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Zbigniew Fiedorowicz, H. Hauschild, J. P. May

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EQUIVARIANT ALGEBRAIC K-THEORY

by Z. Fiedorowicz, H. Hauschild, and J.P. May

There are many ways that group actions enter into algebraic K-theory and there are various theories that fit under the rubric of our title. To anyone familiar with both equivariant topological K-theory and Quillen's original definition and calculations of algebraic K-theory, there is a perfectly obvious program for the definition of the equivariant algebraic K-theory of rings and its calculation for finite fields. While this program surely must have occurred to others, there are no published accounts and the technical details have not been worked out before. That part of the program which pertains to the complex Adams conjecture was outlined in a letter to one of us from Graeme Segal, and the real analog was assumed without proof by tom Dieck [10,11.3.8]. Negatively indexed equivariant K-groups were introduced by Loday [19].

From a topological point of view, one way of thinking about Quillen's original definition runs as follows. Let Π be a topological group, perhaps discrete. One has a notion of a principal Π-bundle and a classifying space $B\Pi$ for such bundles. When Π is discrete, a principal Π-bundle is just a covering (possibly with disconnected total space) with fibre and group Π. Given any increasing sequence of groups Π_n with union Π, we obtain an increasing sequence of classifying spaces $B\Pi_n$ with union $B\Pi$. We then think of $B\Pi$ as a classifying space for stable bundles. When the Π_n are discrete, $B\Pi$ may have desirable homology groups but will have trivial higher homotopy groups. In the cases of interest, we can use the plus construction to convert $B\Pi$ to a Hopf space (= H-space) $(B\Pi)^+$ with the same homology. When $\Pi_n = GL(n,A)$, we define $\pi_q(B\Pi)^+ = K_q(A)$ for $q > 0$.

There is a more structured way of looking at $(B\Pi)^+$. In practice, we have sum maps $\oplus : \Pi_m \times \Pi_n \to \Pi_{m+n}$ and a corresponding Whitney sum of bundles. While this can be used to give $B\Pi$ a product, it is generally not a Hopf space, although it is so in the classical bundle theory cases obtained from $\Pi_n = U(n)$ and $\Pi_n = O(n)$. When $\mathcal{A} = \coprod_{n\geq 0} \Pi_n$ is a permutative category under $\oplus$, $B\mathcal{A} = \coprod_{n\geq 0} B\Pi_n$ inherits a structure of topological monoid and we can form $\Omega BB\mathcal{A}$, the loop space on the classifying space of $B\mathcal{A}$. Homological analysis of the natural map $\zeta: B\mathcal{A} \to \Omega BB\mathcal{A}$ shows that the basepoint component $\Omega_0 BB\mathcal{A}$ is equivalent to $(B\Pi)^+$. Deeper analysis shows that $BB\mathcal{A}$ also has a classifying space, and so on, so that $\Omega BB\mathcal{A}$ and thus $(B\Pi)^+$ are actually infinite loop spaces.

Let G be a finite group. We mimic the outline just given. For a topological group Π, one has a notion of a principal (G,Π)-bundle $p:E \to B$. This is just a principal Π-bundle and a G-map between G-spaces such that the action by each $g \in G$ is a map of Π-bundles. That is, the actions of G and Π on the total space commute (we think of Π as acting on the right and G on the left). One has a classifying G-space $B(G,\Pi)$ for such (G,Π)-bundles. This space carries information about the representations of G in Π. In particular, one has the following basic fact.

Proposition 0.1. Let H be a subgroup of G. For a homomorphism $\rho: H \to \Pi$, define Π^ρ to be the centralizer of ρ, namely

$$\{\pi \mid \rho(h)\pi = \pi\rho(h) \text{ for all } h \in H\}.$$

Then the fixed point subspace $B(G,\Pi)^H$ has the homotopy type of $\coprod_\rho B\Pi^\rho$, where the union runs over a set $R^+(H,\Pi)$ of representatives for the representations of H in Π.

That is, $R^+(H,\Pi)$ consists of one ρ in each conjugacy class of homomorphisms $H \to \Pi$. Note that, with $H = e$, this says that the underlying nonequivariant homotopy type of $B(G,\Pi)$ is just $B\Pi$.

Again, we are interested in increasing sequences $\{\Pi_n\}$ with union Π. While the third author has constructed an equivariant plus construction for G-connected G-spaces X, namely those X for which all X^H are path connected, we shall not use it. Instead, we shall prove that, in the cases of interest to us, $\coprod_n B(G,\Pi_n)$ is G-equivalent to a topological G-monoid $B\mathcal{A}(G)$. This means that $B\mathcal{A}(G)$ is a G-space and a topological monoid such that its unit is a fixed point and its product is a G-map. Since the standard classifying space functor takes G-monoids to G-spaces and the loop space functor takes G-spaces to G-spaces, this will allow us to construct the G-space $\Omega BB\mathcal{A}(G)$ together with a natural G-map

$$\zeta : B\mathcal{A}(G) \to \Omega BB\mathcal{A}(G).$$

These ideas lead us to the following definition.

Definition 0.2. For a discrete ring A, let

$$K(A,G) = \Omega BB\mathcal{J}\mathcal{L}(A,G)$$

be the G-space obtained by setting $\Pi_n = GL(n,A)$ in the discussion above. For an algebraically closed field A of characteristic unequal to 2, let

$$KO(A,G) = \Omega BB\mathcal{O}(A,G)$$

be the G-space obtained by setting $\Pi_n = O(n,A)$. Let

$$K(G) = \Omega BB\mathcal{U}(G) \quad \text{and} \quad KO(G) = \Omega BB\mathcal{O}(G)$$

be the G-spaces obtained by setting $\Pi_n = U(n)$ and $\Pi_n = O(n)$.

The appropriate definition of KO(A,G) for general commutative rings A requires use of more general orthogonal groups and will be given in section 2.

We shall give the details behind this definition in sections 1 and 2, first giving precise models for the general classifying spaces $B(G,\Pi)$ and verifying Proposition 0.1 and then giving different models for the particular $B(G,\Pi_n)$ relevant to the definition. The second model is needed to obtain the required monoid structures since the first model is not product-preserving in Π. The most important tool in equivariant homotopy theory is the reduction of equivariant problems to non-equivariant ones by passage to fixed point spaces, and we study the fixed points of the simplest examples of the G-spaces introduced in Definition 0.2 in section 3.

For based G-spaces X and Y (with G-fixed basepoints), let $[X,Y]_G$ denote the set of G-homotopy classes of based G-maps $X \to Y$. In particular, for $H \subset G$, it is standard and natural to define

$$\pi_q^H(Y) = [(G/H)_+ \wedge S^q, Y]_G = [S^q, Y^H] = \pi_q(Y^H),$$

where S^q is the q-sphere with trivial G-action. Here and henceforward, X_+ denotes the union of a G-space X and a disjoint G-fixed basepoint. A G-map $f:X \to Y$ is said to be a weak G-equivalence if each $f^H:X^H \to Y^H$ is a weak equivalence. If X and Y have the homotopy types of G-CW complexes, as holds for all G-spaces we shall consider, such an f is necessarily a G-homotopy equivalence [7,33]. It is also natural to consider the "homotopy groups"

$$\pi_V^G(Y) = [S^V, Y]_G,$$

where S^V is the one-point compactification of a real representation V of G.

As we shall see in section 5, equivariant topological K-theory of G-bundles over compact G-spaces X is represented in the form

$$K_G(X) = [X_+, K(G)]_G \quad \text{and} \quad KO_G(X) = [X_+, KO(G)]_G,$$

hence the homotopy groups above are all examples of (reduced) K-groups when Y = K(G) or Y = KO(G). We regard the corresponding invariants of K(A,G) and KO(A,G) as equivariant algebraic K-groups. We write

$$K_q^G A = \pi_q^G K(A,G) \qquad \text{and} \qquad K_V^G A = \pi_V^G K(A,G)$$

and similarly in the orthogonal case. However, we are really more interested in the G-homotopy types K(A;G) than in these invariants. While the general linear case is the central one algebraically, the orthogonal case is important in applications from algebra to topology. We develop formal properties of these definitions in section 4. In particular, we discuss the naturality in A of K(A,G), prove that $K_*^G(A)$ is a commutative graded ring if A is commutative, and prove the projection formula. We also verify that $[X,K(A,G)]_G$ is naturally a module over the Burnside ring A(G).

Algebraically, the obvious next step is to introduce the equivariant Q-construction on exact categories, give the equivariant version of Quillen's second definition of algebraic K-theory, and prove the equivalence of the two notions. This can all be done, and an exposition will appear in Benioff [5]. With this approach, Quillen's dévissage theorem applies directly to the computation of fixed point categories and thus of equivariant algebraic K-groups.

Another obvious step is to introduce the appropriate notion of a permutative G-category and prove that K(A,G) is an infinite loop G-space. We have carried out this step and will present it in [12]. We prefer to be elementary in this paper and so will only make a few remarks about this in passing.

We shall concentrate here on another obvious step, namely the equivariant analogs of Quillen's basic calculations in [28] and [29]. We shall first prove the following result.

<u>Theorem 0.3.</u> Let $\overline{k}_q$ be the algebraic closure of the field of q elements, where q is a prime which does not divide the order of G. Then there are Brauer lift G-maps

$$\beta : K(\overline{k}_q, G) \to K(G) \quad \text{and} \quad \beta : KO(\overline{k}_q, G) \to KO(G)$$

whose fixed point maps β^H induce isomorphisms on mod n cohomology for all integers n prime to q.

The outline of the proof is obvious. We simply use our study of classifying spaces to write down an explicit map and check that on fixed point sets it reduces to a product of maps of the form studied by Quillen. We give the argument in section 6 after first developing general facts about the relationship between representation rings and topological equivariant K-theory in section 5.

As we discuss briefly in section 7, this result and the equivariant Dold theorem mod k of Hauschild and Waner [13] can be used to prove the following equivariant version of the Adams conjecture.

<u>Theorem 0.4.</u> Let k be prime to the order of G and let s be minimal such that $k^s \equiv \pm 1$ modulo the order of G. Then for any stable real G-vector bundle ξ over a compact G-connected base space, there exists an integer $e > 0$ such that $k^e s\xi$ and $k^e s\psi^k(\xi)$ are stably fibre G-homotopy equivalent.

That is, $k^e s(\psi^k\xi - \xi)$ is in the kernel of the real equivariant J-homomorphism. The same assertion is also valid in the complex case. Using different techniques, McClure [25] has recently obtained a sharper result, identifying all of the kernel of the J-homomorphism. The factor s cannot be eliminated: it is already necessary when the base space is a point and one is asking about stable G-homotopy equivalence of G-spheres associated to representations.

Finally, in section 8, we obtain the expected relationship between the equivariant algebraic K-theory of finite fields and the Adams operations in equivariant topological K-theory.

Theorem 0.5. Let k_r be the field with $r = q^a$ elements, where q is a prime which does not divide the order of G. Let $F\psi^r(G)$ and $FO\psi^r(G)$ be the homotopy fibres of the G-maps $\psi^r-1 : K(G) \to K(G)$ and $\psi^r-1 : KO(G) \to KO(G)$. Then Brauer lifting induces G-homotopy equivalences

$$\beta : K(k_r, G) \to F\psi^r(G) \quad \text{and} \quad \beta : KO(k_r, G) \to FO\psi^r(G).$$

By the five lemma, this allows application of the reservoir of known information about $K_G(X)$ to the calculation of the functor $[X, K(k_r, G)]_G$, and similarly in the (technically deeper) orthogonal case.

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§1. The classifying spaces $B(G,\Pi)$

As is well-known (e.g. [17,18]), an appropriate iterated join construction can be used to obtain classifying spaces $B(G,\Pi)$. A nicer construction appears peripherally in Waner [34, §3.2], and we shall give a categorical reformulation of his definition.

Throughout this section, both G and Π can be essentially arbitrary topological groups. (Some minor technical restraints should be imposed; G and Π Lie groups, or discrete groups, more than suffices.)

By a G-category we understand a small topological category $\mathcal{C}$ whose object and morphism spaces are G-spaces and whose source, target, identity, and composition functions are G-maps. We require the identity function to be a G-cofibration. The nerve, or simplicial space, determined by $\mathcal{C}$ is a simplicial G-space. Its geometric realization $B\mathcal{C}$ is therefore a G-space, called the classifying G-space of $\mathcal{C}$. Observe that $(B\mathcal{C})^H = B(\mathcal{C}^H)$, where $\mathcal{C}^H$ denotes the category of H-fixed objects and morphisms of $\mathcal{C}$.

We have the concomitant notions of a (continuous) G-functor and of a natural transformation of G-functors $(\eta_{gx} = g\eta_x$ for objects x). A G-functor $S:\mathcal{C} \to \mathcal{D}$ induces a G-map $BS:B\mathcal{C} \to B\mathcal{D}$ and a natural transformation $\eta:S \to T$ induces a G-homotopy $B\eta:BS \simeq BT$. If each S^H is an equivalence of topological categories or admits a left or right adjoint and if $B\mathcal{C}$ and $B\mathcal{D}$ are of the homotopy type of G-CW complexes, as holds if the object and morphism spaces of $\mathcal{C}$ and $\mathcal{D}$ are so, then BS is a G-homotopy equivalence.

Definition 1.1. Define $B(G,\Pi)$ to be the classifying space of the G-category $\mathcal{C}(G,\Pi)$ specified as follows. The objects are all pairs (ρ,x) such that ρ is a continuous homomorphism $H \to \Pi$ for some closed subgroup H of G and x is an element of the space G/H of right cosets. This object set is regarded as the left G-space $\coprod_\rho G/H$. The morphisms $(\rho,x) \to (\sigma,y)$ are the morphisms of principal (G,Π)-bundles

$$\begin{tikzcd} G \times_H \Pi_\rho \arrow[r, "\beta"] \arrow[d] & G \times_K \Pi_\sigma \arrow[d] \\ G/H \arrow[r, "\alpha"] & G/K \end{tikzcd}$$

such that $\alpha(x) = y$. Here Π_ρ denotes Π with the left H-action induced by $\rho: H \to \Pi$ and similarly for $\sigma: K \to \pi$; α is a G-map and β is a $G \times \Pi$-map, where G and Π act on the left and right of $G \times_H \Pi_\rho$ and $G \times_K \Pi_\sigma$ via the multiplications of G and Π. This morphism set is given trivial G-action, $g(\alpha,\beta) = (\alpha,\beta)$, and is topologized via β as a subspace of the disjoint union over ρ and σ of the function spaces of maps $G \times_H \Pi_\rho \to G \times_K \Pi_\sigma$.

Remark 1.2. A morphism $(\alpha,\beta):(\rho,x) \to (\sigma,y)$ is specified by

$$\alpha(gH) = gfK \quad \text{and} \quad \beta(g,p) = (gf,qp)$$

for $g \in G$ and $p \in \Pi$, where $f \in G$ and $q \in \Pi$ are fixed elements such that

$$fK \in (G/K)^H \quad \text{and} \quad q\rho(H) = \sigma(f^{-1}hf)q$$

for $h \in H$. Two pairs (f,q) and $(fk,k^{-1}q)$ define the same morphism (α,β) and, if $x = g_0H$, then $y = g_0fK$.

Waner [34] proved that $B(G,\Pi)$ is a classifying G-space for principal (G,Π)-bundles. Indeed, if one replaces the object space of $\mathcal{C}(G,\Pi)$ by $\coprod_\rho G \times_H \Pi_\rho$, the same definition gives another G-category $\mathcal{J}(G,\Pi)$ with classifying G-space $E(G,\Pi)$. The evident projection $\mathcal{J}(B,\Pi) \to \mathcal{C}(G,\Pi)$ becomes a universal principal (G,Π)-bundle on passage to classifying spaces. Although much of our motivation comes from bundle theory, we shall work entirely on the classifying space level in this paper.

We regard a group or monoid as a category with a single object. The following result gives a proof of Proposition 0.1.

Proposition 1.3. There is an inclusion of categories

$$i: \coprod_{\rho \in R^+(H,\Pi)} \Pi^\rho \to \mathcal{C}(G,\Pi)^H$$

such that i has a right adjoint. Therefore Bi is a homotopy equivalence.

Proof. Define i by sending the unique object of the category Π^ρ to the object (ρ,eH) and sending an element $q \in \Pi^\rho$ to the morphism $(1,\beta):(\rho,eH) \to (\rho,eH)$, where

$\beta(g,p) = (g,qp)$; compare Remark 1.2. Let $\mathcal{C}(H,G,\Pi)$ be the subcategory of $\mathcal{C}(G,\Pi)^H$ which consists of all objects (σ,eH), $\sigma:H \to \Pi$, and all morphisms $(1,\gamma)$. Since σ is conjugate to some $\rho \in R^+(H,\Pi)$, it is easy to see that i maps $\coprod_{\rho \in R^+(H,\Pi)} \Pi^\rho$ isomorphically onto a skeleton of $\mathcal{C}(H,G,\Pi)$. It therefore suffices to construct a right adjoint k to the inclusion, j say, of $\mathcal{C}(H,G,\Pi)$ in $\mathcal{C}(G,\Pi)^H$.

Let (τ,y), $\tau:K \to \Pi$ and $y = fK \in (G/K)^H$, be a typical object of $\mathcal{C}(G,\Pi)^H$. Define $k(\tau,y) = \sigma$, where $\sigma:H \to \Pi$ is specified by $\sigma(h) = \tau(f^{-1}hf)$. Regarding y as a G-map $G/H \to G/K$, we see that the pullback of $G \times_K \Pi_\tau \to G/K$ along y is $G \times_H \Pi_\sigma \to G/H$ and that the resulting pullback square is a morphism

$$\xi:(\sigma,eH) \to (\tau,y).$$

For a morphism $(\alpha,\beta):(\tau,y) \to (\mu,z)$ in $\mathcal{C}(G,A)^H$, $\tau:L \to A$ and $z \in (G/L)^H$, we obtain the following commutative diagram by passage to pullbacks, where $\mu = j(\nu,z)$:

$$\begin{tikzcd}
& G \times_K \Pi_\tau \arrow[rr] & & G \times_L \Pi_\nu \arrow[dd] \\
G \times_H \Pi_\sigma \arrow[ur] \arrow[rr, dashed] \arrow[dd] & & G \times_H \Pi_\mu \arrow[ur] \arrow[dd] & \\
& G/K \arrow[rr, "\alpha"] & & G/L \\
G/H \arrow[ur, "y"] \arrow[rr, equals] & & G/H \arrow[ur, "z"] &
\end{tikzcd}$$

The dotted arrow gives $k(\alpha,\beta)$. With these specifications, k is a functor and ξ is a natural transformation $jk \to \mathrm{Id}$. Visibly $kj = \mathrm{Id}$ and $\xi \circ j$ and $k \circ \xi$ are identity transformations. This proves the result.

We shall need the following naturality property of the categories $\mathcal{C}(G,A)$.

Lemma 1.4. Let $\sigma:\Pi \to \Psi$ be a continuous homomorphism. Then σ induces a G-functor

$$\sigma_*: \mathcal{C}(G,\Pi) \to \mathcal{C}(G,\Psi)$$

whose restriction to any Π^ρ is $\sigma:\Pi^\rho \to \Psi^{\sigma \circ \rho}$. Moreover, conjugate homomorphisms induce naturally isomorphic G-functors.

Proof. On objects, $\sigma_*(\rho,x) = (\sigma \circ \rho,x)$. On morphisms (α,β) with $\beta(g,p) = (gf,qp)$ as in Remark 1.2,

$$\sigma_*(\alpha,\beta) = (\alpha,\sigma_*\beta), \text{ where } \sigma_*\beta(g,p) = (gf, \sigma(q)p).$$

The requisite verifications are immediate from the definitions.

A defect of the construction is that inclusions $H \subset G$ do not induce functors $\mathcal{C}(G,\Pi) \to \mathcal{C}(H,\Pi)$. A related defect is that the natural G-functor

$$\mathcal{C}(G, \Pi \times \Sigma) \to \mathcal{C}(G,\Pi) \times \mathcal{C}(G,\Sigma)$$

is not an isomorphism of G-categories. However, it does restrict to the obvious identification

$$\coprod_{(\rho,\sigma) \in R^+(H,\Pi \times \Sigma)} (\Pi \times \Sigma)^{(\rho,\sigma)} \cong \Big(\coprod_{\rho \in R^+(H,\Pi)} \Pi^\rho\Big) \times \Big(\coprod_{\sigma \in R^+(H,\Sigma)} \Sigma^\sigma\Big)$$

and therefore induces a G-homotopy equivalence

$$B(G,\Pi \times \Sigma) \to B(G,\Pi) \times B(G,\Sigma).$$

Due to this last defect, an associative sum on $\coprod \Pi_n$ does not induce a G-monoid structure on $\coprod B(G,\Pi_n)$. To remedy this, we introduce different categorical models for the relevant G-homotopy types $B(G,\Pi_n)$.

§2. The general linear and orthogonal G-categories

Except where explicitly specified otherwise, we assume henceforward that the ambient group G is finite. Let A be a topological ring and consider A-representations of G, by which we understand isomorphism classes of finite dimensional A-free left modules over the group ring A[G]. A representation is indecomposable if it is not a proper direct sum of representations.

Definition 2.1. Let $U(A,G)$ be a direct sum of countably many copies of a representative of each indecomposable A-representation of G and let $U(A,G)^n$ be the direct sum of n copies of $U(A,G)$. Define $\mathcal{GL}(n,A,G)$ to be the category whose objects are the n-dimensional A-free sub A-modules of $U(A,G)^n$ and whose morphisms are the A-linear isomorphisms between such modules. Here the set of objects is given the discrete topology; free A-modules are given the product topology and the set of morphisms $M \to N$ is then given the function space topology. Let G act on $\mathcal{GL}(n,A,G)$ by translation of submodules and conjugation of isomorphisms; that is, $V \longmapsto gV$ and $f \longmapsto gfg^{-1}$. Define G-functors

$$\oplus : \mathcal{GL}(m,A,G) \times \mathcal{GL}(n,A,G) \to \mathcal{GL}(m+n,A,G)$$

via direct sums of modules and isomorphisms; this makes sense by virtue of the canonical identifications

$$U(A,G)^m \oplus U(A,G)^n = U(A,G)^{m+n}.$$

The unique object $0 \in \mathcal{GL}(0,A,G)$ is a unit for $\oplus$, and $\oplus$ is strictly associative. It is commutative up to coherent natural isomorphism of G-functors. Define

$$\mathcal{GL}(A,G) = \coprod_{n \geq 0} \mathcal{GL}(n,A,G).$$

Of course, we are primarily interested in the case of discrete rings and indeed discrete fields. However, the topologized real and complex numbers lead to equivariant topological K-theory. Here we could just as well restrict attention to orthogonal and unitary representations and to linear isometric isomorphisms, obtaining G-categories $\mathcal{O}(G)$ and $\mathcal{U}(G)$. Since A need not be commutative, we obtain $\mathcal{S}p(G)$ similarly, by use of quaternions.

The G-category $\mathcal{GL}(A,G)$ is "weakly" permutative. Genuine permutative G-categories appropriate for equivariant infinite loop space theory must have more structure [5,12]. The structure we have prescribed is clearly sufficient to ensure that $B\mathcal{GL}(A,G)$ is a G-homotopy commutative topological G-monoid.

The following result connects this definition to that given in the previous section.

Proposition 2.2. There is a G-functor

$$\nu : \mathcal{C}(G,GL(n,A)) \to \mathcal{SL}(n,A,G)$$

such that each fixed point functor ν^H is an equivalence of categories. Therefore

$$B\nu : B(G,GL(n,A)) \to B\mathcal{SL}(n,A,G)$$

is a G-homotopy equivalence. Moreover, the following diagram commutes up to natural isomorphism of G-functors:

$$\begin{tikzcd}
\mathcal{C}(G,GL(m,A)) \times \mathcal{C}(G,GL(n,A)) \arrow[rr, "\nu \times \nu"] & & \mathcal{SL}(m,A,G) \times \mathcal{SL}(n,A,G) \arrow[dd, "\oplus"] \\
\mathcal{C}(G,GL(m,A) \times GL(n,A)) \arrow[u] \arrow[d, "\oplus_*"'] & & \\
\mathcal{C}(G,GL(m+n,A)) \arrow[rr, "\nu"] & & \mathcal{SL}(m+n,A,G)
\end{tikzcd}$$

Therefore $\coprod_{n \geq 0} B(G,GL(n,A))$ is a Hopf G-space and is equivalent as a Hopf G-space to the G-homotopy commutative topological G-monoid $B\mathcal{SL}(A,G)$.

Proof. For each homomorphism $\rho : H \to GL(n,A)$, let A^n_ρ denote the corresponding A[H]-module and choose an H-isomorphism

$$i_\rho : A^n_\rho \to V_\rho \subset U(A,G)^n.$$

There exists such a V_ρ since U(A,G) contains a copy of the A[G]-module $G \times_H A^n_\rho$. For $x = gH \in G/H$, let $xV_\rho = gV_\rho$. Define ν on objects by $\nu(\rho,x) = xV_\rho$. For a morphism $(\alpha,\beta):(\rho,x) \to (\sigma,y)$, with $\beta(g,p) = (gf,qp)$ as in Remark 1.2, define $\nu(\alpha,\beta)$ to be the following composite:

$$xV_\rho = gV_\rho \xrightarrow{g^{-1}} V_\rho \xrightarrow{i_\rho^{-1}} A^n_\rho \xrightarrow{q} A^n_{\sigma\phi} = A^n_\sigma \xrightarrow{i_\sigma} V_\sigma \xrightarrow{gf} gfV_\sigma = yV_\sigma .$$

Here $\phi : H \to K$ is specified by $\phi(h) = f^{-1}hf$ and $q \in GL(n,A)$ satisfies $q\rho(h) = (\sigma\phi)(h)q$; thus q may be viewed as an H-isomorphism $A^n_\rho \to A^n_{\sigma\phi}$, the target being the K-module A^n_σ regarded as an H-module by pullback along ϕ. Since the composite $fi_\sigma q i_\rho^{-1} : V_\rho \to fV_\sigma$ is an H-map, the total composite is independent of the choice of coset representative $g \in x$. It is easy to see that ν is a well-defined G-functor (continuity on morphisms requiring GL(n,A) to have the function space topology of maps $A^n \to A^n$). Clearly ν^H restricts to an isomorphism from $\coprod_{\rho \in R^+(H,\Pi)} \Pi^\rho$, $\Pi = GL(n,A)$, to a skeleton of $\mathcal{SL}(n,A,G)^H$. For the last statement, composites of the form

$$xV_{\rho \oplus \sigma} \xrightarrow{g^{-1}} V_{\rho \oplus \sigma} \xrightarrow{i^{-1}_{\rho \oplus \sigma}} A^{m+n}_{\rho \oplus \sigma} = A^m_\rho \oplus A^n_\sigma \xrightarrow{i_\rho \oplus i_\sigma} V_\rho \oplus V_\sigma \xrightarrow{g} xV_\rho \oplus xV_\sigma$$

provide the required natural isomorphism.

There are variants of the discussion above in terms of the various classical subgroups of general linear groups. We single out the orthogonal case. Thus assume that A is commutative. If the scalar 2 is non-zero and is not a zero-divisor, then the group O(n,A) of matrices $X \in GL(n,A)$ such that $X^{-1} = X^t$ is the group of automorphisms of A^n which fix the standard quadratic form $Q(\sum a_i e_i) = \sum a_i^2$. We could repeat the discussion above with GL(n,A) replaced by O(n,A). However, it is more appropriate to allow general quadratic forms and, to avoid restrictions concerning 2, to work with bilinear forms rather than quadratic forms. Thus, by an orthogonal A-module, we understand a finite dimensional free A-module V together with a nondegenerate symmetric bilinear form. We then write O(V) for the group of orthogonal isomorphisms $V \to V$. By an orthogonal A-representation of G, we understand an A-representation of G and an orthogonal A-module with a G-invariant form.

Definition 2.3. Let Q(A,G) be a direct sum of countably many copies of a representative of each indecomposable orthogonal A-representation of G. Define $\mathcal{O}(n,A,G)$ to be the G-category of n-dimensional orthogonal sub A-modules of $Q(A,G)^n$ and orthogonal isomorphisms between them. Define

$$\mathcal{O}(A,G) = \coprod_{n \geq 0} \mathcal{O}(n,A,G).$$

Again $\mathcal{O}(A,G)$ is a weakly permutative G-category, hence $B\mathcal{O}(A,G)$ is a G-homotopy commutative topological G-monoid, and we define

$$KO(A,G) = \Omega BB\mathcal{O}(A,G).$$

When A is an algebraically closed field of characteristic $\neq 2$, all orthogonal A-modules are isomorphic to standard ones and this definition reduces to that of Definiton 0.2. The following analog of Proposition 2.2 admits the same proof.

Proposition 2.4. Let $\{V_i\}$ be a set of representatives for the isomorphism classes of orthogonal A-modules. There is a G-functor

$$\nu: \coprod_{\dim V_i = n} \mathcal{C}(G, O(V_i, A)) \to \mathcal{O}(n,A,G)$$

such that each fixed point functor ν^H is an equivalence of categories. Therefore

$$B\nu: \coprod_{\dim V_i = n} B(G, O(V_i, A)) \to B\mathcal{O}(n,A,G)$$

is a G-homotopy equivalence. Moreover, up to natural isomorphism of G-functors, the functors ν are compatible with sums. Therefore $\coprod_i B(G,O(V_i,A))$ is equivalent as a Hopf G-space to $B\mathcal{O}(A,G)$.

Remark 2.5. If $\frac{1}{2} \in A$, any orthogonal A-module is a direct summand of a hyperbolic one, hence a cofinality argument (as in section 4) shows that the basepoint component of $KO(A) = KO(A,e)$ is equivalent to the plus construction on $\varinjlim O(hA^n)$. Thus the groups $\pi_* KO(A)$ are examples of the higher Witt groups of Wall [32], Karoubi [15], and others; see Loday [20] for a survey. Clearly our procedures also yield equivariant generalizations of the other examples of such theories.

Remark 2.6. We could just as well have used finitely generated projective A-modules rather than finite dimensional free ones above. The evident analogs of Propositions 2.2 and 2.4 would hold. Again, by cofinality, only π_0 would be altered in the nonequivariant case.

Remark 2.7. Consider $H \subset G$. In situations where complete reducibility holds, $U(A,G)$ is evidently isomorphic as an H-space to $U(A,H)$. Therefore, when regarded as a weakly permutative H-category by neglect of structure, $\mathscr{IL}(A,G)$ is isomorphic to $\mathscr{IL}(A,H)$. In particular, the fixed point category $\mathscr{IL}(A,G)^H$ is independent of the ambient group G. The same is true in the orthogonal case.

Remarks 2.8. The definitions in this section apply perfectly well to general topological groups G and continuous representations, but the proof of Proposition 2.2 fails because the functor ν fails to be continuous on object spaces. Nevertheless, for suitably restricted G and A, the equivalences of Propositions 2.2 and 2.4 can be recovered by use of equivariant bundle theory. We sketch the idea. Define $\mathscr{E}(n,A,G)$ to be the category whose objects consist of pairs (V,v), where V is an A-free sub A-module of $U(A,G)^n$ and v is an element of V, and whose morphisms $(V,v) \to (W,w)$ are the A-linear isomorphisms $f: V \to W$ such that $f(v) = w$. Here the object set is topologized as $\coprod V$ and the morphism sets are topologized via the triples (v,f,w) as subspaces of $V \times W^V \times W$; the factors V and W ensure continuity of the source and target functions. Again, G acts via translation of objects and conjugation of morphisms, and there is an obvious projection of G-categories

$$\pi : \mathscr{E}(n,A,G) \to \mathscr{IL}(n,A,G).$$

One checks first that $B\pi$ is a (G,A)-bundle with fibre A^n. One then checks that if a closed subgroup H of G acts through any continuous homomorphism $\rho : H \to GL(n,A)$ on the total space E of the principal (G,A)-bundle associated to $B\pi$, then E^H is non-empty and contractible. By consideration of bundles over fixed point spaces, it follows that the classifying G-map

$$B\mathcal{GL}(n,A,G) \to B(G,GL(n,A))$$

induces an equivalence on fixed point subspaces and is therefore a G-equivalence; see [17,2.14]. In particular, such an argument works to prove

$$B\mathcal{U}(n,G) \simeq B(G,U(n)) \quad \text{and} \quad B\mathcal{O}(n,G) \simeq B(G,O(n)).$$

Here again, since any unitary or orthogonal representation of a closed subgroup H of G is contained in a unitary or orthogonal representation of G and we have complete reducibility, $\mathcal{U}(G)^H$ and $\mathcal{O}(G)^H$ are independent of the ambient group G.

§3. Analysis of fixed point categories

The first step in the study of the G-spaces introduced in Definition 0.2 must be the analysis of their fixed points. For any topological G-monoid M, we evidently have $(\Omega BM)^H = \Omega B(M^H)$. When $M = B\mathcal{A}$ for a weakly permutative G-category $\mathcal{A}$, $M^H = B(\mathcal{A}^H)$ has the monoid structure determined by the induced structure of a permutative category on $\mathcal{A}^H$. Thus we must analyze the permutative categories $\mathcal{GL}(A,G)^H$. With $\Pi_n = GL(n,A)$, we have already observed that a skeleton of $\mathcal{GL}(n,A,G)^H$ is isomorphic to $\coprod_\rho \Pi_n^\rho$, where ρ ranges through $R^+(H,\Pi_n)$. By itself, this tells us little about the permutative structure on $\mathcal{GL}(A,G)^H$, and in general the lack of complete reducibility obstructs easy analysis of the Π_n^ρ. By Remark 2.7, we may take H = G without loss of generality when complete reducibility holds.

It will clarify the arguments of this section if we change our point of view slightly. Use of the ambient A[G]-spaces $U(A,G)^n$ in Definition 2.1 served to allow the precise specification of a small weakly permutative G-category. The non-equivariant permutative category $\mathcal{GL}(A,G)^G$ may be viewed as a pedantically careful model for the large symmetric monoidal category of all finite dimensional A-free A[G]-modules and all A[G]-isomorphisms between them. We agree to use the same notation for the latter category. In general, we agree to work with such large categories but draw conclusions for equivalent small permutative categories. There is no loss of rigor since passage to skeleta and then to permutative categories (e.g. via [21, 4.2; 23, VI.3.2]) allows functorial replacement of symmetric monoidal categories with sets of isomorphism classes of objects by equivalent permutative categories. In this spirit, we let $\mathscr{GL}(A)$ mean both the category of finite dimensional free (right) A-modules and their isomorphisms and its skeletal permutative model $\coprod GL(n,A)$ (e.g. [23, p.162]). We let

$$K(A) = \Omega BB\mathscr{GL}(A) \simeq BGL(A)^{+} \times Z.$$

<u>Proposition 3.1.</u> Let F be a field of characteristic prime to the order of G. Let $S = \{V_i\}$ be a set of representatives for the irreducible representations of G over F and let D_i be the division ring $\mathrm{Hom}_{F[G]}(V_i, V_i)$. Then $\mathscr{GL}(F,G)^G$ is equivalent as a permutative category to $\times_i \mathscr{GL}(D_i)$. Therefore $K(F,G)^G$ is equivalent to $\times_i K(D_i)$.

<u>Proof.</u> Define $\rho_i : \mathscr{GL}(D_i) \to \mathscr{GL}(A,G)^G$ by $\rho_i(M) = M \otimes_{D_i} V_i$ and $\rho_i(f) = f \otimes 1$ for a finite dimensional D_i-module M and an isomorphism $f: M \to M'$. Certainly ρ_i commutes with sums. It is immediate from Schur's lemma that ρ_i is an equivalence to the full subcategory of $\mathscr{GL}(A,G)^G$ whose objects are the isotypical F[G]-modules of type V_i and that $\mathscr{GL}(A,G)^G$ is isomorphic to the product of these subcategories. The conclusion follows on passage to small mutative equivalents.

Since $K_q^G(F) = \pi_q K(F,G)^G$, we can e our new equivariant K-groups of F in terms of the nonequivariant K-groups o $_i$. That is, $K_q^G(F) = \bigoplus_i K_q(D_i)$. Recall that, if char $F \neq 0$, each D_i is necessarily a field since V_i is obtained by extension of scalars from an irreducible representation U_i defined over some finite subfield k_i of F and thus

$$D_i = \mathrm{Hom}_{F[G]}(V_i, V_i) \cong F \otimes_{k_i} \mathrm{Hom}_{k_i[G]}(U_i, U_i).$$

We need some preliminaries (related to [28, App]) to obtain the orthogonal analog. Consider the duality operator, $V^* = \mathrm{Hom}_F(V,F)$, on $F[G]$-modules. Let $S_0 \subset S$ consist of one V from each pair (V,V^*) such that V is not isomorphic to V^* and let $S_0^* = \{V^* | V \in S_0\}$. Now consider a self-dual $F[G]$-module $V \in S$ with associated division ring D. Clearly an $F[G]$-isomorphism $V \to V^*$ corresponds by adjunction to a G-invariant nondegenerate bilinear form, abbreviated G-form henceforward. Given one G-form $b_0 : V \otimes_F V \to F$, any other G-form b can be written in the forms

$$b(v,w) = b_0(dv,w) = b_0(v,\overline{d}w)$$

for non-zero elements $d,\overline{d} \in D$; the function $d \to \overline{d}$ specifies an involution of D (which depends on the choice of the fixed initially given G-form b_0). If b_0 and b are both symmetric, then

$$b_0(dv,w) = b_0(dw,v) = b_0(w,\overline{d}v) = b_0(\overline{d}v,w)$$

and thus $d = \overline{d}$. The same conclusion holds if both b_0 and b are skew symmetric. If all b are symmetric or all b are skew symmetric, then the involution is trivial and D is necessarily a field. If b_0 is symmetric and b is not, then $d \neq \overline{d}$; in particular, $\overline{d} = -d$ if b is skew symmetric. Let

$$S_+ = \{V | V \cong V^* \text{ and every } b \text{ is symmetric}\} ,$$

$$S_- = \{V | V \cong V^* \text{ and no } b \text{ is symmetric}\} ,$$

and

$$S_\pm = \{V | V \cong V^* \text{ and some } b \text{ is and some } b \text{ is not symmetric}\}.$$

Visibly, we have a decomposition

$$S = S_0 \amalg S_0^* \amalg S_+ \amalg S_- \amalg S_\pm .$$

If $V \in S_-$ and b is a G-form on V, let $c(v,w) = b(v,w) + b(w,v)$. Since c is symmetric, it must be degenerate. Thus

$$\mathrm{Rad}(V,c) = \{w | h(w,V) = 0\}$$

is a non-zero sub G-space of V and must be all of V, hence c must be identically zero. If char $F = 2$, this is a contradiction and S_- is empty. If char $F \neq 2$, this implies that

$$S_- = \{V | V \cong V^* \text{ and every } b \text{ is skew symmetric }\}.$$

If $V \in S_\pm$ and b is not symmetric, then $a(v,w) = b(v,w) - b(w,v)$ must be nondegenerate and is thus a skew symmetric G-form. If char $F \neq 2$, this gives

$$S_\pm = \{V \mid V \text{ admits both symmetric and skew symmetric G-forms}\}.$$

For an F[G]-module V, let $hV = (V \oplus V^*, h)$ be the hyperbolic orthogonal F[G]-module. The G-form h is specified by

$$h((v,f),(w,g)) = g(v) + f(w) \quad \text{for } v,w \in V \text{ and } f,g \in V^*.$$

Note that $hV = hV^*$. We need the following observations. For simplicity, we assume prematurely that the characteristic of F is prime to the order of G.

Recall that a form b on V is said to be alternate if $b(v,v) = 0$ for all $v \in V$. It follows that V is even dimensional and admits a symplectic basis [14, Thm 19]. Of course, alternate forms are skew symmetric, and conversely if char $F \neq 2$.

Lemma 3.2. Let (V,b) be an irreducible orthogonal F[G]-module.

(i) If char $F \neq 2$, then $(V,b) \oplus (V,-b) \cong hV$.

(ii) If char $F = 2$ and $V \in S_+$, then hV is irreducible and $3(V,b) \cong (V,b) \oplus hV$.

(iii) If char $F = 2$ and $V \in S_\pm$, then $2(V,b) \cong hV$.

(iv) If char $F = 2$ and V is non-trivial, then b is alternate and $V \in S_\pm$ if and only if V admits an invariant nonsingular quadratic form.

Proof. Identify V with V^* via $V \leftrightarrow b(v,?)$. For (i), $(v,v') \to (v + v', v - v')$ specifies an isomorphism $hV \to (V, \frac{1}{2}b) \oplus (V, -\frac{1}{2}b)$, and $\frac{1}{2}b$ runs through all symmetric G-forms as b does. For (ii), $hV \cong (V,b) + (V,b')$ is easily checked to be impossible and $(v,v',v'') \to (v + v' + v'', v + v', v + v'')$ specifies an isomorphism $(V,b) \oplus hV \to 3(V,b)$. For (iii), $(v,v') \to (dv + \bar{d}v, v')$ specifies an isomorphism $2(V,b') \to hV$, where $d \neq \bar{d}$ and $b'(v,w) = (dv + \bar{d}v, w)$, and b' runs through all symmetric G-forms as d runs through all non-invariant elements of D. For (iv), the kernel of the function $v \to b(v,v)$ is an invariant sub F[G]-module of V and is zero only if V is trivial; since any nonsingular quadratic form q can be written as $q(v) = b'(v,v)$ for some nonsymmetric bilinear form b' and since b' can be made invariant by averaging, the last clause is clear.

Lemma 3.3. Any orthogonal $F[G]$-module is an orthogonal direct sum of orthogonal $F[G]$-modules of one of the following forms:

(i) (V,b), where $V \in S_+ \amalg S_\pm$ (and b is a symmetric G-form)

(ii) hV, where $V \in S_0 \amalg S_-$ if char $F \neq 2$ and $V \in S_0 \amalg S_+$ if char $F = 2$.

Proof. By induction on the dimension, it suffices to show that any (W,b) has a direct summand of one of the cited forms. Forgetting b, we may assume that $V \subset W$ for some $V \in S$. If $b|V$ is nondegenerate, then $V \in S_+ \amalg S_\pm$ and $(W,b) = (V,b) \oplus (V^\perp,b)$, where $V^\perp = \{w | b(w,V) = 0\}$. Thus assume that $b|V$ is degenerate. Then $\mathrm{Rad}(V,b) = V$ and thus $b|V = 0$. Therefore $V \subset \mathrm{Rad}(V^\perp,b) \subset (V^\perp)^\perp$ and, since $\dim V = \dim (V^\perp)^\perp$, these are equalities. This implies $(V^\perp,b) = (V,0) \oplus (W',b)$, where $b|W'$ is nondegenerate. Let $U = (W')^\perp \subset W$. Then $(W,b) = (U,b) \oplus (W',b)$. Define $\tau: W \to V^*$ by $\tau(w)(v) = b(w,v)$. We clearly have exact sequences

$$0 \longrightarrow V^\perp \longrightarrow W \xrightarrow{\tau} V^* \longrightarrow 0$$

and, by restriction,

$$0 \longrightarrow V \longrightarrow U \xrightarrow{\tau} V^* \longrightarrow 0 .$$

Let $\sigma: V^* \to U$ split U. Regarding U as $V \oplus V^*$ via σ, we see from $b|V = 0$ and $\tau\sigma = 1$ that $b|U$ is given by

$$b((v,f) + (w,g)) = g(v) + f(w) + b(\sigma f, \sigma g).$$

If $(\sigma V^*, b)$ is degenerate, then $b|\sigma V^* = 0$ and $(U,b) = hV$. If $(\sigma V^*,b)$ is nondegenerate, then $V \in S_+ \amalg S_\pm$ and the splitting $(U,b) = (\sigma V^*,b) \oplus ((\sigma V^*)^\perp, b)$ gives the conclusion.

For a field F, let $\mathcal{O}(F)$ be the symmetric monoidal category of orthogonal F-spaces and orthogonal isomorphisms or some equivalent permutative category, such as $\mathcal{O}(F,e)$ of Definition 2.3. If F is algebraically closed and char $F \neq 2$, then $\amalg\, O(n,F)$ is the most efficient permutative model. An efficient model in the case $F = k_r$, r odd, is given in [11, II.4.8]. See [14, 1-10] for the structure of $\mathcal{O}(F)$ when char $F = 2$. Let $KO(F) = \Omega BB\mathcal{O}(F)$.

If char $F \neq 2$, let $\mathcal{S}p(F)$ be the category of symplectic F-spaces (namely finite dimensional F-spaces with a nondegenerate skew symmetric bilinear form) and symplectic isomorphisms or some equivalent permutative category. Since all symplectic F-spaces are isomorphic to standard ones [14, p. 22], $\coprod \mathrm{Sp}(2n,F)$ is a suitable model [11, p.113]. Let $KSp(F) = \Omega BB\,\mathcal{S}p(F)$.

For a division ring D with involution, let $\mathcal{U}(D)$ be the category of unitary D-spaces (namely finite dimensional right D-spaces with a nondegenerate Hermitian form [14]) and unitary isomorphisms or some equivalent permutative category. Let $KU(D) = \Omega BB\,\mathcal{U}(D)$.

Proposition 3.4. Let F be a field of characteristic prime to the order of G. If char $F \neq 2$, then $\mathcal{O}(F,G)^G$ is equivalent as a permutative category to the product

$$\Big[\underset{V_i \in S_0}{\times} \mathcal{SL}(D_i)\Big] \times \Big[\underset{V_i \in S_+}{\times} \mathcal{O}(D_i)\Big] \times \Big[\underset{V_i \in S_-}{\times} \mathcal{S}p(D_i)\Big] \times \Big[\underset{V_i \in S_\pm}{\times} \mathcal{U}(D_i)\Big].$$

If char $F = 2$, $\mathcal{O}(F,G)^G$ is equivalent to the product

$$\Big[\underset{V_i \in S_0}{\times} \mathcal{SL}(D_i)\Big] \times \Big[\underset{V_i \in S_+}{\times} \mathcal{O}(D_i)\Big] \times \Big[\underset{V_i \in S_\pm}{\times} \mathcal{U}(D_i)\Big].$$

Therefore $KO(F,G)^G$ is equivalent to the corresponding product of spaces $K(D_i)$, $KO(D_i)$, $KSp(D_i)$, and $KU(D_i)$.

Proof. Regard $\mathcal{O}(F,G)^G$ as the category of all orthogonal F[G]-modules and orthogonal G-isomorphisms. First consider $V_i \in S_0$. Passage to duals specifies an isomorphism $D_i^{op} \cong \mathrm{Hom}_{F[G]}(V_i^*, V_i^*)$ and, for a right D_i-space M, $M^* \otimes_{D_i^{op}} V_i^*$ is canonically isomorphic to $(M \otimes_{D_i} V_i)^*$. Define $\rho_i : \mathcal{SL}(D_i) \to \mathcal{O}(F,G)^G$ by sending M to the F[G]-module

$$(M \otimes_{D_i} V_i) \oplus (M^* \otimes_{D_i^{op}} V_i^*)$$

with the hyperbolic symmetric G-form and sending an isomorphism $f: M \to M'$ to $(f \otimes 1) \oplus ((f^{-1})^* \otimes 1)$. Observe that every orthogonal G-isomorphism

$\rho_i(M) \to \rho_i(M')$ is of the form $\rho_i(f)$ for some f. Next, consider $V_i \in S_+ \amalg S_- \amalg S_\pm$ and recall that D_i is a field if $V_i \in S_+ \amalg S_-$. If $V_i \in S_-$, fix a skew symmetric G-form b_i on V_i; otherwise, fix a symmetric G-form b_i. Let M be a right D_i-space with a symmetric form b if $V_i \in S_+$, a skew symmetric form b if $V_i \in S_-$, or a Hermitian form b if $V_i \in S_\pm$. Then, in all three cases, the tensorial G-form

$$(b \otimes b_i)(m \otimes v, m' \otimes v') = b_i(b(m,m')v, v')$$

on $M \otimes_{D_i} V_i$ is symmetric. Moreover, if $f: M \to M'$ is an isometric isomorphism, then so is $f \otimes 1$, and every orthogonal G-isomorphism $M \otimes_{D_i} V_i \to M' \otimes_{D_i} V_i$ is of this form. Thus we obtain functors ρ_i with the appropriate domain and with target $\mathcal{O}(F,G)^G$ via $\rho_i(M) = M \otimes_{D_i} V_i$ and $\rho_i(f) = f \otimes 1$. Note that a decomposition of (M,b) as a sum of one-dimensional forms (in the S_+ or $S_\pm$ case) gives a corresponding decomposition of $\rho_i(M,b)$ as a sum of copies of V_i with different symmetric G-forms, and all such sums are so realized. If char F = 2 and $V_i \in S_+$, then $\rho_i(hD_i) = hV_i$ and hD_i is orthogonally irreducible. By Schur's lemma and Lemma 3.3, $\mathcal{O}(F,G)^G$ is isomorphic to a product of full subcategories such that ρ_i maps via an equivalence to the i^{th} subcategory.

We shall need the corresponding facts about topological equivariant K-theory. The following pair of results do not appear in the literature in quite this form. Their consequences

$$K_G(X) \cong R(G) \otimes K(X)$$

and

$$KO_G(X) \cong [R(G;\mathbf{R}) \otimes KO(X)] \oplus [R(G;\mathbf{C}) \otimes K(X)] \oplus [R(G;\mathbf{H}) \otimes KSp(X)]$$

for a compact space X with trivial G-action are due to Segal [30]. Here and below, G may be a compact Lie group and $R(G;\mathbf{F})$ denotes the free Abelian group generated by those irreducible real orthogonal representations of G with associated division ring $\mathbf{F}$.

Let $\mathcal{U}$, $\mathcal{O}$, and $\mathcal{S}p$ denote either the categories of finite dimensional complex, real, and quaternionic inner product spaces and isometric isomorphisms or their permutative skeleta $\coprod U(n)$, $\coprod O(n)$, and $\coprod Sp(n)$ ([23, p.163]). Let $K = \Omega B B \mathcal{U}$, $KO = \Omega B B \mathcal{O}$, and $KSp = \Omega B B \mathcal{S}p$; these spaces represent complex, real, and quaternionic K-theory.

Recall that the weak product of based spaces X_i is the subspace of $\times_i X_i$ consisting of those points with all but finitely many coordinates the basepoint. Similarly, the weak product of based categories $\mathcal{Q}_i$ has all but finitely many coordinates of each object and morphism the base object and its identity morphism.

Proposition 3.5. Let $S = \{V_i\}$ be a set of representatives for the irreducible unitary representations of a compact Lie group G. Then $\mathcal{U}(G)^G$ is equivalent as a permutative category to the weak product of one copy of $\mathcal{U}$ for each V_i. Therefore $K(G)^G$ is equivalent to the weak product of one copy of K for each V_i.

Proof. Regard $\mathcal{U}(G)^G$ as the category of unitary representations of G and their isomorphisms, define $\rho_i: \mathcal{U} \to \mathcal{U}(G)^G$ by $\rho_i(M) = M \otimes_{\mathbf{C}} V_i$ and $\rho_i(f) = f \otimes 1$, and argue as in the proofs above.

Let $t: R(G) \to R(G)$, $r: R(G) \to RO(G)$, $c': RSp(G) \to R(G)$, $c: RO(G) \to R(G)$, and $q: R(G) \to RSp(G)$ be conjugation and the evident neglect of structure and extension of scalars homomorphisms; of course, $tV \cong V^*$. By [3, 3.57], we may choose sets $\{U_i\}$, $\{V_j\}$, and $\{W_k\}$ of irreducible orthogonal, unitary, and symplectic representations of G such that

$$S = S_0 \coprod S_0^* \coprod S_+ \coprod S_- ,$$

where $S_0 = \{V_j\}$, $S_0^* = \{tV_j\}$, $S_+ = \{cU_i\}$, and $S_- = \{c'W_k\}$. Moreover, complete sets of inequivalent irreducible orthogonal and symplectic representations of G are given by

$$\{U_i\} \coprod \{rV_j\} \coprod \{rc'W_k\}$$

and

$$\{qcU_i\} \coprod \{qV_j\} \coprod \{W_k\} .$$

Proposition 3.6. Let G be a compact Lie group. Then $\mathcal{O}(G)^G$ is equivalent as a permutative category to the weak product of one copy of $\mathcal{O}$ for each U_i, one copy of $\mathcal{U}$ for each rV_j, and one copy of $\mathcal{S}p$ for each $rc'W_k$. Therefore $KO(G)^G$ is equivalent to the weak product of correspondingly indexed copies of KO, K, and KSp.

Proof. Regard $\mathcal{O}(G)^G$ as the category of orthogonal representations of G and their isomorphisms. Via $M \to M \otimes_{\mathbf{F}} Z$ on objects and $f \to f \otimes 1$ on morphisms, where Z is U_i, V_j, or W_k and $\mathbf{F}$ is $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, we obtain functors ρ_i, ρ_j, and ρ_k from $\mathcal{O}$, $\mathcal{U}$, and $\mathcal{S}p$ to $\mathcal{O}(G)^G$. Alternatively, we can use Adams [3, 3.50–3.57] to throw the proof onto an argument just like the case $\mathbf{F} = \mathbf{C}$ of Proposition 3.4.

Remark 3.7. A similar argument applies to equivariant symplectic K-theory, where the conclusion is that $KSp(G)^G$ is equivalent to the weak product of one copy of KSp for each qcU_i, one copy of K for each qV_j, and one copy of KO for each W_k; compare Kawakubo [16].

§4. Group completions; naturality and products

The natural map $\zeta : M \to \Omega BM$ for G-homotopy commutative topological G-monoids M substitutes for an equivariant plus construction in our work, and we begin this section by discussing its universal properties. We then use this information to construct naturality maps and pairings relating the spaces K(A,G). Even non-equivariantly, this approach to products seems cleaner than that based on use of the plus construction [19]. We shall give a much more structured spectrum level treatment, like that of [24], in [12].

Until otherwise specified, G can be an arbitrary topological group. Consider Hopf G-spaces, namely G-spaces X with a G-map $X \times X \to X$ and a G-fixed basepoint which acts as a two-sided unit up to G-homotopy. For present purposes, we require the product to be associative and commutative up to G-homotopy. We say that X is grouplike if $\pi_0(X^H)$ is a group for each closed subgroup H of G. We say that a Hopf G-map $f : X \to Y$ is a group completion if Y is grouplike and if each Hopf map $f^H : X^H \to Y^H$ is a group completion. This means that f^H induces group completion on

π_0 and induces localization of $H_*(X^H;R)$ at its submonoid $\pi_0(X^H)$ for each commutative coefficient ring R; see [21, §1] for discussion. If X itself is grouplike, then each f^H is an equivalence and thus f is a G-equivalence.

For M as above, application of the "group completion theorem" [22, 26, 27] to fixed point spaces implies that $\zeta : M \to \Omega BM$ is a group completion. To exploit this fact, we need the following notion.

Definition 4.1. A cofinal sequence in a Hopf G-space X is a sequence of points $a_t \in X^G$, with a_0 the unit, such that a_{t+1} is in the same path component of X^G as $b_t a_t$ for some $b_t \in X^G$ and such that, for each $H \subset G$ and $c \in X^H$, there exists $d \in X^H$ and $t \geq 0$ such that dc and a_t are in the same path component of X^H.

Given such a sequence, let X_t be a copy of X and let $\overline{X}$ be the telescope of the sequence of left translations $b_t : X_t \to X_{t+1}$. If X is grouplike, then each b_t is a G-equivalence and the natural map $X = X_0 \to \overline{X}$ is a G-equivalence. Indeed, if X_t is given the basepoint a_t and the multiplication obtained by composing the product on X with translation by a point in the component of $\pi_0(X^G)$ inverse to a_t, then each b_t is an equivalence of Hopf G-spaces.

If $f : X \to Y$ is a group completion and we form $\overline{X}$ and $\overline{Y}$ with respect to a cofinal sequence in X and its image in Y, then f commutes up to G-homotopy with the translations and therefore factors up to G-homotopy as a composite

$$X \to \overline{X} \xrightarrow{\overline{f}} \overline{Y} \simeq Y.$$

The construction commutes with passage to fixed point spaces, and it is a direct consequence of the group completion property that each $\overline{f}^H$ induces an isomorphism on homology. By an easy (but not obvious) homotopical argument, this implies the following universal property of f; see [8] for details.

Proposition 4.2. Let $f : X \to Y$ be a group completion, where X contains a cofinal sequence, and let $g : X \to Z$ be any Hopf G-map, where Z is grouplike. Then there is a Hopf G-map $\tilde{g} : Y \to Z$, unique up to G-homotopy, such that $\tilde{g}f$ is G-homotopic to g.

By a function space argument, this result has the following consequence, which is due to Caruso; see [8].

Proposition 4.3. Let $f: X \to Y$ and $f': X' \to Y'$ be group completions, where X and X' contain cofinal sequences, and let $g: X \wedge X' \to Z$ be any G-homotopy bilinear G-map, where Z is a grouplike Hopf G-space. Then there is a G-homotopy bilinear G-map $\tilde{g}: Y \wedge Y' \to Z$, unique up to G-homotopy, such that $\tilde{g}(f \wedge f')$ is G-homotopic to g.

Remarks 4.4. We have deliberately misstated the previous two results: all conclusions concerning G-homotopies really only hold up to "weak" G-homotopy in general. Here two G-maps $f, f': X \to Y$ are weakly G-homotopic if their composites $fk, f'k: K \to Y$ are G-homotopic for any compact G-space K and G-map $k: K \to X$. In other words, the results are correct on the level of represented functors defined on compact G-spaces. The $\lim^1$ exact sequences associated with the telescopes we have used are the source of the ambiguity. We leave it to the reader to insert the word "weak" where needed, doing so once in a while ourselves as a reminder.

The reader who would like to see the simpler nonequivariant proofs is referred to [7]. While the results are true for any G, we shall only use them when G is finite and, peripherally, when G is compact Lie and $\bar{f}$ and $\bar{f}'$ are G-equivalences.

Assume that G is finite in the rest of this section. We restrict attention to the general linear case, but everything we say applies equally well in the orthogonal case. We first verify our cofinality assumptions. Recall that components of $B\mathcal{Q}$ correspond bijectively to isomorphism classes of objects when $\mathcal{Q}$ is a groupoid.

Lemma 4.5. $B\mathcal{J}\mathcal{L}(A,G)$ contains a cofinal sequence.

Proof. Write $U(A,G) = \sum_{i=1}^{r} V_i^{\infty}$ and let $W_t = \sum_{i=1}^{r} V_i^t$. We claim that $\{W_t\}$ is cofinal. Certainly $W_{t+1} = W_1 \oplus W_t$. For $H \subset G$ and an A-free H-fixed $Y \subset U(A,G)$, let $Z = G \times_H Y$. Then $Z \cong Y' \oplus Y$ as an A[H]-space and $Z' \oplus Z \cong W_t$ as an A[G]-space for appropriate Y' and Z' in U(A,G) and some t, hence $(Z' \oplus Y') \oplus Y \cong W_t$.

Remarks 4.6. It would not be useful to form the telescope of copies of $\coprod B(G,GL(n,A))$ with respect to the maps induced by the inclusions $GL(n,A) \to GL(n+1,A)$. This would amount to replacing $\{W_t\}$ by the noncofinal sequence $\{A^t\}$ of trivial representations. The resulting telescope is G-equivalent to $B(G,GLA) \times Z$, and its irrelevance explains why $B(G,GLA)$ plays no explicit role in our theory. By cofinality, we do have a natural G-map from $B(G,GLA) \times Z$ to the telescope $\overline{B}\mathcal{GL}(A,G)$ defined as above the respect to the sequence $\{W_t\}$.

We illustrate the force of Proposition 4.2 by using it to construct naturality maps. Let $f:B \to A$ be a ring homomorphism. Applying f to matrix entries, we obtain homomorphisms

$$f^{\#}:GL(n,B) \to GL(n,A).$$

By Lemma 1.4 and the G-equivalences of Proposition 2.2, these homomorphisms induce G-maps

$$f^{*}:B\mathcal{GL}(B,G) \to B\mathcal{GL}(A,G).$$

By the evident compatibility with sums of the $f^{\#}$, f^{*} is a Hopf G-map, hence so is its composite with ζ. By Proposition 4.2, there results a Hopf G-map

$$f^{*}:K(B,G) \to K(A,G)$$

such that $f^{*}\zeta \simeq \zeta f^{*}$. If A is a d-dimensional free left B-module by pullback along f, then a choice of basis for A over B determines homomorphisms

$$f_{\#}:GL(n,A) \to GL(nd,B).$$

Again, there results a Hopf G-map

$$f_{*}:B\mathcal{GL}(A,G) \to B\mathcal{GL}(B,G)$$

and thus a Hopf G-map

$$f_{*}:K(A,G) \to K(B,G)$$

such that $f_{*}\zeta \simeq \zeta f_{*}$.

These particular maps can also be obtained, and with no weak homotopy ambiguity, by direct use of our categorical models. Indeed, $A \otimes_B U(B,G)$ is isomorphic to a direct summand of U(A,G) and, if A is B-free, U(A,G) is isomorphic as a B[G]-module to U(B,G). Therefore we have evident morphisms of weakly permutative G-categories

$$f^{\#} : \mathcal{SL}(B,G) \to \mathcal{SL}(A,G) \quad \text{and} \quad f_{\#} : \mathcal{SL}(A,G) \to \mathcal{SL}(B,G).$$

The uniqueness claim of Proposition 4.2 implies that the resulting maps of K-theory spaces agree with those constructed in the previous paragraph. Of course, Proposition 4.2 may be applied in situations where no such precise categorical argument is available.

Now let A be a commutative ring and write $\Pi_n = GL(n,A)$. The tensor product homomorphisms $\Pi_m \times \Pi_n \to \Pi_{mn}$ induce G-maps

$$B(G,\Pi_m) \times B(G,\Pi_n) \cong B(G, \Pi_m \times \Pi_n) \to B(G, \Pi_{mn})$$

via Lemma 1.4. since Π_0 is the trivial group, $\mathcal{C}(G,\Pi_0)$ has object space $\coprod_H G/H$ and terminal object $* = G/G$, hence $B(G,\Pi_0)$ is G-contractible. Using the distributivity of $\otimes$ over $\oplus$ and Proposition 2.2, we find easily that these maps give rise to a G-homotopy bilinear G-map

$$\otimes : B\mathcal{SL}(A,G) \wedge B\mathcal{SL}(A,G) \to B\mathcal{SL}(A,G).$$

Composing with ζ and applying Proposition 4.3, we obtain a G-homotopy bilinear G-map

$$\otimes : K(A,G) \wedge K(A,G) \to K(A,G).$$

Indeed, K(A,G) is a Hopf ring G-space in the sense that the axioms for a commutative ring hold up to (weak) G-homotopy. For a ring homomorphism $f : B \to A$, $f^{\#}$ commutes with $\otimes$ and the uniqueness claim of Proposition 4.3 implies that $\otimes \circ (f^* \wedge f^*) \simeq f^* \circ \otimes$. On passage to G-homotopy groups, this implies the following result, its last statement being a consequence of the defintion of $B\mathcal{SL}(A,G)$ and the group completion property.

Let $R^{\oplus}_{A}(G)$ denote the Grothendieck ring of finite dimensional A-free A-representations of G under the additivity relations generated by A[G]-split short exact sequences.

Proposition 4.7. For commutative rings A, $K^G_*(A)$ is naturally a commutative graded ring. In particular, $K^G_0(A)$ is isomorphic to the ring $R^{\oplus}_{A}(G)$.

The appropriate relationship between products and f_* is given by the following "projection formula".

Proposition 4.8. Let $f: B \to A$ be a homomorphism of commutative rings such that A is a finite dimensional free B-module. Then

$$f_*(xf^*(y)) = f_*(x)y$$

in $K^G_{q+r}(B)$ for $x \in K^G_q(A)$ and $y \in K^G_r(B)$.

Proof. The following diagrams commute up to conjugation, where $d = \dim_B A$:

$$\begin{tikzcd}
GL(m,A) \times GL(n,A) \arrow[d, "\otimes"'] & GL(m,A) \times GL(n,B) \arrow[l, "1 \times f^{\#}"'] \arrow[r, "f_{\#} \times 1"] & GL(md,B) \times GL(n,B) \arrow[d, "\otimes"] \\
GL(mn,A) \arrow[rr, "f_{\#}"] & & GL(mnd,B)
\end{tikzcd}$$

Therefore, by Lemma 1.4 and Proposition 2.2, the corresponding diagram with general linear groups replaced by classifying spaces $B\mathcal{J}\mathcal{L}(A,G)$ and $B\mathcal{J}\mathcal{L}(B,G)$ commutes up to G-homotopy. Since both composites in the latter diagram are G-homotopy bilinear, the uniqueness claim in Proposition 4.3 gives the G-homotopy commutativity of the diagram

$$\begin{tikzcd}
K(A,G) \wedge K(A,G) \arrow[d, "\otimes"'] & K(A,G) \wedge K(B,G) \arrow[l, "1 \wedge f^*"'] \arrow[r, "f_* \wedge 1"] & K(B,G) \wedge K(B,G) \arrow[d, "\otimes"] \\
K(A,G) \arrow[rr, "f_{\#}"] & & K(B,G)
\end{tikzcd}$$

The conclusion follows on passage to homotopy groups π^G_*.

Let A(G) denote the Burnside ring of G, namely the Grothendieck ring associated to the semi-ring of isomorphism classes of finite G-sets under disjoint union and Cartesian product. Via permutation representations, there is a ring homomorphism $\eta : A(G) \to R_A^{\oplus}(G)$, and $f^*\eta = \eta$ for a ring homomorphism $f:B \to A$. We have the following consequence of the last statement of Proposition 4.7.

Corollary 4.9. For commutative rings A and compact based G-spaces X, $[X,K(A,G)]_G$ is naturally a module over A(G).

Proof. For $\alpha \in A(G)$, $\eta(\alpha)$ may be viewed as a based G-map $S^0 \to K(A,G)$. For $\chi : X \to K(A,G)$, $\alpha\chi$ is defined to be the composite

$$X = S^0 \wedge X \xrightarrow{\eta(\alpha)\wedge\chi} K(A,G) \wedge K(A,G) \xrightarrow{\otimes} K(A,G).$$

The commutativity and compactness hypotheses are actually unnecessary, and we shall see in [12] that this is only a small part of the full algebraic structure on our functors that is implied by equivariant infinite loop space theory. In particular, that theory will imply that K(A,G) is a ring G-space up to all higher G-homotopies rather than merely up to weak G-homotopy.

§5. Equivariant K-theory and representation rings

We first prove that K(G) and KO(G) represent topological equivariant K-theory of G-bundles over compact G-spaces and then construct natural ring homomorphisms

$$\alpha : R(\Pi) \to K_G B(G,\Pi) \quad \text{and} \quad \alpha : RO(\Pi) \to KO_G B(G,\Pi),$$

where $R(\Pi)$ and $RO(\Pi)$ are the complex and real representation rings of Π. Throughout this section, Π and G are to be compact Lie groups. We shall work in the complex case, but all results and proofs apply equally well in the real case.

Consider $\mathcal{U}(G)$. Write $U(\mathbf{C},G) = \sum_{i\geq 1} V_i^{\infty}$ and let $W_t = \sum_{i=1}^{t} V_i^t$ and $Z_t = (\sum_{i=1}^{t} V_i) \oplus V_{t+1}^{t+1}$. Then $\{W_t \mid t > 0\}$ is a cofinal sequence in $B\mathcal{U}(G)$ with $Z_t \oplus W_t = W_{t+1}$. Let $\bar{B}\mathcal{U}(G)$ be the telescope of the sequence of translations $Z_t : B\mathcal{U}(G)_t \to B\mathcal{U}(G)_{t+1}$ of copies of $B\mathcal{U}(G) = \coprod_{n\geq 0} B\mathcal{U}(n,G)$ and let $\bar{\zeta} : \bar{B}\mathcal{U}(G) \to K(G)$

be the resulting factorization of the group completion $\zeta : B\mathcal{U}(G) \to K(G)$. By Remarks 2.8, $B\mathcal{U}(G)$ classifies (complex) G-vector bundles.

Proposition 5.1. The map $\bar{\zeta} : \bar{B}\mathcal{U}(G) \to K(G)$ is a G-homotopy equivalence, and these spaces represent the functor K_G on compact G-spaces X. The same conclusions hold in the real case.

Proof. $K_G(X)$ is the Grothendieck group associated to equivalence classes of G-vector bundles over X under the addition given by Whitney sum. Define a function

$$\Phi : [X_+, \bar{B}\mathcal{U}(G)]_G \to K_G(X)$$

as follows. By compactness, a G-map $f : X \to \bar{B}\mathcal{U}(G)$ factors through some $B\mathcal{U}(G)_t$, say via f_t. Let f_t classify the bundle ξ_t and define $\Phi[f] = \xi_t - \underline{W}_t$. Here, for a unitary representation V, $\underline{V}$ denotes the trivial G-bundle $X \times V \to X$. Clearly Φ is a well-defined injective function. For a G-bundle η over X, the Peter-Weyl theorem implies the existence of a complementary G-bundle $\eta^\perp$ and a representation V such that $\eta \oplus \eta^\perp$ is equivalent to $\underline{V}$; see Segal [30, 2.4]. Adding on a further trivial bundle, we may take $V = W_t$ for some t. Then a difference $\xi - \eta$ in $K_G(X)$ is equal to $\xi + \eta^\perp - \underline{W}_t$. Therefore Φ is a surjection. As the representing space for an Abelian group valued functor on compact G-spaces, $\bar{B}\mathcal{U}(G)$ is a grouplike weak Hopf G-space. Therefore its fixed point subspaces are grouplike weak Hopf spaces and thus simple spaces. Since each $\bar{\zeta}^H$ is a homology isomorphism, each $\bar{\zeta}^H$ is an equivalence and $\bar{\zeta}$ is a G-equivalence.

It is now clear from the discussion at the start of the previous section that $\bar{\zeta}$ is a weak Hopf G-map and thus $\bar{B}\mathcal{U}(G)$ inherits an actual Hopf G-space structure from K(G). Note that $\zeta : B\mathcal{U}(G) \to K(G)$ represents the natural map from G-bundles to virtual G-bundles. We define $K_G(X) = [X_+, K(G)]_G$ for not necessarily compact G-spaces X.

From our present bundle theoretical point of view, it is the existence of complementary bundles that distinguishes topological from algebraic K-theory. It is the absence of complements in the relevant bundle theories with discrete fibres that causes the need for the plus construction and group completion.

By use of Lemma 1.4 and Remarks 2.8, we see that the tensor products $U(m) \times U(n) \to U(mn)$ give rise to a map $K(G) \wedge K(G) \to K(G)$ which represents the tensor product operation on G-vector bundles over compact G-spaces and which gives $K(G)$ a structure of weak Hopf ring G-space. In fact, $K(G)$ is an actual Hopf ring G-space, with no $\lim^1$ ambiguities, and thus $K_G(X)$ is a commutative ring for any G-space X by virtue of Proposition 5.1 and the following result.

Proposition 5.2. $K_G B(G,\Pi)$ contains no $\lim^1$ term. That is, if $B(G,\Pi)$ is the union of an expanding sequence of compact G-spaces B_n, then $K_G B(G,\Pi) = \lim K_G B_n$.

This is not hard to prove directly when G and Π are finite. When G and Π are compact Lie groups, Haeberle and Hauschild have proven the result as a byproduct of a suitable generalization of the Atiyah-Segal completion theorem [4]. The generalization gives that $K_G B(G,\Pi)$ is naturally isomorphic to the completion of $R(G \times \Pi)$ with respect to the topology generated by the kernels of the restriction homomorphisms $R(G \times \Pi) \to R(H)$ for $H \subset G$.

We shall construct Adams operations ψ^r on $K(G)$ and thus on $K_G(X)$ for any G-space X in the course of the following proof.

Proposition 5.3. There is a ring homomorphism $\alpha: R(\Pi) \to K_G B(G,\Pi)$ which is natural in Π and commutes with the Adams operations ψ^r. The same conclusions hold in the real case.

Proof. For a representation $\tau: \Pi \to U(n)$, define $\alpha(\tau)$ to be the G-homotopy class of the composite

$$B(G,\Pi) \xrightarrow{B\tau_*} B(G,U(n)) \simeq B\mathcal{U}(n,G) \subset B\mathcal{U}(G) \xrightarrow{\zeta} K(G)$$

obtained by use of Lemma 2.4 and Remarks 2.8. Trivial diagram chases show that $\alpha(\sigma + \tau) = \alpha(\sigma) + \alpha(\tau)$, hence α extends over $R(G)$. Naturality is clear. Consider the identity homomorphism of $U(n)$ as an element $i_n \in R(U(n))$. By the very construction of $\otimes: K(G) \wedge K(G) \to K(G)$, we certainly have

$\alpha(i_m \otimes i_n) = \alpha(i_m) \otimes \alpha(i_n)$. It follows formally that $\alpha(\sigma \otimes \tau) = \alpha(\sigma) \otimes \alpha(\tau)$ for representations σ and τ. Therefore by bilinearity, α is a ring homomorphism. By Adams [1,§4], $\psi^r:R(\Pi) \to R(\Pi)$ is a natural ring homomorphism. The maps $\alpha\psi^r(i_n):B(G,U(n)) \to K(G)$ therefore determine a map $\psi^r:B\,\mathcal{U}(G) \to K(G)$ of Hopf ring G-spaces. By Propositions 4.2 and 4.3, ψ^r extends to a map of Hopf ring G-spaces $\psi^r:K(G) \to K(G)$, and $\psi^r\alpha(i_n) = \alpha\psi^r(i_n)$ by construction. Therefore $\psi^r\alpha = \alpha\psi^r$ on representations and thus, by linearity, on $R(\Pi)$.

We need to understand the behavior of α on passage to fixed points. Thus, for $H \subset G$ and a G-space X, let

$$\nu^H:K_G(X) = [X_+,K(G)]_G \to [X_+^H,K(G)^H]$$

denote the restriction homomorphism. By Propositions 3.5 and 3.6 and Remarks 2.8,

$$K(G)^H \simeq \underset{i}{\times} K \text{ and } KO(G)^H \simeq (\underset{i}{\times} KO) \times (\underset{j}{\times} K) \times (\underset{k}{\times} KSp)$$

are the weak products indexed on the appropriate irreducible representations of H. Note that, for a space X and weak product $\underset{i}{\times} Y_i$ of grouplike Hopf spaces, there is an inclusion

$$\sum_i [X,Y_i] \subset [X, \underset{i}{\times} Y_i]$$

which is an isomorphism if X is compact or the product is finite. By Proposition 0.1, we have

$$B(G,\Pi)_+^H = \bigvee_{\rho \in R^+(H,\Pi)} B\Pi_+^\rho ,$$

and of course $[\bigvee_k X_k,Y] = \underset{k}{\times} [X_k,Y]$ for any spaces X_k and Y.

<u>Definition 5.4.</u> Let $\rho:H \to \Pi$ be a morphism of compact Lie groups.

(i) Define a homomorphism

$$\nu^\rho:R(\Pi) \to R(H) \otimes R(\Pi^\rho) = \sum_i R(\Pi^\rho)$$

as follows, where $\{V_i\}$ is the set of irreducible unitary representations of H. Regarding a representation $\tau:\Pi \to U(n)$ as a $\mathbf{C}[H]$-module V by pullback along ρ, we may write $V = \sum_i \mathbf{C}^{n_i} \otimes_{\mathbf{C}} V_i$ and define $\tau_i:\Pi^\rho \to U(n_i)$ by commutativity of the following diagrams for $x \in \Pi^\rho$:

$$\begin{array}{ccc} \sum_i \mathbf{C}^{n_i} \otimes_{\mathbf{C}} V_i & = & V \\ {\scriptstyle \sum_i \tau_i(x) \otimes 1} \downarrow & & \downarrow {\scriptstyle \tau(x)} \\ \sum_i \mathbf{C}^{n_i} \otimes_{\mathbf{C}} V_i & = & V \end{array}$$

That is, τ_i is the composite of $\tau : \Pi^\rho \to U(n)^{\tau\rho}$ and the projection $U(n)^{\tau\rho} \to U(n_i)$. Define $\nu^\rho(\tau) = \sum_i \tau_i$. Additivity is easily checked, hence ν^ρ extends over $R(\Pi)$. Visibly, if $\dim_{\mathbf{C}} V_i = d_i$, we have the character formula

$$\chi_\tau(x) = \sum_i d_i \chi_{\tau_i}(x) \qquad \text{for} \quad x \in \Pi^\rho .$$

(ii) Define a homomorphism

$$\nu^\rho : RO(\Pi) \to [R(H;\mathbf{R}) \otimes RO(\Pi^\rho)] \oplus [R(H;\mathbf{C}) \otimes R(\Pi^\rho)] \oplus [R(H;\mathbf{H}) \otimes RSp(\Pi^\rho)]$$

similarly, where $R(H;\mathbf{F})$ is the free Abelian group generated by those irreducible real representations of H with associated division ring $\mathbf{F}$. (Compare the proof of Proposition 3.6.)

<u>Proposition 5.5</u>. The following diagram commutes, where $R(H) \otimes K(B\Pi^\rho)$ is identified with $\sum_i [B\Pi_+^\rho, K]$ and the products run over $\rho \in R^+(H,\Pi)$:

$$\begin{array}{ccccc} R(\Pi) & & \xrightarrow{(\nu^\rho)} & & \times_\rho R(H) \otimes R(\Pi^\rho) \\ {\scriptstyle \alpha} \downarrow & & & & \downarrow {\scriptstyle \times_\rho 1 \otimes \alpha} \\ K_G B(G,\Pi) & \xrightarrow{\nu^H} & \times_\rho [B\Pi_+^\rho, \times_i K] & \supset & \times_\rho R(H) \otimes K(B\Pi^\rho) \end{array}$$

With the evident modifications, the analogous diagram commutes in the real case.

Proof. On representations, this is immediate by inspection of the definitions. The conclusion follows by additivity. Of course, the real analog requires use of the evident homomorphism

$$\alpha : RSp(\Pi^\rho) \to KSp(\Pi^\rho).$$

We also need the following commutation relation between Adams operations and the ν^ρ.

Propositon 5.6. Let H be finite and let r be prime to the order of H. Then the following diagram commutes for each homomorphism $\rho : H \to \Pi$:

$$\begin{array}{ccc} R(\Pi) & \xrightarrow{\nu^\rho} & R(H) \otimes R(\Pi^\rho) \\ \Big\downarrow{\scriptstyle \psi^r} & & \Big\downarrow{\scriptstyle \psi^r \otimes \psi^r} \\ R(\Pi) & \xrightarrow{\nu^\rho} & R(H) \otimes R(\Pi^\rho) \end{array}$$

The same conclusion holds in the real case except that, on the complex part $R(H;\mathbf{C}) \otimes R(\Pi^\rho)$ of the target of ν^ρ, $\psi^r \otimes \psi^r$ must be replaced by the homomorphism "$\psi^r \otimes \psi^{\pm r}$" which sends the j^{th} summand $R(\Pi^\rho)$ to the $\psi^r(j)^{th}$ summand via ψ^r if $\psi^r(V_j) \in S_0$ and via ψ^{-r} if $\psi^r(V_j) \in S_0^*$.

Proof. By the character formula $\chi_{\psi^r(\tau)}(x) = \chi_\tau(x^r)$ and the standard irreducibility criterion [31,p.29], ψ^r acts as a permutation on the set of irreducible representations of H. Choosing a representative V in each orbit and taking iterated translates of these V under ψ^r, we can arrange that ψ^r acts as a permutation on $\{V_i\}$, say $\psi^r(V_i) = V_{\psi^r(i)}$. For a homomorphism $\tau : \Pi \to U(n)$ with $\tau\rho = \sum V_i^{n_i}$, we have

$$\psi^r(\tau) \circ \rho = \psi^r(\tau\rho) = \sum_i V_{\psi^r(i)}^{n_i} .$$

We also have the character formula

$$\sum_i d_i \, \chi_{\psi^r(\tau_i)}(x) = \sum_i d_i \chi_{\tau_i}(x^r) = \chi_\tau(x^r) = \chi_{\psi^r(\tau)}(x) = \sum d_i \chi_{\psi^r(\tau)_i}(x)$$

for $x \in \Pi^{\rho}$, and of course $d_{\psi^r(i)} = d_i$. It follows that $\psi^r(\tau)_{\psi^r(i)} = \psi^r(\tau_i)$. The argument in the real case is similar, except that the isomorphism $rt \cong r$ intervenes when $V_{\psi^r(j)} \in S_0^*$. The point is that the identification of $\tau\rho = V$ with a linear combination assigns a privileged role to one V_j in each of the pairs $\{V_j, tV_j\}$; changing the choice changes the corresponding component of ν^{ρ} by composition with t, and $\psi^{-r} = \psi^r t$.

Corollary 5.7. Let G be finite and let r be prime to the order of G. Then the following diagram commutes for any subgroup H of G:

$$\begin{array}{ccc} K(G)^H & \simeq & \times_i K \\ {\scriptstyle (\psi^r)^H}\big\downarrow & & \big\downarrow{\scriptstyle (\times_i \psi^r)\omega} \\ K(G)^H & \simeq & \times_i K \end{array}$$

where ω permutes factors as ψ^r permutes the indexing representations of H. The same conclusion holds in the real case except that, on the complex part $\times_j K$, $(\times_j \psi^r)\omega$ must be replaced by the map "$(\times_j \psi^{\pm r})\omega$" which sends the j^{th} factor to the $\psi^r(j)^{th}$ factor via ψ^r if $\psi^r(V_j) \in S_0$ and via ψ^{-r} if $\psi^r(V_j) \in S_0^*$.

Proof. Applying Propositions 5.3, 5.5, and 5.6 with $\Pi = U(n)$, we conclude that, for $i_n \in R(U(n))$,

$$[\psi^r\alpha(i_n)]^H = [\alpha\psi^r(i_n)]^H = \times_{\rho} (1 \otimes \alpha)\nu^{\rho}\psi^r(i_n) = \times_{\rho} (\psi^r \otimes \psi^r\alpha)\nu^{\rho}(i_n).$$

If $\rho \in R^+(H,U(n))$ is $\sum V_i^{n_i}$, then the i^{th} component of $\nu^{\rho}(i_n)$ is the projection $U(n)^{\rho} \to U(n_i)$. Thinking of ρ as a sequence $\{n_i\}$ and collecting terms, we see that, for $\psi^r : B\mathcal{U}(G) \to K(G)$ as specified in the proof of Propositon 5.3,

$$(\psi^r)^H \simeq (\times_i \psi^r)\omega : \times_i B\mathcal{U} \to \times_i KU.$$

The conclusion follows on passage to group completions, the proof in the real case being identical.

§6. Brauer lifting and the proof of Theorem 0.3

We revert to our standing assumption that G is finite. Fix a prime q which does not divide the order of G, let k_r denote the field with $r = q^a$ elements, and let $k = \overline{k}_q$ denote the algebraic closure of k_q. To prove Theorem 0.3, we must construct G-maps

$$\beta: K(k,G) \to K(G) \quad \text{and} \quad \beta: KO(k,G) \to KO(G)$$

whose fixed point maps β^H are products of the Brauer lift maps introduced by Quillen.

The construction of β is based on use of the following result in composition with Proposition 5.3. Note that the Frobenius automorphisms $\phi^r: GL(n,k) \to GL(n,k)$ obtained by raising matrix entries to the r^{th} power induce ring automorphisms ϕ^r of the representation rings $R_k(\Pi)$ and $RO_k(\Pi)$ for any finite group Π.

<u>Proposition 6.1.</u> There is a natural ring homomorphism $\lambda: R_k(\Pi) \to R(\Pi)$ which satisfies $\lambda\phi^r = \psi^r\lambda$ for $r = q^a$. The same conclusions hold in the orthogonal case.

<u>Proof.</u> We sketch the argument. One first fixes an embedding $\mu: k^* \to \mathbf{C}^*$ of the roots of unity. For a representation $\tau: \Pi \to GL(n,k)$, one then defines

$$\chi_{\lambda(\tau)}(x) = \sum_i \mu(\alpha_i), \quad x \in \Pi,$$

where $\{\alpha_1, \ldots, \alpha_n\}$ are the eigenvalues of $\tau(x)$. The Brauer induction theorem implies that $\chi_{\lambda(\tau)}$ is the character of a complex representation $\lambda(\tau)$. Additivity, multiplicativity, and the compatibility between ϕ^r and ψ^r are checked by trivial character computations. The orthogonal case is obtained by restriction. For $q \neq 2$, Quillen [28, App] gave details, and he also showed that λ restricts to a homomorphism $RSp_k(\Pi) \to RSp(\Pi)$. Consider the case $q = 2$. Here [11, p.174] erroneously asserted that λ fails to carry orthogonal representations to orthogonal representations. To see that it does, let the exponent of Π be $2^a h$, where h is odd, choose $b \geq a$ such that $2^b - 1$ is divisible by h, and let $r = 2^b$. By Quillen [22, 5.4.2] and the explicit choice for μ given below, ψ^r is an idempotent operator on $R(\Pi)$ such that $\psi^r\lambda = \lambda$. By Adams [3, 3.6.4], ψ^r carries both orthogonal and symplectic representations to orthogonal representations. In view of Lemma 3.3, it suffices to show that if V is a k[G]-module with a symmetric G-form b, then $\lambda(V)$ is self-dual (and thus a sum of orthogonal and symplectic representations). By Adams [3,3.32], this will hold if $\chi_{\lambda(V)}(x) = \chi_{\lambda(V)}(x^{-1})$ for all $x \in \Pi$. Regarding V as a homomorphism $\rho:\Pi \to GL(n,k)$, we need only show that the eigenvalues of the matrices $\rho(x)$ and $\rho(x^{-1})$ are the same. But this is clear since $\rho(x^{-1})$ is adjoint to $\rho(x)$ with respect to the given form b.

When q does not divide the order of Π, λ is an isomorphism. It is crucial to our work that λ takes honest representations to honest representations in this case. To see this, it is convenient to choose μ so as to arrange precise compatibility between Brauer lifting and the decomposition homomorphism. The same point arose in work of May and Tornehave [23, p.222]; and we recall their solution.

Let P_0 be the ideal (q) in the ring $R_0 = Z$ and let $A_0 = Z_{(q)}$. Inductively, given P_{j-1}, R_{j-1}, and A_{j-1}, let A_j be the localization of the ring of cyclotomic integers

$$R_j = Z[\exp(2\pi i/(q^j-1)(q^{j-1} - 1)\cdots(q-1))]$$

at a chosen prime ideal P_j which contains $P_{j-1} \subset R_{j-1}$. Let $K_j \subset \mathbf{C}$ be the field of fractions of A_j and let $\iota:A_j \to \mathbf{C}$ be the inclusion. The quotient of A_j by its maximal ideal is a field $k_{r(j)}$ of characteristic q and $k = \varinjlim k_{r(j)}$. Let $\pi:A_j \to k_{r(j)}$ be the quotient map. Clearly A_j contains a group $\tilde{\nu}_j$ of $(q^j-1)\cdots(q-1)^{th}$ roots of unity which π maps isomorphically onto the corresponding subgroup ν_j of $k^*_{r(j)}$, these isomorphisms being compatible as j varies. We specify $\mu:k^* \to \mathbf{C}^*$ by letting its restriction to ν_j be $\iota \circ \pi^{-1}$.

For j sufficiently large that K_j contains the $m(\Pi)^{th}$ roots of unity, where $m(\Pi)$ is the least common multiple of the orders of the elements of Π, it is now obvious from Serre [31, 18.4] that

$$\lambda:R_{k_{r(j)}}(\Pi) \cong R_k(\Pi) \to R(\Pi)$$

coincides with

$$\sigma:R_{k_{r(j)}}(\Pi) \to R_{K_j}(\Pi) \cong R(\Pi),$$

where σ is the canonical section of the decomposition homomorphism and the isomorphisms are given by extension of scalars.

Proposition 6.2. If q does not divide the order of Π, then $\lambda:R_k(\Pi) \to R(\Pi)$ is an isomorphism and restricts to an isomorphism $R_k^+\Pi \to R^+(\Pi)$ of semi-rings of honest representations. The same conclusions hold in the orthogonal case, where the three types of irreducible representations (or two types if q = 2) are each mapped bijectively onto the corresponding type.

Proof. The first statement follows from [31, 15.5]. For the last statement, we agree to choose the respective sets $S_0 \subset S_0 \amalg S_0^*$ so that $\lambda S_0 = S_0$ and we note that there are no irreducible representations of type $S_\pm$ over an algebraically closed field. For the case q = 2, we note that all self-dual complex representations of an odd order group are orthogonal since, with r as in the previous proof, $\psi^r\chi = \chi$ for all complex characters χ because all χ are in the image of λ.

We also need the following analog of Definition 5.4.

Definition 6.3. Let $\rho:H \to \Pi$ be a homomorphism of finite groups.

(i) Define a homomorphism

$$\nu^\rho:R_k(\Pi) \to R_k(H) \otimes R_k(\Pi^\rho) = \sum_i R_k(\Pi^\rho)$$

as follows, where $\{V_i\}$ is the set of irreducible k-representations of H. For a representation $\tau:\Pi \to GL(n,k)$, set $\nu^\rho(\tau) = \sum_i \tau_i$, where τ_i is the composite of $\tau:\Pi^\rho \to GL(n,k)^{\tau\rho}$ and the projection $GL(n,k)^{\tau\rho} \to GL(n_i,k)$. As in Definition 5.4, we regard $\tau\rho$ as $\sum_i k^{n_i} \otimes_k V_i$ to specify the projections, and we have the following trace formula, where $d_i = \dim_k V_i$:

$$\mathrm{tr}_\tau(x) = \sum_i d_i\, \mathrm{tr}_{\tau_i}(x) \quad \text{for} \quad x \in \Pi^\rho.$$

(ii) Define a homomorphism

$$\nu^\rho:RO_k(\Pi) \to [R(H;S_+) \otimes RO_k(\Pi^\rho)] \oplus [R(H;S_0) \otimes R_k(\Pi^\rho)] \oplus [R(H;S_-) \otimes RSp_k(\Pi^\rho)]$$

similarly, where R(H;T) denotes the free Abelian group generated by those irreducible orthogonal k-representations of H indexed on the set T. (Compare the proof of Proposition 3.4.)

We have the following analog of Proposition 5.5.

Proposition 6.4. If q does not divide the order of H, then the following diagram commutes, where the products run over $\rho \in R^+(H,\Pi)$:

$$\begin{array}{ccc} R_k(\Pi) & \xrightarrow{(\nu^\rho)} & \times_\rho R_k(H) \otimes R_k(\Pi^\rho) \\ \lambda \downarrow & & \downarrow \lambda\otimes\lambda \\ R(\Pi) & \xrightarrow{(\nu^\rho)} & \times_\rho R(H) \otimes R(\Pi^\rho) \end{array}$$

With the evident modifications, the analogous diagram commutes in the orthogonal case.

Proof. We have that $\lambda : R_k(H) \to R(H)$ is the isomorphism induced by a bijection of irreducibles, and of course λ preserves dimension. By Definitions 5.4 and 6.3, we thus have character formulas of the following form for each $\rho : H \to \Pi$:

$$\sum_i d_i \chi_{\lambda(\tau_i)}(x) = \chi_{\lambda(\tau)}(x) = \sum_i d_i \chi_{\lambda(\tau_i)}(x) \qquad \text{for } x \in \Pi^\rho.$$

The conclusion follows.

With these preliminaries, it is now an easy matter to construct the G-maps β and prove Theorem 0.3. Let

$$\iota_{n,r} : GL(n,k_r) \to GL(n,k) \ , \qquad r = q^a,$$

be the natural inclusion. Regarding $\iota_{n,r}$ as an element of $R_k(GL(n,k_r))$, we obtain an element

$$\beta_{n,r} = \alpha\lambda(\iota_{n,r}) \in K_G B(G,GL(n,k_r)),$$

that is, a G-map $\beta_{n,r} : B(G,GL(n,k_r)) \to K(G)$. By the additivity and multiplicity of α and λ, these maps for fixed r specify a map

$$\beta_r : B\mathscr{J}\mathscr{L}(G,k_r) \to K(G)$$

of Hopf ring G-spaces. Moreover, up to G-homotopy, β_r is the restriction of β_{r+1}.

By Propositions 4.2, 4.3, and 5.2, there result compatible maps

$$\beta_r : K(G,k_r) \to K(G)$$

of Hopf ring G-spaces. Since K(G,k) is clearly G-equivalent to the telescope of the $K(G,k_r)$, there results a Hopf ring G-map $\beta : K(G,k) \to K(G)$. Further, $\beta\phi^r$ is G-homotopic to $\psi^r\beta$. The same assertions hold in the orthogonal case.

Taking G = e, we have $\beta_{n,r} = \alpha\lambda(\iota_{n,r}) \in KBGL(n,k_r)$ and thus $\beta_r : B\mathscr{GL}(k_r) \to K$ and $\beta : K(k) \to K$. By Quillen [28], this map and, when $q \neq 2$, its orthogonal analog are homology isomorphisms away from q. By [11, III §7], the symplectic analog and the orthogonal analog with q = 2 are also homology isomorphisms away from q (Proposition 6.1 leading to a cleaner construction but no substantive change of argument in the latter case).

Consider $H \subset G$. We have $K(G,k)^H \simeq \times_i K(k)$ and $K(G)^H \simeq \times_i K$, where the products are indexed on the irreducible k-representations $\{V_i\}$ of H and their images under λ. We claim that $\beta^H \simeq \times_i \beta$, and similarly in the orthogonal case. Clearly the claim will complete the proof of Theorem 0.3. By Propositions 5.5 and 6.4, we have

$$\beta^H_{n,r} = \nu^H(\beta_{n,r}) = \times_i (\lambda \otimes \alpha\lambda)\nu^\rho(\iota_{n,r}) ,$$

where ρ runs through $R^+(H,GL(n,k_r))$. When k_r contains the $m(H)^{th}$ roots of unity, $R_{k_r}(H) \cong R_k(H)$ and $V_i \cong k \times_{k_r} W_i$ for irreducible k_r-representations W_i of H. If $\rho = \sum W_i^{n_i}$, then $\iota_{n,r}\rho = \sum V_i^{n_i}$ and thus the i^{th} component of $\nu^\rho(\iota_{n,r})$ is just the composite of $\iota_{n_i,r}$ and the projection $GL(n,k_r)^\rho \to GL(n_i,k_r)$. Thinking of ρ as a sequence $\{n_i\}$ and collecting terms, we see that this implies

$$\beta^H_r \simeq \times_i \beta_r : \times_i B\mathscr{GL}(k_r) \to \times_i K,$$

and our claim follows from the uniqueness clause of the nonequivariant version of Proposition 4.2.

§7. The equivariant Adams conjecture

A G-map $\xi:E \to X$ is a G-fibration if it satisfies the G-CHP. It is a (linear) spherical G-fibration if each fibre $\xi^{-1}(x)$ has the G_x-homotopy type of the 1-point compactification of a real representation of G_x, where G_x is the isotropy subgroup of x. (We require the basepoints of fibres to specify a fibrewise cofibration $X \to E$ [22, 5.2]). There is a fibrewise smash product between spherical G-fibrations (as in [22, 5.6]). In particular, $\xi \wedge \underline{V} = \Sigma^V \xi$ is the fibrewise suspension of ξ by V, where V is a real representation of G and $\underline{V}$ is the trivial G-fibration $X \times S^V \to X$. Two spherical G-fibrations ξ and ξ' are stably equivalent if $\Sigma^V \xi$ and $\Sigma^V \xi'$ are fibre G-homotopy equivalent for some V.

For a compact G-space X, define $Sph_G(X)$ to be the Grothendieck group of fibre G-homotopy equivalence classes of spherical G-fibrations over X. The elements of $Sph_G(X)$ are formal differences of stable equivalence classes of spherical G-fibrations. Via fibrewise 1-point compactification, real G-vector bundles over X give rise to spherical G-fibrations, and this procedure converts Whitney sums to fibrewise smash products. The resulting natural homomorphism

$$J_G:KO_G(X) \to Sph_G(X)$$

is the real equivariant J-homomorphism. $Sph_G(X)$ is represented by a Hopf G-space Sph(G) and J_G is induced by a Hopf G-map $j:K(G) \to Sph(G)$; see Waner [34] and also [12], where it will be shown that j is actually an infinite loop G-map.

Theorem 0.4 asserts that, for any $\xi \in KO_G(X)$, there exists $e > 0$ such that $k^e s J_G(\psi^k \xi - \xi) = 0$, where k is prime to the order of G and s is minimal such that $k^s \equiv \pm 1$ modulo the order of G.

A necessary condition for $J_G(\eta - \xi) = 0$ is that the isotropy group representations of the fibres of η and ξ be stably equivalent. The point of multiplying by s is that this suffices to arrange this necessary condition; compare [10, 9.7] and [13, §2]. It is easy to see that $k^e \psi^k V$ and $k^e V$ can be stably inequivalent for all e. For example, this happens when $G = Z_5$, $k = 3$, and V is the canonical 2-dimensional representation. Here there is a degree k G-map $V \to \psi^3 V$, so this example also shows that one obvious generalization of Adams' Dold theorem mod k fails. However, because there can be other linear combinations $\sum s_i(\psi^{k_i}\xi_i - \xi_i)$ with stably equivalent fibre representations, our version of the equivariant Adams conjecture fails to detect all of the kernel of J_G. See McClure [25] for further discussion.

Theorem 0.4 was proven for one and two dimensional G-vector bundles by Hauschild and Waner [13], their procedure being to formulate and prove the appropriate equivariant Dold theorem mod k and apply the arguments of Adams [2]. It follows that if $f:Y \to X$ is a finite G-cover and $\tau:KO_G(Y) \to KO_G(X)$ is the associated transfer, then Theorem 0.4 holds for $\tau(\xi)$ when ξ is one or two dimensional. The implication requires the relation $\psi^k\tau(\xi) = \tau\psi^k(\xi)$ in $KO_G(X)[1/k]$, which is proven in [13, §5] and amounts to the verification that, away from k, ψ^k is an infinite loop G-map. For the rest, one can either use Quillen's argument [28, 2.3] to deduce the implication from the explicit proofs in [13] or one can use the general relation $J_G\tau = \tau J_G$, which can either be verified directly [25] or deduced from the fact that j is an infinite loop G-map.

Next, consider a finite group Π and a principal (G,Π)-bundle $E \to X = E/\Pi$. For a real representation V of Π, $E \times_\Pi V \to X$ is a G-vector bundle over X. This construction is additive in V and induces a homomorphism $RO(\Pi) \to KO_G(X)$. The elements in the image of any such homomorphism are the elements of $KO_G(X)$ with a reduction of their structural group to Π. As in Quillen [28,§2], Theorem 0.4 holds for all such elements. To see this, let Λ be a subgroup of Π, let $Y = E/\Lambda$, and let $f:Y \to X$ be the associated G-cover. If V is induced from a representation W of V, then

$$\tau(E \times_\Lambda W) = E \times_\Pi V,$$

as is easily verified from the usual fibrewise description

$$\tau(E \times_\Lambda W)_x = \bigotimes_{f(y) = x} (E \times_\Lambda W)_y \;.$$

Since every element of $RO(\Pi)$ is an integral linear combination of representations induced from one and two dimensional representations of subgroups [28, 2.4], this implies the assertion.

Thus it remains to reduce the general case of Theorem 0.4 to the finite structural group case. Again, Quillen's arguments in [28, §1] generalize to the equivariant setting. Since $\psi^j\psi^k = \psi^{jk}$, we may as well assume that k is a prime, say $k = q$. (Since we are in possession of the orthogonal case of Theorem 0.3 for $q = 2$ as well as for $q > 2$, we need not follow Quillen in handling the real case for $q = 2$ by reducing it to the complex case.)

The Brauer lift map β of Theorem 0.3 clearly factors through a G-map

$$\beta:\overline{B}\mathcal{O}(k,G) \to \overline{B}\mathcal{O}(G) \simeq KO(G)$$

such that each β^H is a mod n homology isomorphism for n prime to q. This more elementary version of Theorem 0.3 is appropriate here since passage to group completion would obscure the relationship to bundle theory. Since $\overline{B}\mathcal{O}(k,G)$ is the telescope of copies of $B\mathcal{O}(k,G)$ under translations and $B\mathcal{O}(k,G)$ is G-equivalent to the telescope of the $B\mathcal{O}(k_r,G)$, an element of $KO_G(X)$ admits a reduction of its structural group to a finite group if its classifying G-map $X \to KO(G)$ lifts to $\overline{B}\mathcal{O}(k,G)$.

We need some general facts about "G-connected covers"; see [8] for proofs. Let Y be a based G-space. Then there is a G-connected G-space Y_0 and a G-map $\gamma:Y_0 \to Y$ such that $\gamma_*:\pi_q(Y_0^H) \to \pi_q(Y^H)$ is an isomorphism for all $H \subset G$ and $q > 0$. Thus, up to homotopy, Y_0^H is the basepoint component of Y^H. Y_0 is a functor of Y and γ is natural. If Y is a Hopf G-space, then Y_0 is a Hopf G-space and γ is a Hopf G-map. For a G-connected G-space X,

$$\gamma_*:[X,Y_0]_G \to [X,Y]_G$$

is a bijection.

For a G-connected Hopf G-space Z, let $Z[1/q]$ be the telescope of a sequence of copies of Z under the q^{th} power map $Z \to Z$ and let $i:Z \to Z[1/q]$ be the inclusion of the first copy of Z. Clearly $Z[1/q]^H$ is a localization of Z^H away from q.

Now consider the sequence of maps of G-connected G-spaces

$$\overline{B}\,\mathcal{O}(k,G)_0 \xrightarrow{\beta_0} KO(G)_0 \xrightarrow{s(\psi^q-1)_0} KO(G)_0 \xrightarrow{j_0} Sph(G)_0 \xrightarrow{i} Sph(G)_0[1/q] ,$$

where s is minimal such that $q^s \equiv \pm 1$ modulo the order of G. The composite is null G-homotopic since it is so when precomposed with $f:X \to \overline{B}\,\mathcal{O}(k,G)_0$ for any G-map f defined on a compact G-connected G-space X and since there are no $\lim^1$ terms here. The homotopy groups of the fixed point spaces $Sph(G)_0^H$ are all finite (see [12]), and it follows from Bredon's equivariant obstruction theory [6] and the absence of $\lim^1$ terms that

$$\beta_0^*:[KO(G),\ Sph(G)_0[1/q]]_G \to [\overline{B}\,\mathcal{O}(k,G)_0\ Sph(G)_0[1/q]]_G$$

is a bijection. Therefore $ij_0s(\psi^q - 1)_0$ is also null G-homotopic and Theorem 0.4 follows.

§8. The equivariant algebraic K-theory of finite fields

We revert to the notation of section 6; k_r denotes the field with $r = q^a$ elements, where q is a prime not dividing the order of G, and k denotes the algebraic closure of k_r. Let ι denote both the extension of scalars functors induced by the inclusion of k_r in k and the maps of K-theory spaces induced by these functors (as in section 4). Similarly, let ϕ^r denote both the permutative functors induced by the Frobenius automorphism and the maps they induce on K-theory spaces. We are heading towards the proof of Theorem 0.5, but we shall first prove the following analog.

Theorem 8.1. The following are G-fibration sequences:

$$K(k_r,G) \xrightarrow{\iota} K(k,G) \xrightarrow{\phi^r-1} K(k,G)$$

and

$$KO(k_r,G) \xrightarrow{\iota} KO(k,G) \xrightarrow{\phi^r-1} KO(k,G).$$

That is, $K(k_r,G)$ is G-equivalent to the homotopy theoretical fibre $F\phi^r(G)$ of the G-map $\phi^r - 1 = \mu(\phi^r,\chi)\Delta$, where μ and χ are the standard loop product and inverse map on $K(k,G) = \Omega BB\mathscr{SL}(k,G)$ and Δ is the diagonal map. Since $\phi^r\iota = \iota$ on the category level, the maps ϕ^r and 1 restrict to precisely the same map $K(k_r,G) \to K(k,G)$. Therefore ι factors canonically as the composite of the natural map $F\phi^r(G) \to K(k,G)$ and a lifting $\gamma: K(k_r,G) \to F\phi^r(G)$. With this construction, γ is a Hopf G-map since ϕ^r and 1 are. We shall prove that γ is a G-equivalence by verifying that γ^H is an equivalence for each subgroup H of G. Note for this purpose that the construction of homotopy fibres (as the fibre over the basepoint of the associated mapping path fibration) commutes with passage to fixed points and with products. Of course, homotopy fibres depend only on the basepoint component of the base space.

Thus fix H. Since $\mathscr{SL}(F,G)^H = \mathscr{SL}(F,H)^H$ if char F is prime to the order of G, by Remark 2.7, we may quote the results of section 3 with G replaced by H. We shall first analyze $(\phi^r)^H$ and ι^H on the level of permutative categories. Let

$$S = S_0 \amalg S_0^* \amalg S_+ \amalg S_-$$

and

$$S^r = S_0^r \amalg (S_0^r)^* \amalg S_+^r \amalg S_-^r \amalg S_\pm^r$$

be sets of representatives for the irreducible representations of H over k and k_r, S_- and S_-^r being empty if q = 2. Propositions 3.1 and 3.4 give the following equivalences of permutative categories.

(1) $\mathscr{SL}(k,G)^H \approx \underset{V \in S}{\times} \mathscr{SL}(k)$

(2) $\mathscr{O}(k,G)^H \approx [\underset{V \in S_0}{\times} \mathscr{SL}(k)] \times [\underset{V \in S_+}{\times} \mathscr{O}(k)] \times [\underset{V \in S_-}{\times} \mathscr{Sp}(k)]$

(3) $\mathscr{SL}(k_r,G)^H \approx \underset{U \in S^r}{\times} \mathscr{SL}(k_{r^u})$

(4) $\mathscr{O}(k_r)^H \approx [\underset{U \in S_0^r}{\times} \mathscr{SL}(k_{r^u})] \times [\underset{U \in S_+^r}{\times} \mathscr{O}(k_{r^u})] \times [\underset{U \in S_-^r}{\times} \mathscr{Sp}(k_{r^u})] \times [\underset{U \in S_\pm^r}{\times} \mathscr{U}(k_{r^u})]$.

Here, in (3) and (4), we have used the fact that any finite division ring is a field to write $\mathrm{Hom}_{k_r[G]}(U,U) = k_{r^u}$ for some u depending on U.

Clearly ϕ^r acts as a permutation on the set of irreducible representations of H over k, hence we can arrange that ϕ^r acts as a permutation on the set S of representatives. We let [V] denote the orbit of V and let $\overline{V}$ denote the ϕ^r-invariant representation $\sum_{W \in [V]} W$. For a ϕ^r-invariant subset T of S, we let

$$T/\phi^r = \{\overline{V} \mid [V] \subset T\}.$$

The duality operator on representations commutes with the action of ϕ^r, and the subsets S_+, S_-, and $S_0 \amalg S_0^*$ of S are ϕ^r-invariant. Moreover, we can write

$$S_0 \amalg S_0^* = \overline{S}_0 \amalg \overline{S}_0^* \amalg S_\pm$$

as a ϕ^r-set, where

$$\overline{S}_0 = \{V \mid V \in S_0 \text{ and } V^* \notin [V]\}$$

$$\overline{S}_0^* = \{V^* \mid V \in \overline{S}_0\} = \{V \mid V \in S_0^* \text{ and } V^* \notin [V]\}$$

and

$$S_\pm = \{V \mid V \in S_0 \text{ or } V \in S_0^* \text{ and } V^* \in [V]\}.$$

Let $t\colon \mathscr{GL}(k) \to \mathscr{GL}(k)$ be the functor which is the identity on objects and sends a nonsingular matrix to its transpose inverse. In analogy with topological K-theory, we agree to let $\phi^{-r} = \phi^r t = t\phi^r$. The following analysis of $(\phi^r)^H$ is precisely analogous to the analysis of $(\psi^r)^H$ in Corollary 5.7.

<u>Proposition 8.2.</u> The following diagram of functors commutes:

$$\begin{array}{ccc}
\mathscr{GL}(k,G)^H & \approx & \times_{V \in S} \mathscr{GL}(k) \\
\Big\downarrow (\phi^r)^H & & \Big\downarrow (\times_V \phi^r)\omega \\
\mathscr{GL}(k,G)^H & \approx & \times_{V \in S} \mathscr{GL}(k)
\end{array}$$

where ω permutes factors as ϕ^r permutes the indexing representations. The same conclusion holds in the orthogonal case except that, on the general linear part

$\underset{V \in S_0}{\times} \mathcal{BL}(k)$, $(\underset{V}{\times} \phi^r)\omega$ must be replaced by the functor "$(\underset{V}{\times} \phi^{\pm r})\omega$" which sends the V^{th} factor to the $\phi^r(V)^{th}$ factor via ϕ^r if $\phi^r(V) \in S_0$ and via ϕ^{-r} if $\phi^r(V) \in S_0^*$.

Proof. The equivalence carries a k-space k^n in the V^{th} factor $\mathcal{BL}(k)$ to $k^n \otimes V$ and carries a nonsingular matrix f to $f \otimes 1$, while ϕ^r carries $k^n \otimes V$ to $k^n \otimes \phi^r(V)$ and $f \otimes 1$ to $\phi^r(f) \otimes 1$. This is the same as first shifting to the $\phi^r(V)^{th}$ factor, next applying ϕ^r, and then applying the equivalence. The argument in the orthogonal case is the same except that, when $V \notin S_0$, the equivalence carries k^n to $(k^n \otimes V) \oplus (k^n \otimes V^*)$ with the hyperbolic G-form and carries f to $(f \otimes 1) \oplus ((f^{-1})^t \otimes 1)$. Here if $\phi^r(V) \in S_0^*$, then $\phi^r(V^*) = \phi^r(V)^* \in S_0$ and, after application of ϕ^r, the two summands appear in reverse order in the factor of $\mathcal{BL}(k)$ indexed on $\phi^r(V^*)$.

Clearly the product $(\underset{V}{\times} \phi^r)\omega$ breaks up into the product over $\overline{V} \in S/\phi^r$ of its restrictions $(\underset{W \in [V]}{\times} \phi^r)\omega$, and similarly in the orthogonal case.

To compute ι^H, we need the following general observation.

Lemma 8.3. Let L be a Galois extension of a field K with Galois group Π and let M be an L-space with basis $\{m_1,\ldots,m_n\}$. For $\sigma \in \Pi$, define a K-map $\sigma: M \to M$ by $\sigma(\sum y_j m_j) = \sum \sigma(y_j) m_j$ for $y_j \in L$. Then the composite

$$(*) \qquad L \otimes_K M \xrightarrow{\Sigma(1 \otimes \sigma)} \sum_{\sigma \in \Pi} L \otimes_K M \xrightarrow{\Sigma \mu} \sum_{\sigma \in \Pi} M$$

is an isomorphism of L-spaces, where μ is the action map. If $f \in GL(n,L)$ is regarded as an L-map $M \to M$ via $\{m_j\}$ and $\sigma: GL(n,L) \to GL(n,L)$ is the automorphism induced entrywise by σ, then the following diagram commutes.

$$\begin{tikzcd} L \otimes_K M \arrow[r, "1 \otimes \sigma"] \arrow[d, "(1 \otimes f)"'] & L \otimes_K M \arrow[r, "\mu"] & M \arrow[d, "\sigma f"] \\ L \otimes_K M \arrow[r, "1 \otimes \sigma"] & L \otimes_K M \arrow[r, "\mu"] & M \end{tikzcd}$$

In particular, if a finite group H acts on M via $\rho: H \to GL(n,L)$ and σM denotes M with action $\sigma\rho$, then (*) is an isomorphism of $L[H]$-modules

$$L \underset{K}{\times} M \longrightarrow \sum_{\sigma \in \Pi} \sigma M .$$

Proof. Clearly (*) will be an isomorphism in general if it is so when $M = L$ and here, by comparison of dimensions, it suffices to show that (*) is a monomorphism. Consider the trace $tr:L \to K$, $tr(y) = \sum \sigma y$. Give L the trace form $b(y,y') = tr(yy')$, give $L \otimes_K L$ the tensor product form, and give $\sum_\sigma L$ the sum form. A simple calculation shows that (*) is an isometry and thus a monomorphism.

Now consider the extension of scalars homomorphism $\iota:R_{k_r}(H) \to R_k(H)$. By [31, 14.6], ι is a split monomorphism. Moreover, its image is clearly contained in the subgroup $R_k(H)^{\phi^r}$ of representations invariant under ϕ^r, and S/ϕ^r is a basis for the latter. Since any irreducible representation V over k is a summand of $\iota(U)$ for some irreducible representation U over k_r, by [31,14.6] again,

$$\iota:R_{k_r}(H) \to R_k(H)^{\phi^r}$$

is an isomorphism. In view of the natural isomorphisms

$$k \otimes_{k_r} \mathrm{Hom}_{k_r}(U,k_r) \cong \mathrm{Hom}_{k_r}(U,k) \cong \mathrm{Hom}_k(k \otimes_{k_r} U,k),$$

ι commutes with the duality operators. The following result analyzes the indexing sets and the fields k_{r^u} in (3) and (4) in terms of the action of ϕ^r on the indexing sets in (1) and (2). In particular, it shows that ι restricts to an isomorphism

$$\iota:RO_{k_r}(H) \to RO_k(H)^{\phi^r}.$$

Lemma 8.4. S^r and S can be so chosen that extension of scalars specifies a bijection $S^r \to S/\phi^r$ which restricts to bijections

$$S^r_+ \to S_+/\phi^r \quad , \quad S^r_- \to S_-/\phi^r$$

$$S^r_0 \to \overline{S}_0/\phi^r \quad , \quad (S^r_0)^* \to \overline{S}^*_0/\phi^r$$

and

$$S^r_{\pm} \to S_{\pm}/\phi^r.$$

Moreover, if $U \in S^r$ with $\iota(U) = \overline{V}$ and if $\mathrm{Hom}_{k_r[H]}(U,U) = k_{r^u}$, then u is minimal such that $\phi^{r^u}(V) = V$ and is thus the number of elements in the orbit [V]. Further, if $U \in S^r_{\pm}$ and t is minimal such that $V^* = \phi^{r^t}(V)$, then $u = 2t$ and we may therefore choose $[V] \cap S_0$ to be $\{\phi^{r^i}(V) \mid 0 \leq i < t\}$.

Proof. Fix $U \in S^r$. To simplify notation, write $K = k_r$ and $L = k_{r^u}$. We may regard U as a representation over L. As such, it is absolutely irreducible since $\mathrm{Hom}_{L[H]}(U,U) \cong L$. Therefore $V = k \otimes_L U$ is irreducible over k. Applying the functor $k \otimes_L (?)$ to the isomorphism

$$L \otimes_K U \cong \sum_{i=0}^{u-1} \phi^{r^i} U$$

of Lemma 8.3, we obtain an isomorphism of k[H]-modules

$$k \otimes_K U \cong \sum_{i=0}^{u-1} \phi^{r^i}(V) = m\overline{V},$$

where m is the multiplicity of V in $\{\phi^{r^i}(V) \mid 0 \leq i < u\}$. By our observations about ι above, $m = 1$ and u is as stated. Regard U as $L \otimes_L U \subset V$. If V admits a G-form $\tilde{b}_0$, then nondegeneracy implies the existence of elements $u_0, u_0' \in U$ such that $\tilde{b}_0(u_0,u_0') \neq 0$. Choose a K-linear functional $f: k \to K$ such that $f\tilde{b}_0(u_0,u_0') \neq 0$ and define $b_0(u,u') = f\tilde{b}_0(u,u')$ for $u,u' \in U$. Then b_0 is a K-bilinear G-form on U. It is nondegenerate since it is not identically zero and U is irreducible. Any other G-form on U is specified by

$$b_0(du,u') = f\tilde{b}_0(du,u') = f(d\tilde{b}_0(u,u'))$$

for some $d \in L$. Visibly, if $\tilde{b}_0$ is symmetric or skew symmetric, then all G-forms on U are symmetric or skew symmetric. Now suppose that $V \in S_0$ and $V^* \in [V]$. Let t be minimal such that $V^* = \phi^{r^t}(V)$. Certainly $0 < t < u$. Since $\phi^{r^{2t}}(V) = V^{**} = V$, u divides 2t. This implies $u = 2t$. Let $W = \sum_{i=0}^{t-1} \phi^{r^i}(V)$. Then $\overline{V} = W \oplus W^*$ admits both the symmetric hyperbolic G-form and the non-symmetric G-form

$$b_x((w,\omega),\ (w',\omega')) = \omega(w') + x\,\omega'(w)$$

for any $x \in k$, $x \neq 0$ and $x \neq 1$. Regard U as $K \otimes_K U \subset \overline{V}$ (and observe that these forms are not L-bilinear on U). Choosing appropriate K-linear functionals $k \to K$, we can compress the restrictions to U of these G-forms on $\overline{V}$ to G-forms on U with the same symmetry properties. Thus $U \in S^r_{\pm}$. The rest should be clear.

<u>Proposition 8.5.</u> The following diagram of functors commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathscr{SL}(k_r,G)^H \arrow[r, phantom, "\approx"] \arrow[d, "\iota^H"'] & \displaystyle\times_{U\in S^r} \mathscr{SL}(k_{r^u}) \arrow[d, "\times_U(\times_i \phi^{r^i}\iota)"] \\
\mathscr{SL}(k,G)^H \arrow[r, phantom, "\approx"] & \displaystyle\times_{U\in S^r}\ \times_{i=0}^{u-1} \mathscr{SL}(k)
\end{tikzcd} .$$

The same conclusion holds in the orthogonal case except that, on the unitary factor $\mathscr{U}(k_{r^u})$ indexed on $U \in S^r_{\pm}$, the right arrow must be replaced by

$$\times_{i=0}^{t-1} \phi^{r^i}\iota : \mathscr{U}(k_{r^u}) \to \times_{i=0}^{t-1} \mathscr{SL}(k),$$

where ι is interpreted as neglect of form followed by extension of scalars.

<u>Proof.</u> We regard S as the union of its orbits [V] and so index it on $U \in S^r$ and $0 \le i < u$, (U,i) corresponding to $\phi^{r^i}(V)$ if $\iota(U) = \overline{V}$. Fix U and again write $K = k_r$ and $L = k_{r^u}$. The top equivalence carries the L-space L^n in the U^{th} factor $\mathscr{SL}(L)$ to $L^n \otimes_L U$ and carries $f \in GL(n,L)$ to $f \otimes 1$. The functor ι^H carries $L^n \otimes_L U$ to

$$k \otimes_K (L^n \otimes_L U) \cong (k \otimes_L L^n) \otimes_k (k \otimes_K U) \cong \sum_{i=0}^{u-1} (k \otimes_L L^n) \otimes_k \phi^{r^i}(V).$$

Here the first isomorphism is specified by

$$\sum_j x \otimes (e_j \otimes u) \to \sum_j (x \otimes e_j) \otimes (1 \otimes u)$$

in terms of the canonical basis $\{e_1,\dots,e_n\}$ of L^n. Note that this is highly non-invariant: we are using this basis to identify both sides with the sum of n copies of $k \otimes_K U$, and scalars from L in $L^n \otimes_L U$ must be pushed over to U before evaluating the isomorphism. The second isomorphism is derived from that displayed in the previous proof. Writing $f(e_j) = \sum f_{jk}e_k$, $f_{jk} \in L$, and calculating, we see that

$1 \otimes (f \otimes 1)$ corresponds under the composite isomorphism above to $\sum_i (1 \otimes \phi^{r^i} f) \otimes 1$. This implies the first statement. On the general linear, orthogonal, and symplectic factors of $\mathcal{O}(k_r,G)^H$, the orthogonal case is an obvious elaboration. Consider $U \in S^r_{\pm}$ with $u = 2t$. Then ϕ^{r^t} specifies the involution on L determined by a symmetric G-form on U. Up to isomorphism, the only Hermitian form on L^n is the standard one [14, Thm 2.8], namely

$$b\left(\sum_j y_j e_j, \sum_j y'_j e_j\right) = \sum_j y_j \phi^{r^t}(y'_j).$$

Thus if $f \in U(n,L)$, then $f^{-1} = \phi^{r^t}(f^t)$, where f^t is the transpose of f. Now $[V] \cap S_0 = \{\phi^{r^i}(V) \mid 0 \leq i < t\}$, and these are the elements of [V] which index copies of $\mathcal{GL}(k)$ in $\mathcal{O}(k,G)^H$. We have $\phi^{r^{i+t}}(V) = \phi^{r^i}(V)^*$ and $\phi^{r^{i+t}} f = (\phi^{r^i} f^{-1})^t$. Writing the right side of the isomorphism above as

$$\sum_{i=0}^{t-1} [(k \otimes_L L^n) \otimes_k \phi^{r^i}(V)] \oplus [(k \otimes_L L^n) \otimes_k \phi^{r^i}(V)^*],$$

we see that the last assertion follows from the specification of the $\phi^{r^i}(V)^{th}$ functor $\mathcal{GL}(k) \to \mathcal{O}(k,G)^H$ in the proof of Proposition 3.4.

Propositions 8.2 and 8.5 imply that the sequence

$$K(k_r,G)^H \xrightarrow{\iota^H} K(k,G)^H \xrightarrow{(\phi^r)^H-1} K(k,G)^H$$

breaks up into the product of sequences, one for each $U \in S^r$,

$$K(k_{r^u}) \xrightarrow{\times_i \phi^{r^i} \iota} \times_{i=0}^{u-1} K(k) \xrightarrow{[(\times_i \phi^r)\omega]-1} \times_{i=0}^{u-1} K(k) ,$$

where ω is the cyclic permutation of factors. A similar conclusion holds in the orthogonal case except that, for $U \in S^r_{\pm}$, the resulting sequence is of the form

$$KU(k_r) \xrightarrow{\times_i \phi^{r^i} \iota} \times_{i=0}^{t-1} K(k) \xrightarrow{[(\phi^{-r} \times \times_{i=0}^{t-1} \phi^r)\omega]-1} \times_{i=0}^{t-1} K(k) .$$

For $U \in S^r$, let $F\phi^r(U)$ denote the homotopy fibre of the right-hand map. Using the commutation of homotopy fibres with passage to fixed point sets and with products, we see that our original lifting $\gamma : K(k_r, G) \to F\phi^r(G)$ of ι restricts to a lift $\gamma : K(k_{r^u}) \to F\phi^r(U)$ of $\times_i \phi^{r^i}\iota$ for each U. We thus have the following commutative diagrams, where i_0 is the inclusion of the 0^{th} factor and the commutativity of the right-hand square is a trivial computation:

$$\begin{tikzcd}
K(k_{r^u}) \arrow[r, "\iota"] \arrow[d, "\gamma"] & K(k) \arrow[r, "\phi^{r^u} - 1"] \arrow[d, "\times_i \phi^{r^i}"] & K(k) \arrow[d, "i_0"] \\
F\phi^r(U) \arrow[r] & \times_{i=0}^{u-1} K(k) \arrow[r, "{[(\times_i \phi^r)\omega]-1}"] & \times_{i=0}^{u-1} K(k)
\end{tikzcd}$$

For $U \in S_0^r$, S_+^r, or S_-^r, we have a precisely analogous diagram with K replaced by K,KO, or KSp throughout. For $U \in S_{\pm}^r$, the analogous diagram has the bottom right map displayed above and the top row

$$KU(k_{r^u}) \xrightarrow{\iota} K(k) \xrightarrow{\phi^{-r^t} - 1} K(k) .$$

By Quillen [29] in the general linear case and Fiedorowicz and Priddy [11] in the orthogonal, symplectic, and unitary cases, the top rows are all fibration sequences. To prove Theorem 8.1, it suffices to show that the Hopf maps γ induce isomorphisms on π_0, which is clear, and on π_n of the basepoint components for $n > 0$. To show the latter, it suffices to check that the right-hand squares induce isomorphisms on kernels and cokernels of the maps of higher homotopy groups induced by the horizontal arrows. The required calculations are immediate from the following result, which determines the relevant maps of homotopy groups. The homotopy groups of K(k), KO(k), and KSp(k) are tabulated in [11, p.246], the non-trivial groups being

$$\pi_{2n-1}K(k) \cong \pi_{4n-1}KO(k) \cong \pi_{4n-1}KSp(k) \cong \sum_{p \neq q} Z_{p^\infty}$$

and, if $q \neq 2$,

$$\pi_{8n+1}KO(k) \cong \pi_{8n+2}KO(k) \cong \pi_{8n+5}KSp(k) \cong \pi_{8n+6}KSp(k) \cong Z_2.$$

(Note that, when $q = 2$, our space KO(k) has basepoint component equivalent to the space $\Gamma_0 B\,\mathfrak{O}p(k)$ of [11]; see [11,II.7.6].) Recall that

$$\sum_{p \neq q} Z_{p^\infty} \cong Z[1/q]/Z \cong Q/Z_{(q)} .$$

Lemma 8.6. The maps ϕ^r and ϕ^{-r} on $K(k)$ induce multiplication by r^n and $(-r)^n$ on $\pi_{2n-1}K(k)$. The maps ϕ^r on $KO(k)$ and $KSp(k)$ induce multiplication by r^{2n} on $\pi_{4n-1}KO(k)$ and $\pi_{4n-1}KSp(k)$ and induce the identity on the homotopy groups equal to Z_2.

Proof. By [23, VIII, 2.9] and [11, p.170-175], Brauer lift gives rise to the following commutative diagram of completions of basepoint components away from q, the map β being an equivalence:

$$\begin{tikzcd} \hat{K}(k)_0[1/q] \arrow[r, "\phi^r"] \arrow[d, "\beta"', "\simeq"] & \hat{K}(k)_0[1/q] \arrow[d, "\simeq"', "\beta"] \\ \hat{K}_0[1/q] \arrow[r, "\psi^r"] & \hat{K}_0[1/q] \end{tikzcd}$$

The same assertion holds with r replaced by -r and with K replaced by KO or KSp. The behavior of ψ^r on homotopy groups was computed by Adams [1, 5.2] (see also [23, V.2.9]). By Bousfield and Kan [6, p.153], we have a natural exact sequence

$$0 \to \mathrm{Ext}(\sum_{p\neq q} Z_{p^\infty}, \pi_n X) \to \pi_n \hat{X}[1/q] \to \mathrm{Hom}(\sum_{p\neq q} Z_{p^\infty}, \pi_{n-1}X) \to 0$$

for simple spaces X. For $X = KO(k)_0$ amd $X = KSp(k)_0$, the homotopy groups $\pi_n X = Z_2$ give Ext groups $Z_2 = \pi_n \hat{X}[1/q]$, and these groups are mapped isomorphically since their images under β are mapped isomorphically by ψ^r. In the remaining cases, the homotopy groups $\pi_{2n-1}X = \sum_{p\neq q} Z_{p^\infty}$ (n even for KO(k) or KSp(k)) give Hom groups $\hat{Z}[1/q] = \pi_{2n}\hat{X}[1/q]$, and ϕ^r and ϕ^{-r} map these groups by r^n or r^{-n} since ψ^r and ψ^{-r} so map their images under β. The conclusions follow.

We have now proven Theorem 8.1, and we turn to the proof of Theorem 0.5. Consider the following diagram:

$$\begin{tikzcd} F\phi^r(G) \arrow[r] \arrow[d, "\delta"] & K(G,k) \arrow[r, "\phi^r - 1"] \arrow[d, "\beta"] & K(G,k) \arrow[d, "\beta"] \\ F\psi^r(G) \arrow[r] & K(G) \arrow[r, "\psi^r - 1"] & K(G) \end{tikzcd}$$

As noted in section 6, the right square is G-homotopy commutative. Thus there exists a lift δ. In view of Theorem 8.1, it suffices to prove that δ is a G-equivalence to conclude the complex case of Theorem 0.5. For this, it suffices to show that each δ^H is an equivalence. In view of Corollary 5.7 and Proposition 6.2, δ^H restricts to a lift δ in the following diagram of fibrations for each $U \in S^r$:

$$\begin{tikzcd}
F\phi^r(U) \arrow[r] \arrow[d, "\delta"'] & \underset{i=0}{\overset{u-1}{\times}} K(k) \arrow[r, "{[(\times_i \phi^r)\omega]-1}"] \arrow[d, "\times_i \beta"] & \underset{i=0}{\overset{u-1}{\times}} K(k) \arrow[d, "\times_i \beta"] \\
F\psi^r(U) \arrow[r] & \underset{i=0}{\overset{u-1}{\times}} K \arrow[r, "{[(\times_i \psi^r)\omega]-1}"] & \underset{i=0}{\overset{u-1}{\times}} K
\end{tikzcd}$$

Because of the odd and even degrees in which the nontrivial homotopy groups occur on the right, we cannot conclude directly that δ is an equivalence. However, comparing top rows to top rows and bottom rows to bottom rows, we can convert the diagram to one of the form

$$\begin{tikzcd}
F\phi^{r^u} \simeq K(k_{r^u}) \arrow[r] \arrow[d, "\delta"'] & K(k) \arrow[r, "\phi^{r^u}-1"] \arrow[d, "\beta"] & K(k) \arrow[d, "\beta"] \\
F\psi^{r^u} \arrow[r] & K \arrow[r, "\psi^{r^u}-1"] & K
\end{tikzcd}$$

Here δ is an equivalence by Quillen's results [29].

The proof of the orthogonal case of Theorem 0.5 is exactly the same, modulo one highly non-trivial point. In the diagram just given, there is a unique lift δ. This remains true with r^u replaced by $-r^u$. However, this is not true with K replaced by KSp or KO and, in the latter case, δ will fail to be an equivalence if it is wrongly chosen. Fiedorowicz and Priddy [11] proved that δ is an equivalence if it is a Hopf map. May [23] proved that there is a lift which is a Hopf map by proving the existence of a lift which is an infinite loop map (by an argument involving pulling back Bott periodicty along the equivalence $\beta:\hat{KO}(k)_0[1/q] \to \hat{KO}_0[1/q]$ and analyzing the relationship between space and spectrum level periodicity). Presumably a more direct proof is possible. In our situation, we must prove that the original lift $\delta: FO\phi^r(G) \to FO\psi^r(G)$ can be chosen as a Hopf G-map in order to ensure that each δ^H and thus each $\delta: FO\phi^r(U) \to FO\psi^r(U)$, $U \in S_+^r$, is a Hopf map (passage to the last diagram above with K replaced by KO presenting no difficulty). Again, while a more direct proof should be possible, the only argument we know is the equivariant infinite loop space version of May's argument just cited; see [12].

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