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# EQUIVARIANT BORDISM OF MAPS

BY

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ABSTRACT. This note computes the bordism classification of equivariant maps between closed manifolds with action of a cyclic group of prime order.

1. **Introduction.** Being given a finite group $G$, one would like to classify equivariant maps $f: (N^n, \phi) \to (M^m, \psi)$ between closed manifolds with $G$ action. Roughly, this would combine the analysis of $G$ actions introduced by Conner and Floyd [1] and the analysis of maps in [2].

This paper is intended to illustrate that the Conner and Floyd fixed point methods may be used to classify equivariant maps up to bordism. Attention will be restricted to the case of $G$ cyclic of prime order, leaving open the generalization to bordism groups of equivariant maps needed for a full scale study. Notation will follow [3].

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2. **Definition of the $G$ bordism groups.** Let $G$ be a finite group. A family $\mathfrak{F}$ in $G$ is a collection of subgroups of $G$ so that $H \in \mathfrak{F}$, $g \in G$ and $K \subset H$ imply $gHg^{-1} \in \mathfrak{F}$ and $K \in \mathfrak{F}$. Let $\mathfrak{F}' \subset \mathfrak{F}$ be a pair of families. An $(\mathfrak{F}, \mathfrak{F}')$-free map of dimension $(m, n)$ is a triple $((M^m, \psi), (N^n, \phi), f)$ where $M^m$ and $N^n$ are compact manifolds with boundary of dimensions $m$ and $n$, $\psi: G \times M \to M$ and $\phi: G \times N \to N$ are differentiable $G$ actions which are $(\mathfrak{F}, \mathfrak{F}')$-free; i.e. each isotropy group $G_x$, $x \in M$ or $N$, belongs to $\mathfrak{F}$, and each isotropy group $G_x$, $x \in \partial M$ or $\partial N$, belongs to $\mathfrak{F}'$, and $f: (N, \partial N, \phi) \to (M, \partial M, \psi)$ is a differentiable map equivariant with respect to the given actions. Two $(\mathfrak{F}, \mathfrak{F}')$-free maps of dimension $(m, n)$, $((M, \psi), (N, \phi), f)$ and $((M', \psi'), (N', \phi'), f')$ are equivalent if there is a triple $((V, V^+, \Psi), (W, W^+, \Phi), F)$ where $V$ and $W$ are compact manifolds with boundary; $M$, $M'$, $V^+$ being regularly imbedded submanifolds of $\partial V$ with $\partial V = M \cup M' \cup V^+$, $M \cap M' = \varnothing$, $V^+ \cap M = \partial M$, $V^+ \cap M' = \partial M'$, $V^+ \cap (M \cup M') = \partial V^+$; $N$, $N'$, $W^+$ being similarly related in $\partial W$; $\Psi$ and $\Phi$ are differentiable $G$ actions extending $\psi$, $\psi'$, $\phi$, $\phi'$ with $V^+$, $W^+$ being invariant, so that $V$ and $W$ are $\mathfrak{F}$-free and $V^+$ and $W^+$ are $\mathfrak{F}'$-free; and $F: (W, W^+, \Phi) \to (V, V^+, \Psi)$ is a differentiable equivariant map extending $f$ and $f'$.

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The disjoint union of maps makes the set of equivalence classes into an abelian group ($Z_2$ vector space) denoted $\mathfrak{N}^G_{m,n}(\mathcal{F}, \mathcal{F}')$.

All of the general structure of $G$ actions with families may be carried through for maps. Thus if $j: (\mathcal{F}_0, \mathcal{F}'_0) \subset (\mathcal{F}, \mathcal{F}')$ one has an induced homomorphism

$$j_*: \mathfrak{N}^G_{m,n}(\mathcal{F}_0, \mathcal{F}'_0) \to \mathfrak{N}^G_{m,n}(\mathcal{F}, \mathcal{F}')$$

by weakening family restrictions, and one has a boundary homomorphism

$$\partial: \mathfrak{N}^G_{m,n}(\mathcal{F}, \mathcal{F}') \to \mathfrak{N}^G_{m-1,n-1}(\mathcal{F}', \phi)$$

given by $\partial((M, \psi), (N, \phi), f) = ((\partial M, \psi|_{G\times\partial M}), (\partial N, \phi|_{G\times\partial N}), f|_{\partial N})$.

One easily obtains: If $\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F}$ are families in $G$, the sequence

$$\begin{tikzcd} \mathfrak{N}^G_{*,*}(\mathcal{F}', \mathcal{F}'') \arrow[rr, "j_{1*}"] & & \mathfrak{N}^G_{*,*}(\mathcal{F}, \mathcal{F}'') \arrow[dl, "j_{2*}"] \\ & \mathfrak{N}^G_{*,*}(\mathcal{F}, \mathcal{F}') \arrow[ul, "j_{3*}\partial"] & \end{tikzcd}$$

is exact, where $j_i$ are the inclusions.

3. **The case** $G = Z_2$. For the group $Z_2$, there are three families, $\mathcal{A}ll = \{Z_2, \{1\}\}$, $\mathcal{F}ree = \{\{1\}\}$, and $\phi$, and hence three nontrivial pairs of families related by the exact sequence

$$\begin{tikzcd} \mathfrak{N}^{Z_2}_{*,*}(\mathcal{A}ll, \phi) \arrow[rr, "i_*"] & & \mathfrak{N}^{Z_2}_{*,*}(\mathcal{A}ll, \mathcal{F}ree). \arrow[dl, "j_*\partial"] \\ & \mathfrak{N}^{Z_2}_{*,*}(\mathcal{F}ree, \phi) \arrow[ul, "k_*"] & \end{tikzcd}$$

*Note.* An $\phi$-free action is interpreted as an action on an empty manifold.

Beginning with the free case, one has

**Proposition 3.1.** *$\mathfrak{N}^{Z_2}_{m,n}(\mathcal{F}ree, \phi)$ is isomorphic to the bordism group $\mathfrak{N}_m(\Omega^\infty TBO_{\infty+m-n} \times BZ_2)$, where $\Omega^\infty TBO_{\infty+m-n}$ is a representing space for $(m-n)$-dimensional cobordism. Further, $k_*$ is zero since there is a splitting $D: \mathfrak{N}^{Z_2}_{*,*}(\mathcal{F}ree, \phi) \to \mathfrak{N}^{Z_2}_{*,*}(\mathcal{A}ll, \mathcal{F}ree)$ assigning to $((M, \psi), (N, \phi), f)$ the induced map $((\overline{M}, \overline{\psi}), (\overline{N}, \overline{\phi}), \overline{f})$ where $\overline{M} = M \times [-1, 1]/(m, t) \sim (\psi(-1, m), -t)$ and $\overline{\psi}(g, (m, t)) = (\psi(g, m), t)$ and $\overline{f}(n, t) = (f(n), t)$.*

**Proof.** Clearly $D$ is a splitting, so $k_* = 0$. To compute $\mathfrak{N}^{Z_2}_{*,*}(\mathcal{F}ree, \phi)$ one defines for $((M, \psi), (N, \phi), f)$ the maps $\tilde{f}: N/Z_2 \to M/Z_2$ and $\alpha: M/Z_2 \to BZ_2$ where $\tilde{f}$ is induced by passing to quotients and $\alpha$ classifies the double cover

$M \to M/Z_2$. As in [2], the map $\tilde{f}$ induces a map into the representing space for cobordism (the limit over the spectral maps of $\Omega^r TBO_{r+m-n}$), which will be denoted $\hat{f}: M/Z_2 \to \Omega^\infty TBO_{\infty+m-n}$. One associates to $((M, \psi), (N, \phi), f)$ the bordism class of $(M/Z_2, \hat{f} \times \alpha)$. $\square$

Turning to $\mathfrak{N}^{Z_2}_{m,n}(\mathfrak{All}, \mathfrak{Free})$, consider an equivariant bordism element $((M, S), (N, T), f)$ where $S$ and $T$ are involutions defining the $Z_2$ actions.

The fixed point set of $S$ on $M$, $F_S$, is the disjoint union of closed submanifolds $F^k_S$, of dimension $k$, imbedded in the interior of $M$. $F^k_S$ has normal bundle $\nu_{m-k}$ (of fiber dimension $m-k$) and one may equivariantly identify a collection of disjoint tubular neighborhoods of the sets $F^k_S$ with the disc bundles $D(\nu_{m-k})$ on which $S$ acts as multiplication by $-1$.

Similarly, the fixed point set of $T$ on $N$, $F_T$, is the disjoint union of closed submanifolds $F^{j,k}_T$, the $j$-dimensional part which is mapped into $F^k_S$ by $f$, imbedded in the interior of $N$, with normal bundle $\tilde{\nu}^k_{n-j}$ (of fiber dimension $n-j$). Let $\nu: F^{j,k}_T \to BO_{n-j}$ classify $\tilde{\nu}^k_{n-j}$ and consider the diagram

$$\begin{array}{ccc} F^{j,k}_T & \xrightarrow{f} & F^k_S \\ {\scriptstyle\nu}\big\downarrow & & \\ BO_{n-j} & & \end{array}$$

the "map" bordism class of which is obviously an invariant of the class $f$. By slight modification of the prodecure in [2] this map is realized by a map

$$\hat{f}_j: F^k_S \to \Omega^\infty(TBO_{\infty+k-j} \wedge BO^+_{n-j}).$$

Specifically, let $i: F^{j,k}_T \to S^r$ (not hitting the base point) be an imbedding giving an imbedding $i \times f: F^{j,k}_T \to S^r \times F^k_S$ and hence a collapse $S^r \times F^k_S \to T\nu_{i\times f}$ onto the Thom space of the normal bundle of the imbedding. As in the Pontrjagin-Thom construction for bordism, $\nu_{i\times f} \times \nu: F^{j,k}_T \to BO_{r+k-j} \times BO_{n-j}$ Thomifies to a map $T\nu_{i\times f} \to TBO_{r+k-j} \wedge BO^+_{n-j}$. By pushing off the sphere factor in the composite

$$S^r \times F^k_S \to T\nu_{i\times f} \to TBO_{r+k-j} \wedge BO^+_{n-j}$$

and letting $r$ go to infinity, one has the map $\hat{f}_j$.

Thus the fixed point set of $T$ gives a map

$$\hat{f}_T = \prod_{j\neq n-1} \hat{f}_j: F^k_S \to \prod_{j\neq n-1} \Omega^\infty(TBO_{\infty+k-j} \wedge BO^+_{n-j})$$

whose bordism class depends only on the class of $f$, ignoring the term $j = n-1$.

In addition, another invariant exists. The involution $S$ is a free action on a neighborhood of the sphere bundles $S(\nu_{m-k}) = \partial D(\nu_{m-k})$ and by a small equivariant deformation of $f$, one may suppose $f$ is transverse regular to $\bigcup_k S(\nu_{m-k})$. (If preferred, $\bigcup_k \{D(\nu_{m-k}) - F_S^k\} \to (0, 1]$ by means of the radial distance and the composite with $f$ has a regular value $r \in (0, 1)$, so that $f$ is transverse regular to $r(\bigcup_k S(\nu_{m-k}))$ for some $r$.) Then $f^{-1}(S(\nu_{m-k})) = P^{n-1,k}$ is a closed submanifold of $N$ on which $T$ acts freely with $f\colon (P^{n-1,k}, T) \to (S(\nu_{m-k}), -1)$ being an equivariant map. Thus one has a diagram

$$\begin{array}{ccc} P^{n-1,k} & \xrightarrow{f} & S(\nu_{m-k}) \\ & & \Big\downarrow \pi \\ & & F_S^k \end{array}$$

and the "map" bordism class of this is an invariant of $f$.

Techniques for handling such objects are discussed in [4]. Briefly, one may proceed as follows. Passing to orbits of the $Z_2$ action gives $\bar{f}\colon P/Z_2 \to RP(\nu_{m-k})$, the projective space bundle of $\nu_{m-k}$, or $\bar{\bar{f}}\colon RP(\nu_{m-k}) \to \Omega^\infty TBO_{\infty+m-n}$ $((m-1) - (n-1) = m-n)$. Letting $\pi\colon U \to F_S^k$ be the principal bundle of $\nu_{m-k}$, one has

$$\bar{\bar{f}}\colon U \times_{O_{m-k}} RP(m-k-1) \to \Omega^\infty TBO_{\infty+m-n}$$

which may be interpreted as on $O_{m-k}$ equivariant map

$$U \to (\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-1)}.$$

Thinking of this as a free $O_{m-k}$ bordism element one applies the standard technique of crossing with the universal free $O_{m-k}$ space and dividing out the group action to obtain

$$f_P\colon F_S^k = U/O_{m-k} \to \{(\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-1)} \times EO_{m-k}\}/O_{m-k}.$$

*Note.* Projecting this onto $EO_{m-k}/O_{m-k} = BO_{m-k}$ classifies $\nu_{m-k}$, so that one has lifted a classifying map for $\nu_{m-k}$ to the associated bundle with fiber $(\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-k)}$. For $k = m-1$, $RP(0)$ is a point and $f_P\colon F_S^{m-1} \to \Omega^\infty TBO_{\infty+m-n} \times BO_1$.

One then has

**Proposition 3.2.** *Assigning to* $((M, S), (N, T), f)$ *the classes* $(F_S^k, \hat{f}_T \times f_P)$ *defines an isomorphism of* $\mathfrak{N}_{m,n}^{Z_2}(\mathfrak{All}, \mathfrak{Free})$ *with*

$$\bigoplus_{k=0}^{m} \mathfrak{N}_k\left(\left\{\prod_{j\neq n-1} \Omega^\infty(TBO_{\infty+k-j} \wedge BO_{n-j}^+)\right\} \times \left\{\frac{(\Omega^\infty TBO_{\infty+m-n})^{RP(m-k-1)} \times EO_{m-k}}{O_{m-k}}\right\}\right).$$

**Proof.** Being given a $k$-dimensional bordism class $(F, \hat{f}_T \times f_P)$ in this product space is equivalent to being given diagrams

$$\begin{array}{ccc} P^{n-1} & \xrightarrow{h} & S(\xi_{m-k}) \\ & & \downarrow{\scriptstyle \pi} \\ & & F \end{array}$$

where $h$ is an equivariant map of $Z_2$ spaces, $\xi_{m-k}$ being an $m-k$ plane bundle, and

$$\begin{array}{ccc} F^j & \xrightarrow{h_j} & F \\ {\scriptstyle \rho}\downarrow & & \\ BO_{n-j} & & \end{array}$$

where $\rho$ classifies an $n-j$ plane bundle $\rho_{n-j}$ over $F^j$, $j \neq n-1$. Combining these, one has an $(\mathfrak{All}, \mathfrak{Free})$ $m$-dimensional manifold $D(\xi_{m-k})$ with involution $-1$ in the fibers together with an $n$-dimensional $(\mathfrak{All}, \mathfrak{Free})$ manifold which is the union of $P \times [-1, 1]/(p, t) \sim (Tp, -t)$ with involution $T \times 1$ (where $T$ is the involution on $P$) and the manifolds $D(\rho_{n-j})/(x \sim -x;\ x \in S(\rho_{n-j}))$ with involution induced by $-1$ in the fibers, together with an equivariant map into $D(\xi_{m-k})$ given by sending $(p, t) \in P \times [-1, 1]$ to $t \cdot h(p)$ and given by $h_j \circ \pi_j$ on $D(\rho_{n-j})$, where $\pi_j$ is the projection. This constructs an inverse $Q$ to the homomorphism $R$: $\mathfrak{N}^{Z_2}_{m,n}(\mathfrak{All}, \mathfrak{Free}) \to A$, where $A$ is the given direct sum.

By taking fixed sets and applying transverse regularity, $RQ = 1$ trivially. To see that $QR = 1$, one considers the bordism element $((M, S), (N, T), f)$. After deformation of $f$ as before, one may suppose $f$ is transverse regular to $\bigcup_k S(\nu_{m-k})$ with $f^{-1}(S(\nu_{m-k})) = P^{n-1,k}$. Then one has $f\colon f^{-1}(\bigcup_k D(\nu_{m-k})) \to \bigcup_k D(\nu_{m-k})$ imbedded in $f\colon N \to M$ with the complementary part being free as $Z_2$ map, so one may suppose $M = \bigcup_k D(\nu_{m-k})$ by excising the complementary portion. By radial deformation in the fibers of $D(\nu_{m-k})$, one may suppose that $P^{n-1,k} = \partial(f^{-1}(D(\nu_{m-k})))$ maps to $\partial D(\nu_{m-k})$ with a tubular neighborhood $P^{n-1,k} \times [1, 0]$ of $P^{n-1,k}$ mapping by radial extension into $D(\nu_{m-k})$ $(f(p, t) = t \cdot f(p))$ and with the remainder of $f^{-1}(D(\nu_{m-k}))$ being mapped into the zero section $F^k_S$. By splitting $f^{-1}(D(\nu_{m-k}))$ along $P \times 0$ and dividing out the free involution $T$ on the two copies of $P$ introduced, one replaces $f$ by the map of $P \times [-1, 1]/(p, t) \sim (Tp, -t)$ into $D(\nu_{m-k})$ given as part of $QR$ together with a map of a closed $Z_2$ manifold $L^n$ into $F^k_S$. This splitting may be accomplished via an equivalence: specifically one may suppose $f$ maps a neighborhood $P \times [1, -\epsilon]$ of $P$ by $f(p, t) = tf(p)$ if $t \geq 0$ and $f(p, t) = 0 \cdot f(p)$ if $t \leq 0$ by a small deformation, and letting $V = f^{-1}(D(\nu_{m-k})) \times [0, 1]$ with $(p, t, 1) \sim (Tp, t, 1)$ for $t \in [0, -\epsilon]$ with map $f\cdot\pi$ into $D(\nu_{m-k})$ gives an equivalence of $f^{-1}(D(\nu_{m-k}))\times 0$ with the split map at the $(\times 1)$-edge. The splitting along $P\times 0$ gives $L$ a fixed submanifold of codimension one, so that the maps $L\to F^k_S$ and $\bigcup_{j\neq n-1} D(\rho_{n-j})/(x\sim -x;\ x\in S(\rho_{n-j}))\to F^k_S$ which occurs in the image of $QR$ are $Z_2$ bordism elements of the fixed space $F^k_S$ having the same fixed data of codimension not equal to one, hence are bordant in $F^k_S$. Combining all of these constructions gives an equivalence of $((M, S), (N, T), f)$ and its image under $QR$. □

One may completely analyze the exact sequence for $Z_2$ equivariant maps in terms of the decomposition given by Proposition 3.2. Applying the splitting $D$ of Proposition 3.1, one may identify

$$D\colon \mathfrak{N}^{Z_2}_{m-1,n-1}(\mathfrak{Free}, \phi)\to \mathfrak{N}^{Z_2}_{m,n}(\mathfrak{All}, \mathfrak{Free})$$

with the homomorphism

$$\mathfrak{N}_{m-1}(\Omega^\infty TBO_{\infty+m-n}\times BO_1)$$
$$\to \mathfrak{N}_{m-1}\left(\left\{\coprod_{j\neq n-1}\Omega^\infty(TBO_{\infty+m-1-j}\wedge BO^+_{n-j})\right\}\times \Omega^\infty TBO_{\beta+m-n}\times BO_1\right)$$

induced by the slice at a point in the first factor. Corresponding to this, one has a splitting $P\colon \mathfrak{N}^{Z_2}_{m,n}(\mathfrak{All}, \mathfrak{Free})\to \mathfrak{N}^{Z_2}_{m,n}(\mathfrak{All}, \phi)$ assigning to $((M, S), (N, T), f)$ the class obtained as follows. Let $f$ be made transverse regular to $\bigcup_k S(\nu_{m-k})$ and consider the map

$$\tilde f\colon \bigcup_k f^{-1}(D(\nu_{m-k}))/(x\sim Tx;\ x\epsilon\partial f^{-1}(D(\nu_{m-k})))$$
$$\to \bigcup_k D(\nu_{m-k})/(x\sim Sx;\ x\epsilon S(\nu_{m-k}))$$

induced by $f$. $P((M, S), (N, T), f)$ is given by the class of $\tilde f$. Clearly $i_*P(x)$ and $x$ have the same fixed data except for the portion of the fixed set $\bigcup_k S(\nu_{m-k})/Z_2$ introduced, which is codimension one, with the inverse image fixed set being $\bigcup_k \partial f^{-1}(D(\nu_{m-k}))/Z_2$ which is also codimension one. The new invariants introduced are all given by the transverse regularity construction as the maps

$$\begin{array}{ccc}\partial f^{-1}(D(\nu_{m-k})) & \to & S(\nu_{m-k})\\ & & \downarrow\\ & & S(\nu_{m-k})/Z_2\end{array}$$

which are the classes $D(f\colon \partial f^{-1}(D(\nu_{m-k}))\to S(\nu_{m-k}))$. Thus $i_*Px=x$ mod image $D$. Further, image $D\subset$ kernel $P$, for being given a free map $g\colon (A, T)\to (B, S)$, $PD(g)$ is the map

$$\overline{g \times 1}\colon A \times [-1, 1]/\{(a, t) \sim (Ta, t),\ t = \pm 1\}$$
$$\longrightarrow B \times [-1, 1]/\{(b, t) \sim (Sb, t),\ t = \pm 1\}$$

with involutions $T \times 1$ and $S \times 1$. Letting $g'\colon \bar{A} \to \bar{B}$ be the map constructed for $D$, $\overline{g \times 1}$ is the boundary of the map $g' \times 1\colon \bar{A} \times [0, 1] \to \bar{B} \times [0, 1]$. The sequence

$$0 \to \mathfrak{N}_{*,*}^{Z_2}(\mathcal{F}ree, \phi) \xrightarrow{D} \mathfrak{N}_{*,*}^{Z_2}(\mathcal{A}ll, \mathcal{F}ree) \xrightarrow{P} \mathfrak{N}_{*,*}^{Z_2}(\mathcal{A}ll, \phi) \to 0$$

is then exact with $D$ splitting $j_*\partial$ and $P$ splitting $i_*$.

Thus, one sees that $Z_2$ equivariant maps may be analyzed in a fashion which is pure analogy with the Conner and Floyd analysis of $Z_2$ actions in [1, §28].

4. **The case** $G = Z_p$, $p$ **odd.** Let $p$ be an odd prime and consider $G = Z_p$. Again there are three families: $\mathcal{A}ll$, $\mathcal{F}ree$, and $\phi$, giving three nontrivial pairs of families related by the exact sequence

$$\begin{tikzcd}
\mathfrak{N}_{*,*}^{Z_p}(\mathcal{F}ree, \phi) \arrow[rr, "k_*"] & & \mathfrak{N}_{*,*}^{Z_p}(\mathcal{A}ll, \phi) \arrow[dl, "i_*"] \\
& \mathfrak{N}_{*,*}^{Z_p}(\mathcal{A}ll, \mathcal{F}ree). \arrow[ul, "j_*\partial"] &
\end{tikzcd}$$

As with the $Z_2$ case one has

**Proposition 4.1.** *Assigning to* $((M, \psi), (N, \phi), f)$ *the diagram* $N/Z_p \xrightarrow{\bar{f}} M/Z_p \xrightarrow{\alpha} BZ_p$, *where* $\alpha$ *classifies the cover* $M \to M/Z_p$ *defines an isomorphism of* $\mathfrak{N}_{m,n}^{Z_p}(\mathcal{F}ree, \phi)$ *with* $\mathfrak{N}_m(\Omega^\infty TBO_{\infty + m - n} \times BZ_p)$.

Before beginning the analysis of the $(\mathcal{A}ll, \mathcal{F}ree)$ case recall that there is a classifying space $F'_{Z_p}(BO_k)$ for $k$-plane bundles with linear $Z_p$ action over $Z_p$ fixed spaces having the property that only the zero section is pointwise fixed by $Z_p$; i.e. if $\xi \to X$ is a $k$-plane bundle with $Z_p$ action over the $Z_p$ fixed space $X$ and no fiber in $\xi$ contains a copy of the trivial representation, then $\xi$ is classified by a map into $F'_{Z_p}(BO_k)$. From [1, §38], one has $F'_{Z_p}(BO_{2k}) = \bigcup BU_{k_1} \times \cdots \times BU_{k_{(p-1)/2}}$, the union being for $k_1 + \cdots + k_{(p-1)/2} = k$, and $F'_{Z_p}(BO_{2k+1}) = \phi$.

Now consider a class $((M, \psi), (N, \phi), f)$ in $\mathfrak{N}_{m,n}^{Z_p}(\mathcal{A}ll, \mathcal{F}ree)$. The fixed set $F_{Z_p}(M)$ is a union of closed submanifolds $F_{Z_p}^{m-2k}(M)$ of codimension $2k$ imbedded in the interior of $M$ with normal bundle $\nu_{2k}$ classified by a map $\nu_{2k}$: $F_{Z_p}^{m-2k}(M) \to F'_{Z_p}(BO_{2k})$, and with a collection of disjoint tubular neighborhoods of the sets $F_{Z_p}^{m-2k}(M)$ being equivariantly identified with the disc bundles $D(\nu_{2k})$.

Similarly, the fixed set of $\phi$ on $N$ is a disjoint union of closed submanifolds

$F_{Z_p}^{n-2j,k}(N)$ of codimension $2j$ imbedded in the interior of $N$ and mapping into $F_{Z_p}^{m-2k}(M)$ under $f$. Letting $\tilde{\nu}_{2j}^{k}$ denote the normal bundle of $F_{Z_p}^{n-2j,k}(N)$, one has a diagram

$$\begin{array}{ccc} F_{Z_p}^{n-2j,k}(N) & \xrightarrow{f} & F_{Z_p}^{m-2k}(M) \\ \downarrow{\scriptstyle \tilde{\nu}} & & \\ F'_{Z_p}(BO_{2j}) & & \end{array}$$

inducing a map

$$\hat{f}_j\colon F_{Z_p}^{m-2k}(M) \to \Omega^\infty(TBO_{\infty+m+2j-n-2k} \wedge F'_{Z_p}(BO_{2j})^+).$$

One may then identify disjoint tubular neighborhoods of the sets $F_{Z_p}^{n-2j,k}(N)$ with the disc bundles $D(\tilde{\nu}_{2j}^{k})$ and by deforming radially in the fibers of these discs, may assume $f|_{D(\tilde{\nu}_{2j}^k)} = f|_{F_{Z_p}^{n-2j,k}(N)} \circ \pi$, where $\pi$ is the projection. One may then cut $N$ along the sets $S(\tilde{\nu}_{2j}^{k})$ and identify the resulting boundaries under the antipodal map (which commutes with the $Z_p$ action by bundle maps and is compatible with $f$ since $f$ factors through the projection). This replaces $f\colon N \to M$ by an equivalent map $f'\colon N' \cup \bigcup_{k,j} D(\tilde{\nu}_{2j}^{k})/(x \sim -x;\ x \in S(\tilde{\nu}_{2j}^{k})) \to M$ where $f'(\partial N') \subset \partial M$ and the $Z_p$ action on $N'$, call it $\phi'$, is a free action.

Since $(N', \phi')$ is free, one may deform $f'$ to be transverse regular to $\bigcup_k F_{Z_p}^{m-2k}(M)$ with $f'^{-1}(F_{Z_p}^{m-2k}(M)) = \bar{N}^{n-2k}$ being a free $Z_p$ manifold with action $\phi'$, and with normal bundle $\bar{\nu}_{2k}$ being induced from $\nu_{2k}$. By a further deformation of $f'$, one may suppose $f'$ is given on the collection of disjoint tubular neighborhoods $D(\bar{\nu}_{2k})$ by the bundle maps covering $f'|_{\bar{N}^{n-2k}}$.

Then $f'\colon N' \cup \bigcup D(\tilde{\nu}_{2j}^{k})/(Z_2 \mid S(\tilde{\nu}_{2j}^{k})) \to M$ contains the map $f'\colon \bigcup D(\bar{\nu}_{2k}) \cup \bigcup D(\tilde{\nu}_{2j}^{k})/(Z_2 \mid S(\tilde{\nu}_{2j}^{k})) \to \bigcup D(\nu_{2k})$ and the complement is a free $Z_p$ map, which may be excised.

Corresponding to the diagram

$$\begin{array}{ccc} \bar{N}^{n-2k}/Z_p & \xrightarrow{\tilde{f}'} & F_{Z_p}^{m-2k}(M) \\ \downarrow{\scriptstyle \alpha_k} & & \\ BZ_p & & \end{array}$$

where $\alpha_k$ classifies the cover by $\bar{N}$ and $\tilde{f}'$ is induced by $f'$, one has

$$\beta_k\colon F_{Z_p}^{m-2k}(M) \to \Omega^\infty(TBO_{\infty+m-n} \wedge BZ_p^+).$$

**Proposition 4.2.** *Assigning to* $((M, \psi), (N, \phi), f)$ *the classes* $(F^{m-2k}_{Z_p}(M), \nu_{2k} \times (\times_j \hat{f}_j) \times \beta_k)$ *gives an isomorphism* $\Theta$ *of* $\mathfrak{N}^{Z_p}_{m,n}(\mathfrak{All}, \mathfrak{Free})$ with

$$\bigoplus_{k=0}^{m/2} \mathfrak{N}_{m-2k}\left(F'_{Z_p}(BO_{2k}) \times \left\{\bigtimes_j \Omega^\infty(TBO_{\infty+m+2j-n-2k} \wedge F'_{Z_p}(BO_{2j})^+)\right\} \times \Omega^\infty(TBO_{\infty+m-n} \wedge BZ_p^+)\right).$$

Rather than brutalize the entire thing, let $A$ denote this sum, and define a map $P: A \to \mathfrak{N}^{Z_p}_{m,n}(\mathfrak{All}, \phi)$ as follows. Being given $\alpha = (F^{m-2k}, \nu \times (\times_j \hat{f}_j) \times \beta)$ in $A$, one has diagrams $\bar{N}^{n-2k} \xrightarrow{f'} F^{m-2k}$ with $\bar{N}$ a free $Z_p$ space, and

$$\begin{array}{ccc} H^{n-2j} & \xrightarrow{f} & F^{m-2k} \\ \rho\downarrow & & \\ F'_{Z_p}(BO_{2j}) & & \end{array}$$

with $\rho$ inducing a bundle $\rho^k_{2j}$ with $Z_p$ action over $H^{n-2j}$, and a bundle $\xi_{2k}$ over $F^{m-2k}$ with $Z_p$ action, induced by the map $\nu$. One may then form the $Z_p$ manifold $M' = D(\xi_{2k})/(x \sim -x|\ x \in S(\xi_{2k}))$ with $Z_p$ action given by the bundle action, and the manifold $N' = D(f'^*\xi_{2k})/(x \sim -x|\ x \in S(f'^*\xi_{2k}))$ with induced bundle $Z_p$ action and map $f': N' \to M'$ arising from the map of the induced bundle. One lets $E_j = D(\rho^k_{2j})/(x \sim -x|\ x \in S(\rho^k_{2j}))$ with bundle $Z_p$ action and map $f': E_j \to M'$ given by $f \circ \pi$, $\pi$ being the projection on $H^{n-2j}$.

Let $p(\alpha) = (M', N' \cup \bigcup_j E_j, f')$. From all of the above constructions one clearly has $\Theta i_* P(\alpha) = \alpha$ and have shown the equivalence of $i_* P(\Theta(M, N, f))$ and $(M, N, f)$. Thus $\Theta$ is an isomorphism with inverse $i_* P$.

Finally, $P\Theta$ defines a splitting for $i_*$ and one has

**Proposition 4.3.** *The exact sequence for* $Z_p$ *families splits and one has*

$$\mathfrak{N}^{Z_p}_{m,n}(\mathfrak{All}, \phi) \cong \mathfrak{N}^{Z_p}_{m,n}(\mathfrak{Free}, \phi) \oplus \mathfrak{N}^{Z_p}_{m,n}(\mathfrak{All}, \mathfrak{Free}).$$

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