EQUIVALENT LOW-PASS REPRESENTATIONS FOR BANDPASS VOLTERRA SYSTEMS

by

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I. INTRODUCTION

The attraction for the use of the Volterra series method in the solution of nonlinear communications problems originates in the desire for a straightforward, systematic tool that can be applied to a large number of nonlinear problems. The Volterra series method supplies the need for a systematic technique, but is generally avoided due to the complexity of the solution.

Since the introduction of Volterra methods into nonlinear circuit analysis by Wiener in 1942 [1], applications of the approach have been numerous. In the early 1960's, Van Trees [2] used Volterra series to define nonlinearities in the operation of phase locked loops. Shortly thereafter, Maurer and Narayanan [3] used this method to describe the effect on noise-power ratio of a third order nonlinearity in a single stage transistor amplifier. Later, Mircea and Sinnreich [4] applied the series to distortion analysis. The early 1970's brought Bedrosian and Rice's tutorial work [5] on applications to nonlinear systems with memory. Later, Benedetto, et al. [6] applied the Volterra approach to baseband digital systems.

Even with these demonstrations of the wide range of applications for the use of the Volterra series method, the approach has not gained widespread use due to the enormous task of carrying out numerical evaluations of the resulting multiple sums. When only the first few terms of the series are computed, the solutions are simplified but the problems are thereby limited to the solution of some "better behaved" nonlinearities such as might be described by a fourth or fifth order polynomial.

The point of attack for this report is the simplification of the task of carrying out solutions to these multiple sums. This simplification is accomplished for a class of Volterra systems that include linear zonal input and output filters such as the case of a T.W.T. amplifier in a communications satellite transponder. The inherent symmetry in the Volterra kernels and certain observations concerning intermediate results will allow a significant simplification of the Volterra method as the result of the development of a low-pass equivalent Volterra kernel.

This report will approach the subject beginning in Section II with a discussion of the general Volterra systems approach to the solution of nonlinear problems. Section III develops the model for the general case of a nonlinear system with arbitrary input and output filters. Section IV reviews the applications of equivalent low-pass representations to linear bandpass systems. In Section V, techniques similar to those described in Section IV are applied to the Volterra system model developed in Section III. The intermediate results obtained in Section V are simplified in two parts in Appendix A and Appendix B. The combination and further simplification of these two parts are shown in Section V. The final low-pass representation for the Volterra kernel and output signal are presented at the end of Section V and are applied to some simple examples in Section VI.

II. VOLTERRA SYSTEMS

An example used in Van Trees [10] of a second order nonlinear system with memory is shown in Figure 2.1. The output of the filter can be represented by the convolution

$$y(t) = h_1(t) \otimes x(t) = \int_{-\infty}^{\infty} h_1(\tau)x(t-\tau)d\tau \ . \tag{2.1}$$

The system output is

$$z(t) = y^2(t) \ . \tag{2.2}$$

Substituting (2.1) into (2.2) and rearranging,

$$z(t) = \iint_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 \, d\tau_2 \tag{2.3}$$

which appears much like a double convolution. This expression can be used to define the second order Volterra kernel as

$$h_2(\tau_1,\tau_2) \equiv h_1(\tau_1)h_1(\tau_2) \ . \tag{2.4}$$

The output becomes

$$z(t) = \iint_{-\infty}^{\infty} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 . \tag{2.5}$$

This result can be extended to higher orders of nonlinearity. If $f(y)$ is analytic in some region, the example may be extended to a case for which $z(t)$ is some power series with coefficients γ_k which shall be considered real constants. Therefore, $z(t)$ is given by

$$z(t) = f(y(t)) = \gamma_0 + \gamma_1 y(t) + \gamma_2 y^2(t) + \cdots \ . \tag{2.6}$$

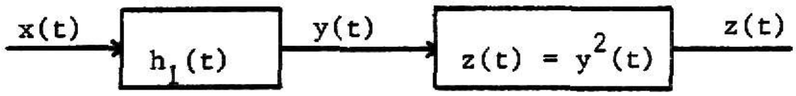

Figure 2.1. Second Order Nonlinearity With Memory

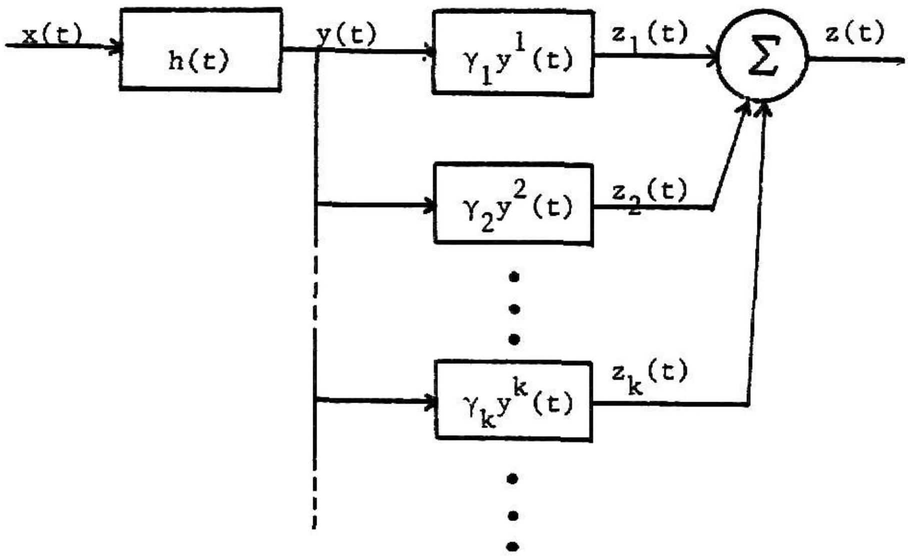

Figure 2.2. General Volterra Series Nonlinear System With Memory

Using the procedure just outlined on each term of (2.6), the output becomes

$$z(t) = \sum_{k=1}^{\infty} \gamma_k \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_k(\tau_1, \cdots, \tau_k) x(t-\tau_1) \cdots x(t-\tau_k) d\tau_1 \cdots d\tau_k . \quad (2.7)$$

This expression is a series of terms, each representing a separate order of the nonlinearity and each contributing to the output z(t). A helpful conceptual model representing (2.7) is shown in Figure 2.2. This concept is important to the development of a general result for the kth order Volterra kernel because it allows each order of nonlinearity to be treated in a separate expression as part of (2.7).

It can be seen even in these simpler models that the complexity of a solution grows at an enormous rate with the addition of each higher order of nonlinearity.

III. MODELING NONLINEAR CHANNELS WITH MEMORY

The basic model under consideration for the remainder of this report will be termed a nonlinear communication channel with memory. Although this model occurs more often in practice than the conceptual model presented in the previous section, it is further complicated by the addition of a second memory block in the form of an output linear filter. A block diagram of the new model is shown in Figure 3.1.

The desired result is an expression for the Volterra kernel in terms of the impulse response of the linear filters and the power series describing the nonlinearities. By convolution, each order of nonlinearity in the sum

$$v(t) = \sum_{k=1}^{\infty} \gamma_k u^k(t)$$

contributes the amount

$$y_k(t) = \int_{-\infty}^{\infty} h''(\tau) v_k(t-\tau) d\tau \tag{3.1}$$

to the output y(t), as implied by Figure 3.2.

The signal u(t) can also be obtained by convolving the input signal with the impulse response of the input filter

$$u(t) = \int_{-\infty}^{\infty} h'(\tau) x(t-\tau) d\tau \quad . \tag{3.2}$$

and, since

$$v_k(t-\tau) = \gamma_k u^k(t-\tau) \quad , \tag{3.3}$$

it is necessary to form u(t-τ). The result is

$$u(t-\tau) = \int_{-\infty}^{\infty} h'(\tau)\, x(t-\tau-u)\, du \quad . \tag{3.4}$$

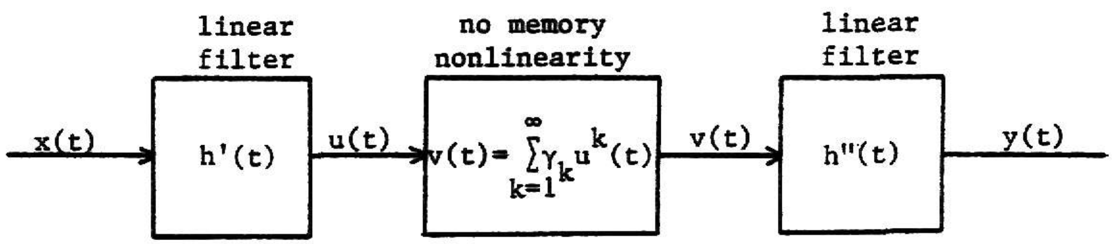

Figure 3.1. Nonlinear System With Memory

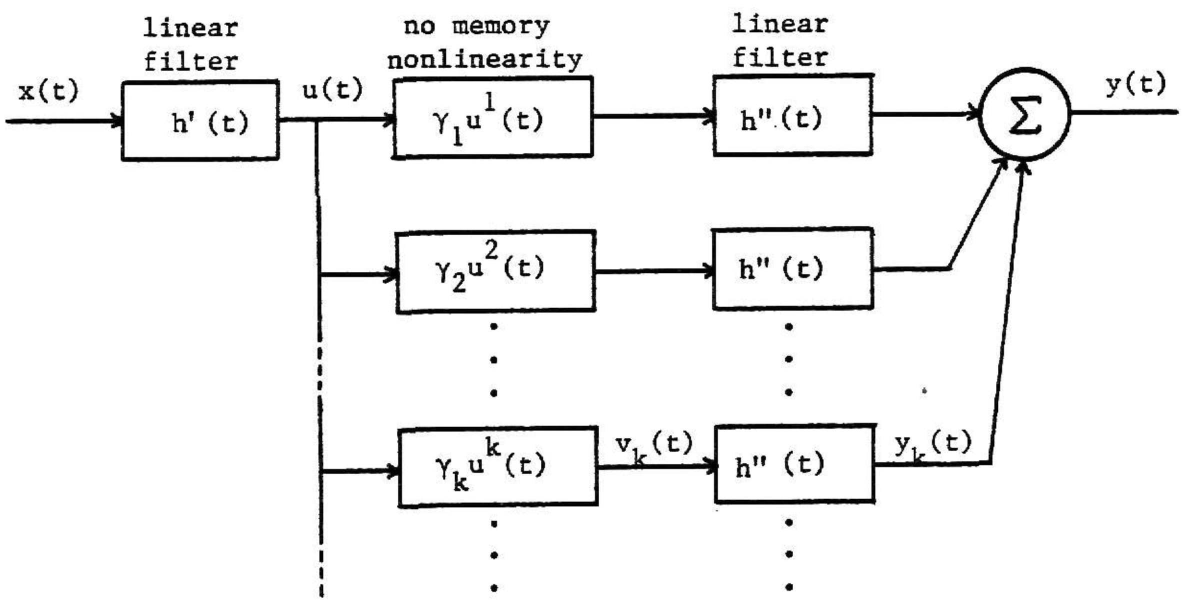

Figure 3.2. Nonlinear System With Memory (Parallel Model)

A more convenient form is obtained by making the change of variables $\tau_i = \tau + u$, yielding

$$u_i(t-\tau) = \int_{-\infty}^{\infty} h'(\tau_i-\tau)x(t-\tau_i)d\tau_i \ . \tag{3.5}$$

Substitution of (3.5) into (3.3) allows (3.1) to be rewritten as

$$y\ (t) = \int_{-\infty}^{\infty} h''(\tau)\gamma_k\left(\int_{-\infty}^{\infty} h'(\tau_i-\tau)x_i(t-\tau_i)d\tau_i\right)^k d\tau \tag{3.6}$$

or, finally

$$y_k(t) = \int\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\gamma_k\int_{-\infty}^{\infty} h''(\tau)\prod_{i=1}^{k} h'(\tau_i-\tau)d\tau\prod_{i=1}^{k} x(t-\tau_i)\right)d\tau_1\cdots d\tau_k \ . \tag{3.7}$$

The resulting Volterra kernel of this system is

$$h_k(\tau_1,\cdots,\tau_k) = \gamma_k\int_{-\infty}^{\infty} h''(\tau)\prod_{i=1}^{k} h'(\tau_i-\tau)d\tau \ . \tag{3.8}$$

Using (3.8) in (3.7)

$$y_k(t) = \int\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_k(\tau_1,\cdots,\tau_k)\prod_{i=1}^{k} x(t-\tau_i)d\tau_1\cdots d\tau_i \ . \tag{3.9}$$

Again, solution of this representation is time consuming and extremely involved, even with close attention to efficient programming and simplified algorithms. Notice also that to this point no frequency restrictions have been placed on the input and output filters or the signal input. Later, the development of a bandpass model and its equivalent low-pass representation will require these to be linear zonal filters and narrowband bandpass signals. The development of a low-pass representation will begin in the next section with a review of linear bandpass systems.

IV. LOW-PASS EQUIVALENT REPRESENTATION OF LINEAR BANDPASS SYSTEMS

One method of simplifying expressions for output signals for a bandpass linear system is to reduce the entire system to an equivalent low-pass form by complex envelope representation of the narrow-band bandpass signals. This method is used extensively in linear systems and serves to simplify both the calculations and the statement of the results. A typical transformation for the linear case seen in Figure 4.1 if from Stein and Jones [7] .

The linear case assumes a narrow-band bandpass signal input and output with the unit impulse response of the system in terms of complex equivalents. The linear bandpass system is described by its unit impulse response g(t) and its frequency function G(f) by

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \tag{4.1}$$

It can be shown [7] that for narrow-band systems where f_0 is the band center frequency and

$$G(f)\Big|_{f>0} = H(f-f_0) \qquad \text{and} \qquad H(f-f_0) = 0 \quad f<0$$

with H(f) a low-pass frequency function nonvanishing only near the origin, the unit impulse response is given by

$$g(t) = 2\mathrm{Re}\{ e^{j2\pi f_0 t}\int_{-\infty}^{\infty} H(\lambda)e^{j2\pi\lambda t}d\lambda\} = 2\mathrm{Re}\{ h(t)e^{j2\pi f_0 t} \} \tag{4.2}$$

where $h(t) \Longleftrightarrow H(f)$. Thus, the linear bandpass system is described in terms of its band center frequency f_0 and its equivalent low-pass system given by H(f) and h(t) which are transform pairs.

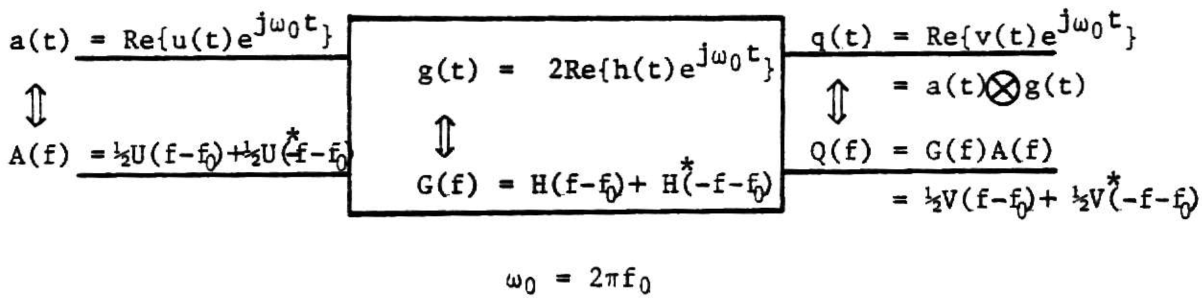

Figure 4.1a. Linear Bandpass Model.

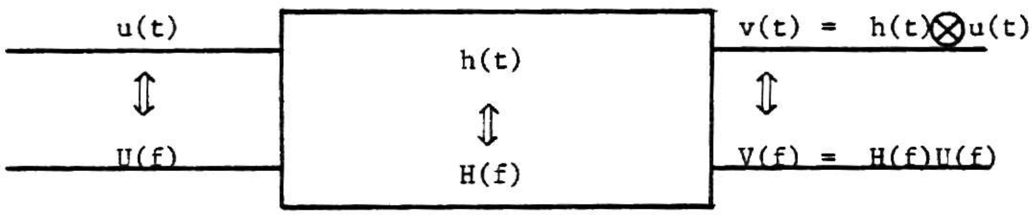

Figure 4.1b. Equivalent Low-Pass Model.

Figure 4.1. Complex Signal Representation of Narrow-band, Bandpass Volterra Signals and Systems.

In a linear system driven by a bandpass signal with carrier frequency f_0 at the band center frequency of the system, the output q(t) is also a bandpass signal which can be given as

$$q(t) = \text{Re}\{ v(t)e^{j2\pi f_0 t}\} \tag{4.3}$$

or, as

$$q(t) = \tfrac{1}{2}v(t)e^{j2\pi f_0 t} + \tfrac{1}{2}v^*(t)e^{-j2\pi f_0 t} \tag{4.4}$$

where v(t) is the complex envelope of q(t). The bandpass frequency spectrum Q(f) is related to V(f), the frequency spectrum of the complex envelope by

$$Q(f) = \tfrac{1}{2}V(f-f_0) + \tfrac{1}{2}V^*(-f-f_0) \tag{4.5}$$

with the bandpass system as shown in Figure 4.1a, the output q(t) is related to the impulse response g(t) and the input signal a(t) by the convolution

$$q(t) = a(t)\otimes g(t) = \int_{-\infty}^{\infty} a(\tau)g(t-\tau)d\tau \tag{4.6}$$

and the product in the frequency domain is

$$Q(f) = G(f)A(f) \; . \tag{4.7}$$

Expressing both A(f) and G(f) in their low-pass spectra forms U(f) and H(f) , (4.7) becomes

$$Q(f) = \tfrac{1}{2}\left(H(f-f_0) + H^*(-f-f_0)\right)\left(U(f-f_0) + U^*(-f-f_0)\right) \tag{4.8}$$

Since U(f) and H(f) are nonoverlapping, the product terms involving one conjugated term and one nonconjugated term from (4.8) are zero in in the resulting expression

$$Q(f) = \tfrac{1}{2}H(f-f_0)U(f-f_0) + \overset{*}{H}(-f-f_0)\overset{*}{U}(-f-f_0) \qquad (4.9)$$

Comparision of (4.9) with (4.5) shows that

$$V(f) = H(f)U(f) \ . \qquad (4.10)$$

It follows that the complex envelope of the output is the convolution of the complex envelope of the input and the impulse response of the equivalent low-pass system

$$v(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau . \qquad (4.11)$$

Substituting (4.11) into (4.3), the output bandpass signal becomes

$$q(t) = \mathrm{Re}\{ v(t)e^{j2\pi f_0 t}\} = \mathrm{Re}\{ e^{j2\pi f_0 t}\int_{-\infty}^{\infty} u(\tau)h(t-\tau)\, d\tau \} \ . \qquad (4.12)$$

This relationship completely describes the signal shaping effects of the system on the input signal. It also allows the removal of the term $e^{j2\pi f_0 t}$ from the integral (4.12), simplifying calculation of the result.

The application of this scheme to the Volterra system of Figure 3.2 will reduce the complexity of the solution for the output $y_k(t)$. This is an advantage if it can be shown that the number of terms has not increased in the process of introducing the complex envelope representations for the signals and filters. Intuitively, the number of terms would seem to increase as two complex terms are exchanged for each real term in the original expression (3.7).

However, it will be seen that by considering each intermediate result as they are derived in Section V and by taking advantage of the symmetry inherent in these Volterra series representations, no additional terms will be involved in the soltuion and the indicated advantage of suppression of the carrier term will be realized.

V. EQUIVALENT LOW-PASS REPRESENTATION OF BANDPASS VOLTERRA SYSTEMS

A class of nonlinear systems which occurs frequently in communication equipment consists of a bandpass filter input to a nonlinear element followed by a zonal output filter. The output filter might be tuned to the same frequency or some integer multiple of the input center frequency. Some examples of this type system are receiver mixers or frequency converters and satellite communication channels which use T.W.T. type amplifiers. If low-pass output filters are allowed, then various signal detection schemes provide some additional examples.

The model developed in Section III and shown in Figure 3.1 will be used to represent this class of systems. The impulse responses of the input and output linear filters are h'(t) and h"(t) respectively. The nonlinearity is assumed to be a no-memory nonlinear transformation which can be represented by a power series expansion. Various forms of this model have been used [2,6,8] to characterize Volterra systems. The input-output relationship for this type system is known [5,9] to be

$$y(t) = \sum_{k=1}^{\infty} y_k(t) \tag{5.1}$$

where

$$y_k(t) = \int_{\infty}^{\infty}\int \cdots \int_{\infty}^{\infty} h_k(\tau_1,\tau_2,\cdots,\tau_k) \prod_{i=1}^{k} x(t-\tau_i)d\tau_1\cdots d\tau_k \ . \tag{3.9}$$

Here, x(t) is the input signal, y(t) is the output signal, $y_k(t)$ is the contribution from the kth order portion of the nonlinearity to the system output and $h_k(\tau_1,\cdots,\tau_k)$ for $k = 1, 2,\cdots$ are the kernels of the Volterra system. In the previous section it was shown that the kernels for the system of Figure 3.1 are given by

$$h_k(\tau_1,\tau_2,\cdots,\tau_k) = \gamma_k \int_{-\infty}^{\infty} h''(\tau) \prod_{i=1}^{k} h'(\tau_i-\tau)\, d\tau \qquad (5.3)$$

where γ_k is the coefficient of the kth order term in the series expansion of the nonlinearity. The objective of this report is to present a low-pass equivalent representation for the model of Figure 3.1.

The first step in the development of the low-pass representation is the expression of $x(t)$, $h'(t)$ and $h''(t)$ in terms of their complex envelopes similar to the development in Section IV. For the bandpass input signal $x(t)$,

$$x(t) = \text{Re}\,\{\, \tilde{x}(t)e^{j\omega_0 t} \,\} = \tfrac{1}{2}\tilde{x}(t)\, e^{j\omega_0 t} + \tfrac{1}{2}\tilde{x}^*(t)e^{-j\omega_0 t} \qquad (5.4)$$

where ω_0 is the center frequency of $x(t)$, $\tilde{x}(t)$ is the slowly varying complex envelope of $x(t)$ and the symbol $*$ denotes the complex conjugate. Similarly, for $h'(t)$ and $h''(t)$

$$h'(t) = 2\text{Re}\,\{\, \tilde{h}'(t)e^{j\omega_0 t} \,\} = \tilde{h}'(t)e^{j\omega_0 t} + \tilde{h}'^*(t)e^{-j\omega_0 t} \qquad (5.5)$$

and

$$h''(t) = 2\text{Re}\,\{\, \tilde{h}''(t)e^{jn\omega_0 t}\} = \tilde{h}''(t)e^{jn\omega_0 t} + \tilde{h}''^*(t)e^{-jn\omega_0 t}. \qquad (5.6)$$

Note the notation in (5.6) indicates the zonal output filter is centered on $n\omega_0$, $n = 1, 2,\cdots$. The case $n = 0$ corresponds to a low-pass output filter resulting in a baseband output signal.

Using the complex equivalent forms (5.5) and (5.6) in equation (5.3) for the Volterra kernels along with the form (5.4), substitution into equation (5.2) for the kth output contribution yields the low-pass representation

$$y_k(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\gamma_k\int_{-\infty}^{\infty} h''(\tau)\prod_{i=1}^{k} h'(\tau_i-\tau)d\tau \prod_{m=1}^{k} x(t-\tau_m)d\tau_1\cdots d\tau_k$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\gamma_k\int_{-\infty}^{\infty} \left(\tilde{h}''(\tau)e^{jn\omega_0\tau} + \tilde{h}''^*(\tau)e^{-jn\omega_0\tau}\right)$$

$$\cdot\prod_{i=1}^{k}\left(\tilde{h}'(\tau_i-\tau)e^{j\omega_0(\tau_i-\tau)} + \tilde{h}'^*(\tau_i-\tau)e^{-j\omega_0(\tau_i-\tau)} \right)d\tau$$

$$\cdot\prod_{m=1}^{k}\tfrac{1}{2}\left(\tilde{x}(t-\tau_m)e^{j\omega_0(t-\tau_m)} + \tilde{x}^*(t-\tau_m)e^{-j\omega_0(t-\tau_m)} \right)d\tau_1\cdots d\tau_k \tag{5.7}$$

which appears rather cumbersome. However, most of the numerous products implied do not contribute to the output of the system and may be screened by inspection. For example, consider the product similar to the last term in (5.7)

$$\prod_{m=1}^{k}\left(\tfrac{1}{2}\tilde{x}(t-\tau_m)e^{j\omega_0(t-\tau_m)} + \tfrac{1}{2}\tilde{x}^*(t-\tau_m)e^{-j\omega_0(t-\tau_m)}\right) . \tag{5.8}$$

Expanding (5.8) for arbitrary k, say k equal 3, a few of the resulting terms are

$$\frac{1}{8}\,\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)\tilde{x}(t-\tau_3)e^{j\omega_0(3t - \tau_1 - \tau_2 - \tau_3)}$$

$$+\ \frac{1}{8}\,\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)\tilde{x}^*(t-\tau_3)e^{j\omega_0(1t - \tau_1 - \tau_2 + \tau_3)}$$

$$+ \frac{1}{8}\tilde{x}(t-\tau_1)\tilde{x}^*(t-\tau_2)\tilde{x}^*(t-\tau_3)e^{-j\omega_0(1t+\tau_1-\tau_2-\tau_3)}$$

$$+ \cdots . \qquad (5.9)$$

A table of all such terms for k = 3 shows only terms with $\pm 3\omega_0 t$ or $\pm 1\omega_0 t$ in the exponent, indicating possible contributions to the system output can only be at center frequencies of $3\omega_0$ or $1\omega_0$. None of these terms will survive the output filtering unless the narrow-band output filter is centered at either $3\omega_0$ or $1\omega_0$. This same reasoning allows the screening of terms for other resulting products for arbitrary choice of k. After the selection of the output harmonic determined by n, the corresponding $n\omega_0$ terms are retained and the remaining terms are discarded.

It is observed from (5.9) that there is some relationship between the number of conjugated terms and the resulting harmonic associated with each of the products. A table may be constructed for a few values of k from which this relationship between the number of conjugated terms, the order of nonlinearity k and the output center frequency $n\omega_0$ may be determined as

$$N = N(n,k) = \binom{k}{\frac{k+n}{2}} = \binom{k}{\frac{k-n}{2}} = \frac{k!}{\left(\frac{k+n}{2}\right)!\left(\frac{k-n}{2}\right)!} \qquad (5.10)$$

where N(n,k) is the number of conjugate pairs of terms contributing to the output at $n\omega_0$. This scheme is developed more fully in Appendix A.

A similar method but different argument is used to screen terms from the remaining product in the τ integral

$$\left(\tilde{h}''(\tau)e^{jn\omega_0\tau} + \tilde{h}''^*(\tau)e^{-jn\omega_0\tau}\right)\prod_{i=1}^{k}\left(\tilde{h}'(\tau_i-\tau)e^{j\omega_0(\tau_i-\tau)} + \tilde{h}'(\tau_i-\tau)e^{-j\omega_0(\tau_i-\tau)}\right) \qquad (5.11)$$

from which the derivation of a complex kernel may be anticipated.

Again, as for (5.8), (5.11) can be expanded for k = 3 and arbitrary n to obtain expressions such as

$$\tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)\tilde{h}'(\tau_3-\tau)e^{j\omega_0((n-3)\tau\ +\tau_1+\tau_2+\tau_3)}$$

and

$$\tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)\tilde{h}'^{*}(\tau_3-\tau)e^{j\omega_0((n-1)\tau\ +\tau_1+\tau_2-\tau_3)} \tag{5.12}$$

It can be shown that there are a total of 2^4 possible terms like those shown in (5.12) or 2^{k+1} terms for arbitrary orders of nonlinearities. This number of terms is reduced considerably when the indicated integration with respect to τ is carried out. The complex envelopes represented by $\tilde{h}'(\tau)$ and $\tilde{h}''(\tau)$ are assumed to be slowly varying functions of τ. Thus, when they are used with factors such as $\exp(j(n-m)\omega_0\tau)$ for which (n-m) does not equal zero, the entire term will be insignificant. It is not difficult to establish a method of determining the number of terms for which (n-m) is zero (Appendix B). This requirement forces $\exp(j(n-m)\omega_0\tau)$ equal to one and allows the combination of these surviving terms from (5.11) with the previous results from (5.8).

The terms resulting from this combination are at frequency $(n-m)\omega_0$ plus $n\omega_0$ or, for the simplifications indicated, $n\omega_0$. Since these terms are all at the center frequency of the output filter, they have the potential for being passed to the output. A few of the terms resulting from this combination for the k=3 example are

$$\tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)\tilde{h}'(\tau_3-\tau)\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)\tilde{x}(t-\tau_3)e^{j3\omega_0 t}\ ,$$

$$\tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)\tilde{h}'^{*}(\tau_3-\tau)\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)\tilde{x}^{*}(t-\tau_3)e^{j1\omega_0 t}$$

and

$$\tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)\tilde{h}'^{*}(\tau_3-\tau)\tilde{x}(t-\tau_1)\tilde{x}^{*}(t-\tau_2)\tilde{x}(t-\tau_3)e^{j1\omega_0 t}e^{j\omega_0(\tau_2-\tau_3)}\ . \tag{5.13}$$

Again, since the complex envelopes $\tilde{h}'(\tau)$ and $\tilde{x}(t)$ vary so much slower than the carrier frequency terms like $\exp(j\omega_0(\tau_2-\tau_1))$, the only terms that will survive integration by τ_1,τ_2 and τ_3 are terms like the first and second terms in (5.13), which do not involve τ_1,τ_2 or τ_3 in the exponent. All terms like the third one in which the conjugated factors in the signal part of the product do not "match" those in the input filter product will not contribute to the final system output.

With a simple change of variables, it can be shown that all of the terms with the same number of conjugated parts yield the same result when the integrals with respect to τ_1, τ_2 and τ_3 are carried out. This allows rearranging and matching of surviving terms in the general form of the complex representation described by (5.7). The number of terms in this simplified form of (5.7) is given by the coefficient $N(n,k)$ from (5.10). Any one of the N number of terms could be used to describe all the terms, but the notation here will associate all the lowest number subscripts of τ with the proper number of conjugated terms given by

$$NC = NC(n,k) = \frac{k-n}{2} \tag{5.14}$$

Thus, if $y_k^{(n)}(t)$ denotes the kth order output at the nth harmonic, then the result from (5.7) is the surprisingly more simple relationship

$$y_k^{(n)}(t) = 2\mathrm{Re}\Big\{2^{-k}\gamma_k N\int_{-\infty}^{\infty}\int\cdots\int_{-\infty}^{\infty}\tilde{h}''(\tau)\tilde{h}'^{*}(\tau_1-\tau)\cdots\tilde{h}'^{*}(\tau_{NC}-\tau)\tilde{h}'(\tau_{NC+1}-\tau)\cdots \\ \tilde{h}'(\tau_k-\tau)d\tau\ \tilde{x}^{*}(t-\tau_1)\cdots\tilde{x}^{*}(t-\tau_{NC})\tilde{x}(t-\tau_{NC+1})\cdots\tilde{x}(t-\tau_k) \\ d\tau_1 d\tau_2\cdots d\tau_k e^{jn\omega_0 t}\Big\} \quad . \tag{5.15}$$

The definition of an equivalent low-pass Volterra kernel is seen to be

$$\tilde{h}_k^{(n)}(\tau_1,\tau_2,\cdots\tau_k) = 2^{-k+1}\gamma_k N \int_{-\infty}^{\infty} \tilde{h}''(\tau)\tilde{h}'^*(\tau_1-\tau)\cdots\tilde{h}'^*(\tau_{NC}-\tau) \cdot\tilde{h}'(\tau_{NC+1}-\tau)\cdots\tilde{h}'(\tau_k-\tau)\, d\tau \quad (5.16)$$

Also, defining the kth order complex output envelope $\tilde{y}_k^{(n)}(t)$ in

$$y_k^{(n)}(t) = \mathrm{Re}\left\{ \tilde{y}_k^{(n)}(t)e^{jn\omega_0 t} \right\} , \quad (5.17)$$

the desired result is obtained as

$$\tilde{y}_k^{(n)}(t) = \int\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\tilde{h}_k^{(n)}(\tau_1,\tau_2,\cdots,\tau_k)\tilde{x}^*(t-\tau_1)\cdots\tilde{x}^*(t-\tau_{NC})\tilde{x}(t-\tau_{NC+1})\cdots \tilde{x}(t-\tau_k)\, d\tau_1\, d\tau_2 \cdots d\tau_k . \quad (5.18)$$

The system output at $n\omega_0$ is the sum of the contributions from the individual orders of nonlinearity given by

$$\tilde{y}^{(n)}(t) = \sum_{k=1}^{\infty} \tilde{y}_k^{(n)}(t) . \quad (5.19)$$

This form demonstrates the desired similarity to the familiar linear system output

$$y(t) = \mathrm{Re}\left\{ \tilde{y}(t)e^{jn\omega_0 t} \right\} . \quad (5.20)$$

VI. APPLICATIONS: SQUARE LAW DEVICES

One possible application of this low-pass representation for a Volterra system is the solution of a problem for which there is only one order of nonlinearity present. The system considered here has an input which may be represented as a bandpass signal. The nonlinearity contains a square law device and some constant gain factor γ_2. The entire system, including the output filter and input filter is shown in Figure 6.1.

There are really two configurations for the output filter in this system for which the output signal will be of interest. One form is for when the output filter is a low-pass filter and the other is for when the output filter is centered at $2\omega_0$. If the output filter is at $2\omega_0$, then equation (5.15) results in

$$y_2^{(2)}(t) = 2\mathrm{Re}\left\{ 2^{-2}\gamma_k \iiint_{-\infty}^{\infty} \tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)d\tau\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2) \cdot d\tau_1 d\tau_2 e^{j2\omega_0 t} \right\}, \quad (6.1)$$

for which N(n,k) is equal to one and NC(n,k) is equal to zero, indicating no conjugate products are involved in this solution. If the solution calls for the complex kernel representation, it is clear that the kernel is

$$\tilde{h}_2^{(2)}(\tau_1,\tau_2) = \tfrac{1}{2}\gamma_2 \int_{-\infty}^{\infty} \tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)d\tau \quad . \quad (6.2)$$

And, the output complex envelope is

$$\tilde{y}_2^{(2)}(t) = \iint_{-\infty}^{\infty} \tilde{h}_2^{(2)}(\tau_1,\tau_2)\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)d\tau_1 d\tau_2 \quad . \quad (6.3)$$

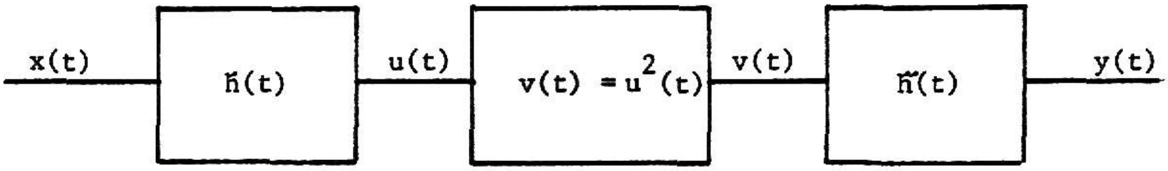

Figure 6.1a. Square Law Bandpass System

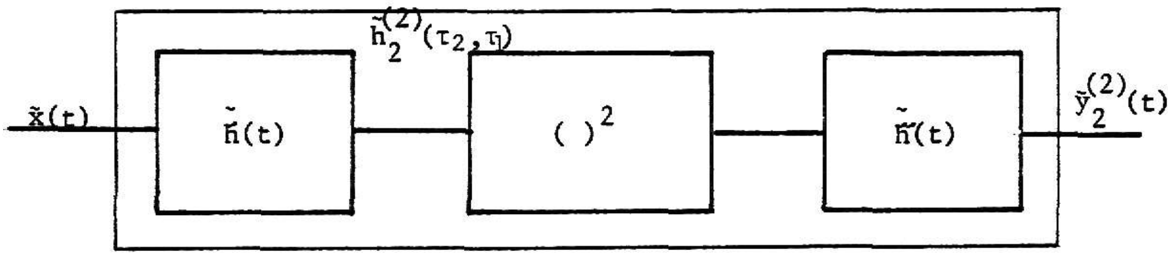

Figure 6.1b. Complex Square Law Bandpass Output Filter

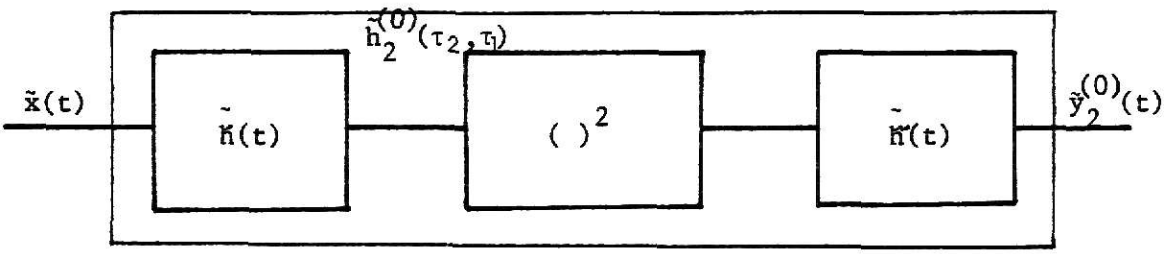

Figure 6.1c. Complex Square Law Low-pass Output Filter

Figure 6.1. Square Law System

The resulting output signal is

$$y^{(2)}(t) = \mathrm{Re}\{ \tilde{y}_2^{(2)}(t)e^{j2\omega_0 t} \} .$$

If the output of the system is desired at baseband as in Figure 6.1c, then substitution of $\frac{1}{2}h''(\tau)$ for $h''(\tau)$, because $h''(\tau)$ is real if n=0, yields

$$\tilde{h}_2^{(0)}(\tau_1,\tau_2) = \tfrac{1}{4}\, \gamma_2 \int_{-\infty}^{\infty} \tilde{h}''(\tau)\tilde{h}'^{*}(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)d\tau \quad . \tag{6.5}$$

Also,

$$\tilde{y}_2^{(0)}(t) = \iint_{-\infty}^{\infty} \tilde{h}_2^{(0)}(\tau_1,\tau_2)\tilde{x}^{*}(t-\tau_1)\tilde{x}(t-\tau_2)\, d\tau_1\, d\tau_2 \tag{6.6}$$

and

$$y^{(0)}(t) = \mathrm{Re}\{ \tilde{y}_2^{(0)}(t)e^{j0} \} = \mathrm{Re}\{ \tilde{y}^{(0)}(t) \}, \tag{6.7}$$

which is the desired baseband result.

VII. CONCLUSIONS

The result of this effort to manipulate the basic Volterra series representation of bandpass nonlinear systems into the familiar low-pass form is the presentation of a systematic and possibly more familiar looking approach to the solution of a broad class of problems termed here as nonlinear systems with memory. The enormous task of solving the multiple integrals inherent in the Volterra series method is reduced by removing the carrier dependence from the integrals representing the kernels and system inputs and outputs. This simplification allows the integration to be carried out over the slowly varying envelopes rather than a carrier frequency modified by a signal or filter response. Since all of the functions integrated involve only low-pass slowly varying functions, fewer samples may be used to represent the function in a numerical solution and a considerable reduction in computer time can result. The bandpass representation is summarized in equations (5.10) and (5.14) through (5.19).

The versitility of the approach is demonstrated in the two examples of Section VI. Each of the examples could have had identical input and filter envelopes. If so, the only difference in the solutions would have been multiplication by a conversion constant to adjust for a different number of terms at the second harmonic and a change of carrier frequency in the final result. The shape of the output signals would have been similar.

APPENDIX A

COMPLEX MULTIPLICATION AND HARMONICS GENERATED EQUATION (5.8)

The intermediate result from Section V was given as a product of terms such as

$$\prod_{i=1}^{k} \left(\tfrac{1}{2}\tilde{x}(t-\tau_i)e^{j\omega_0(t-\tau_i)} + \tfrac{1}{2}\tilde{x}^*(t-\tau_i)e^{-j\omega_0(t-\tau_i)} \right) . \qquad \text{(A.1)}$$

This expression is simplified by multiplying out a few terms and verifying that the result can be predicted by the application of some simple mathematical expression. The object of this method is to determine the number of terms that will result at a particular harmonic of ω_0 and to determine some relationship between the order k of nonlinearity and n, the harmonic selected by the output filter.

For example, if a term is desired for the second order (k=2) and the output filter center frequency is $2\omega_0$, multiplication of the expression (A.1) would result in the sum of four terms, two of which are

$$\frac{1}{4}\,\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)e^{j\omega_0(2t-\tau_1-\tau_2)}$$

and

$$\frac{1}{4}\,\tilde{x}(t-\tau_1)\tilde{x}^*(t-\tau_2)e^{j\omega_0(0t-\tau_1+\tau_2)} \quad .$$

The other two terms are simply the conjugates of these two terms. Of the four terms, it is easily seen that only two of them are at the $\pm 2\omega_0$ frequencies and two are at zero frequency. Since the only interesting terms are those that will be passed by the narrow output filter centered at $2\omega_0$, the zero frequency terms will be stopped and will not contribute to the output of the system.

Similarly, for k = 3 and n = 1, the terms after multiplication of (A.1) are the four terms

$$\frac{1}{8}\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)\tilde{x}(t-\tau_3)e^{j\omega_0(3t-\tau_1-\tau_2-\tau_3)} ,$$

$$\frac{1}{8}\tilde{x}(t-\tau_1)\tilde{x}(t-\tau_2)\tilde{x}^*(t-\tau_3)e^{j\omega_0(1t-\tau_1-\tau_2+\tau_3)} ,$$

$$\frac{1}{8}\tilde{x}(t-\tau_1)\tilde{x}^*(t-\tau_2)\tilde{x}(t-\tau_3)e^{j\omega_0(1t-\tau_1+\tau_2-\tau_3)}$$

and

$$\frac{1}{8}\tilde{x}^*(t-\tau_1)\tilde{x}(t-\tau_2)\tilde{x}(t-\tau_3)e^{j\omega_0(1t+\tau_1-\tau_2-\tau_3)} .$$

The remaining four terms are of course the conjugates of the above terms. The conjugates of the above will be the negative frequency terms at $\pm 3\omega_0$ and $\pm 1\omega_0$. Some effort will be made to list only the positive frequency terms and remember there will always be corresponding conjugates that are so easily obtained that they need not be listed.

Using these and similar examples for higher orders of nonlinearities, a table can be built using a simple shorthand notation to condense the information. In this notation, a nonconjugated term such as $x(t-\tau_i)$ is represented by a zero, while a conjugated term like $x^*(t-\tau_i)$ is a one. Although there are 2^k combinations of conjugated or nonconjugated parts, half of the terms may be represented by a binary count from zero to 2^{k-1}. The other half are the conjugates of the listed terms. To obtain the code for any of the conjugate terms, the listed code is complimented.

Also listed in the table is the resulting coefficient of $\omega_0 t$ in the exponent also represented by each code. This listing represents the harmonics generated and transmitted to the output filter by the nonlinearity.

This harmonic indicator is determined by simply subtracting the number of nonconjugated terms, zeros, from the number of conjugated terms, ones, in each code listed. It can also be seen at this point that the sign on each of the $\tau_1, \tau_2, \ldots \tau_k$ terms in the exponent represented is dependent upon which of the terms in the code are conjugates. A one code such as $\overset{*}{x}(t-\tau_i)$ indicates a $+\tau_i$. A zero code indicates a $-\tau_i$. The codes for the k=3 example are, for instance

(0 0 0) , (0 0 1) , (0 1 0) and (1 0 0) .

Tabulating the first few columns and projecting the results to the higher orders of nonlinearities, Table A represents a method of displaying the possible terms resulting from (A.1) for a few orders of nonlinerities.

It is desirable to represent only positive frequency terms in the table. However, when a particular entry for a given k contains more conjugated than nonconjugated terms, this code would represent a negative entry in the harmonic column. For these special cases, a correction code is given as the code for the conjugate term representing a positive frequency in these cases. The correction bits (5) and (6) are ones only for k=5 and k=6 respectively.

This small table is enough to indicate a pattern of correspondence between the number of entries in any one column at a particular harmonic n and the order of nonlinearity k. This relationship is given by the expression

$$N(n,k) = \binom{k}{\frac{k+n}{2}} , \qquad \text{(A.2)}$$

Table A

Complex Multiplication and Harmonics Generated

	Codes: Correction (6 5 4 3 2 1)	Codes: Count (6 5 4 3 2 1)	Harmonics generated by codes given: k=2	k=3	k=4	k=5	k=6
		0 0 0 0 0 0	2	3	4	5	6
k=2		0 0 0 0 0 1	0	1	2	3	4
		0 0 0 0 1 0		1	2	3	4
k=3	1 0 0	0 0 0 0 1 1		1	0	1	2
		0 0 0 1 0 0			2	3	4
		0 0 0 1 0 1			0	1	2
		0 0 0 1 1 0			0	1	2
k=4	(5) 1 0 0 0	0 0 0 1 1 1			2	1	0
		0 0 1 0 0 0				3	4
		0 0 1 0 0 1				1	2
		0 0 1 0 1 0				1	2
	1 0 1 0 0	0 0 1 0 1 1				1	0
		0 0 1 1 0 0				1	2
	1 0 0 1 0	0 0 1 1 0 1				1	0
	1 0 0 0 1	0 0 1 1 1 0				1	0
k=5	(6) 1 0 0 0 0	0 0 1 1 1 1				3	2
		0 1 0 0 0 0					4
		0 1 0 0 0 1					2
		0 1 0 0 1 0					2
		0 1 0 0 1 1					0
		0 1 0 1 0 0					2
		0 1 0 1 0 1					0
		0 1 0 1 1 0					0
	1 0 1 0 0 0	0 1 0 1 1 1					2
		0 1 1 0 0 0					2
		0 1 1 0 0 1					0
		0 1 1 0 1 0					0
	1 0 0 1 0 0	0 1 1 0 1 1					2
		0 1 1 1 0 0					0
	1 0 0 0 1 0	0 1 1 1 0 1					2
	1 0 0 0 0 1	0 1 1 1 1 0					2
k=6	1 0 0 0 0 0	0 1 1 1 1 1					4

where N(n,k) is the number of positive frequency terms at the nth harmonic resulting from the kth order nonlinearity. Again, it is understood that there is an equal number of negative frequency terms at $-n\omega_0$ except for the case n = 0 which is a special case.

APPENDIX B

HARMONICS MODIFIED BY OUTPUT CENTER FREQUENCY

The second part of the intermediate result from Section V involves the complex representations for the input and output filters in a product such as

$$\left(\tilde{h}''(\tau)e^{jn\omega_0\tau}+\tilde{h}''^{*}(\tau)e^{-jn\omega_0\tau}\right)\prod_{i=1}^{k}\left(\tilde{h}'(\tau_i-\tau)e^{j\omega_0(\tau_i-\tau)}+\tilde{h}'^{*}(\tau_i-\tau)e^{-j\omega_0(\tau_i-\tau)}\right). \quad (B.1)$$

The portion of (B.1) involving only the input filter is very similar to the signal portion described in Appendix A. In fact, a table can be built from this product. In this case, a { 0 0 1 } code would indicate the term

$$\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)\tilde{h}'^{*}(\tau_3-\tau)e^{j\omega_0(-\tau\ +\tau_1+\tau_2-\tau_3)} \ .$$

It is noted that the only difference between this term and the one that would result from Table A is the sign on all the exponent terms. For this reason, in the new table, the difference between the number of non-conjugate and conjugate terms has the opposite sign of the harmonic indicator listed in Table A.

Furthermore, Table B includes the additional combinations generated by the multiplication of the remaining portion of (A.1) involving the output filter

$$\tilde{h}''(\tau)e^{jn\omega_0\tau}+\tilde{h}''^{*}(\tau)e^{-jn\omega_0\tau} \ .$$

This multiplication expands the original table to twice as many entries, adding one more column on the left of the counting and correction codes.

Table B

Complex Multiplication Codes and Harmonics Generated by Output Filter Center Frequency

	Codes: Correction M	6	5	4	3	2	1	Codes: Count M	6	5	4	3	2	1	Harmonics generated by codes given: k=2	k=3	k=4	k=5	k=6
								0	0	0	0	0	0	0	-2+n	-3+n	-4+n	-5+n	-6+n
								1	0	0	0	0	0	0	-2-n	-3-n	-4-n	-5-n	-6-n
								0	0	0	0	0	0	1	0+n	-1+n	-2+n	-3+n	-4+n
↑ k=2								1	0	0	0	0	0	1	0-n	-1-n	-2-n	-3-n	-4-n
								0	0	0	0	0	1	0		-1+n	-2+n	-3+n	-4+n
								1	0	0	0	0	1	0		-1-n	-2-n	-3-n	-4-n
	1				1	0	0	0	0	0	0	0	1	1		-1+n	0+n	-1+n	-2+n
↑ k=3	0				1	0	0	1	0	0	0	0	1	1		-1-n	0-n	-1-n	-2-n
								0	0	0	0	1	0	0			-2+n	-3+n	-4+n
								1	0	0	0	1	0	0			-2-n	-3-n	-4-n
								0	0	0	0	1	0	1			0+n	-1+n	-2+n
								1	0	0	0	1	0	1			0-n	-1-n	-2-n
								0	0	0	0	1	1	0			0+n	-1+n	-2+n
								1	0	0	0	1	1	0			0-n	-1-n	-2-n
	1		(5)	1	0	0	0	0	0	0	0	1	1	1			-2+n	-1+n	0+n
↑ k=4	0		(5)	1	0	0	0	1	0	0	0	1	1	1			-2-n	-1-n	0-n
								0	0	0	1	0	0	0				-3+n	-4+n
								1	0	0	1	0	0	0				-3-n	-4-n
								0	0	0	1	0	0	1				-1+n	-2+n
								1	0	0	1	0	0	1				-1-n	-2-n
								0	0	0	1	0	1	0				-1+n	-2+n
								1	0	0	1	0	1	0				-1-n	-2-n
	1		1	0	1	0	0	0	0	0	1	0	1	1				-1+n	0+n
	0		1	0	1	0	0	1	0	0	1	0	1	1				-1-n	0-n
								0	0	0	1	1	0	0				-1+n	-2+n
								1	0	0	1	1	0	0				-1-n	-2-n
	1		1	0	0	1	0	0	0	0	1	1	0	1				-1+n	0+n
	0		1	0	0	1	0	1	0	0	1	1	0	1				-1-n	0-n
	1		1	0	0	0	1	0	0	0	1	1	1	0				-1+n	0+n
	0		1	0	0	0	1	1	0	0	1	1	1	0				-1-n	0-n
	1	(6)	1	0	0	0	0	0	0	0	1	1	1	1				-3+n	-2+n
↑ k=5	0	(6)	1	0	0	0	0	1	0	0	1	1	1	1				-3-n	-2-n
								0	0	1	0	0	0	0					-4+n

This multiplication also modifies the resulting harmonic columns for different orders of nonlinearities by +n for a zero or nonconjugated $\tilde{h}''(\tau)$ and by -n for a one or conjugated $\tilde{h}''^{*}(\tau)$.

The table as shown in Table B can only be completed upon the specific choice of output center frequency $n\omega_0$. Once n is determined, it is substituted into the relationship listed for the harmonic generated. Because only odd values of n are allowed for odd orders of nonlinearity k, once n is given, half of the columns in the table may be ignored. It is also noted at this point that for the soltuion of the Volterra series problems as stated in Section V, the only entries in the table that are of interest are those that result in a zero harmonic. This is due to the fact that these are the only terms that will combine with those obtained in Appendix A to produce an output at the frequency $n\omega_0$.

Product terms can be easily reconstructed from Table B. For example, a code {0|0 0 1} represents the product term

$$\tilde{h}''(\tau)\tilde{h}'(\tau_1-\tau)\tilde{h}'(\tau_2-\tau)\tilde{h}'^{*}(\tau_3-\tau)e^{j\omega_0(-1\tau+\tau_1+\tau_2-\tau_3)}.$$

Again, it can be seen from this example that the signs on $\tau_1, \tau_2, \cdots, \tau_k$ are the opposite of those indicated by a similar code in Table A. This fact is important to a further simplification when the two product terms (A.1) and (B.1) are combined in Section V.

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EQUIVALENT LOW-PASS REPRESENTATIONS FOR BANDPASS VOLTERRA SYSTEMS

by

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ABSTRACT

The complex envelope representation is used to describe a narrow-band bandpass nonlinear communication channel with memory. The Volterra series method is applied to determine the output of the bandpass system at various harmonics of the input signal center frequency. The entire system is reduced to its equivalent low-pass form to simplify the Volterra solution by suppressing carrier frequency terms.