THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA VOLUME 35, NUMBER 11 NOVEMBER 1963

S4

Equivalent Linearization Techniques

THOMAS K. CAUGHEY

California Institute of Technology, Pasadena, California

The method of equivalent linearization of Kryloff and Bogoliubov is generalized to the case of nonlinear dynamic systems with random excitation. The method is applied to a variety of problems, and the results are compared with exact solutions of the Fokker–Planck equation for those cases where the Fokker–Planck technique may be applied. Alternate approaches to the problem are discussed, including the characteristic function method of Rice.

INTRODUCTION

IN the first paper of this symposium,[1] the derivation and application of the Fokker–Planck equation to discrete nonlinear dynamic systems was discussed. This method was used when the nonlinearities depended only on the displacements in the system. It was also pointed out that, if the nonlinearities involved the velocities as well as the displacements, or if the excitation was not white, then the Fokker–Planck method was inapplicable. The purpose of this paper is to discuss an *approximate* technique that will allow us to obtain solutions to the problem of the response of weakly or slightly nonlinear systems to random excitation where (a) the nonlinearities involve both the velocity and displacement, or (b) the nonlinearity is of hereditary type, or (c) the excitation is nonwhite. This method is based on the well-known technique of equivalent linearization of Kryloff and Bogoliubov.[2] The extension of this technique to problems of random excitation was made independently and more or less simultaneously by Booton[3] and Caughey,[4] and has been used extensively by Caughey.[5–7]

I. EQUIVALENT LINEARIZATION OF THE SINGLE-DEGREE-OF-FREEDOM NONLINEAR SYSTEM

To illustrate the development of the theory, let us consider the following nonlinear oscillator subjected to stationary Gaussian random excitation, which does not necessarily have a white spectrum:

$$\ddot{x}+\beta\dot{x}+\omega_0^2x+\eta g(x,\dot{x},t)=f(t). \tag{1.1}$$

It is assumed that β and η are small in some sense, such that the system is lightly damped and weakly nonlinear. In addition, the nonlinearity $g(x,\dot{x},t)$ may contain both velocity and displacement terms and may depend on the past history of the system; i.e., the system may have hereditary characteristics. A typical hereditary system has the characteristics shown in Fig. 1. Rewriting Eq. (1.1),

$$\ddot{x}+\beta_{\text{eq}}\dot{x}+\omega_{\text{eq}}^2x+e(x,\dot{x},t)=f(t), \tag{1.2}$$

where β_{eq} is the "equivalent linear damping" coefficient per unit mass, ω_{eq}^2 is the "equivalent linear stiffness"

[1] T. K. Caughey, "Derivation and Application of the Fokker–Planck Equation to Discrete Nonlinear Systems Subjected to White Random Excitation," J. Acoust. Soc. Am. 35, 1683 (1963). [*This issue.*]

[2] N. Minorsky, *Nonlinear Mechanics* (J. W. Edwards, Ann Arbor, Mich., 1947).

[3] R. C. Booton, "The Analysis of Nonlinear Control Systems with Random Inputs," in *Proceedings of the Symposium on Nonlinear Circuit Analysis* (Polytechnic Inst. Brooklyn, New York, 1953), Vol. 2.

[4] T. K. Caughey, "Response of Nonlinear Systems to Random Excitation," Lecture Note, California Inst. Technol. (1953) (unpublished).

[5] T. K. Caughey, "Response of Nonlinear String to Random Loading," J. Appl. Mech. 26, 341–344 (1953).

[6] T. K. Caughey, "Random Excitation of a Loaded Nonlinear String," J. Appl. Mech. 27, 575–578 (1960).

[7] T. K. Caughey, "Random Excitation of a System with Bilinear Hysteresis," J. Appl. Mech. 27, 649–652 (1960).

coefficient per unit mass, and $e(x,\dot{x},t)$ is the error or equation deficiency term.

If $e(x,\dot{x},t)$ is neglected, Eq. (1.2) is linear and may be readily solved. The smaller that the error term is, the smaller the error in neglecting it. The logical choice of ω_{eq} and β_{eq} is, therefore, those values that make $e(x,\dot{x},t)$ a minimum. The choice of minimization procedure is somewhat arbitrary, but, by analogy with Galerkin's method and for mathematical expediency, it is desirable to use the minimization of the mean-squared error. From the above equations,

$$e(x,\dot{x},t)=(\beta-\beta_{eq})\dot{x}+(\omega_0^2-\omega_{eq}^2)x+\eta g(x,\dot{x},t). \quad (1.3)$$

The mean-squared error is given by

$$\overline{e^2}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^2(x,\dot{x},t)dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[(\beta-\beta_{eq})\dot{x}+(\omega_0^2-\omega_{eq}^2)x+\eta g(x,\dot{x},t)]^2dt. \quad (1.4)$$

Now, let us minimize $\overline{e^2}$ with respect to β_{eq} and ω_{eq}^2:

$$\frac{\partial\overline{e^2}}{\partial\beta_{eq}}=2[(\beta-\beta_{eq})\overline{\dot{x}^2}+(\omega_0^2-\omega_{eq}^2)\overline{x\dot{x}}+\eta\overline{\dot{x}g(x,\dot{x},t)}]^2=0;$$

$$\frac{\partial\overline{e^2}}{\partial\omega_{eq}^2}=2[(\beta-\beta_{eq})\overline{x\dot{x}}+(\omega_0^2-\omega_{eq}^2)\overline{x^2}+\eta\overline{xg(x,\dot{x},t)}]^2=0. \quad (1.5)$$

If the process is stationary, $\overline{x\dot{x}}=0$. Hence,

$$\beta_{eq}=\beta+\eta\overline{\dot{x}g(x,\dot{x},t)}/\overline{\dot{x}^2};$$
$$\omega_{eq}^2=\omega_0^2+\eta\overline{xg(x,\dot{x},t)}/\overline{x^2}. \quad (1.6)$$

From Eq. (1.5),

$$\frac{\partial^2\overline{e^2}}{\partial\beta_{eq}^2}=2\overline{\dot{x}^2}>0;$$

$$\frac{\partial^2\overline{e^2}}{\partial(\omega_{eq}^2)^2}=2\overline{x^2}>0;$$

$$\frac{\partial^2\overline{e^2}}{\partial\omega_{eq}^2\partial\beta_{eq}}=2\overline{x\dot{x}}=0. \quad (1.7)$$

Thus, Eq. (1.6) does indeed define a minimum for $\overline{e^2}$. Under the assumption that the system is lightly damped and weakly nonlinear, the motion of the system will approximate that of sinusoid with a slow random modulation of amplitude and frequency. Thus,

$$x\approx a(t)\sin[\omega_{eq}t+\phi(t)]=a\sin\theta, \quad (1.8)$$

where the envelope $a(t)$ and the phase $\phi(t)$ are both slowly varying functions of time. Hence,

$$\beta_{eq}=\beta+\eta[\overline{\omega_{eq}a\cos\theta g(a\sin\theta,\omega_{eq}a\cos\theta,\theta)}]/\overline{[\omega_{eq}^2a^2\cos^2\theta]};$$
$$\omega_{eq}^2=\omega_0^2+\eta[\overline{a\sin\theta g(a\sin\theta,\omega_{eq}a\cos\theta,\theta)}]/\overline{[a^2\sin^2\theta]}; \quad (1.9)$$

which may be written as

$$\beta_{eq}=\beta+\eta\frac{\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}\frac{1}{2\pi}\int_{i2\pi}^{(i+1)2\pi}a\cos\theta g(a\sin\theta,\omega_{eq}a\cos\theta,\theta)d\theta}{\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}\frac{1}{2\pi}\int_{i2\pi}^{(i+1)2\pi}\omega_{eq}a^2\cos^2\theta d\theta};$$
$$\omega_{eq}^2=\omega_0^2+\eta\frac{\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}\frac{1}{2\pi}\int_{i2\pi}^{(i+1)2\pi}a\sin\theta g(a\sin\theta,\omega_{eq}a\cos\theta,\theta)d\theta}{\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}\frac{1}{2\pi}\int_{i2\pi}^{(i+1)2\pi}a^2\sin^2\theta d\theta}. \quad (1.10)$$

Since a and ϕ are slowly varying functions of time, they do not change appreciably over one cycle. Thus,

$$\frac{1}{2\pi}\int_{i2\pi}^{(i+1)2\pi}a\cos\theta g(x,\dot{x},t)d\theta\approx C(a_i);$$
$$\frac{1}{2\pi}\int_{i2\pi}^{(i+1)2\pi}a\sin\theta g(x,\dot{x},t)d\theta\approx S(a_i); \quad (1.11)$$

where

$$C(a_i)=\frac{1}{2\pi}\int_0^{2\pi}a_i\cos\theta g(x,\dot{x},t)d\theta;$$
$$S(a_i)=\frac{1}{2\pi}\int_0^{2\pi}a_i\sin\theta g(x,\dot{x},t)d\theta. \quad (1.12)$$

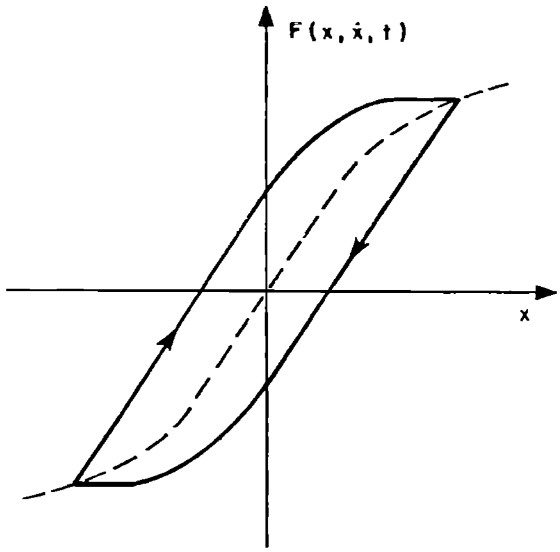

FIG. 1. Force vs displacement characteristic for a typical hereditary system.

Thus,

$$\beta_{\mathrm{eq}}=\beta+\frac{2\eta}{\omega_{\mathrm{eq}}}\left[\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}C(a_i)\right]\times\left[\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}a_i^2\right]^{-1};$$

$$\omega_{\mathrm{eq}}^2=\omega_0^2+2\eta\left[\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}S(a_i)\right]\times\left[\lim_{N\to\infty}\frac{1}{2N}\sum_{i=-N}^{N}a_i^2\right]^{-1}. \tag{1.13}$$

By the assumption that a_i varies slowly, the summations may be replaced by integrals without serious error. Therefore,

$$\beta_{\mathrm{eq}}=\beta+(2\eta/\omega_{\mathrm{eq}})[\overline{C(a)}/\overline{a^2}]; \qquad \omega_{\mathrm{eq}}^2=\omega_0^2+2\eta[\overline{S(a)}/\overline{a^2}]. \tag{1.14}$$

If the process is ergodic, time averages may be replaced by ensemble averages:

$$\beta_{\mathrm{eq}}=\beta+\frac{2\eta}{\omega_{\mathrm{eq}}}\frac{\langle C(a)\rangle}{\langle a^2\rangle}; \qquad \omega_{\mathrm{eq}}^2=\omega_0^2+2\eta\frac{\langle S(a)\rangle}{\langle a^2\rangle}. \tag{1.15}$$

If $e(x,\dot{x},t)$ is neglected in Eq. (1.2) and if β_{eq} is small, the distribution of the envelope may be approximated by the Rayleigh distribution of peaks for the corresponding linear system, whose probability density is

$$p_p(a)=(a/\langle x^2\rangle)\exp\{-a^2/2\langle x^2\rangle\}. \tag{1.16}$$

Thus,

$$\langle a^2\rangle=\int_0^\infty[a^3/\langle x^2\rangle]\exp\{-a^2/2\langle x^2\rangle\}da=2\langle x^2\rangle;$$

$$\langle C(a)\rangle=\int_0^\infty[aC(a)/\langle x^2\rangle]\times\exp\{-a^2/2\langle x^2\rangle\}da=\mathcal{C}(\langle x^2\rangle); \tag{1.17}$$

$$\langle S(a)\rangle=\int_0^\infty[aS(a)/\langle x^2\rangle]\times\exp\{-a^2/2\langle x^2\rangle\}da=\mathcal{S}(\langle x^2\rangle).$$

By neglecting $e(x,\dot{x},t)$ in Eq. (1.2), the spectral density of the response is given by

$$W_{xx}(\omega)=[(\omega_{\mathrm{eq}}^2-\omega^2)^2+(\omega\beta_{\mathrm{eq}})^2]^{-1}W_{ff}(\omega), \tag{1.18}$$

where the spectral density of the excitation is $W_{ff}(\omega)=W_{ff}(f)/2\pi$. Therefore, the mean square displacement of the response is

$$\langle x^2\rangle=\int_0^\infty[(\omega_{\mathrm{eq}}^2-\omega^2)^2+(\omega\beta_{\mathrm{eq}})^2]^{-1}W_{ff}(\omega)d\omega. \tag{1.19}$$

If β_{eq} is small and if the spectral density of the excitation is a relatively smooth function, then Eq. (1.19) may be approximated by

$$\langle x^2\rangle\approx\frac{\pi}{2}\frac{W_{ff}(\omega_{\mathrm{eq}})}{\beta_{\mathrm{eq}}\omega_{\mathrm{eq}}^2}. \tag{1.20}$$

Using Eqs. (1.15) and (1.17),

$$\langle x^2\rangle[\omega_0^2+2\eta\mathcal{S}(\langle x^2\rangle)/\langle x^2\rangle]\times[\beta+2\eta\mathcal{C}(\langle x^2\rangle)/\langle x^2\rangle(\omega_0^2+2\eta\mathcal{S}(\langle x^2\rangle)/\langle x^2\rangle)^{\frac{1}{2}}]\approx\frac{\pi}{2}W_{ff}(\omega_{\mathrm{eq}}). \tag{1.21}$$

Equation (1.21) may then be solved graphically or numerically to obtain $\langle x^2\rangle$.

A. Nonhereditary Nonlinearity

If $g(x,\dot{x},t)$ in Eq. (1.1) does not depend on the past history of the motion, the analysis is somewhat simpler. In Eq. (1.6), the time averages may be replaced by ensemble averages:

$$\beta_{\mathrm{eq}}=\beta+\eta\langle\dot{x}g(x,\dot{x})\rangle/\langle\dot{x}^2\rangle; \qquad \omega_{\mathrm{eq}}^2=\omega_0^2+\eta\langle xg(x,\dot{x})\rangle/\langle x^2\rangle. \tag{1.22}$$

If $e(x,\dot{x},t)$ is neglected in Eq. (1.2), the response is

Gaussian when the excitation is Gaussian. Therefore,

$$p(x,\dot{x})=(2\pi)^{-1}(\langle x^2\rangle\langle \dot{x}^2\rangle)^{-\frac{1}{2}} \times\exp\left\{-\tfrac{1}{2}\left(\frac{x^2}{\langle x^2\rangle}+\frac{\dot{x}^2}{\langle \dot{x}^2\rangle}\right)\right\}. \quad (1.23)$$

The distribution takes this simple form since $\langle x\dot{x}\rangle=0$ for a stationary process. Hence,

$$\langle x^2\rangle=\int_0^\infty [(\omega_{eq}^2-\omega^2)^2+(\omega\beta_{eq})^2]^{-1}W_{ff}(\omega)d\omega;$$
$$\langle \dot{x}^2\rangle=\int_0^\infty [(\omega_{eq}^2-\omega^2)^2+(\omega\beta_{eq})^2]^{-1}\omega^2W_{ff}(\omega)d\omega. \quad (1.24)$$

For any specific $W_{ff}(\omega)$, $\langle x^2\rangle$ and $\langle \dot{x}^2\rangle$ may be numerically determined by substituting Eqs. (1.22) into Eqs. (1.24). In the case where β_{eq} is small and $W_{ff}(\omega)$ is smooth, Eqs. (1.24) may be approximated by

$$\langle x^2\rangle\approx\frac{\pi}{2}\frac{W_{ff}(\omega_{eq})}{\beta_{eq}\omega_{eq}^2};$$
$$\langle \dot{x}^2\rangle\approx\frac{\pi}{2}\frac{W_{ff}(\omega_{eq})}{\beta_{eq}}. \quad (1.25)$$

Thus, we have two equations relating the two unknowns from which these quantities may be determined.

B. Example

To illustrate the equivalent linearization technique, let us show that the mean square displacement of a "hardening spring" oscillator is always less than that of the corresponding linear oscillator when both systems are exposed to the same white spectral density of the excitation. Assuming that $\eta=\omega_0^2\epsilon$, $g(x,\dot{x})=g(x)$, and certain "hardening spring" restrictions,[8] the equivalent linear damping and stiffness per unit mass are

$$\beta_{eq}=\beta$$

and

$$\omega_{eq}^2=\omega_0^2[1+\epsilon\langle xg(x)\rangle/\langle x^2\rangle]. \quad (1.26)$$

From Eqs. (1.25),

$$\langle x^2\rangle=(W_0/4\beta\omega_0^2)[1+\epsilon\langle xg(x)\rangle/\langle x^2\rangle]^{-1}$$

and

$$\langle \dot{x}^2\rangle=W_0/4\beta=\sigma_x^2, \quad (1.27)$$

where W_0 is the white spectral density of $W_{ff}(f)=2\pi W_{ff}(\omega)$. Thus,

$$\langle x^2\rangle=\sigma_x^2-\epsilon\langle xg(x)\rangle<\sigma_x^2, \quad (1.28)$$

which is the desired result.

[8] Reference 1, Eq. (2.12).

C. Calculation of Mean Square Displacement

If the exact probability density $p(x)$ is used, then Eq. (1.28) is identical with the exact solution obtained by using the Fokker–Planck equation.[9]

However, $p(x)$ may be approximated by the Gaussian probability density given in Eq. (1.23). Applying this to the Duffing oscillator, where $g(x)=x^3$, the mean square displacement may be determined from Eq. (1.28):

$$3\epsilon(\langle x^2\rangle)^2+\langle x^2\rangle-\sigma_x^2=0, \quad (1.29)$$

where it may be shown that $\langle x^4\rangle=3(\langle x^2\rangle)^2$ for a Gaussian process.[10] Solving this quadratic and utilizing only the positive root (since $\langle x^2\rangle>0$),

$$\langle x^2\rangle=(6\epsilon)^{-1}[(1+12\epsilon\sigma_x^2)^{\frac{1}{2}}-1]. \quad (1.30)$$

$$\langle x^2\rangle\approx\sigma_x^2-3\epsilon\sigma_x^4 \quad (\epsilon\sigma_x^2\ll1). \quad (1.31)$$

Comparison of Eq. (1.30) with the exact solution shows good agreement for $\epsilon\sigma_x^2<1$, while Eq. (1.31) concurs with the approximate solution obtained by the perturbation method.[11]

II. EXTENSION TO n-DEGREE-OF-FREEDOM SYSTEMS

The extension of the foregoing technique to the general n-degree-of-freedom system is quite involved so we shall not discuss the most general case, but restrict our discussion to the case where the nonlinearity is a function of the displacements alone.

Consider the following system of equations in matrix notation:

$$I\{\ddot{x}\}+\beta I\{\dot{x}\}+[\Omega^2]\{x\}+\mu\left\{\frac{\partial V^\dagger}{\partial x}\right\}=\{f(t)\}, \quad (2.1)$$

where (a) $[\Omega^2]$ is an $N\times N$ symmetric positive definite matrix, (b) $\mu>0$, (c) $[\theta]^T[\theta]=I$, (d) $[\theta]^T[\Omega^2][\theta]=[\![\omega^2]\!]$, and (e) $f_i(t)$ are uncorrelated Gaussian processes with means of zero; i.e.,

$$\langle f_i(t)\rangle=0 \quad (i=1,2,\cdots,n);$$
$$\langle f_i(t_1)f_i(t_2)\rangle=R_f(t_2-t_1); \quad (2.2)$$
$$\langle f_i(t_1)f_j(t_2)\rangle=0 \quad (i\neq j).$$

Letting $\{x\}=[\theta]\{\xi\}$ and $\{Q(t)\}=[\theta]^T\{f(t)\}$,

$$I\{\ddot{\xi}\}+\beta I\{\dot{\xi}\}+[\![\omega^2]\!]\{\xi\}+\mu[\theta]^T\left\{\frac{\partial V^\dagger}{\partial x}\right\}=\{Q(t)\}. \quad (2.3)$$

It may be shown that $[\theta]^T\{\partial V^\dagger/\partial x\}=\{\partial V^\dagger/\partial\xi\}$. Also, the correlation matrix $\langle\{Q(t_1)\}\{Q(t_2)\}^T\rangle$ is a diagonal matrix. Hence, the Q_i's are uncorrelated. Thus, the

[9] Reference 1, Eqs. (2.10) and (2.17).

[10] H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, N. J., 1946), p. 184.

[11] S. H. Crandall, "Perturbation Techniques for Random Vibration of Nonlinear Systems," J. Acoust. Soc. Am. 35, 1700 (1963), Eq. (2.14). [*This issue.*]

ith row of Eq. (2.3) is

$$\ddot{\xi}_i+\beta\dot{\xi}_i+\omega_i^2\xi_i+\mu\partial V^\dagger/\partial\xi_i=Q_i(t), \quad (2.4)$$

which may be written as

$$\ddot{\xi}_i+\beta\dot{\xi}_i+\omega_{i\mathrm{eq}}^2\xi_i+e_i(\xi_1,\xi_2,\cdots,\xi_n)=Q_i(t). \quad (2.5)$$

Minimizing $\overline{e^2}$ with respect to $\omega_{i\mathrm{eq}}^2$ yields

$$\omega_{ieq}^2=\omega_i^2+\mu\overline{\xi_i\frac{\partial V^\dagger}{\partial\xi_i}}\Big/\overline{\xi_i^2}. \quad (2.6)$$

If the process is ergodic, the time averages may be replaced by the ensemble averages:

$$\omega_{ieq}^2=\omega_i^2+\mu\langle\xi_i\partial V^\dagger/\partial\xi_i\rangle/\langle\xi_i^2\rangle. \quad (2.7)$$

If $e_i(\xi_1,\xi_2,\cdots,\xi_n)$ is neglected in Eq. (2.5), then the equation is linear and may be solved by standard techniques. Thus,

$$W_{\xi_i\xi_i}(\omega)=[(\omega_{ieq}^2-\omega^2)^2+(\omega\beta)^2]^{-1}W_{Q_iQ_i}(\omega); \quad (2.8)$$

$$\langle\xi_i^2\rangle=\int_0^\infty[(\omega_{ieq}^2-\omega^2)^2+(\omega\beta)^2]^{-1}W_{Q_iQ_i}(\omega)d\omega; \quad (2.9)$$

$$p(\xi_1,\xi_2,\cdots,\xi_n)=[(2\pi)^n\prod_{i=1}^n\langle\xi_i^2\rangle]^{-\frac{1}{2}} \times\exp\{-\sum_{i=1}^n\xi_i^2/2\langle\xi_i^2\rangle\}. \quad (2.10)$$

If β is small and $W_{Q_iQ_i}(\omega)$ is a smooth function, then Eq. (2.9) becomes

$$\langle\xi_i^2\rangle\approx\frac{\pi}{2}\frac{W_{Q_iQ_i}(\omega_{ieq})}{\beta\omega_{ieq}^2}. \quad (2.11)$$

Using Eq. (2.7), Eq. (2.11) reduces to

$$\langle\xi_i^2\rangle\approx\frac{\pi}{2}\frac{W_{Q_iQ_i}(\omega_{ieq})}{\beta\omega_i^2}-\frac{\mu}{\omega_i^2}\left\langle\xi_i\frac{\partial V^\dagger}{\partial\xi_i}\right\rangle. \quad (2.12)$$

A. Example

Using equivalent linearization, let us show that the mean square displacements in a nonlinear n-degree-of-freedom system are less than those of the corresponding linear system when the nonlinearities are of the "hardening spring" type and when both systems are subjected to the same spectral density of the excitation.

If $\xi_i(\partial V^\dagger/\partial\xi_i)\geq0$ for all ξ_i, then from Eq. (2.12),

$$\langle\xi_i^2\rangle<\frac{\pi}{2}\frac{W_{Q_iQ_i}(\omega_{ieq})}{\beta\omega_i^2}. \quad (2.13)$$

In addition, if the f_i's are white random excitations, then the Q_i's are also white and Eq. (2.9) yields

$$\langle\xi_i^2\rangle=\sigma_{\xi_i}^2-\frac{\mu}{\omega_i^2}\left\langle\xi_i\frac{\partial V^\dagger}{\partial\xi_i}\right\rangle<\sigma_{\xi_i}^2. \quad (2.14)$$

Utilizing the procedure discussed in the first paper of this symposium,[12]

$$\langle x_i^2\rangle=\sum_{j=1}^N(\theta_i^j)^2\langle\xi_j^2\rangle<\sum_{j=1}^N(\theta_i^j)^2\sigma_{\xi_j}^2=\sigma_{x_i}^2, \quad (2.15)$$

which is the desired result.

B. Calculation of Mean Square Displacement

Since the nonlinearity is a function of the displacements alone, then

$$\left\langle\xi_i\frac{\partial V^\dagger}{\partial\xi_i}\right\rangle=\langle\xi_i^2\rangle H_i(\xi_1,\xi_2,\cdots,\xi_n). \quad (2.16)$$

Inserting Eq. (2.16) into Eq. (2.12),

$$\langle\xi_i^2\rangle\approx\frac{\pi}{2}\frac{W_{Q_iQ_i}(\omega_{ieq})}{\beta\omega_i^2}(1+\mu H_i/\omega_i^2)^{-1}, \quad (2.17)$$

where $\omega_{ieq}^2=\omega_i^2+\mu H_i(\xi_1,\xi_2,\cdots,\xi_n)$. When $\mu H_i/\omega_i^2\ll1$ and the f_i's are white, a good approximation is given by

$$\langle\xi_i^2\rangle\approx\sigma_{\xi_i}^2(1-\mu H_i/\omega_i^2). \quad (2.18)$$

III. N-DEGREE-OF-FREEDOM QUASILINEAR SYSTEM

Again, consider the system studied in the first paper[13]:

$$I\{\ddot{x}\}+\beta I\{\dot{x}\}+[1+2\lambda V^*][\Omega^2]\{x\}=\{f(t)\}, \quad (3.1)$$

where $f_i(t)$'s are stationary, Gaussian, white random excitation with means of zero, and

$$\langle f_i(t_1)f_j(t_2)\rangle=\tfrac{1}{2}W_0\delta_i^j(t_2-t_1). \quad (3.2)$$

Using the transformation $\{x\}=[\theta]\{\xi\}$ on Eq. (3.1),

$$I\{\ddot{\xi}\}+\beta I\{\dot{\xi}\}+[1+2\lambda V^*][\![\omega^2]\!]\{\xi\}=\{Q(t)\}. \quad (3.3)$$

The ith row of Eq. (3.3) is

$$\ddot{\xi}_i+\beta\dot{\xi}_i+[\omega_i^2\xi_i+\lambda\partial U/\partial\xi_i]=Q_i(t), \quad (3.4)$$

where

$$U=\tfrac{1}{4}\sum_{j=1}^N\sum_{k=1}^N\omega_j^2\omega_k^2\xi_j^2\xi_k^2.$$

Let us now apply the equivalent linearization technique to Eq. (3.4):

$$\ddot{\xi}_i+\beta\dot{\xi}_i+\omega_{ieq}^2\xi_i+e_i(x,\dot{x},t)=Q_i(t). \quad (3.5)$$

[12] Reference 1, Eqs. (2.46)–(2.50).
[13] Reference 1, Eqs. (2.35)–(2.40).

Minimization of $\overline{e_i^2}$ yields for an ergodic process

$$\omega_{i\mathrm{eq}}^2=\omega_i^2+\lambda\left\langle \xi_i\frac{\partial U}{\partial\xi_i}\right\rangle \Big/ \langle\xi_i^2\rangle. \tag{3.6}$$

Hence,

$$\langle\xi_i^2\rangle=\int_0^\infty \left[(\omega_{i\mathrm{eq}}^2-\omega^2)^2+(\omega\beta)^2\right]^{-1}W_{Q_iQ_i}(\omega)d\omega$$

$$=\frac{\pi}{2}\frac{W_{Q_iQ_i}}{\beta\omega_{i\mathrm{eq}}^2}, \tag{3.7}$$

since the Q_i's are white. Substituting Eq. (3.6) into Eq. (3.7),

$$\langle\xi_i^2\rangle=\sigma_{\xi_i}^2-\frac{\lambda}{\omega_i^2}\left\langle \xi_i\frac{\partial U}{\partial\xi_i}\right\rangle. \tag{3.8}$$

It should be noted that, if the exact probability density is used to calculate $\langle\xi_i\partial U/\partial\xi_i\rangle$, then Eq. (3.8) is identical with the exact solution. In general, however, p is not known and the assumption is made that it is Gaussian, with variances to be determined. Thus,

$$p(\xi_1,\xi_2,\cdots,\xi_N)=\left[(2\pi)^N\prod_{i=1}^{N}\langle\xi_i^2\rangle\right]^{-\frac{1}{2}} \times\exp\left\{-\sum_{i=1}^{N}\xi_i^2/2\langle\xi_i^2\rangle\right\}. \tag{3.9}$$

Using this probability,

$$\left\langle \xi_i\frac{\partial U}{\partial\xi_i}\right\rangle=\omega_i^2\langle\xi_i^2\rangle\left[\left(\sum_{j=1}^{N}\omega_j^2\langle\xi_j^2\rangle\right)+2\omega_i^2\langle\xi_i^2\rangle\right]. \tag{3.10}$$

If $S=\lambda\sum_{j=1}^{N}\omega_j^2\langle\xi_j^2\rangle$, then, as Caughey has shown,[6]

$$S=\left\{-1+\left[1+4(N+2)\lambda\sigma_{\xi_k}^2\right]^{\frac{1}{2}}\right\}/2(1+2/N) \tag{3.11}$$

and

$$\langle\xi_k^2\rangle=\left\{-(1+S)+\left[(1+S)^2+8\lambda\sigma_{\xi_k}^2\right]^{\frac{1}{2}}\right\}/4\lambda\omega_k^2, \tag{3.12}$$

which for small nonlinearities gives

$$\langle\xi_k^2\rangle\approx\left[1-(N+2)\lambda\sigma_{\xi_k}^2\right]\sigma_{\xi_k}^2. \tag{3.13}$$

If, however, we make the approximate evaluation of Eq. (3.8) by evaluating $\langle\xi_i\partial U/\partial\xi_i\rangle$ using the Gaussian distribution for the linear system, then as before

$$\langle\xi_k^2\rangle\approx\left[1-(N+2)\lambda\sigma_{\xi_k}^2\right]\sigma_{\xi_k}^2. \tag{3.14}$$

This corresponds with the first-order solution obtained by Ariaratnam[14] by expanding the exact solution from the Fokker–Planck equation.

IV. OTHER TECHNIQUES FOR EQUIVALENT LINEARIZATION

The Booton–Caughey technique is only one of many different methods for obtaining approximate solutions to nonlinear random excitation problems. Rice[15] has developed a number of techniques for analyzing frequency-independent nonlinear elements. These techniques can easily be modified for use in equivalent linearization, as done by Sewaragi and Takahashi[16] and Caughey.[17] They are somewhat more difficult to use and, for this reason, have not been used very often for the class of problems discussed in this paper. However, they are ideally suited to problems involving both random and deterministic parts in the solution.

---

[14] S. T. Ariaratnam, "Random Vibration of Non-Linear Suspensions," J. Mech. Eng. Sci. **2**, 195–201 (1960).

[15] S. O. Rice, "Mathematical Analysis of Random Noise," Bell System Tech. J. **23**, 282–332 (1944); **24**, 46–156 (1945). [Also, N. Wax *et al.*, *Selected Papers on Noise and Stochastic Processes* (Dover Publications, Inc., New York, 1954), pp. 133–294.]

[16] Y. Sewaragi and S. Takahashi, "Statistical Analysis of Control Systems Containing Zero-Memory Non-Linearization under Random Inputs," Proc. Japan Natl. Congr. Appl. Mech. 5th (1955).

[17] T. K. Caughey, "Response of Van der Pol's Oscillator to Random Excitation," J. Appl. Mech. **26**, 345–348 (1959).