}{\sim_0} H \quad (\text{for } H \neq x_i)$

Balancing (the crucial tool for “synchronizing”)



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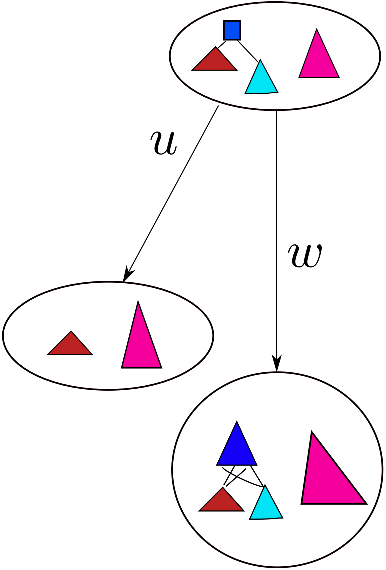

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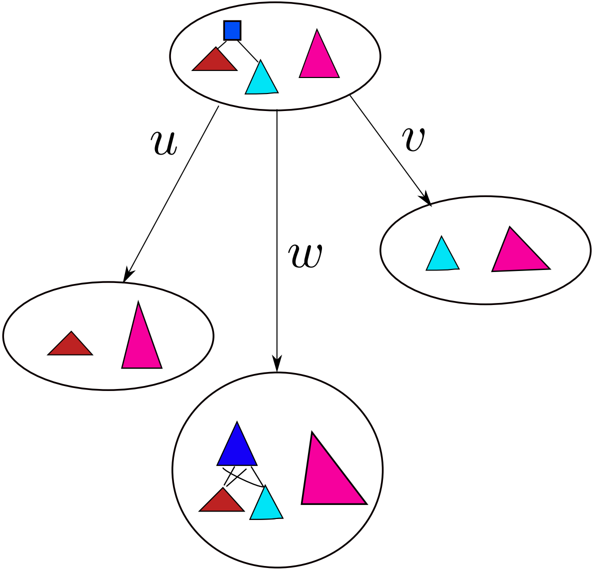

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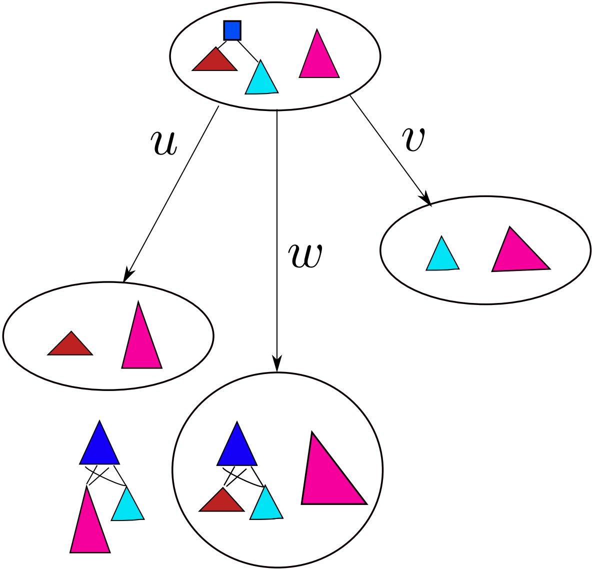

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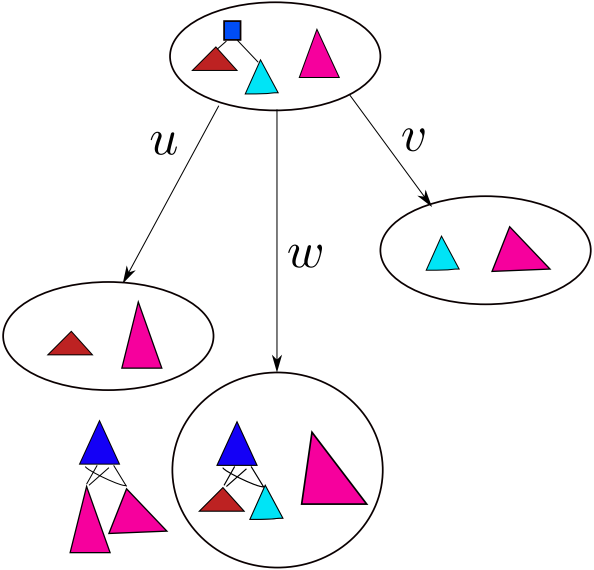

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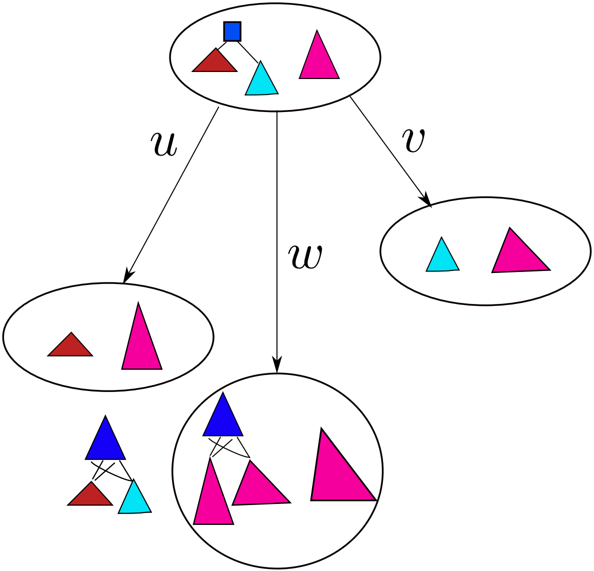

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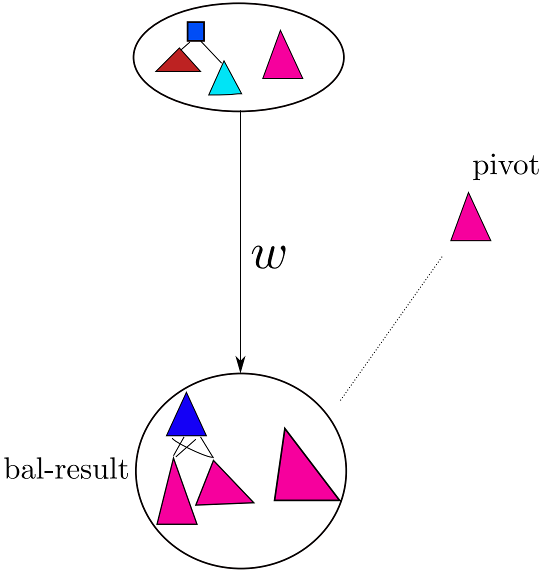

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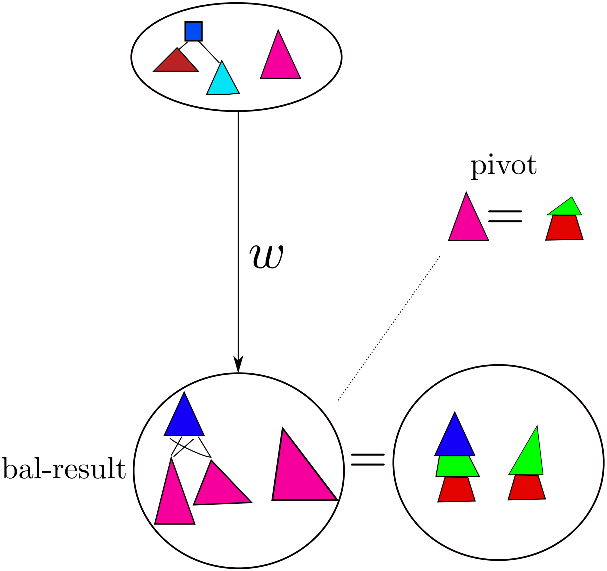
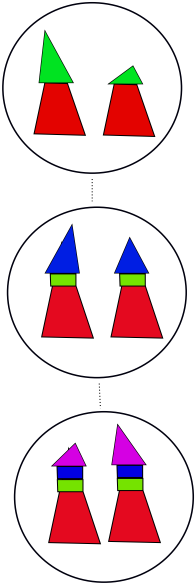
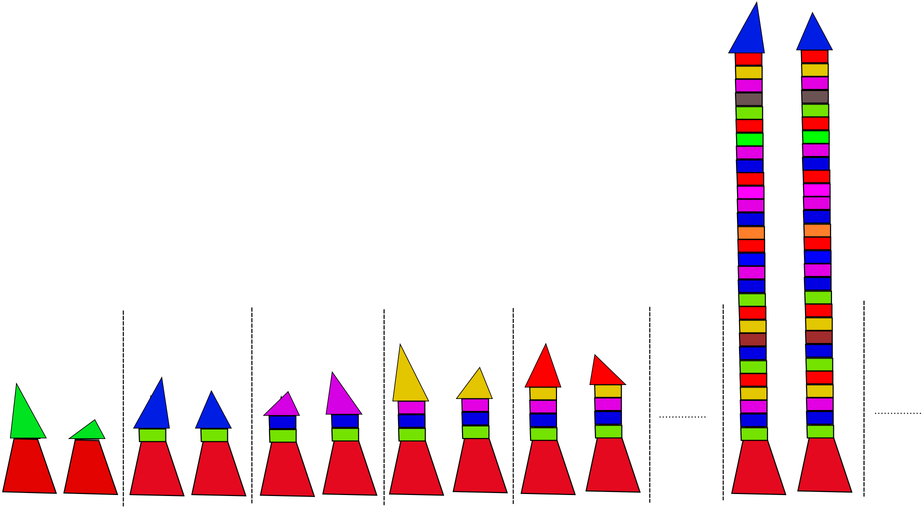


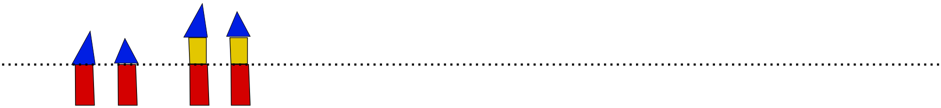
“Stair subsequence” of pairs (on balanced witness path)



Stair subsequence of pairs (written horizontally)



(ℓ, n)-(sub)sequences, with 2^ℓ pairs



(ℓ, n)-(sub)sequences, with 2^ℓ pairs

$(1, n)$-sequence

2^1 pairs

n ... thickness

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There is no EL-decreasing $(1, 0)$-sequence.

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There is no EL-decreasing $(1, 0)$-sequence.

$(1, n)$-sequence

q ... cardinality of "alphabet"

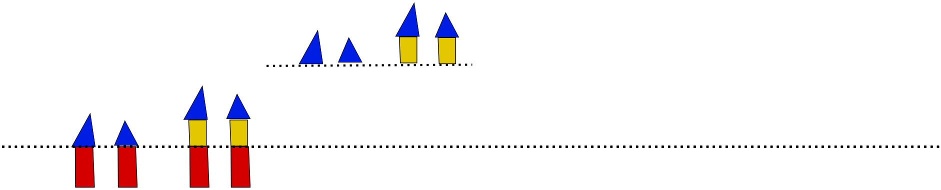
2^1 pairs

In $\boxed{h(1) = 1 + q}$ pairs (of thickness n)

n ... thickness

there is some $(1, n)$-sequence.

(ℓ, n)-(sub)sequences, with 2^ℓ pairs



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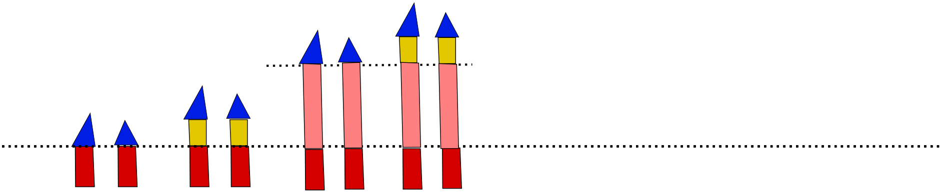
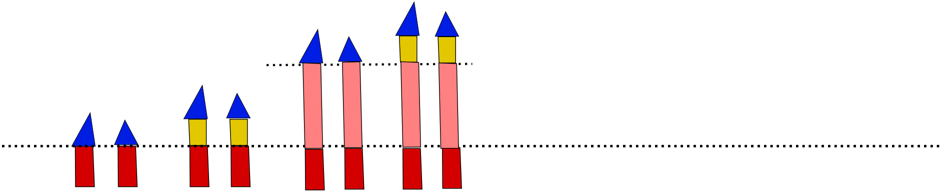


(ℓ, n)-(sub)sequences, with 2^ℓ pairs

$(2, n)$-sequence

$2^2 = 4$ pairs

n ... thickness



(ℓ, n)-(sub)sequences, with 2^ℓ pairs

q ... cardinality of "alphabet"

$(2, n)$-sequence

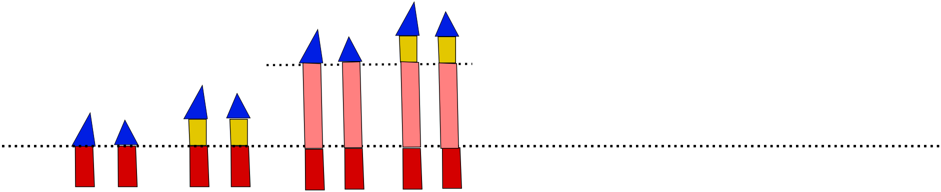
$h(1) = 1 + q$ ... $(1, n)$-sequence

$2^2 = 4$ pairs

In $\boxed{h(2) = h(1) \cdot (1 + q^{h(1)})}$ pairs

n ... thickness

there is some $(2, n)$-sequence.



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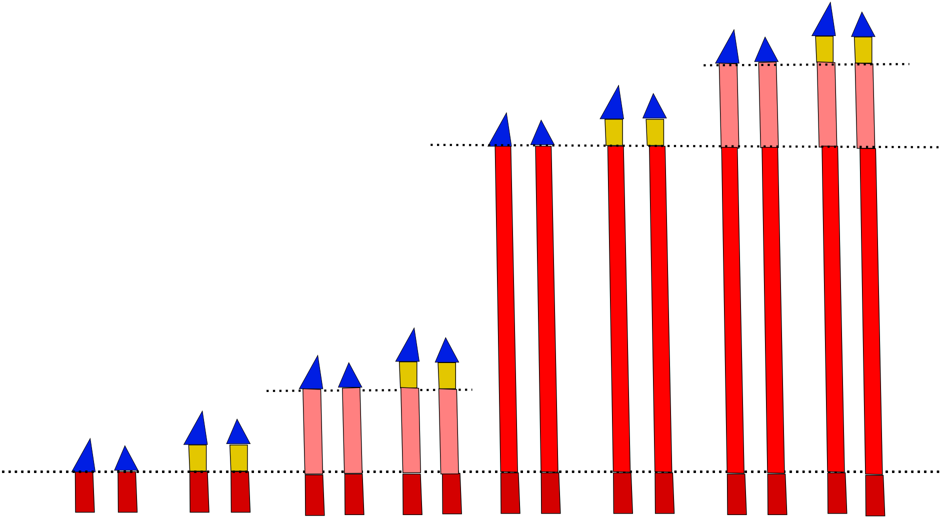
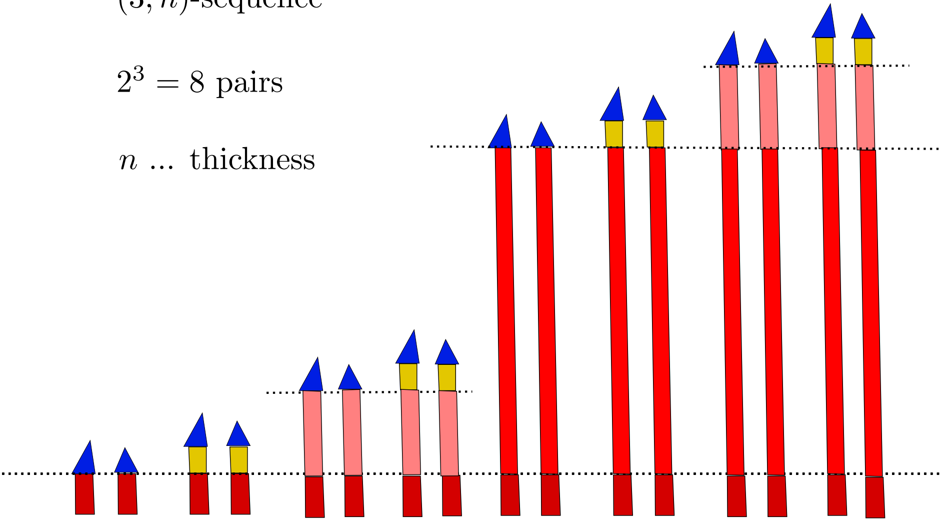


(ℓ, n)-(sub)sequences, with 2^ℓ pairs

$(3, n)$-sequence

$2^3 = 8$ pairs

n ... thickness



(ℓ, n)-(sub)sequences, with 2^ℓ pairs

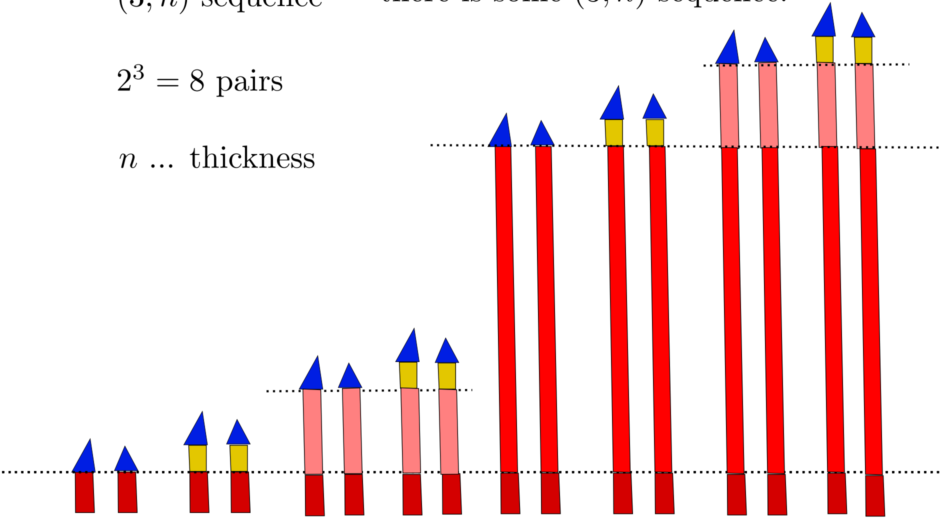
In $\boxed{h(3) = h(2)\cdot(1+q^{h(2)})}$ pairs

there is some $(3, n)$-sequence.

$(3, n)$-sequence

$2^3 = 8$ pairs

n ... thickness



Final (conditional) step of the "TOWER-proof"

Recall: There is no EL-decreasing $(1,0)$-sequence.

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Claim. Any EL-decreasing $(\ell{+}1, n{+}1)$-sequence gives rise to an EL-decreasing (ℓ, n)-sequence.

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Recall that

$$h(1) = 1 + q,$$
$$h(j{+}1) = h(j) \cdot (1 + q^{h(j)})$$

and that $h(j)$ "stairs" gives rise to (j, n)-sequence (n being the "small" thickness).

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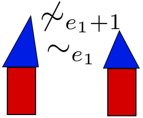
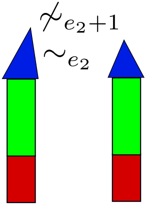
Corollary. There are less than $h(n+1)$ stairs, and $h(n+1) \leq tower(g(n))$.

Repeating heads yield an "equation"

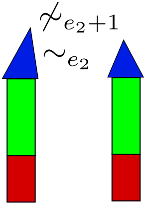
$e_1 > e_2$

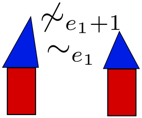


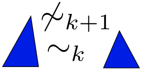


Repeating heads yield an "equation"

$e_1 > e_2$

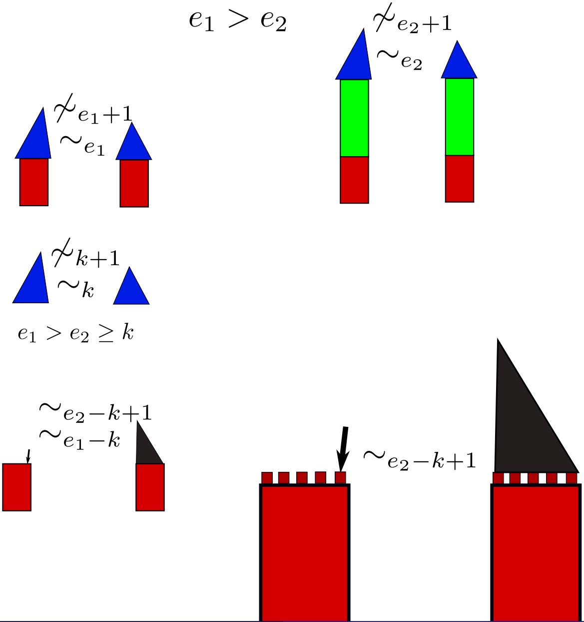




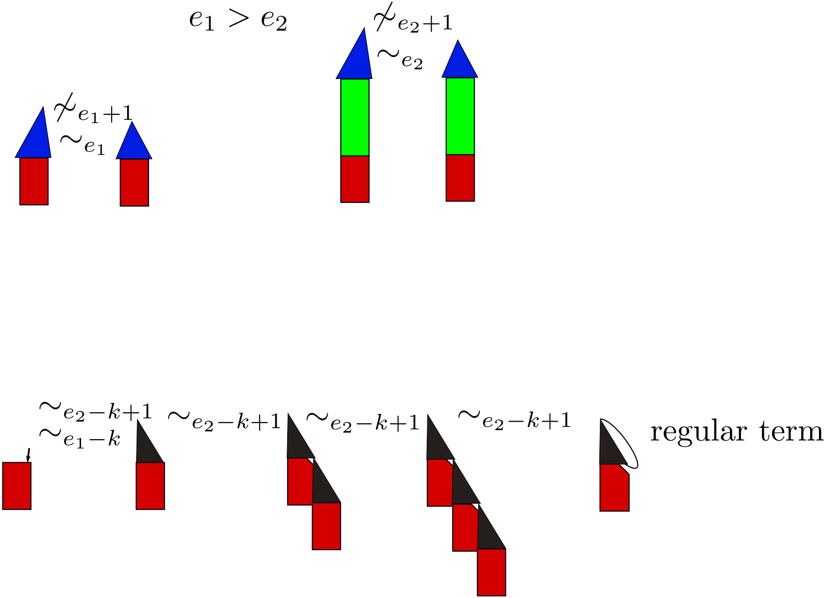


$e_1 > e_2 \geq k$

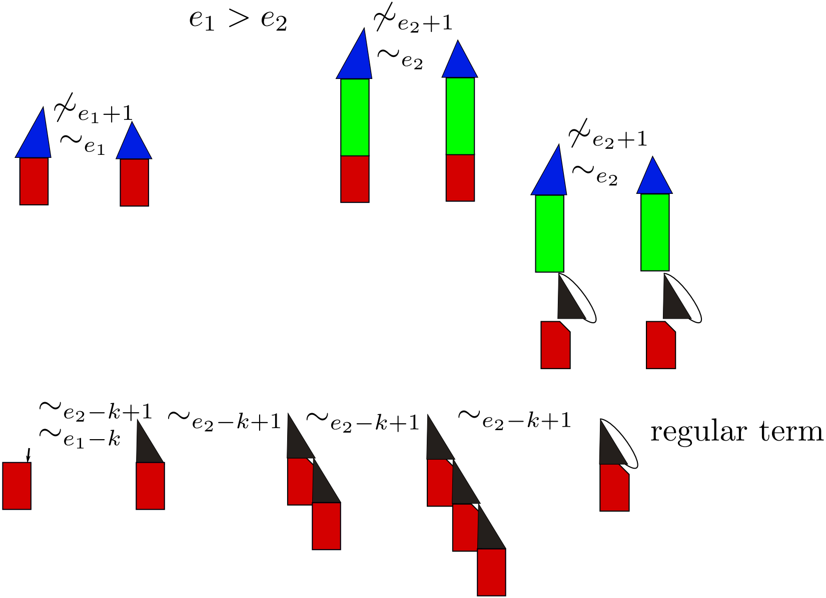
Repeating heads yield an "equation"



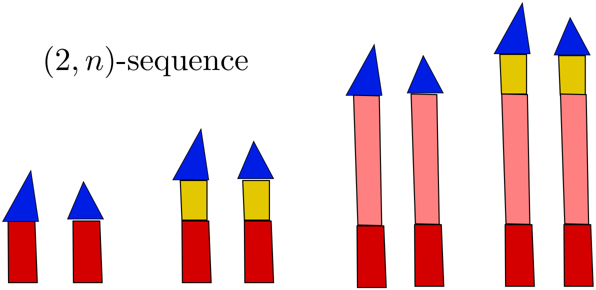
Repeating heads yield an "equation"



Repeating heads yield an "equation"



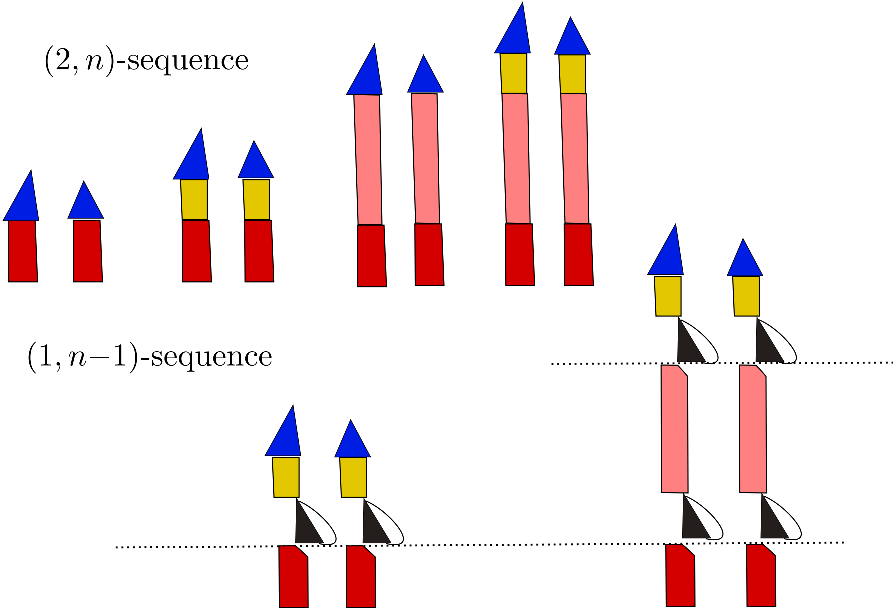
From (ℓ, n) to $(\ell{-}1, n{-}1)$ ... decreasing thickness



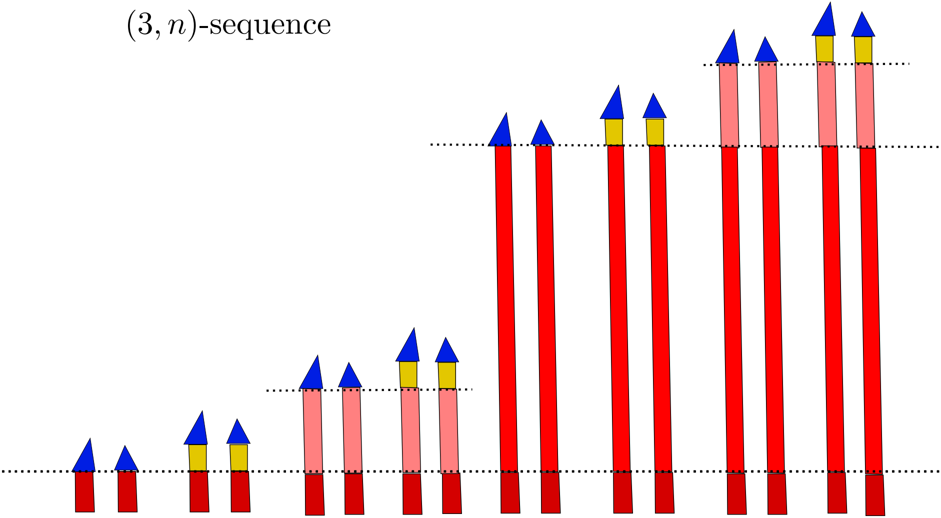
From (ℓ, n) to $(\ell{-}1, n{-}1)$ ... decreasing thickness

$(2, n)$-sequence

$(1, n{-}1)$-sequence



From (ℓ, n) to $(\ell-1, n-1)$ ... decreasing thickness



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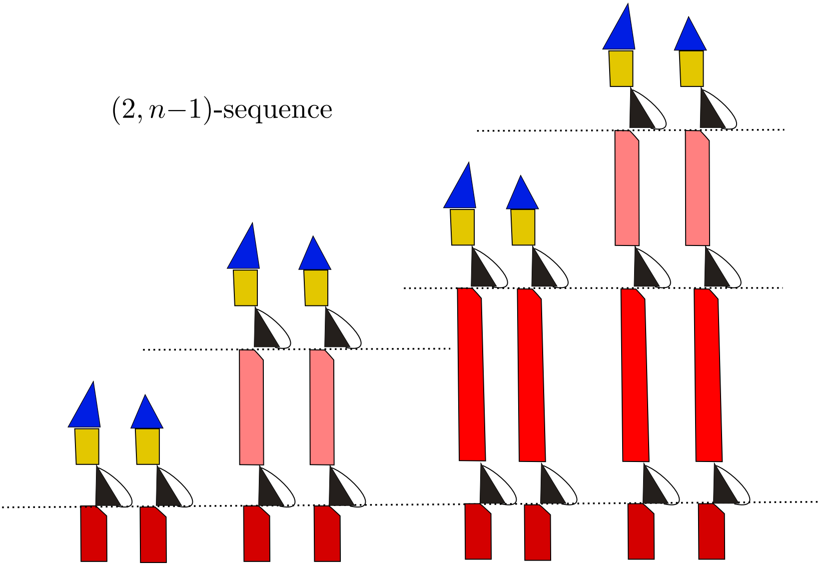
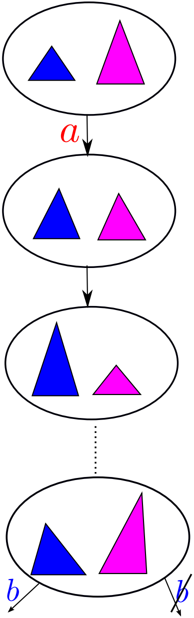


Bounding lengths of witnesses (End of Part 1)



Theorem.

> There is an elementary function g such that for any
> det-FO grammar $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ and $T \not\sim U$ of size n we have
>
> $$EL(T, U) \leq tower(g(n)).$$

Proof is based on two ideas:

1. "Synchronize" the growth of lhs-terms and rhs-terms *while not changing the respective eq-levels*. (Hence no repeat.)
2. Derive a tower-bound on the size of terms in the (modified) sequence.

Part 2

Bisimulation equivalence for FO-grammars is Ackermann-hard.

Note:
Benedikt M., Göller S., Kiefer S., Murawski A.S.:
Bisimilarity of Pushdown Automata is Nonelementary. LICS 2013
(no ε-transitions)

Ackermann function, class ACK, ACK-completeness

Family $f_0, f_1, f_2, \ldots$ of functions:

$$f_0(n) = n{+}1$$
$$f_{k+1}(n) = f_k(f_k(\ldots f_k(n)\ldots)) = f_k^{(n+1)}(n)$$

Ackermann function f_A: $f_A(n) = f_n(n)$.

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ACK ... class of problems solvable in time $f_A(g(n))$ where g is a primitive recursive function.

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where g is a primitive recursive function.

Ackermann-budget halting problem (AB-HP):

Instance: Minsky counter machine M.
Question: does M halt from the zero initial configuration
within $f_A(size(M))$ steps ?

Fact. AB-HP is ACK-complete.

Control state reachability in reset counter machines

Reset counter machines (RCMs).

nonnegative counters $c_1, c_2, \ldots, c_d$,
control states $1, 2, \ldots, r$,

configuration $(\ell, (n_1, n_2, \ldots, n_d))$, initial conf. $(1, (0, 0, \ldots, 0))$,

(nondeterministic) instructions of the types

$\ell \xrightarrow{inc(c_i)} \ell'$ *(increment c_i),*

$\ell \xrightarrow{dec(c_i)} \ell'$ *(decrement c_i, if $c_i > 0$),*

$\ell \xrightarrow{reset(c_i)} \ell'$ *(reset c_i, i.e., put $c_i = 0$).*

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CS-reach problem for RCM:

Instance: an RCM M, a control state ℓ_{FIN} .

Question: is $(1, (0, 0, \dots, 0)) \longrightarrow^* (\ell_{\text{FIN}}, (\dots))$?

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Question: is $(1, (0, 0, \ldots, 0)) \longrightarrow^* (\ell_{\mathrm{FIN}}, (\ldots))$?

Fact. CS-reach problem for RCM is ACK-complete.
(See [Schnoebelen, MFCS 2010].)

Bisimulation equivalence as a game

Assume LTS $\mathcal{L} = (\mathcal{S}, \mathcal{A}, (\xrightarrow{a})_{a \in \mathcal{A}})$.

In a position (s, t),

1. Attacker chooses either some $s \xrightarrow{a} s'$ or some $t \xrightarrow{a} t'$.
2. Defender responses by some $t \xrightarrow{a} t'$ or some $s \xrightarrow{a} s'$, respectively.

The new position is (s', t').

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These rounds are repeated. If a player is stuck, then (s)he loses.
An infinite play is a win of Defender.

We put $s \sim t$ (s, t are bisimulation equivalent) if Defender has a winning strategy from position (s, t).

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Observation. For deterministic LTSs, bisimulation equivalence coincides with trace equivalence.

Reduction of CS-reach for RCM to FO-bisimilarity

Given an RCM M, i.e.,

counters $c_1, c_2, \ldots, c_d$,
control states $1, 2, \ldots, r$,

and instructions of the types

$\ell \xrightarrow{inc(c_i)} \ell'$ *(increment c_i),*
$\ell \xrightarrow{dec(c_i)} \ell'$ *(decrement c_i, if $c_i > 0$),*
$\ell \xrightarrow{reset(c_i)} \ell'$ *(reset c_i, i.e., put $c_i = 0$),*

and ℓ_{FIN},

we construct $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ and E_0, F_0 so that

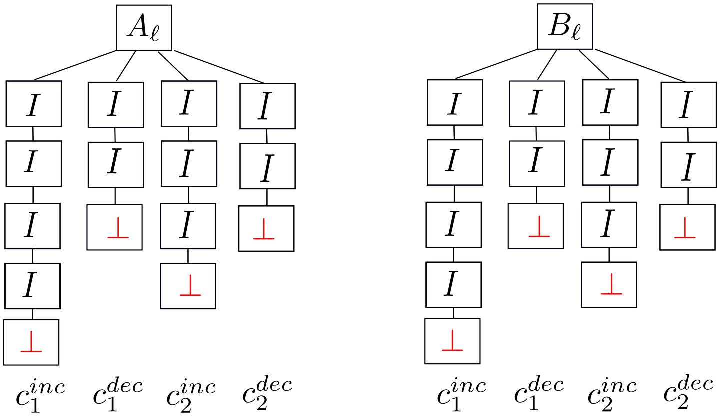
$$(1, (0, 0, \ldots, 0)) \longrightarrow^* (\ell_{\mathrm{FIN}}, (\ldots)) \quad \textit{iff} \quad E_0 \not\sim F_0.$$

CS-reachability as bisimulation game

Example with counters c_1, c_2; we start with the pair

$$(A_1(\perp,\perp,\perp,\perp,), B_1(\perp,\perp,\perp,\perp)).$$

The pair after mimicking $(1,(0,0)) \longrightarrow^* \boxed{(\ell,(2,1))}$ might be



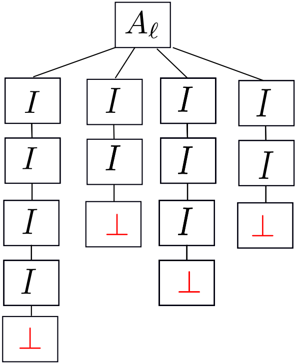
Attacker wins in

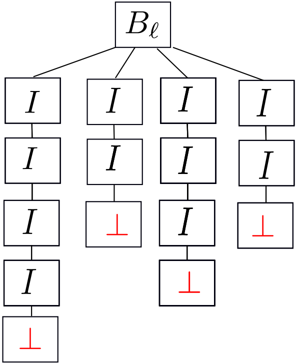
$$(A_{\ell_{\text{FIN}}}(\dots), B_{\ell_{\text{FIN}}}(\dots))$$

due to the rule $A_{\ell_{\text{FIN}}}(x_1, x_2, x_3, x_4) \xrightarrow{a} \dots$ (while there is no rule for $B_{\ell_{\text{FIN}}}$).

Counter increment

For $\boxed{ins = \ell \xrightarrow{inc(c_2)} \ell'}$
we have rules
$A_\ell(x_1, x_2, x_3, x_4) \xrightarrow{ins} A_{\ell'}(x_1, x_2, I(x_3), x_4)$,
$B_\ell(x_1, x_2, x_3, x_4) \xrightarrow{ins} B_{\ell'}(x_1, x_2, I(x_3), x_4)$,





Counter increment

For $\boxed{ins = \ell \xrightarrow{inc(c_2)} \ell'}$

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$A_\ell(x_1, x_2, x_3, x_4) \xrightarrow{ins} A_{\ell'}(x_1, x_2, I(x_3), x_4)$,

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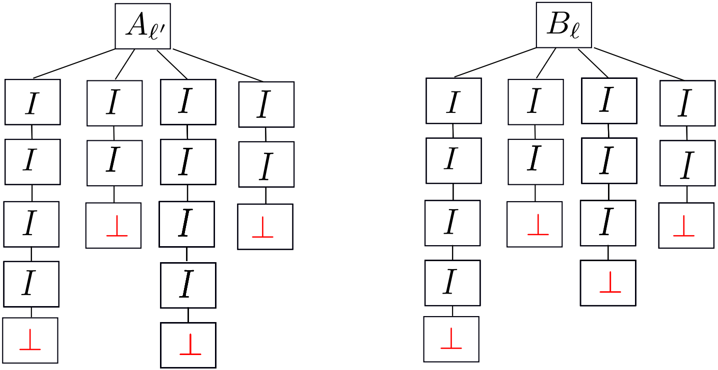

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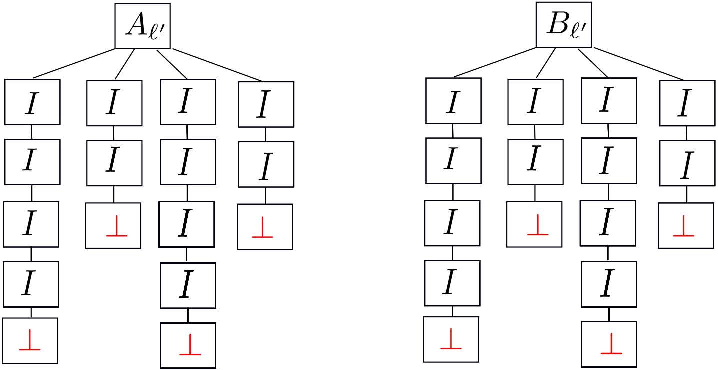
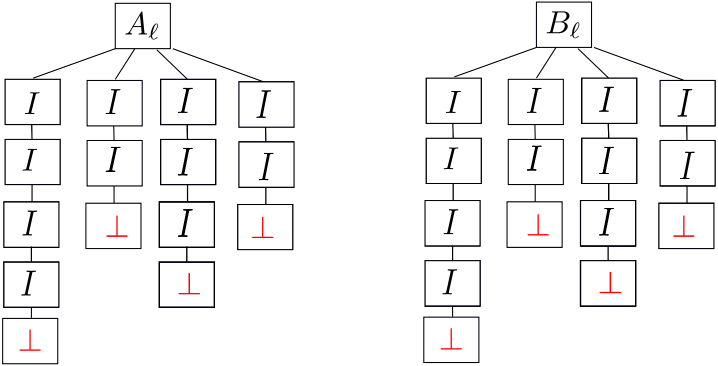


Counter reset

For $\boxed{ins = \ell \xrightarrow{reset(c_2)} \ell'}$
we have rules
$A_\ell(x_1, x_2, x_3, x_4) \xrightarrow{ins} A_{\ell'}(x_1, x_2, \bot, \bot)$,
$B_\ell(x_1, x_2, x_3, x_4) \xrightarrow{ins} B_{\ell'}(x_1, x_2, \bot, \bot)$,



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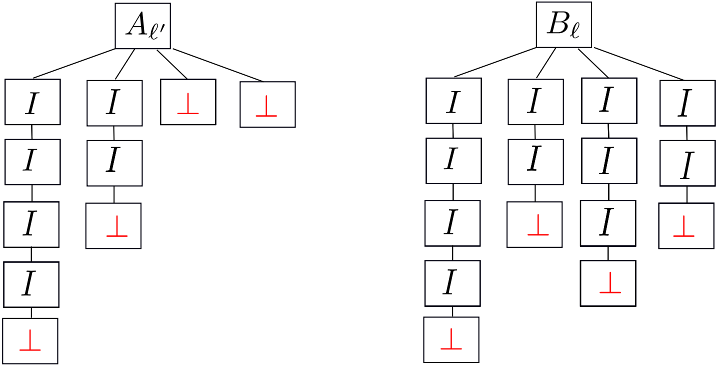

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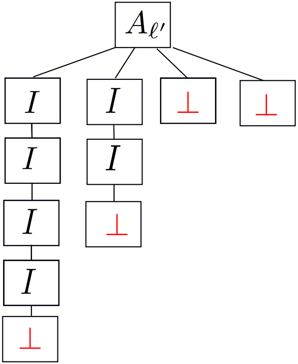
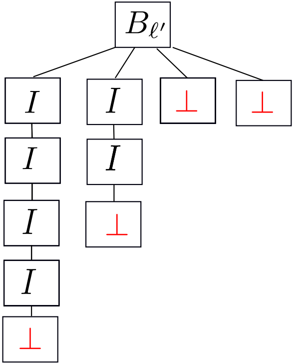


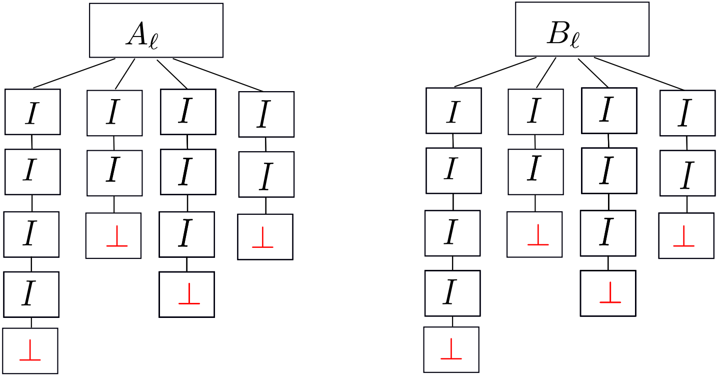


Counter decrement

For $\boxed{ins = \ell \xrightarrow{dec(c_2)} \ell'}$ we have two phases; the first-phase rules are

$$A_\ell \xrightarrow{ins} A_{(\ell',2)},\ A_\ell \xrightarrow{ins} B_{(\ell',2,a)},\ A_\ell \xrightarrow{ins} B_{(\ell',2,b)},$$

$$B_\ell \xrightarrow{ins} B_{(\ell',2,a)},\ B_\ell \xrightarrow{ins} B_{(\ell',2,b)},$$



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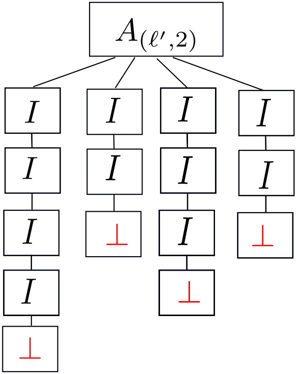

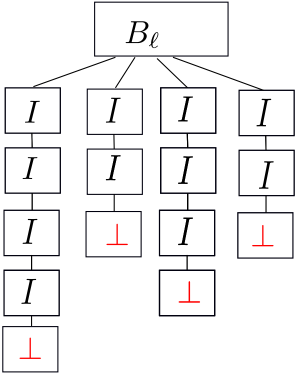

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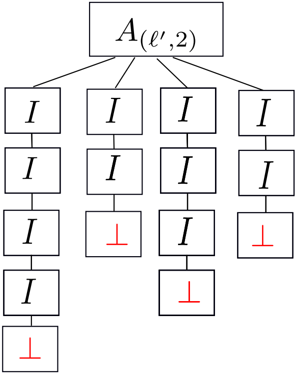
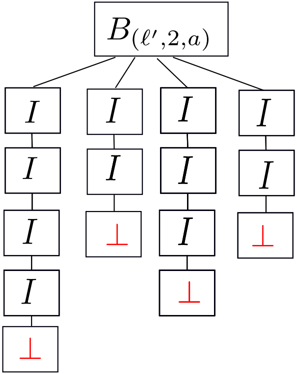
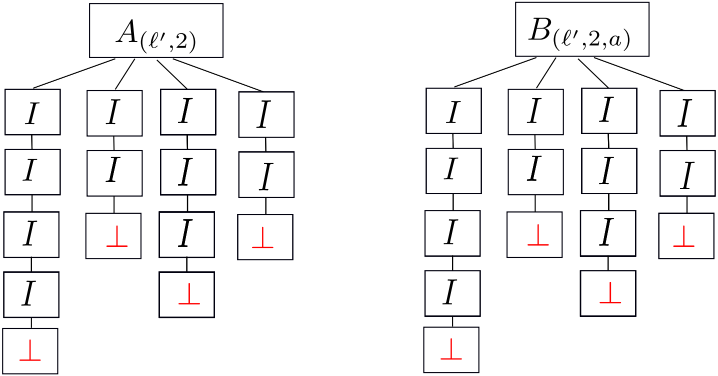




Counter decrement (option a)

$A_{(\ell',2)}(x_1,x_2,x_3,x_4) \xrightarrow{a} A_{\ell'}(x_1,x_2,x_3,I(x_4))$, $A_{\ell',2}(x_1,x_2,x_3,x_4) \xrightarrow{b} x_3$,
$B_{(\ell',2,a)}(x_1,x_2,x_3,x_4) \xrightarrow{a} B_{\ell'}(x_1,x_2,x_3,I(x_4))$,

$B_{(\ell',2,a)}(x_1,x_2,x_3,x_4) \xrightarrow{b} x_3$,



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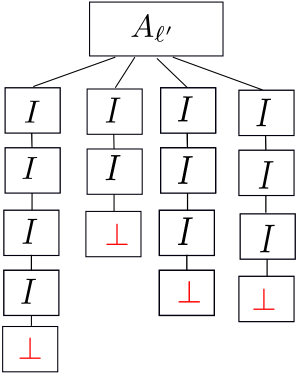
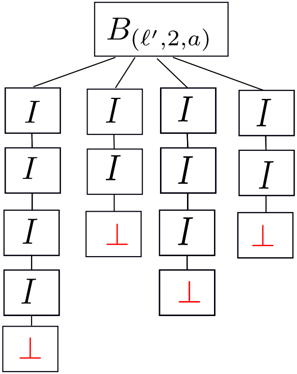
$B_{(\ell',2,a)}(x_1, x_2, x_3, x_4) \xrightarrow{b} x_3$,





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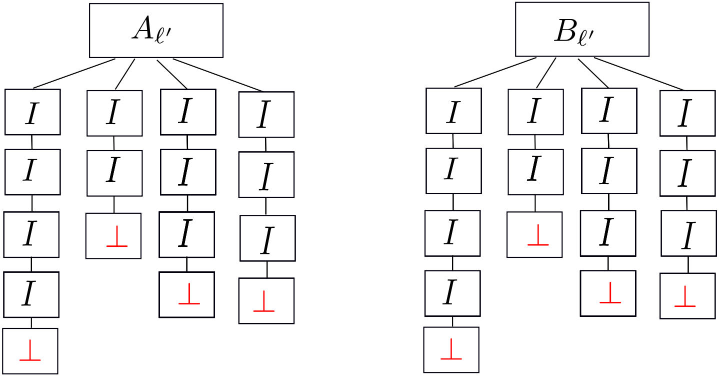
$$B_{(\ell',2,a)}(x_1,x_2,x_3,x_4) \xrightarrow{a} B_{\ell'}(x_1,x_2,x_3,I(x_4)),$$

$$B_{(\ell',2,a)}(x_1,x_2,x_3,x_4) \xrightarrow{b} x_3,$$

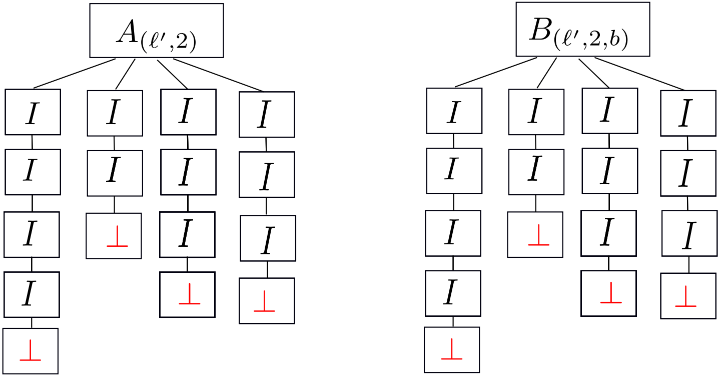


Counter decrement (option b)

$A_{(\ell',2)}(x_1, x_2, x_3, x_4) \xrightarrow{a} A_{\ell'}(x_1, x_2, x_3, I(x_4))$, $A_{\ell',2}(x_1, x_2, x_3, x_4) \xrightarrow{b} x_3$,
$B_{(\ell',2,b)}(x_1, x_2, x_3, x_4) \xrightarrow{a} A_{\ell'}(x_1, x_2, x_3, I(x_4))$,

$B_{(\ell',2,b)}(x_1, x_2, x_3, x_4) \xrightarrow{b} x_4$,

$I(x_1) \xrightarrow{c} x_1$



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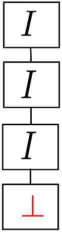

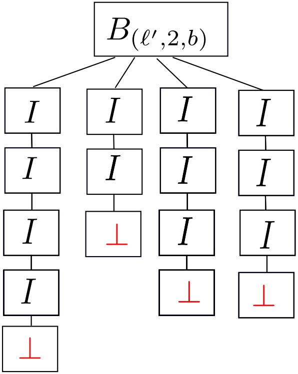

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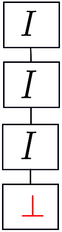
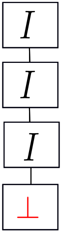
$$A_{(\ell',2)}(x_1,x_2,x_3,x_4) \xrightarrow{a} A_{\ell'}(x_1,x_2,x_3,I(x_4)),\ A_{\ell',2}(x_1,x_2,x_3,x_4) \xrightarrow{b} x_3,$$

$$B_{(\ell',2,b)}(x_1,x_2,x_3,x_4) \xrightarrow{a} A_{\ell'}(x_1,x_2,x_3,I(x_4)),$$

$$B_{(\ell',2,b)}(x_1,x_2,x_3,x_4) \xrightarrow{b} x_4,$$

$$I(x_1) \xrightarrow{c} x_1$$





Concluding remarks

We have shown

- (Trace) equivalence of deterministic first-order grammars is in TOWER.
- Bisimulation equivalence of first-order grammars is Ackermann-hard.

Questions/problems/related results:

- more precise complexity bounds ...
- subcases (simple grammars, one-counter automata, ...)
- higher orders ...
- ....