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EQUIVARIANT ALEXANDER-SPANIER COHOMOLOGY FOR ACTIONS OF COMPACT LIE GROUPS

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Introduction.

In [3] we constructed for a finite group G an equivariant cohomology theory, defined on all G-pairs and having properties analogous to those of ordinary Alexander-Spanier cohomology. The purpose of this paper is to extend the construction of this "equivariant Alexander-Spanier cohomology theory" to the case of a compact Lie group G.

Recall that the ordinary Alexander-Spanier cohomology groups ([7], 6.4), as well as the equivariant groups of [3], are computed from a cochain complex which is obtained by first defining certain cochain groups and then deviding out "locally zero" cochains. The main difficulty in the generalization from a finite group G to a compact Lie group G is the proper formulation of the notion of a locally zero cochain. We give this definition in section 3 below. The cochains themselves can be defined in almost the same way as in [3], cf. section 2. In these two sections it appears useful to keep in mind certain aspects of the correspondence between G-maps from a homogeneous space G/H to a G-space X, and H-fixed points of X. Therefore we devote a short section 1 to these matters; this section is essentially based on [5], 3.2.i).

The remaining two sections show that the properties of the equivariant Alexander-Spanier cohomology proved in [3] remain valid after the generalization to a compact Lie group G. In section 4 we indicate how the proofs given in [3] can be modified to show that the equivariant Eilenberg-Steenrod axioms still hold. Section 5 contains three results: a) In a paracompact G-space any closed G-subspace is taut; b) the equivariant Alexander-Spanier cohomology of a paracompact G-space X is isomorphic to the ordinary cohomology of the orbit space X/G with coefficients in a certain sheaf; c) if the paracompact G-space X is locally sufficiently nice, then the equivariant Alexander-Spanier cohomology of X is isomorphic to its equivariant singular cohomology, as defined in [4].

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Notation and terminology.

In this paper G is a compact Lie group. The notation $H \leqq G$ means that H is a *closed* subgroup of G. Recall the *discrete orbit category* $\mathrm{Or}_d G$ of G: its objects are the homogeneous spaces G/H, $H \leqq G$, and its morphisms are the G-homotopy classes of G-maps between them. A *contravariant G-coefficient system* is a contravariant functor $\mathrm{Or}_d G \to \mathrm{Ab}$; we typically denote such a coefficient system by m. Finally, all G-spaces are assumed to be Hausdorff, as usual.

1. G-maps $G/H \to X$.

Let X be a G-space and $H \leqq G$ a closed subgroup of G. In this preliminary section we recall some facts about G-maps $G/H \to X$; the set of these maps is denoted by $\mathrm{Map}_G(G/H, X)$.

First of all, the formula $f \mapsto f(eH)$ defines a natural bijection

$$(1.1) \qquad \mathrm{Map}_G(G/H, X) \xrightarrow{\simeq} X^H.$$

Both sides of 1.1 have a natural left action of the Weyl group $WH = NH/H$ such that 1.1 is WH-equivariant; in $\mathrm{Map}_G(G/H, X)$ the action is given by

$$(1.2) \qquad (nH)\cdot f\colon gH \mapsto f(gnH),\ n \in NH, f \in \mathrm{Map}_G(G/H, X),\ g \in G.$$

Let WH_0 be the identity component of WH. From 1.1 we get the bijection $\mathrm{Map}_G(G/H, X)/WH_0 \xrightarrow{\simeq} X^H/WH_0$. It is clear from the considerations in 3.2.i) of [5] that two G-maps $f_1, f_2\colon G/H \to X$ are in the same WH_0-orbit of $\mathrm{Map}_G(G/H, X)$ if and only if they are G-homotopic by a homotopy h_t satisfying $h_t(G/H) = f_1(G/H) = f_2(G/H)$ for every $t \in [0, 1]$. In this case we say that f_1 and f_2 are *G-fiber homotopic* (as maps from $G/H \to *$ to $X \to X/G$). Hence 1.1 induces a natural bijection

$$(1.3) \qquad \left\{\begin{matrix}G\text{-fiber homotopy classes}\\ \text{of } G\text{-maps } G/H \to X\end{matrix}\right\} \xrightarrow{\simeq} X^H/WH_0.$$

Assume now that $u\colon G/K \to G/H$ is a G-map; then $u(gK) = gaH$ for every $g \in G$, where $u(eK) = aH$ and $a \in G$ satisfies $a^{-1}Ka \leqq H$. The map u induces the map $X(u)\colon X^H \to X^K, x \mapsto ax$ (the notation $X(u)$ is from [5]). By 1.3 it is clear that $X(u)$ further induces the arrow $\bar{X}(u)$ making the diagram

$$(1.4) \qquad \begin{array}{ccc} X^H & \xrightarrow{X(u)} & X^K \\ \downarrow & & \downarrow \\ X^H/WH_0 & \xrightarrow{\bar{X}(u)} & X^K/WK_0 \end{array}$$

commutative, and in addition $\bar{X}(u)$ depends only on the G-homotopy class of u. Altogether we have now reviewed the definition of the functor

$$\bar{X}\colon \operatorname{Or}_d G \to \mathrm{Top},\ \ G/H \mapsto X^H/WH_0$$

of [5], 3.2.i).

The commutativity of 1.4 has the following immediate consequence:

LEMMA 1.5. *In the preceding notation, if $A \subset X^K$ is WK_0-invariant, then $X(u)^{-1}(A) \subset X^H$ is WH_0-invariant. As a special case (with $K = \{e\}$), if $A \subset X$ is G_0-invariant, then $A \cap X^H$ is WH_0-invariant.*

2. Definition of cochains.

Let again X be a G-space. Given $n \in \mathrm{N}$ and $H \leqq G$, we denote by $V_n^H(X)$ the set of all $(n+1)$-tuples $\varphi = (\varphi_0, \ldots, \varphi_n)$, where each φ_i is a G-map $G/H \to X$, i.e. $V_n^H(X) = \operatorname{Map}_G(G/H, X)^{n+1}$. Further we set

$$V_n(X) = \bigcup_{H \leqq G} V_n^H(X) = \bigcup_{H \leqq G} \operatorname{Map}_G(G/H, X)^{n+1}.$$

Let $m\colon \operatorname{Or}_d G \to \mathrm{Ab}$ be a contravariant coefficient system and $M = \bigoplus_{H \leqq G} m(G/H)$. We can now define the n^{th} *cochain group $C^n(X; m)$ of X with coefficients m* in the same way as in [3], namely

$$C^n(X; m) = \{c\colon V_n(X) \to M \mid c(\varphi) \in m(G/H) \text{ if } \varphi \in V_n^H(X)\}.$$

These form the cochain complex $C^*(X; m)$ with the usual coboundary operator.

A cochain $c \in C^n(X; m)$ is *equivariant*, if the following condition holds: given $\varphi = (\varphi_0, \ldots, \varphi_n) \in V_n^H(X)$ and $\alpha = (\alpha_0, \ldots, \alpha_n) \in V_n^K(G/H)$ such that the G-maps α_i: $G/K \to G/H$ are all G-homotopic,

$$c(\varphi \circ \alpha) = m(\alpha_0)(c(\varphi)),$$

where of course $\varphi \circ \alpha = (\varphi_0 \circ \alpha_0, \ldots, \varphi_n \circ \alpha_n)$; note that $m(\alpha_0) = \ldots = m(\alpha_n)$. The equivariant cochains form a cochain subcomplex $C_G^*(X; m) \subset C^*(X; m)$.

We point out the following functoriality property: any equivariant function $f\colon X \to Y$, not necessarily continuous, between G-spaces induces a cochain map $f^*\colon C^*(Y; m) \to C^*(X; m)$ by the formula

$$(f^*(c))(\varphi_0, \ldots, \varphi_n) = c(f \circ \varphi_0, \ldots f \circ \varphi_n);\ \ c \in C^n(Y; m),\ \varphi \in V_n(X).$$

Namely, the composite $f \circ \varphi$: $G/H \to Y$ is a G-map if φ: $G/H \to X$ is one, even if f is not continuous. It is also clear that $f^* C_G^*(Y; m) \subset C_G^*(X, m)$.

Next we give another description of the cochain complex $C_G^*(X; m)$. We begin with

LEMMA 2.1. *If* $c \in C^n_G(X;m)$, $\varphi, \varphi' \in V^H_n(X)$ *and* φ_i *is G-fiber homotopic to* φ'_i *for every* $i \in \{0,\dots,n\}$, *then* $c(\varphi) = c(\varphi')$.

PROOF. By 1.3 there are elements $n_i H \in WH_0$ such that $\varphi'_i(gH) = \varphi_i(gn_iH)$ for $i \in \{0,\dots,n\}$, $g \in G$. This means that $\varphi'_i = \varphi_i \circ \alpha_i$, where $\alpha_i\colon G/H \to G/H$ is the G-map $gH \mapsto gn_iH$, and $\alpha_i \underset{G}{\simeq} \mathrm{id}$ for all $i \in \{0,\dots,n\}$. By the equivariance of c we have

$$c(\varphi') = c(\varphi \circ \alpha) = m(\alpha_0)(c(\varphi)) = m(\mathrm{id})(c(\varphi)) = c(\varphi).$$

This shows that in the definition of $C^n(X;m)$ above, instead of $V_n(X) = \bigcup_{H \leqq G} V^H_n(X)$, we could have used $\bar{V}_n(X) = \bigcup_{H \leqq G} \bar{V}^H_n(X)$, where

$$\bar{V}^H_n(X) = \{(n+1)\text{-tuples of } G\text{-fiber homotopy classes of } G\text{-maps } G/H \to X\}.$$

Let C_* be the functor which to a topological space A associates the chain complex $C_*(A)$, $C_n(A)$ = free abelian group with basis A^{n+1}. Because

$$\bar{V}^H_n(X) \cong (X^H/WH_0)^{n+1}$$

by 1.3, we get the following identification (where $\mathbf{Z}\text{-}\mathrm{Or}_d G$ is the abelian category of contravariant coefficient systems):

PROPOSITION 2.2. $C^*_G(X;m) = \mathrm{Hom}_{\mathbf{Z}\text{-}\mathrm{Or}_d G}(C_*(\bar{X}), m)$.

In the rest of this paper, however, we prefer to use the original definition of $C^*_G(X;m)$ to keep our notation consistent with [3].

3. Locally zero cochains.

To get an adequate notions of locally zero cochains, when G is a compact Lie group, we must replace the open G-coverings used in [3] with a more complicated concept.

DEFINITION 3.1. An *open G-covering* of the G-space X is a pair $(\mathscr{U}, \mathscr{P})$, where $\mathscr{U}$ is a covering of X by G_0-invariant open sets and $\mathscr{P}$ is a function assigning to every $U \in \mathscr{U}$ and $H \leqq G$ a partition $\mathscr{P}_H(U)$ of $U \cap X^H$ into disjoint WH_0-invariant open subsets. Furthermore we require:

i) $\mathscr{U}$ is G-invariant, i.e. $g \in G$, $U \in \mathscr{U} \Rightarrow gU \in \mathscr{U}$;
ii) $\mathscr{P}_{\{e\}}(U) = \{U\}$ for every $U \in \mathscr{U}$;
iii) $U \in \mathscr{U}$, $H \leqq G$, $g \in G \Rightarrow g \cdot \mathscr{P}_H(U) = \mathscr{P}_{gHg^{-1}}(gU)$;
iv) $H \leqq K \leqq G$, $U \in \mathscr{U} \Rightarrow \mathscr{P}_K(U)$ is a refinement of $\mathscr{P}_H(U) \cap X^K$;
v) Assume $U \in \mathscr{U}$; then x has an open neighborhood V_x in X with the following property: whenever $x \in X^H$, i.e. $H \leqq G_x$, the set $V_x \cap X^H$ is contained in some set of $\mathscr{P}_H(U)$ (in particular $V_x \subset U$).

REMARK 3.2. i) The sets $U \cap X^H$ for $U \in \mathscr{U}$ are WH_0-invariant by 1.5.
ii) If a) X is a locally smooth G-manifold, or b) G is finite, then condition v) is automatically satisfied.

PROOF a) In the notation of 3.1.v), let $K = G_x$. Because X is also a locally smooth K-manifold, x has an open K-neighborhood V_x such that V_x is K-homeomorphic to a linear representation of K. Now $V_x \cap X^H$ is connected and hence contained in some set of $\mathscr{P}_H(U)$, for every $H \leqq K$.

b) Let again $x \in U \in \mathscr{U}$. For each $H \leqq G_x$ choose an open set $V_H \subset U$ such that $x \in V_H \cap X^H \in \mathscr{P}_H(U)$. Then $V_x = \bigcap_{H \leqq G_x} V_H$ is the required open neighborhood of x, the intersection being finite, because G is finite.

EXAMPLE 3.3. For a locally smooth G-manifold X, take $\mathscr{U}$ = set of components of X and $\mathscr{P}_H(U)$ = set of components of $U \cap X^H$ $(U \in \mathscr{U}, H \leqq G)$. Then $(\mathscr{U}, \mathscr{P})$ is an open G-covering of X.

Using this concept of a G-covering we can now define locally zero cochains. Note that if $H \leqq G$ and $\varphi \in V_n^H(X)$, then $\{\varphi_0(eH), \ldots, \varphi_n(eH)\}$ is contained in X^H by 1.1.

DEFINITION 3.4. A cochain $c \in C^n(X; m)$ is *locally zero*, if there exists an open G-covering $(\mathscr{U}, \mathscr{P})$ of X with the property that $c(\varphi) = 0$ whenever $\varphi \in V_n^H(X)$ $(H \leqq G)$ and $\{\varphi_0(eH), \ldots, \varphi_n(eH)\}$ is contained in a set of $\mathscr{P}_H(U)$ for some $U \in \mathscr{U}$.

In the situation of 3.4 we also say that c is locally zero with respect to (w.r.t.)$(\mathscr{U}, \mathscr{P})$.

It is clear that two open G-coverings $(\mathscr{U}, \mathscr{P})$ and $(\mathscr{U}', \mathscr{P}')$ of X have a common refinement (intersect the sets of $(\mathscr{U}, \mathscr{P})$ with those of $(\mathscr{U}', \mathscr{P}')$). Therefore the locally zero n-cochains form a subgroup $C_0^n(X; m)$ of $C^n(X; m)$, and in fact $C_0^*(X; m)$ is a subcomplex of $C^*(X; m)$. We also denote

$$C_{G,0}^*(X; m) = C_G^*(X; m) \cap C_0^*(X; m),$$

$$\bar{C}^*(X; m) = C^*(X; m)/C_0^*(X; m) \text{ and } \bar{C}_G^*(X; m) = C_G^*(X; m)/C_{G,0}^*(X; m)$$

as in [3].

If $f\colon X \to Y$ is a (continuous) G-map, then $f^* C_0^*(Y; m) \subset C_0^*(X; m)$; namely, if $c \in C^n(Y; m)$ is locally zero w.r.t. $(\mathscr{V}, \mathscr{Q})$, then $f^*(c) \in C^n(X; m)$ is locally zero w.r.t. $(f^{-1}\mathscr{V}, f^{-1}\mathscr{Q})$, where

$$f^{-1}\mathscr{V} = \{f^{-1}(V) \mid V \in \mathscr{V}\},$$

$$(f^{-1}\mathscr{Q})_H(f^{-1}(V)) = \{f^{-1}(V_H) \mid V_H \in \mathscr{Q}_H(V)\} \quad (H \leqq G, V \in \mathscr{V}).$$

Thus we get $f^*\colon \bar{C}^*(Y; m) \to \bar{C}^*(X; m)$ and $f^*\colon \bar{C}_G^*(Y; m) \to \bar{C}_G^*(X; m)$.

To end this section, we show that in case G is finite, definition 3.4 agrees with the definition of locally zero cochains given in [3].

PROPOSITION 3.5. *Suppose G is a finite group. Then a cochain $c \in C^n(X;m)$ is locally zero in the sense of* 3.4, *if and only if it is locally zero in the sense of* [3].

PROOF. The "if"-part is trivial, for an open G-covering $\mathscr{U}$ of X in the sense of [3] can also be regarded as an open G-covering $(\mathscr{U}, \mathscr{P})$, with $\mathscr{P}_H(U) = \{U \cap X^H\}$ $(U \in \mathscr{U}, H \leqq G)$.

To prove the "only if"-part, assume that c is locally zero w.r.t. $(\mathscr{U}, \mathscr{P})$. Let $x \in X$ and pick $U \in \mathscr{U}$ such that $x \in U$. Choose an open neighborhood V_x of x as in 3.1.v) (cf. 3.2.ii) b)). Because $\bigcup_{H \nleqq G_x} X^H$ is closed (since X is Hausdorff) and $x \notin \bigcup_{H \nleqq G_x} X^H$, we may assume that $V_x \cap X^H = \emptyset$ for $H \nleqq G_x$. Further we may assume that V_x is G_x-invariant (take $\bigcap_{g \in G_x} gV_x$ if necessary). Then, if $y = gx$ $(g \in G)$ is in the orbit of x, define $V_y = gV_x$.

Now we have constructed an open neighborhood V_y of every point y in the orbit of x. Performing this construction for every orbit we obtain an open G-covering $\mathscr{V} = \{V_x \mid x \in X\}$ of X in the sense of [3], and c is locally zero w.r.t. $\mathscr{V}$ in the sense of [3].

Note that here the Hausdorff condition on X was needed, while in [3] it played no role until section 5.

4. Definition of the cohomology theory.

Let X be a G-space and $A \subset X$ a G-subspace. Let $i\colon A \hookrightarrow X$ be the inclusion. We give the same definition as in [3]:

DEFINITION 4.1. $\bar{C}^*_G(X, A; m) = \ker[i^*\colon \bar{C}^*_G(X;m) \to \bar{C}^*_G(A;m)]$ is the *equivariant Alexander-Spanier cochain complex* of the G-pair (X, A) with coefficients m. Its cohomology groups $\bar{H}^n_G(X, A; m) = H^n(\bar{C}^*_G(X, A; m))$ are the *equivariant Alexander-Spanier cohomology groups* of (X, A) with coefficients m.

THEOREM 4.2. *The functors $\bar{H}^n_G$ satisfy all the Eilenberg-Steenrod axioms for an equivariant cohomology theory, including the dimension axiom.*

As in [3], *exactness* is immediate from the definition of $\bar{H}^*_G$, and the proof of the *excision axiom* needs only marginal modifications. As for the *dimension axiom*, the essential point is that lemma 2.2 of [3] holds in this more general setting, too:

LEMMA 4.3. Assume $H \leqq G$ *and* $c \in C^n_G(G/H; m)$. *Then c is locally zero if and only if* $c(\mathrm{id}_{G/H}, \ldots, \mathrm{id}_{G/H}) = 0$.

PROOF. Let $c(\mathrm{id}, \ldots, \mathrm{id}) = 0$. The equivariance of c then implies that $c(\varphi_0, \ldots, \varphi_n) = 0$, if the G-maps $\varphi_i\colon G/K \to G/H$ are G-homotopic. Let $(\mathscr{U}, \mathscr{P})$ be the open G-covering of G/H described in 3.3. Now c is locally zero w.r.t. $(\mathscr{U}, \mathscr{P})$, for if $\varphi = (\varphi_0, \ldots, \varphi_n) \in V_n^K(G/H)$ is such that $\{\varphi_0(eK), \ldots, \varphi_n(eK)\}$ is contained in a set of $\mathscr{P}_K(U)$, $U \in \mathscr{U}$, then the G-maps $\varphi_i\colon G/K \to G/H$ are G-homotopic, and so $c(\varphi) = 0$.

The reader may convince himself that the proof of the *homotopy axiom* given in section 4 of [3] can also be modified to hold in the present situation. However, we want to be more specific about two particular modifications needed; here we restrict to the absolute case to keep notation more simple, and leave the case of G-pairs in peace.

Firstly, given a G-space X and an open G-covering $(\mathscr{U}, \mathscr{P})$ of X, we define a simplicial complex $X(\mathscr{U}, \mathscr{P})$ as follows: its set of vertices is $V_0(X)$, and the vertices $\varphi_0, \ldots, \varphi_n$ span a simplex, if $\varphi_0, \ldots, \varphi_n \in V_0^H(X)$ for some $H \leqq G$ and $\{\varphi_0(eH), \ldots, \varphi_n(eH)\}$ is contained in some set of $\mathscr{P}_H(U)$ for some $U \in \mathscr{U}$. The complex $X(\mathscr{U}, \mathscr{P})$ is clearly a natural generalization of the complex $X(\mathscr{U})$ considered in [3].

Secondly, let X be a G-space, $I = [0, 1]$ the unit interval with trivial G-action and $(\mathscr{U}, \mathscr{P})$ an open G-covering of $X \times I$. The core of the proof of the homotopy axiom is, as in [3], to construct a suitable open G-covering $(\mathscr{V}, \mathscr{Q})$ of X and a suitable chain homotopy $C_*(X(\mathscr{V}, \mathscr{Q})) \to C_{*+1}((X \times I)(\mathscr{U}, \mathscr{P}))$. We want to explain the construction of $(\mathscr{V}, \mathscr{Q})$.

Let $x \in X$. By the compactness of I we can find an open neighborhood V_x of x in X satisfying

(4.4) there is an $n = n_x \in \mathrm{N}$ with the property that for each $k \in \{0, 1, \ldots, 2^n - 1\}$, $V_x \times [k/2^n, (k+1)/2^n] \subset U_{x,k}$ for some $U_{x,k} \in \mathscr{U}$.

Because the sets $U_{x,k}$ are G_0-invariant, $G_0 \cdot V_x$ also satisfies 4.4; thus we may assume that V_x is G_0-invariant. In addition, V_x may be assumed G_x-invariant (replace it with a smaller neighborhood, if necessary, by [1], Exercise I.9 applied to the group $\langle G_0, G_x \rangle$). Finally, if $y = gx$ $(g \in G)$ is in the orbit of x, define $V_y = gV_x$.

In this way we obtain a G-invariant covering $\mathscr{V} = \{V_x \mid x \in X\}$ of X by G_0-invariant open subsets. Let $H \leqq G$. Then

$$(V_x \cap X^H) \times [k/2^n, (k+1)/2^n] \subset U_{x,k} \cap (X^H \times I) \quad (n = n_x),$$

and the partitions $\mathscr{P}_H(U_{x,k})$ of $U_{x,k} \cap (X^H \times I)$ $(0 \leqq k \leqq 2^n - 1)$ induce a partition $Q_H(V_x)$ of $V_x \cap X^H$; this is due to the fact that, by connectedness, every slice $\{y\} \times [k/2^n, (k+1)/2^n]$ is contained in some set of $\mathscr{P}_H(U_{x,k})$. This finishes the construction of the required G-covering $(\mathscr{V}, \mathscr{Q})$ of X.

5. Some properties.

In this section we explain, how the proofs of the results of sections 5, 6 and 7 of [3] carry over to the present situation.

a) *Tautness*

PROPOSITION 5.1. *Suppose X is a paracompact G-space and $A \subset X$ a closed G-subspace. Then the canonical homomorphism*

$$\varinjlim_N \bar{H}^n_G(N; m) \to \bar{H}^n_G(A; m)$$

is an isomorphism (N runs through the G-neighborhoods of A).

As in [3], the proof of 5.1 is similar to the classical case ([7] 6.6.2), and is based on the following two lemmas:

LEMMA 5.2. *Let X be a paracompact G-space and $(\mathscr{U}, \mathscr{P})$ an open G-covering of X. Then there exists an open G-covering $(\mathscr{V}, \mathscr{Q})$ of X such that for each $H \leqq G$, the covering $\mathscr{V}_H = \bigcup_{V \in \mathscr{V}} \mathscr{Q}_H(V)$ is a star refinement of $\mathscr{U}_H = \bigcup_{U \in \mathscr{U}} \mathscr{P}_H(U)$.*

PROOF. We first show that there is a locally finite open refinement $\mathscr{U}'$ of $\mathscr{U}$ such that $\mathscr{U}'$ is G-invariant and consists of G_0-invariant subsets of X.

The sets U/G_0 $(U \in \mathscr{U})$ form an open covering of X/G_0. Because X/G_0 is paracompact by the theorem of E. Michael ([2], p. 165), this covering has an open locally finite refinement. Taking inverse images in X we obtain an open locally finite refinement $\mathscr{W}$ of $\mathscr{U}$ such that the sets of $\mathscr{W}$ are G_0-invariant. Then, because G/G_0 is finite,

$$\mathscr{U}' = \bigcap_{gG_0 \in G/G_0} g\mathscr{W}$$

is furthermore G-invariant (compare [1], p. 133).

Let $U' \in \mathscr{U}'$ and choose some $U \in \mathscr{U}$ such that $U' \subset U$. If $H \leqq G$, we define

$$\mathscr{P}'_H(U') = \left(\bigcap_{gG_0} \mathscr{P}_H(gU) \right) \cap U',$$

the first (finite) intersection being taken over those $gG_0 \in G/G_0$ for which $gU' = U'$ (note: if $\mathscr{A}$ and $\mathscr{B}$ are collections of subsets of a certain set Y, we denote $\mathscr{A} \cap \mathscr{B} = \{A \cap B \mid A \in \mathscr{A}, B \in \mathscr{B}\}$). For a set gU' in the orbit of U' in $\mathscr{U}'$, we can now define $\mathscr{P}'_H(gU') = g \cdot \mathscr{P}'_{g^{-1}Hg}(U')$. In this way we obtain an open G-covering $(\mathscr{U}', \mathscr{P}')$ of X, and if we replace $(\mathscr{U}, \mathscr{P})$ with $(\mathscr{U}', \mathscr{P}')$, we can (and henceworth will) assume that the original $\mathscr{U}$ is locally finite.

Next we construct, using the method of [2], p. 167, a star refinement $\mathscr{V}$ of $\mathscr{U}$ such that $\mathscr{V}$ is G-invariant and consists of G_0-invariant open subsets of X. Fix $V \in \mathscr{V}$, $V \neq \emptyset$. The set

$$\mathscr{U}_V = \{U \in \mathscr{U} \mid V \subset U\}$$

is finite, because $\mathscr{U}$ is locally finite. For $H \leqq G$ define

$$\mathscr{Q}_H(\mathscr{V}) = \left(\bigcap_{U \in \mathscr{U}_V} \mathscr{P}_H(U) \right) \cap V.$$

Then $(\mathscr{V}, \mathscr{Q})$ is an open G-covering of X.

To prove that $(\mathscr{V}, \mathscr{Q})$ has the required property, let $H \leqq G$ and $W \in \mathscr{Q}_H(V)$ for some $V \in \mathscr{V}$. Choose $U \in \mathscr{U}$ such that $\mathrm{st}(V, \mathscr{V}) \subset U$. Then $U \in \mathscr{U}_V$. Assume that $W' \in \mathscr{Q}_H(V')$, $V' \in \mathscr{V}$, and $W \cap W' \neq \emptyset$. Now $V \cap V' \neq \emptyset$, $V' \subset \mathrm{st}(V, \mathscr{V}) \subset U$ and so $U \in \mathscr{U}_{V'}$. By the definition of $\mathscr{Q}_H(V)$ and $\mathscr{Q}_H(V')$, W and W' are each contained in a set of $\mathscr{P}_H(U)$, and in fact $W \cap W' \neq \emptyset$ implies that they are both contained in the same set of $\mathscr{P}_H(U)$. It follows that $\mathrm{st}(W, \mathscr{V}_H)$ is contained in this set of $\mathscr{P}_H(U)$.

LEMMA 5.3. *Let X be a completely regular* (e.g. *paracompact*) *G-space and $A \subset X$ a G-subspace. Given an open G-covering $(\mathscr{V}, \mathscr{Q})$ of X, there is an open G-neighborhood N of A and an equivariant function $f: N \to A$* (*not necessarily continuous*) *satisfying*

i) $f(x) = x$ for $x \in A$, and
ii) if $H \leqq G$ and $W \in \mathscr{V}_H$, then $f(W \cap N) \subset \mathrm{st}(W, \mathscr{V}_H)$.

PROOF. Let

$$N = \{x \in X \mid \text{there is a } V \in \mathscr{V} \text{ with } x \in V, \text{ and a } G\text{-map } f_x: Gx \to A \text{ such that } f_x(x) \in W \cap A \text{ whenever } x \in X^H\ (H \leqq G) \text{ and } x \in W \in \mathscr{Q}_H(V)\}.$$

Clearly $A \subset N$ and N is a G-subset. We show that N is open. Let $x \in N$, and choose $V \in \mathscr{V}$ and f_x: $Gx \to A$ as above. Also, let $V_x \subset V$ be an open neighborhood of x as in v) of 3.1. Pick a slice $S_x \subset V_x$ at x ([1], II 5.4). For every $y \in S_x$ there is a G-map h_y: $Gy \to Gx$ with $h_y(y) = x$. Now the set G_0S_x is an open neighborhood of x, and $G_0S_x \subset G_0V = V$. We claim that $G_0S_x \subset N$.

Let $z \in G_0S_x$, $z = gy$ $(g \in G_0, y \in S_x)$. Then $z \in V$ and $f_x \circ h_y$: $Gy \to Gx \to A$ is a G-map. If $z \in X^K$, $z \in W \in \mathscr{Q}_K(V)$, then $y \in X^{g^{-1}Kg}$, $y \in g^{-1} W \in \mathscr{Q}_{g^{-1}Kg}(g^{-1}V) = \mathscr{Q}_{g^{-1}Kg}(V)$. Because $G_y \leqq G_x$ and $y \in S_x \subset V_x$, also $x \in g^{-1}W$, and thus

$$(f_x \circ h_y)(z) = g \cdot f_x(x) \in g \cdot ((g^{-1}W) \cap A) = W \cap A.$$

Therefore $z \in N$. This completes the proof that N is open.

We now construct $f: N \to A$. Set $f(x) = x$ for $x \in A$. To define $f \mid N \setminus A$, let S be a set of representatives for the G-orbits of $N \setminus A$. Given $y \in S$, we choose a $V_y \in \mathscr{V}$ with $y \in V_y$ and a G-map f_y: $Gy \to A$ as in the definition of N. We define $f \mid Gy = f_y$.

By definition, f satisfies i). To prove ii), assume that $H \leqq G$, $W \in \mathscr{Q}_H(V)$, $V \in \mathscr{V}$ and $x \in W \cap N$; we claim that $f(x) \in \mathrm{st}(W, \mathscr{V}_H)$. This is obvious if $x \in A$. Let then $x \in N \setminus A$, $x = gy$ with $g \in G$, $y \in S$. Because $x \in X^H$, we have $y \in X^{g^{-1}Hg}$; assume

$y \in W' \in \mathcal{Q}_{g^{-1}Hg}(V_y)$ ($V_y \in \mathscr{V}$ was chosen in the preceding paragraph). Then $gW' \in \mathcal{Q}_H(gV_y)$ and $x = gy \in W \cap gW'$, whence $gW' \subset \mathrm{st}(W, \mathscr{V}_H)$. Finally $f(x) = g \cdot f(y) = g \cdot f_y(y) \in g \cdot (W' \cap A) \subset gW' \subset \mathrm{st}(W, \mathscr{V}_H)$, so $f(x) \in \mathrm{st}(W, \mathscr{V}_H)$.

b) *Interpretation as sheaf cohomology of X/G*

Let X be a paracompact G-space. Because the canonical projection π: $X \to X/G$ is a closed surjection, the theorem of E. Michael referred to above implies that X/G is also paracompact. Exactly as in section 6 of [3] we obtain the exact sequence

$$(5.4) \qquad \bar{\mathcal{C}}_G^0 \xrightarrow{d} \bar{\mathcal{C}}_G^1 \xrightarrow{d} \bar{\mathcal{C}}_G^2 \to \dots$$

of fine sheaves on X/G, where $\bar{\mathcal{C}}_G^n$ is the sheaf associated to the presheaf $U \mapsto \bar{C}_G^n(\pi^{-1}U; m)$, $U \subset X/G$ open. Recall in particular that the exactness of 5.4 is a consequence of the tautness result 5.1. The proof of 6.3 in [3] can also be modified to show that the global sections of $\bar{\mathcal{C}}_G^n$ are

$$\Gamma(X/G, \bar{\mathcal{C}}_G^n) = \bar{C}_G^n(X; m).$$

Define $A = \ker[\bar{\mathcal{C}}_G^0 \xrightarrow{d} \bar{\mathcal{C}}_G^1]$, a sheaf on X/G. As in [3], the above remarks suffice to prove

THEOREM 5.5. *$H^n(X/G; A) \cong \bar{H}_G^n(X; m)$ for all $n \in \mathbb{N}$.*

We say that a 0-cochain $c \in C_G^0(X; m)$ is *locally constant*, if there exists an open G-covering $(\mathscr{V}, \mathscr{P})$ of X with the property that $c(\varphi) = c(\varphi')$ for C-maps φ, φ': $G/H \to X$ ($H \leqq G$) whenever $\{\varphi(eH), \varphi'(eH)\}$ is contained in a set of $\mathscr{P}_H(V)$ for some $V \in \mathscr{V}$. As in 6.5.a) of [3], the sections of A on an open set $U \subset X/G$ are given by

$$(5.6) \qquad \Gamma(U, A) = \{c \in C_G^0(\pi^{-1}U; m) \mid c \text{ is locally constant}\}.$$

As to the stalks of A, the result of 6.5.b) of [3] holds, but the proof needs some modifications:

PROPOSITION 5.7. *If $y \in X/G$, $x \in \pi^{-1}(y)$ and $H = G_x \leqq G$, then the stalk A_y of A at y is isomorphic to $m(G/H)$.*

PROOF. We have the canonical homomorphism

$$\gamma\colon A_y = \varinjlim_U \Gamma(U, A) \to m(G/H), y \in U \subset X/G, U \text{ open},$$

defined as follows: Let $\varphi_x\colon G/H \to Gx$ be the G-homeomorphism $gH \mapsto gx$. If $U \subset X/G$ is an open neighborhood of y, we denote the composite $G/H \xrightarrow{\varphi_x} Gx \hookrightarrow \pi^{-1}U$ by the same symbol φ_x. Now γ is the direct limit of the homomorphisms γ_U: $\Gamma(U, A) \to m(G/H)$, $\gamma_U(c) = c(\varphi_x)$. We claim that γ is an isomorphism.

Let W be a tube around the orbit $\pi^{-1}(y) = Gx$ and $r: W \to Gx$ a G-retraction. For the surjectivity of γ, let $a \in m(G/H)$. We define $c \in C_G^0(W; m)$ by $c(\varphi) = m(r \circ \varphi)(a) \in m(G/K)$ for a G-map $\varphi: G/K \to W$. Let $(\mathscr{U}, \mathscr{P})$ be the open G-covering of $Gx \cong G/H$ described in 3.3; then c is locally constant w.r.t. the open G-covering $(r^{-1}\mathscr{U}, r^{-1}\mathscr{P})$ of W. Thus $c \in \Gamma(\pi W, A)$, and clearly $\gamma_{\pi W}: c \mapsto a$, proving surjectivity.

For injectivity, let $c \in \Gamma(U, A)$, $U \subset X/G$ an open neighborhood of y, and assume that $\gamma_U(c) = c(\varphi_x) = 0$. By assumption, c is locally w.r.t. an open G-covering $(\mathscr{V}, \mathscr{P})$ of $\pi^{-1}U$. Choose $V \in \mathscr{V}$ such that $x \in V$. By 3.1.v) there is an open neighborhood V_x of x, which we may assume H-invariant, with the property that $V_x \cap X^K$ is contained in some set of $\mathscr{P}_K(V)$, whenever $K \leqq H$.

Let $S_x = r^{-1}(x)$ be the slice at x corresponding to the above tube W and retraction r. Then $G(S_x \cap V_x)$ is also a tube around Gx; to simplify notation we denote this smaller tube again by W. Now we have $r^{-1}(x) \subset V_x$. We intend to show that $c \mid \pi W = 0 \in \Gamma(\pi W, A)$.

Let $\varphi: G/K \to W$ be a G-map, and let $\varphi': G/K \to W$ be the composite

$$\varphi': G/K \xrightarrow{\varphi} W \xrightarrow{r} Gx \hookrightarrow W.$$

We have $\varphi'(\mathrm{eK}) = gx \in gV_x$ for some $g \in G$, $g^{-1}Kg \leqq H$ and $\varphi(eK) \in r^{-1}(gx) = g \cdot r^{-1}(x) \subset gV_x$. By the choice of V_x, $gV_x \cap X^K$ is contained in a set of $\mathscr{P}_K(gV)$. Therefore $\{\varphi(eK), \varphi'(eK)\}$ is contained in a set of $\mathscr{P}_K(gV)$, and the local constantness of c w.r.t. $(\mathscr{V}, \mathscr{P})$ implies that $c(\varphi) = c(\varphi')$.

Finally, $\varphi' = \varphi_x \circ \alpha$, where $\alpha: G/K \to G/H$ is the G-map given by $g_1K \mapsto g_1gH$ ($g_1 \in G$; g as in the preceding paragraph), and thus $c(\varphi) = c(\varphi') = m(\alpha)(c(\varphi_x)) = 0$.

c) *Comparison with equivariant singular cohomology*

The result of section 7 of [3] (Theorem 7.2) is true in the present context, too:

THEOREM 5.8. *Suppose that X is a paracompact G-space and every orbit $Gx \subset X$ is taut with respect to equivariant singular cohomology $H_G^*(\cdot\,; m)$. Then there is a natural isomorphism*

$$\bar{H}_G^*(X; m) \cong H_G^*(X; m).$$

In fact section 7 of [3] can be read almost word for word to get a proof of 5.8. The following two comments are in order, however:

Firstly, we must define an equivariant singular cochain $c \in S_G^n(X; m)$ to be *locally zero*, if there is an open G-covering $(\mathscr{V}, \mathscr{P})$ of X with the property that $c(\sigma) = 0$ for any equivariant singular simplex $\sigma: G/H \times \Delta^n \to X$ such that $\sigma(\{eH\} \times \Delta^n)$ is contained in a set of $\mathscr{P}_H(V)$ for some $V \in \mathscr{V}$. Again the proof of Prop. I.6.4 in [4] essentially shows that the cohomology of $S_G^*(X; m)$ does not change, if we divide out the locally zero cochains.

Secondly, in the article [6] of Piacenza referred to in [3], it is assumed that G is a discrete group. However, the result we need, i.e. that $H_G^*(X; m)$ can be computed from the exact sequence $S_G^0 \to S_G^1 \to S_G^2 \to \dots$ of fine sheaves on X/G, clearly holds under the assumptions of 5.8, too.

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