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EQUIVALENT MIXING CONDITIONS FOR RANDOM FIELDS[1]

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For strictly stationary random fields indexed by $\mathbb{R}^d$ or $\mathbb{Z}^d$, certain versions of the "strong mixing" condition are equivalent to corresponding versions of the "ρ-mixing" condition.

1. Introduction. Dobrushin [(1968), page 205] and Zhurbenko [(1984) and (1986), page 8] each gave an example to show that, for strictly stationary random fields indexed by $\mathbb{Z}^2$, apparently natural versions of the "ϕ-mixing" condition turn out to be extremely restrictive. Adapting the insights in their examples, the author [Bradley (1989)] showed that, for such random fields, such versions of ϕ-mixing or even "absolute regularity" are in fact equivalent to corresponding versions of "m-dependence". By Rosenblatt [(1985), page 73, Theorem 7], this does not apply to corresponding versions of the "strong mixing" or "ρ-mixing" conditions. In this note it will be shown that, for stationary random fields, certain versions of these latter two conditions are equivalent to each other.

Suppose $d \geq 2$ is an integer. Suppose $X := (X_t, t \in \mathbb{R}^d)$ is a (real) strictly stationary random field on a probability space $(\Omega, \mathscr{F}, P)$. For any $t \in \mathbb{R}^d$, let $\|t\|$ denote its usual Euclidean norm. Let us give one formulation of the ordinary notion of "mixing" for X: For any elements $t_1, \dots, t_m \in \mathbb{R}^d$, any elements $u_1, \dots, u_n \in \mathbb{R}^d$, any Borel set $A \subset \mathbb{R}^m$ and any Borel set $B \subset \mathbb{R}^n$, one has that

$$P\big((X_{t(1)}, \dots, X_{t(m)}) \in A \text{ and } (X_{u(1)+v}, \dots, X_{u(n)+v}) \in B\big)$$
$$\to P\big((X_{t(1)}, \dots, X_{t(m)}) \in A\big) P\big((X_{u(1)}, \dots, X_{u(n)}) \in B\big)$$

as $\|v\| \to \infty$, $v \in \mathbb{R}^d$. [Here of course $t(i)$ and $u(i)$ just mean t_i and u_i.]

Now for any two σ-fields $\mathscr{A}$ and $\mathscr{B} \subset \mathscr{F}$, define the following measures of dependence:

$$(1.1)\quad \alpha(\mathscr{A}, \mathscr{B}) := \sup |P(A \cap B) - P(A)P(B)|, \qquad A \in \mathscr{A}, B \in \mathscr{B};$$

$$(1.2)\quad \rho(\mathscr{A}, \mathscr{B}) := \sup |\operatorname{Corr}(V, W)|, \qquad V \in \mathscr{L}_2(\mathscr{A}), W \in \mathscr{L}_2(\mathscr{B}).$$

The following inequality is elementary:

$$(1.3)\qquad \alpha(\mathscr{A}, \mathscr{B}) \leq (1/4)\rho(\mathscr{A}, \mathscr{B}).$$

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The "distance" between any two disjoint nonempty subsets S and $T \subset \mathbb{R}^d$ will be denoted $\operatorname{dist}(S,T) := \inf_{s \in S,\, t \in T} \|s - t\|$. For disjoint nonempty sets S and T, we use the abbreviations

$$\alpha(S,T) := \alpha(\sigma(X_t, t \in S), \sigma(X_t, t \in T)),$$
$$\rho(S,T) := \rho(\sigma(X_t, t \in S), \sigma(X_t, t \in T)).$$

[The notation $\sigma(\cdots)$ denotes the σ-field generated by $(\cdots)$.] For any real number $r > 0$, define the following dependence coefficients for the given random field X:

$$\alpha(r) := \sup \alpha(S,T), \tag{1.4}$$
$$\rho(r) := \sup \rho(S,T), \tag{1.5}$$

where in both (1.4) and (1.5) the supremum is taken over all pairs of disjoint closed d-dimensional *half-spaces* S and $T \subset \mathbb{R}^d$ such that $\operatorname{dist}(S,T) \ge r$. [A "closed half-space" means of course a set of the form $\{t \in \mathbb{R}^d\colon \langle t - s, u \rangle \ge 0\}$, where $s \in \mathbb{R}^d$, $u \in \mathbb{R}^d$, $u \ne (0,\dots,0)$ and $\langle \cdot , \cdot \rangle$ denotes the dot product.] For any real $r > 0$, define also

$$\alpha^*(r) := \sup \alpha(S,T), \qquad S \subset \mathbb{R}^d, T \subset \mathbb{R}^d, \operatorname{dist}(S,T) \ge r, \tag{1.6}$$
$$\rho^*(r) := \sup \rho(S,T), \qquad S \subset \mathbb{R}^d, T \subset \mathbb{R}^d, \operatorname{dist}(S,T) \ge r; \tag{1.7}$$

that is, here there are no restrictions on the sets S and T except that they be nonempty and satisfy $\operatorname{dist}(S,T) \ge r$. Obviously, $\alpha(r) \le \alpha^*(r)$ and $\rho(r) \le \rho^*(r)$. Conditions such as $\alpha(r) \to 0$ (as $r \to \infty$), $\rho^*(r) \to 0$, and so on, have been used in the study of limit theory for random fields [see, e.g., Gorodetskii (1984), Rosenblatt (1985), Ivanov and Leonenko (1989) or Goldie and Morrow (1986) and the references therein].

Our main result is as follows.

THEOREM 1. *Suppose $d \ge 2$. Suppose $X := (X_t,\ t \in \mathbb{R}^d)$ is a strictly stationary random field which is mixing and $r > 0$ is a real number. Then the following statements hold*:

(a) $\alpha(r) \le \rho(r) \le 2\pi\alpha(r)$.
(b) $\alpha(r) = \frac{1}{4}$ *if and only if* $\rho(r) = 1$.
(c) $\alpha^*(r) \le \rho^*(r) \le 2\pi\alpha^*(r)$.
(d) $\alpha^*(r) = \frac{1}{4}$ *if and only if* $\rho^*(r) = 1$.

REMARK 1. Of course the condition $\alpha(r) \to 0$ implies mixing. Hence by Theorem 1 the conditions $\alpha(r) \to 0$ and $\rho(r) \to 0$ are equivalent to each other for strictly stationary random fields $X := (X_t,\ t \in \mathbb{R}^d)$, $d \ge 2$. Similarly, for such random fields the conditions $\alpha^*(r) \to 0$ and $\rho^*(r) \to 0$ are equivalent to each other.

REMARK 2. By essentially the same proof, obvious analogs of Theorem 1(a)–(d) hold for strictly stationary mixing random fields $(X_k,\ k \in \mathbb{Z}^d)$, $d \ge 2$, that is, indexed by the d-dimensional integer lattice.

REMARK 3. Again by the same proofs, analogs of parts (c) and (d) of Theorem 1 still hold when $d = 1$, that is, for strictly stationary mixing random processes $(X_t,\ t \in \mathbb{R})$ or $(X_k,\ k \in \mathbb{Z})$. This is of course not true in general for parts (a) and (b).

REMARK 4. One can trivially reformulate some theorems involving $\rho^*(r)$. For example, from Theorems 3 and 4 of Bradley (1992) and Remark 3 above, one has the following central limit theorem.

THEOREM 2. *If $(X_k,\ k \in \mathbb{Z})$ is a strictly stationary sequence of real centered square-integrable random variables such that $E(X_1 + \cdots + X_n)^2 \to \infty$ and $\alpha^*(n) \to 0$ as $n \to \infty$, then $(X_1 + \cdots + X_n)/\|X_1 + \cdots + X_n\|_2$ converges to $N(0,1)$ in distribution as $n \to \infty$.*

Here of course $\alpha^*(r)$ is as defined by (1.6), with $\mathbb{R}^d$ replaced by $\mathbb{Z}$.

2. Proof of Theorem 1. We shall prove parts (c) and (d) first and then just indicate the (trivial) changes that are needed to obtain (a) and (b).

PROOF OF (c) AND (d). Denote $\rho := \rho^*(r)$. The case $\rho = 0$ is trivial, so we assume $\rho > 0$. Because of (1.3), to prove both (c) and (d) it suffices to prove that $\alpha^*(r) \geq (2\pi)^{-1} \arcsin \rho$. Let $\varepsilon \in (0, \rho)$ be arbitrary but fixed. To prove both (c) and (d) it suffices to prove

$$\alpha^*(r) \geq (2\pi)^{-1} \arcsin(\rho - \varepsilon). \tag{2.1}$$

There exist real simple random variables Y and Z with

$$EY = EZ = 0, \qquad EY^2 = EZ^2 = 1, \quad \text{and} \quad q := EYZ \geq \rho - \varepsilon \tag{2.2}$$

such that Y is $\sigma(X_{t(1)}, \dots, X_{t(I)})$-measurable and Z is $\sigma(X_{u(1)}, \dots, X_{u(J)})$-measurable, where I and J are positive integers and $t_1, \dots, t_I, u_1, \dots, u_J$ are elements of $\mathbb{R}^d$ such that $\operatorname{dist}(\{t_1, \dots, t_I\}, \{u_1, \dots, u_J\}) \geq r$. There exist Borel simple functions $f\colon \mathbb{R}^I \to \mathbb{R}$ and $g\colon \mathbb{R}^J \to \mathbb{R}$ such that $Y = f(X_{t(1)}, \dots, X_{t(I)})$ and $Z = g(X_{u(1)}, \dots, X_{u(J)})$. (Let all of these things be fixed.)

Let H be a positive number such that

$$H \geq r + \max\|v - w\|, \tag{2.3}$$

where the maximum is taken over all pairs of elements v, w in the set $\{t_1, \dots, t_I,\ u_1, \dots, u_J\}$. Define the random variables Y_k and Z_k, $k \in \mathbb{Z}$, as follows:

$$Y_k := f\big(X_{t(1)+(kH,0,\dots,0)}, \dots, X_{t(I)+(kH,0,\dots,0)}\big),$$

$$Z_k := g\big(X_{u(1)+(kH,0,\dots,0)}, \dots, X_{u(J)+(kH,0,\dots,0)}\big).$$

Note that $Y_0 = Y$ and $Z_0 = Z$. By (2.3), $\sigma(Y_k,\ k \in \mathbb{Z}) \subset \sigma(X_t,\ t \in S)$ and $\sigma(Z_k,\ k \in \mathbb{Z}) \subset \sigma(X_t,\ t \in T)$ for some pair of sets $S, T \in \mathbb{R}^d$ such that

$\operatorname{dist}(S, T) \geq r$. Hence

$$\alpha(\sigma(Y_k, k \in \mathbb{Z}), \sigma(Z_k, k \in \mathbb{Z})) \leq \alpha^*(r). \tag{2.4}$$

LEMMA 1. *Suppose $\mathscr{G}$ is a finite σ-field $\subset \sigma(X_t, t \in \mathbb{R}^d)$. Then* $\lim_{j \to \infty} \alpha(\mathscr{G}, \sigma(Y_j, Z_j)) = 0$.

PROOF. There are finitely many sets $G \in \mathscr{G}$ and finitely many sets $Q \subset$ [range of f] $\times$ [range of g]. For each such pair G, Q, one has that

$$\left|P\left(G \cap \{(Y_j, Z_j) \in Q\}\right) - P(G)P((Y, Z) \in Q)\right| \to 0 \quad \text{as } j \to \infty,$$

by the assumption that X is mixing. Lemma 1 follows. □

Define the strictly increasing sequence of positive integers $J_1, J_2, J_3, \dots$ recursively as follows:

To begin, define $J_1 := 1$.

Now suppose that $n \geq 2$ is an integer and that $J_1, \dots, J_{n-1}$ have already been defined. Note that $\sigma((Y_j, Z_j), j \in \{J_1, \dots, J_{n-1}\})$ is a finite σ-field. Using Lemma 1, let J_n be a positive integer such that $J_n > J_{n-1}$ and

$$\alpha\left(\sigma\left((Y_j, Z_j), j \in \{J_1, \dots, J_{n-1}\}\right), \sigma\left(Y_{J(n)}, Z_{J(n)}\right)\right) \leq 1/n^2. \tag{2.5}$$

This completes the recursive definition of the sequence $J_1, J_2, J_3, \dots$.

LEMMA 2. *As $N \to \infty$, the random vector*

$$(N+1)^{-1/2}\left(\sum_{n=N}^{2N} Y_{J(n)}, \sum_{n=N}^{2N} Z_{J(n)}\right)$$

converges in distribution to the centered normal law on $\mathbb{R}^2$ with covariance matrix $\begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$, where q is as in (2.2).

PROOF. This is a well-known type of result. The random vectors $(Y_{J(n)}, Z_{J(n)})$, $n = 1, 2, 3, \dots,$ have the same distribution [namely, that of (Y, Z)], with mean vector $[0, 0]$ and covariance matrix $\begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$. Also, by (2.5),

$$\begin{aligned} &\sum_{n=N}^{2N-1} \alpha\left(\sigma\left((Y_j, Z_j), j \in \{J_N, J_{N+1}, \dots, J_n\}\right), \sigma\left(Y_{J(n+1)}, Z_{J(n+1)}\right)\right) \\ &\qquad \leq \sum_{n=N}^{2N-1} \frac{1}{(n+1)^2} \\ &\qquad < \frac{1}{N} \\ &\qquad \to 0 \text{ as } N \to \infty. \end{aligned} \tag{2.6}$$

This last fact allows us to reduce Lemma 2 to the case of independent random vectors, by an old, well-known argument. That argument was spelled out in the Appendix of Bradley (1991) in a particular form with our context in mind, but it is essentially the argument given in, for example, Ibragimov and Linnik [(1971), page 338, from line 4 to the bottom of the page]. One shows that the characteristic function of $(\Sigma_{n=N}^{2N} Y_{J(n)}, \Sigma_{n=N}^{2N} Z_{J(n)})$, differs from the $(N + 1)$th power of that of (Y, Z) by uniformly at most $16 \cdot$ [l.h.s. of (2.6)]. Thus from the classic CLT for i.i.d. random vectors, Lemma 2 follows. □

Now we adapt a simple argument from Kolmogorov and Rozanov (1960). By a standard calculation, the centered normal law on $\mathbb{R}^2$ with covariance matrix $\begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$ assigns probability $(1/4)[1 + (2/\pi)(\arcsin q)]$ to the first quadrant $(0, \infty) \times (0, \infty)$. For each $N = 1, 2, 3, \dots$ define the events $A_N := \{\Sigma_{n=N}^{2N} Y_{J(n)} > 0\}$ and $B_N := \{\Sigma_{n=N}^{2N} Z_{J(n)} > 0\}$. By Lemma 2,

$$\lim_{N \to \infty} [P(A_N \cap B_N) - P(A_N)P(B_N)] = (2\pi)^{-1}(\arcsin q).$$

Hence, by (2.4) and (2.2), one has that (2.1) holds. This completes the proof of (c) and (d). □

Sketch of proof of (a) and (b). By using translations and rotations of the coordinate system on $\mathbb{R}^d$, one can easily derive (a) and (b) of Theorem 1 from the following lemma.

Lemma 3. *Suppose $d \geq 2$. Suppose $X := (X_t, t \in \mathbb{R}^2)$ is a strictly stationary random field which is mixing and $r > 0$ is a real number. Define the closed half-planes*

$$S^* := \{s := (s_1, \dots, s_d) \in \mathbb{R}^d : s_d \leq 0\},$$

$$T^* := \{s := (s_1, \dots, s_d) \in \mathbb{R}^d : s_d \geq r\}.$$

Then $\alpha(S^*, T^*) \geq (2\pi)^{-1} \arcsin \rho(S^*, T^*)$.

To prove Lemma 3, one simply carries out the proof of statements (c) and (d) of Theorem 1, with appropriate modifications. In particular, in the definition of the random variables Y and Z, one takes the indices $t_1, \dots, t_I$ from S^* and the indices $u_1, \dots, u_J$ from T^*. Then (later on) one automatically has $\sigma(Y_k, k \in \mathbb{Z}) \subset \sigma(X_t, t \in S^*)$ and $\sigma(Z_k, k \in Z) \subset \sigma(X_t, t \in T^*)$.

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REFERENCES

Bradley, R. C. (1989). A caution on mixing conditions for random fields. *Statist. Probab. Lett.* **8** 489–491.

Bradley, R. C. (1991). Equivalent mixing conditions for random fields. Technical Report 336, Center for Stochastic Processes, Dept. Statistics, Univ. North Carolina, Chapel Hill.

Bradley, R. C. (1992). On the spectral density and asymptotic normality of weakly dependent random fields. *J. Theoret. Probab.* **5** 355–373.

DOBRUSHIN, R. L. (1968). The description of a random field by means of conditional probabilities and conditions of its regularity. *Theory Probab. Appl.* **13** 197–224.

GOLDIE, C. M. and MORROW, G. J. (1986). Central limit questions for random fields. In *Dependence in Probability and Statistics* (E. Eberlein and M. S. Taqqu, eds.) 275–289. Birkhäuser, Boston.

GORODETSKII, V. V. (1984). The central limit theorem and an invariance principle for weakly dependent random fields. *Soviet Math. Dokl.* **29** 529–532.

IBRAGIMOV, I. A. and LINNIK, YU. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.

IVANOV, A. V. and LEONENKO, N. N. (1989). *Statistical Analysis of Random Fields*. Kluwer, Boston.

KOLMOGOROV, A. N. and ROZANOV, YU. A. (1960). On strong mixing conditions for stationary Gaussian processes. *Theory Probab. Appl.* **5** 204–208.

ROSENBLATT, M. (1985). *Stationary Sequences and Random Fields*. Birkhäuser, Boston.

ZHURBENKO, I. G. (1984). On mixing conditions for random processes with values in a Hilbert space. *Soviet Math. Dokl.* **30** 465–467.

ZHURBENKO, I. G. (1986). *The Spectral Analysis of Time Series*. North-Holland, Amsterdam.

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