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EQUIVARIANT COFIBRATIONS AND NILPOTENCY

BY

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ABSTRACT. Let $f\colon B \to Y$ be a cofibration whose cofiber is a Moore space. We give necessary and sufficient conditions for f to be induced by a map of the desuspension of the cofiber into B. These conditions are especially simple if B and Y are nilpotent.

We obtain some results on the existence of equivariant Moore spaces, and use them to construct examples of noninduced cofibrations between nilpotent spaces. Our machinery also leads to a cell structure proof of the characterization of pre-nilpotent spaces due to Dror and Dwyer [7], and to a simple proof, for finite fundamental group, of the result of Brown and Kahn [4] that homotopy dimension equals simple cohomological dimension in nilpotent spaces.

0. Introduction. Much work has recently been done on the structure of nonsimply connected spaces, particularly nilpotent spaces. It has been shown that in many ways they are just as tractible as simply connected spaces. For example, Brown and Kahn [4] have shown that for nilpotent spaces homotopy dimension equals simple cohomological dimension.

In one significant respect, at least, nilpotent spaces remain as obscure as other nonsimply connected spaces: one cannot visualize how the cell structure relates to the homology. The reason is that nilpotency has so far been analyzed using fibrations, not cofibrations. By "visualize" I take as paradigm the homology decomposition of [10], whereby it is possible to picture a simply connected space as arising by attaching Moore spaces together, one for each homology group. (A Moore space is a simply connected CW complex having a single nonvanishing homology group.) Dually, it is possible to kill the homology dimension by dimension by successively attaching Moore spaces.

This paper is a step toward producing analogous ideas for nilpotent spaces.

The first section derives many algebraic lemmas, several of interest in their own right. They all concern finitely generated modules over finitely generated nilpotent groups π. In Proposition 1.5 we show that if the trivializing map $M \to M/IM$ splits over Z (the integers) and if $H_1(\pi, M) = 0$, then $H_i(\pi, M) = 0\ \forall\, i \geqslant 1$. As a corollary, we have the theorem that if $\exists n \geqslant 1$ such that $H_n(\pi, M)$ is a free Abelian group and $H_{n+1}(\pi, M) = 0$ then $H_i(\pi, M) = 0\ \forall\, i \geqslant n + 1$. Additional corollaries relating to nilpotent modules are derived. The main technique here is the idea of resolving a module by a free chain complex of length one that is not acyclic, but whose homology modules are perfect.

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In the second section we are concerned with the equivariant version of cofibrations whose cofiber is a Moore space. By an *equivariant cofibration* we mean a pair (Y, B), usually a CW pair, which some group π acts upon (cellularly). We supply several criteria for deciding when such a cofibration is induced. Theorem 2.5 determines exactly when this occurs in the nilpotent case.

The third section opens with a discussion of one of the fundamental problems about equivariant cofibrations, the existence of equivariant Moore spaces. This problem was posed by Steenrod around 1960. We show that such a complex $K'(M, n)$ exists for π and M nilpotent if M satisfies certain stringent homology conditions (relating back to §1) (Theorem 3.4). We also have a theorem (3.1) stating that topological actions on Moore spaces can be made cellular in a nice way. The section closes with the construction of many examples of equivariant cofibrations.

§§4 and 5 are applications of the machinery developed earlier. In §4 we give a cell structure proof of the theorem due to Dror and Dwyer **[7]** characterizing pre-nilpotent spaces. For π a finite p-group this is an equivariant version of killing homology by attaching Moore spaces. §5 provides a short easy proof of the theorem **[4]** that homotopy dimension of a nilpotent space equals simple cohomological dimension. Our proof is for the most difficult case, finite fundamental group, but would work in general if a certain algebraic result (next paragraph) were proven.

This research points out two questions about the homology of finitely generated modules over finitely generated nilpotent groups:

(1) If π is finite and M is nilpotent with $H_1(\pi, M) = H_2(\pi, M) = 0$, must M be cohomologically trivial? (See Theorems 3.4–3.6.)

(2) If M is torsion free and $H_i(\alpha, M) = 0\ \forall\, i \geqslant 1$, $\forall\, \alpha$ normal in π, must M be projective? (See §5.)

I believe that the answer to (2) is yes.

Some comments on notation. "$\tilde{A}$" means the universal cover of A. "Space" means CW complex whenever convenient. All tensor products are taken over the ground ring Z, the integers. All modules are left modules. "I" is the augmentation ideal of $Z\pi$ and IM is the submodule of M generated by all elements of the form $\alpha m - m$, $\alpha \in \pi$, $m \in M$. Refer to **[5]** and **[14]** for basic homological algebra.

I thank the referee for the present (more powerful) version of 1.9 and 1.10, and for pointing out 3.6.

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1. Algebraic preliminaries. All of the results of this section are corollaries of the following theorem due to Dwyer, proven in **[8]**.

THEOREM 1.1. *If π is a finitely generated nilpotent group and M is a finitely generated perfect π-module, then $H_i(\pi, M) = 0$, $\forall\, i \geqslant 0$.*

Recall that a π-module is said to be perfect if $IM = M$, or equivalently if $H_0(\pi, M) = 0$. For the rest of this section, M will be a finitely generated π-module and π will be a finitely generated nilpotent group. $Z\pi$ is then (left and right) Noetherian [**12**].

Let $\hat{M} = \varprojlim M/I^kM$. In [3] Brown and Dror show that the natural map $M \to \hat{M}$ preserves exact sequences of finitely generated modules. Several of the results of this section can be derived very quickly from their techniques. We have chosen to take the much more elementary 1.1 as our starting point (even the proof in [**8**] is unnecessarily complex) in order to emphasize the elementary nature of the results.

COROLLARY 1.2. *A submodule of a finitely generated perfect module is perfect. Thus, finitely generated perfect modules form a Serre class.*

PROOF. If M is perfect and $A \subset M$, look at the long exact homology sequence of $0 \to A \to M \to M/A \to 0$. It is elementary that the image of a perfect module is perfect. By Theorem 1.1, $H_1(\pi, M/A) = 0$ and the result follows.

COROLLARY 1.3. *Given an exact sequence of π-modules $0 \to N \to M \to P \to 0$ with P perfect, $\exists$ an exact sequence $0 \to IN \to IM \to P \to 0$. Inductively, $0 \to I^nN \to I^nM \to P \to 0$.*

PROOF. From the long exact homology sequence associated with the given sequence, it follows from Theorem 1.1 that $H_0(\pi, N) \xrightarrow{\approx} H_0(\pi, M)$. Complete the proof by applying the Nine-Lemma, or similar diagram chase, to the following.

$$\begin{array}{ccccccccc}
0 & \to & IN & \to & N & \to & N/IN & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\approx & & \\
0 & \to & IM & \to & M & \to & M/IM & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & IM/IN & \to & P & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Recall that a π-module M is said to be pre-nilpotent if I^nM is perfect for some $n \geqslant 0$. Thus, the class of pre-nilpotent modules includes both the nilpotent and perfect modules.

COROLLARY 1.4. *Finitely generated pre-nilpotent π-modules form a Serre class.*

PROOF. Given $0 \to A \to M \to B \to 0$, suppose first that M is pre-nilpotent. Then I^nM is perfect for some n. The induced map $I^nM \to I^nB$ is a surjection so I^nB is perfect. Furthermore, $I^nA \subset I^nM$ so I^nA is perfect by Corollary 1.2.

On the other hand, suppose A and B are pre-nilpotent. Standard methods yield the sequence $0 \to A \cap I^nM \to I^nM \to I^nB \to 0$, where we take n so that I^nB is perfect. By Corollary 1.3 we derive, for every $k \geqslant 0$, $0 \to I^k(A \cap I^nM) \to I^{n+k}M \to I^nB \to 0$. But $I^k(A \cap I^nM) \subset I^kA$ and so is perfect for large enough k. Thus, for this value of k, I^{n+k} is perfect.

PROPOSITION 1.5. *If the map $M \to M/IM$ splits over Z (i.e., as a group homomorphism) and $H_1(\pi, M) = 0$, then $H_i(\pi, M) = 0\ \forall\ i \geqslant 1$.*

PROOF. Let $f\colon M/IM \to M$ be a Z-splitting and map the induced module $Z\pi \otimes M/IM \xrightarrow{h} M$ by $h(\lambda \otimes x) = \lambda f(x)$. This homomorphism induces an isomorphism on $H_0(\pi, -)$. Denote its image in M by I_0.

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & I_0 & & & & \\
 & & & & \nearrow h' & & & \searrow i & & & \\
 & & Z\pi \otimes M/IM & & & & \xrightarrow{h} & & M & & \\
 & \nearrow & & & & & & & & \searrow & \\
K & & & & & & & & & & M/I_0 \\
\nearrow & & & & & & & & & & \searrow \\
0 & & & & & & & & & & 0
\end{array}$$

The map induced on $H_0(\pi, -)$ by h' is surjective because h' is surjective, and it is injective because h induces an isomorphism on $H_0(\pi, -)$. It follows that M/I_0 is perfect, and so $H_1(\pi, I_0) = H_1(\pi, M) = 0$.

Now, the induced module $Z\pi \otimes M/IM$ has, of course, no homology above dimension zero. But the vanishing of $H_1(\pi, I_0)$ forces K to be perfect, so it too has no homology above dimension zero.

Therefore, all the higher homology of both I_0 and M/I_0 vanish, and the result follows.

The same technique is used in the following theorem, which could be called "resolving M using frees and perfects".

THEOREM 1.6. *Suppose that $H_1(\pi, M)$ is a free Abelian group and $H_2(\pi, M) = 0$. Then $\exists$ finitely generated free π-modules F_0 and F_1 and maps*

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0,$$

where ∂_0 is onto, $\ker \partial_0/\operatorname{Im} \partial_1$ *is perfect, and* $\ker \partial_1$ *is perfect.*

PROOF. Let $F_0 \to M$ be any surjection of a finitely generated free module onto M, and let K_0 be the kernel. Observe that since $H_1(\pi, M)$ is free Abelian and $H_2(\pi, M) = 0$, $H_0(\pi, K_0)$ is free Abelian and $H_1(\pi, K_0) = 0$. We may therefore apply the method of Proposition 1.5 to K_0, and the result follows as a porism of Proposition 1.5.

COROLLARY 1.7. *If $H_n(\pi, M)$ is free Abelian and $H_{n+1}(\pi, M) = 0$ then $H_i(\pi, M) = 0\ \forall\, i \geqslant n + 1$.*

PROOF. By induction on n, using Theorem 1.6. Given such an M, take a (finitely generated) free F mapping onto M, call the kernel K, and look at the long exact homology sequence of $0 \to K \to F \to M \to 0$.

Finally, we shall add nilpotency assumptions to Proposition 1.5.

COROLLARY 1.8. *Let M be as in Proposition* 1.5 *and assume in addition that M and $Z\pi \otimes M/IM$ are nilpotent. Then $M \cong Z\pi \otimes M/IM$.*

PROOF. Examine the proof of Proposition 1.5. The new assumptions obviously force M/I_0 and K to vanish.

If only M is nilpotent, then $Z\pi \otimes M/IM$ is pre-nilpotent. The next two results show that induced modules are not often pre-nilpotent or nilpotent.

PROPOSITION 1.9. *Let π be a nontrivial finitely generated nilpotent group and A be a nontrivial finitely generated Abelian group. Then $Z\pi \otimes A$ is pre-nilpotent iff π and A are both finite.*

PROOF. If: trivial, since $Z\pi \otimes A$ has only finitely many subgroups.

Only if: We first verify the theorem for the special cases $\pi = Z$ and $\pi = Z_p$, p prime.

If $\pi = Z_p$, decompose A as $F \oplus S$ where F is free and S consists of torsion. Then $I^n(Z\pi \otimes F)$ is a direct summand of $I^n(Z\pi \otimes A)$. But for $\pi = Z_p$, $I^n(Z\pi)$ is strictly contained in $I^{n-1}(Z\pi)$, for every n (Gruenberg **[9]**), so $Z\pi \otimes A$ cannot be pre-nilpotent unless $F = 0$, as claimed.

If $\pi = Z$, then $I(Z\pi) \cong Z\pi$, and it is easy to see that $I^{n-1}(Z\pi \otimes A)/I^n(Z\pi \otimes A) \cong A$, for all n.

In the general case, given any nontrivial π find an epimorphism $\pi \to Z_p$, some prime p. If σ is the kernel of this map, then $Z\pi \otimes A/(\sigma\text{-action})$ is isomorphic as a π- (or Z_p-) module to $Z(Z_p) \otimes A$. Thus, if $I^n(Z\pi \otimes A)$ is π-perfect then $I^n(Z(Z_p) \otimes A)$ is Z_p-perfect, so A must be finite.

If π is infinite argue as in the preceeding paragraph, but with an epimorphism $\pi \to Z$.

PROPOSITION 1.10. *With π and A as in* 1.9, *$Z\pi \otimes A$ is nilpotent iff there is some prime p such that π and A are both finite and of exponent p.*

PROOF. Only if: π and A are finite by 1.9. For primes p and q let Z_p be a normal subgroup of π and A_q the q-torsion subgroup of A. Then $Z(Z_p) \otimes A_q \subset Z\pi \otimes A$ must be nilpotent over $Z(Z_p)$. It is easy to check then (a simple proof is in **[11]**) that $q \neq p \Rightarrow A_q = 0$.

If: By 1.9 $Z\pi \otimes A$ is pre-nilpotent, so contains a perfect module that consists entirely of p-torsion. But such a module must be trivial (prove it first for $\pi = Z_p$, then use induction on $|\pi|$).

Another theorem about induced modules is in **[2]**.

2. Induced cofibrations. We are chiefly concerned here with cofibrations $A \to X \to X/A$, where the pair (X, A) is $(n-1)$-connected, $n \geq 3$. We set $\pi = \pi_1 X$, $G = H_n(X, A)$, and $\bar{G} = \pi_n(X, A) = H_n(\tilde{X}, \tilde{A})$. By the relative Hurewicz Theorem, $\bar{G} \to G$ is just $\bar{G} \to \bar{G}/I\bar{G}$. In most of the applications, X/A is a Moore space $K'(G, n)$.

DEFINITION. The cofibration $A \to X \to K'(G, n)$ is said to be *induced* if X arises from A by attaching the cone on $K'(G, n-1)$ via a map $K'(G, n-1) \to A$.

If A is simply connected then every cofibration $A \to X \to K'(G, n)$ is induced (see Hilton **[10]**). This is obviously not the case if A is not simply connected. For example, consider $RP^2 \to RP^4 \to K'(Z_2, 3)$. Here $\bar{G} = Z$ with Z_2-action $1 \mapsto -1$, and $\tilde{X}/\tilde{A}$ is not a Moore space.

The two main results of this section are

THEOREM A. *$A \to X \to K'(G, n)$ is induced iff $\tilde{X}/\tilde{A} = K'(\bar{G}, n)$ and $\bar{G}$ is an induced π-module.*

THEOREM B. *If $A \to X$ is as in Theorem A and A and X are nilpotent, then $A \to X$ is induced iff $\bar{G} \to G$ splits over Z and $Z\pi \otimes G$ is nilpotent.*

We begin with

THEOREM 2.1 (EASY HALF OF THEOREM A). *If the cofibration $A \to X \to K'(G, n)$ is induced then $\tilde{X}/\tilde{A} = K'(\bar{G}, n)$ and $\bar{G}$ is an induced π-module.*

PROOF. Since $A \to X$ is induced, $\exists f$: $K'(G, n-1) \to A$ such that the induced map $(A_f, A) \to (X, A)$ is a homotopy equivalence of pairs

$$\left(A_f = A \cup_f CK'(G, n-1)\right).$$

The map f has a lift f_α for each element α of π. If $\tilde{A}_f$ denotes the indicated pushout

$$\begin{array}{ccccc} \bigoplus_\alpha K'(G, n-1) & \to & \tilde{A} & & \\ \downarrow & & \downarrow & \searrow & \\ & & & & A \\ \bigoplus_\alpha CK'(G, n-1) & \to & \tilde{A}_f & & \downarrow \\ & & & \searrow & \\ & & & & A_f \end{array}$$

it is easy to see that the induced map $\tilde{A}_f \to A_f$ is a universal covering. The result follows.

The next proposition generalizes a construction due to Wall [**18**]. It allows induced modules to be used instead of free modules to kill homotopy groups of a pair.

PROPOSITION 2.2. *Given the $(n-1)$-connected cofibration $A \to X \to X/A$, $n \geqslant 3$, suppose that there is an induced module $\bar{F}$ and a π-epimorphism $\bar{F} \xrightarrow{f} H_n(\tilde{X}, \tilde{A})$. Then $\exists$ an induced cofibration $A \to X_1 \to K'(G, n)$ and a map $X_1 \to X$ which is the identity on A, such that (X, X_1) is n-connected, $H_n(\tilde{X}_1, \tilde{A})$ is $\bar{F}$, and $H_n(\tilde{X}_1, \tilde{A}) \to H_n(\tilde{X}, \tilde{A})$ is f.*

PROOF. Write $\bar{F} = Z\pi \otimes F$ and let h: $F \to Z\pi \otimes F$ be $x \mapsto 1 \otimes x$, the natural splitting of the map of $\bar{F}$ to its trivialization. From the Universal Coefficient Theorem for homotopy groups (Hilton [**10**]), we have the diagram:

$$\begin{array}{ccccc} \pi_n(F; X, A) & \to & \operatorname{Hom}(F, \pi_n(X, A)) & \to & 0 \\ \downarrow & & \downarrow & & \\ \pi_n(F; X/A) & \to & \operatorname{Hom}(F, \pi_n(X/A)) & \to & 0 \end{array}$$

Pick a pair of maps (ϕ, ψ) representing an element of $\pi_n(F; X, A)$ whose image in $\mathrm{Hom}(F, \pi_n(X, A))$ is fh. Let X_1 be the indicated pushout

$$\begin{array}{ccccc} K'(F, n-1) & & \overset{\phi}{\rightarrow} & & A \\ & & & \swarrow & \\ \downarrow & & X_1 & & \downarrow \\ & \nearrow & & \searrow & \\ CK'(F, n-1) & & \underset{\psi}{\rightarrow} & & X \\ \downarrow & & & & \downarrow \\ K'(F, n) & & \rightarrow & & X/A \end{array}$$

The proof is completed by examining the homology exact sequences of the triples (X, X_1, A) and $(\tilde{X}, \tilde{X}_1, \tilde{A})$.

DEFINITION. A *pseudo-induced cofibration*, $A \to X \to K'(G, n)$, $n \geq 3$, (X, A) $(n-1)$-connected, is one in which $\tilde{X}/\tilde{A}$ is a Moore space.

Problem. Given A and $\overline{G}$, does there exist a pseudo-induced cofibration? If so, classify all such up to equivariant homotopy type. The next theorem, as well as Theorem 3.8, shed some light on the question.

THEOREM 2.3. *In our standard situation of $A \to X \to K'(G, n)$, it is always true that $H_1(\pi, \overline{G}) = 0$. If the cofibration is pseudo-induced then $H_i(\pi, \overline{G}) = 0\ \forall\, i > 1$. If in addition the pair (X, A) is finite dimensional, then $\overline{G}$ has a finite free resolution.*

PROOF. The last statement is obvious from the exact sequence

$$\cdots \to H_{n+1}(\tilde{X}^{n+1}, \tilde{X}^n) \to H_n\tilde{X}^n \to H_n(\tilde{X}, \tilde{A}) \to 0,$$

in which $\tilde{X}^k = (\tilde{X}, \tilde{A})^k$.

Consider the spectral sequence $E^2_{st} = H_s(\pi, H_t(\tilde{X}, \tilde{A})) \Rightarrow H_{s+t}(X, A)$. If $E^2_{1,n}$ were not zero it would survive to $E^\infty_{1,n}$, which would imply that $H_{n+1}(X, A) \neq 0$.

If the cofibration is pseudo-induced then $E^2_{st} = 0\ \forall\, t > n+1$. Thus, any non-zero entry in the row E^2_{sn} would survive into E^∞, which is impossible for $s > 1$ since $H_{n+s}(X, A) = 0$.

THEOREM 2.4 (OTHER HALF OF THEOREM A). *If $\overline{G}$ is an induced π-module then the pseudo-induced cofibration is actually induced.*

PROOF. In Proposition 2.2 choose $f\colon \overline{F} \to H_n(\tilde{X}, \tilde{A})$ to be the identity map. The induced map $X_1 \to X$ is easily checked to be a homotopy equivalence.

In combination with Theorem 2.1, this last result says that a pseudo-induced cofibration is induced iff $\overline{G}$ is induced. Examples of pseudo-induced cofibrations which are not induced will be constructed at the end of §3.

Given the standard situation $A \to X \to K'(G, n)$, we remarked earlier that if A (and therefore X) is simply connected then the cofibration is induced. This suggests the conjecture that if A and X are nilpotent then the cofibration is induced. The conjecture is false; examples will be given at the end of §3. However, we can now prove Theorem B.

PROOF OF THEOREM B. Only if: trivial, by Theorem 2.1.

If: By Theorem 2.3, $H_1(\pi, \overline{G}) = 0$. The hypotheses of Corollary 1.8 are now satisfied and so $\overline{G}$ is induced.

Consider again the spectral sequence $E^2_{st} = H_s(\pi, H_t(\tilde{X}, \tilde{A})) \Rightarrow H_{s+t}(X, A)$. $E^2_{st} = 0$ if $t = n$ and $s > 0$. $E^2_{0,n+1} = 0$: otherwise $E^{\infty}_{0,n+1}$ would be nontrivial and so then would $H_{n+1}(X, A)$. $H_{n+1}(\tilde{X}, \tilde{A})$ is therefore both perfect and nilpotent. Its vanishing allows us to repeat the argument with $E^2_{0,n+2}$. We conclude inductively that $H_{n+i}(\tilde{X}, \tilde{A}) = 0 \ \forall\, i \geq 1$, so the cofibration is pseudo-induced. Applying Theorem 2.4 completes the proof.

Proposition 1.10 shows that $Z\pi \otimes G$ can be nilpotent in interesting cases.

3. Equivariant cell complex constructions.

DEFINITION. A (free) π-complex is a CW complex X on which π acts cellularly (and freely). It will usually be simply connected, and the action will usually be either free or free-based (which means there is precisely one point of X which is fixed, and everything in π fixes it).

If a universal cover cofibration $\tilde{A} \to \tilde{X} \to K'(M, n)$ exists, as in the previous section, then $\tilde{X}/\tilde{A}$ is a simply connected π-complex having the π-module M as the only nonvanishing homology group. We shall refer to such a complex as an *equivariant Moore space of type* (M, n, π). It behooves us to discuss, first of all, whether such a complex exists for a given triple (M, n, π), and, if so, whether it can be realized as the cofiber of such a cofibration, given A.

The first question is a fairly well-known problem, first asked by Steenrod in 1960 and considered by Swan [17] (who required in addition a finite complex). We answer the question affirmatively in Theorem 3.4 for certain nilpotent modules M over finitely generated nilpotent π. However, the preliminary nature of 3.4 is emphasized by 3.6, which shows that the conditions imposed on the nilpotent module M are quite stringent. Our approach is quite different from that of Arnold [1] and Smith [15] who have recently obtained partial results.

As for the second question, the answer is "yes" in three cases. If $\exists$ an equivariant Moore space of type (M, n, π) and $A = K(\pi, 1)$ then we can find an $\tilde{X}$ such that $\tilde{X}/\tilde{A}$ is the desired Moore space. Secondly, if M has a free resolution of length ≤ 1 then A can be any complex with $\pi_1 A = \pi$. Thirdly, if A and M are nilpotent and π and M satisfy the sort of finite generation and homology conditions of §1, then we may create the desired $\tilde{X}$. The question of how many $\tilde{X}$ exist up to equivariant homotopy type is left open.

Returning now to the first question, the obvious way to begin is to generalize the simple construction which shows that ordinary $K'(G, n)$ Moore spaces exist for arbitrary Abelian groups G. Given a π-module M and a free π-module resolution $\cdots \to F_1 \to F_0 \to M \to 0$, the procedure would be to realize F_0 with a wedge of n-spheres and then attach $n+1, n+2, \ldots$ cells equivariantly according to the maps in the resolution, thereby creating a π-complex X *geometrically realizing* the resolution, i.e., with $H_m(X^m, X^{m-1}) \cong F_{m-n}$ and boundary maps corresponding.

The problem with this inductive construction is that in order to kill homology by attaching cells we must know that the map $\pi_m X^m \to H_m X^m$ is a surjection, and there is no simple way to show that, as m gets large. (However the method does work if M has a resolution of length $\leqslant 2$.)

There appear, then, to be two distinct questions: Given the triple (M, n, π),

(1) Is there some particular resolution of M which can be realized?

(2) Can *every* resolution be geometrically realized?

Our first theorem shows that these questions are equivalent.

DEFINITION. A topological action of a group π on a space Y is a homomorphism from π into the group of homeomorphisms $Y \to Y$.

THEOREM 3.1. *Given the triple (M, n, π), $n \geqslant 2$, suppose there exists an ordinary Moore space $Y = K'(M, n)$ and a topological action of π on Y such that $H_n Y \cong M$. Then any free π-resolution of M can be geometrically realized.*

PROOF. Let

$$\cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

be a free π-resolution of M. The idea is to construct inductively the desired π-complex X, using Y to guarantee that the construction can proceed from step to step.

Assume then that X^m, $m \geqslant n + 1$, has been constructed such that X^m realizes the resolution up to F_{m-n}, in particular

(1) $H_n X^m \cong M$, $H_i X^m = 0$ for $m > i > n$, $H_m X^m = \ker \partial_{m-n}$.

(2) $\exists$ a map f_m: $X^m \to Y$ commuting with the π-action on each space and inducing an isomorphism on H_n.

Note that f_m is m-connected. The initial construction of X^{n+1} and f_{n+1} is routine.

The key observation is that $\pi_m X^m \to H_m X^m$ is a π-split surjection. This is evident from the diagram:

$$\begin{array}{ccccccccc}
\cdots & \to & \pi_{m+1} f_m & \to & \pi_m X^m & \to & \pi_m Y & \to & 0 \\
 & & \downarrow \approx & & \downarrow & & & & \\
0 & \to & H_{m+1} f_m & \overset{\approx}{\to} & H_m X^m & \to & 0 & &
\end{array}$$

Use this π-splitting to equivariantly attach $m + 1$ cells, creating X^{m+1}, such that $H_{m+1}(X^{m+1}, X^m) \cong F_{m-n+1}$. Since the image of $\pi_{m+1}(X^{m+1}, X^m) \to \pi_m X^m$ is precisely that of $\pi_{m+1} f_m \to \pi_m X^m$, the map f_m may be extended to a π-map f_{m+1}, completing the inductive step.

COROLLARY 3.2. *If M is π-trivial then any resolution of M can be geometrically realized.*

PROOF. Take Y to be an ordinary $K'(M, n)$ with trivial homotopy action.

The next construction is the basic technique which allows us to kill homology.

THEOREM 3.3. *Let X be a π-complex, $n \geqslant 2$. Suppose M is a submodule of $H_n X$ which can be split back to $\pi_n X$, i.e., $\exists M' \subset \pi_n X$ carried isomorphically onto M by $\pi_n X \to H_n X$. Any free chain complex*

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

can be used to attach $n + 1$ and $n + 2$ cells to X, creating a π-complex Y with $H_n Y \cong H_n X/M$. If the chain complex is acyclic, no new homology will be introduced in dimensions $n + 1$ and $n + 2$.

PROOF. Essentially routine. For each π-basis element b_i of F_0 attach an $n + 1$ cell to X via a map f_i: $S^n \to X$ obtained as the composite $F_0 \to M \to M' \subset \pi_n X$. Then attach a cell via αf_i for each $\alpha \in \pi$. Call this new complex Y^{n+1}. Since $\operatorname{Im} \partial_1 \subset \ker \partial_0$, ∂_1: $F_1 \to F_0 \cong \pi_{n+1}(Y^{n+1}, X)$ can be lifted to $F_1 \to \pi_{n+1} Y^{n+1}$. We can therefore attach $n + 2$ cells in the same manner as for $n + 1$ and create the desired complex Y.

The exact sequence of the pair (Y, X) breaks into the two sequences

$$0 \to H_{n+2}X \to H_{n+2}Y \to \ker \partial_1 \to H_{n+1}X \to H_{n+1}Y \to \ker \partial_0/\operatorname{Im} \partial_1 \to 0,$$
$$0 \to M \to H_n X \to H_n Y \to 0,$$

establishing the proposition.

THEOREM 3.4. *Suppose that π is a finitely generated nilpotent group, M is a finitely generated nilpotent π-module, and $n \geqslant 2$. If $H_1(\pi, M)$ is free Abelian and $H_2(\pi, M) = 0$ then $\exists$ an equivariant Moore space of type (M, n, π).*

PROOF. Using Theorem 1.6 we find a chain complex

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

with $\ker \partial_0/\operatorname{Im} \partial_1$ and $\ker \partial_1$ perfect. Realize F_0 with a wedge of n-spheres and attach $n + 1$ cells equivariantly using ∂_1. If the resulting π-complex is X^{n+1}, observe that $H_{n+1}X^{n+1}$ is perfect and that $H_n X^{n+1}$ appears in the sequence $0 \to P \to H_n X^{n+1} \to M \to 0$ with $P = \ker \partial_0/\operatorname{Im} \partial_1$ a perfect module.

Now P certainly satisfies the hypotheses of Theorem 1.6, so we resolve it with $F_1' \to F_0' \to P \to 0$. Since $\pi_n X^{n+1} \overset{\approx}{\twoheadrightarrow} H_n X^{n+1}$ we may use Theorem 3.3 to attach $n + 1$ and $n + 2$ cells to X^{n+1}, forming a π-complex X^{n+2}. From the proof of Theorem 3.3 we see that $H_n X^{n+2} \cong M$ and that $H_{n+1}X^{n+2}$ and $H_{n+2}X^{n+2}$ are perfect (by Corollary 1.2).

We proceed inductively to construct a π-complex X^{n+m}, $m \geqslant 2$, satisfying

(1) $H_n X^{n+m} \cong M$, $H_i X^{n+m} = 0$ for $n + m - 2 \geqslant i \geqslant n + 1$, $H_{n+m}X^{n+m}$ and $H_{n+m-1}X^{n+m}$ are perfect π-modules.

Given X^{n+m-1}, the idea is to use Theorem 3.3 to kill $H_{n+m-2}X^{n+m-1}$ by attaching $n + m - 1$ and $n + m$ cells according to a resolution obtained by Theorem 1.6. This creates X^{n+m} satisfying (1). The crucial point is that the homology to be killed is perfect, and all lower dimensional homology is nilpotent (being either 0 or M). Thus, the isomorphism hypothesis of Theorem 3.3 is satisfied because of the follwoing lemma:

LEMMA 3.5. *Let X be a simply connected π-complex and suppose that $H_i X$ is nilpotent for $2 \leqslant i \leqslant n - 1$. If M is any finitely generated perfect submodule of $H_n X$, then $\exists$ a submodule $M' \subset \pi_n X$ carried isomorphically onto M by the Hurewicz map.*

PROOF. Let $\mathfrak{N}$ denote the Serre class of nilpotent π-modules. By elementary $\mathcal{C}$-theory the map $\pi_n X \to H_n X$ is a $\mathfrak{N}$-isomorphism. This means that $\exists$ nilpotent modules N_1 and N_2 and an exact sequence

$$0 \to N_1 \to \pi_n X \xrightarrow{h} H_n X \to N_2 \to 0.$$

M goes to 0 in N_2 so it lies in the image of h. Let $B = h^{-1}(M) \subset \pi_n X$. We have then the exact sequence $0 \to N_1 \to B \xrightarrow{h} M \to 0$, which yields $\forall j > 0$,

$$0 \to I^j N_1 \to I^j B \to M \to 0,$$

by Corollary 1.3. But $I^j N_1 = 0$ for large enough j, and the proof is complete.

PROPOSITION 3.6. *Suppose that π is a finitely generated nilpotent group and M is a finitely generated nilpotent π-module such that $H_1(\pi, M)$ is free Abelian and $H_2(\pi, M) = 0$. Then one of the following three alternatives holds*:

(1) *π is infinite cyclic.*

(2) *π is infinite but not infinite cyclic, and $M = 0$.*

(3) *π is finite and $H_1(\pi, M) = 0$.*

PROOF. If π is finite then it is not hard to prove that $H_1(\pi, M)$ is finite for any finitely generated M (show it first for $\pi = Z_p$, then use the Serre spectral sequence of $0 \to Z_p \to \pi \to \sigma \to 0$ and induction). Hence case (3).

If π is infinite but not infinite cyclic, find a surjection $\pi \to Z$ with kernel $\sigma \neq 0$. The Serre spectral sequence of this fibration is $E^2_{st} = H_s(Z, H_t(\sigma, M)) \Rightarrow H_{s+t}(\pi, M)$. Since $H_2(\pi, M) = 0$, $H_1(Z, H_1(\sigma, M)) = 0$. But for any $Z(Z)$-module N, $H_1(Z, N)$ is the submodule consisting of elements fixed by the $Z(Z)$ action. Since $H_1(\sigma, M)$ is nilpotent over $Z(Z)$, this submodule is nontrivial if $H_1(\sigma, M)$ is nontrivial. We conclude that $H_1(\sigma, M) = 0$.

Consequently, $E^2_{01} = 0$, so that $E^2_{10} = H_1(Z, H_0(\sigma, M))$ is free Abelian. By Lemma 3.7 (below), $H_0(\sigma, M)$ is free Abelian. But now we can apply Corollary 1.9 to M and σ. σ is nontrivial so M must be 0.

Notice that only case (3) is of interest in constructing equivariant Moore spaces. Must M be cohomologically trivial in that case? If so, it has a free resolution of length $\leqslant 1$, and we do not need 3.4 to construct the equivariant Moore space.

LEMMA 3.7. *If M is a finitely generated $Z(Z)$-module and $H_1(Z, M)$ is free Abelian, then M is free Abelian.*

PROOF. Note first that any submodule of M has free Abelian H_1. Thus, any submodule of M consisting only of fixed points must be free Abelian.

Now proceed by induction on the nilpotency length of M. Since there is an epimorphism $M \to IM$ whose kernel consists of fixed points, the proof is complete.

The remainder of this section is concerned with the relative case: constructing equivariant cofibrations whose cofiber is a space of type (M, n, π).

THEOREM 3.8. *Let A be a CW complex with $\pi_1 A = \pi$, M a π-module, $n > 2$. If either of the following conditions hold then $\exists$ an equivariant cofibration $\tilde{A} \to \tilde{X} \to K'(M, n)$.*

(1) *M has a free resolution of length $\leqslant 1$.*

(2) *π is finite nilpotent and M is finitely generated nilpotent with $H_1(\pi, M) = H_2(\pi, M) = 0$.*

Note. The homology condition in (2) is always satisfied in pseudo-induced cofibrations (Theorem 2.3). This theorem says that, conversely, in the nilpotent case any module satisfying $H_i(\pi, M) = 0\ \forall\, i \geq 1$ can appear as the cofiber of a pseudo-induced cofibration.

PROOF OF 3.8. Take a resolution

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0.$$

In case (1) it will be a free resolution; in case (2) it will be a resolution by frees and perfects (Theorem 1.6).

Call one of the points of the fiber of $\tilde{A} \to A$ "the" basepoint, and pick a maximal tree T in the one-skeleton of $\tilde{A}$. T contains all of the fiber. Attach to the basepoint one n-sphere for each basis element of F_0, then put the same bouquet of spheres at every other point of the fiber. There is an obvious free π-action on the resulting space $\tilde{X}^n$, and $H_n(\tilde{X}^n/\tilde{A}) = F_0$.

F_0 is a direct summand of $\pi_n\tilde{X}^n$, so we may use the map ∂_1: $F_1 \to F_0$ and the maximal tree T to attach one $n+1$ cell for each π-basis element of F_1. Then attach other $n+1$ cells to create $\tilde{X}^{n+1}$ with $\pi_{n+1}(\tilde{X}^{n+1}, \tilde{X}^n) = F_1$ and π acting freely on $\tilde{X}^{n+1}$. In case (1) we are now done: take $\tilde{X} = \tilde{X}^{n+1}$.

In case (2) the argument now mimics that of Theorem 3.4. $H_n\tilde{X}^{n+1} = H_n\tilde{A} \oplus F_0/\mathrm{Im}\ \partial_1$. $F_0/\mathrm{Im}\ \partial_1$ has the perfect submodule $\ker \partial_0/\mathrm{Im}\ \partial_1$. Find a resolution by frees and perfects for this module and use Proposition 3.3 to kill it, obtaining, say, $\tilde{X}^{n+2}$. Since M and A are nilpotent, $H_i\tilde{X}^{n+2}$ is nilpotent for $2 \leq i < n$, while $H_{n+1}\tilde{X}^{n+2}$ and $H_{n+2}\tilde{X}^{n+2}$ are pre-nilpotent. The perfect homology which has been introduced into $H_{n+1}\tilde{X}^{n+2}$ can, by Lemma 3.5, be killed by the method of Proposition 3.3, returning the homology to $H_{n+1}\tilde{A}$. The inductive argument proceeds exactly as in Theorem 3.4, yielding $\tilde{X}$ with $\tilde{X}/\tilde{A} = K'(M, n)$.

PROPOSITION 3.9. *If $A = K(\pi, 1)$ and M is any π-module for which $\exists$ a space of type (M, n, π), then $\exists$ an equivariant cofibration $\tilde{A} \to \tilde{X} \to K'(M, n)$.*

PROOF. Use Theorem 3.1 to obtain a $K'(M, n)$ with free-based π-action, and give $\tilde{A} \times K'(M, n)$ the diagonal action. Then the cofibration $\tilde{A} \to \tilde{A} \times K'(M, n) \to K'(M, n)$ is π-equivariant.

We can now construct examples of pseudo-induced cofibrations. The easiest examples of modules M with vanishing homology occur over finite π when M is cohomologically trivial.

For example, if $\pi = Z_2$ one can make Z_8 a nilpotent cohomologically trivial π-module by the action $1 \mapsto 5$. Many similar examples can be constructed. Since cohomologically trivial modules over finite groups have resolutions of length ≤ 1, equivariant Moore spaces of type (M, n, π) need have cells only in dimensions n and $n+1$.

The following examples show that the hypotheses of Theorem B, §2, cannot be weakened.

EXAMPLES 3.10. Let $\pi = Z_2$, A any nilpotent CW complex with $\pi_1 A = \pi$. If $M = Z_8$ with action $1 \mapsto 5$, $M/IM = Z_4$ and $Z\pi \otimes Z_4$ is nilpotent. Construct the pseudo-induced cofibration $\tilde{A} \to \tilde{X} \to K'(M, n)$, yielding $A \to X \to K'(Z_4, n)$ with A and X nilpotent. The cofibration fails to be induced because $M \to M/IM$ is not Z-split.

On the other hand, suppose M is Z_3 with trivial (therefore nilpotent) Z_2-action. $M \to M/IM$ is certainly Z-split. Construct again $\tilde{A} \to \tilde{X} \to K'(M, n)$ yielding $A \to X \to K'(Z_3, n)$. The cofibration fails to be induced because $Z\pi \otimes Z_3$ is not nilpotent.

4. Pre-nilpotent spaces. In this section we use the techniques of the previous sections to prove a theorem due to Dror and Dwyer **[7]** characterizing pre-nilpotent spaces. Our proof has the advantage of being very geometrical.

DEFINITION. A space X is said to be pre-nilpotent if $\exists$ a homology equivalence $X \to N$ where N is nilpotent.

DEFINITION. We will say that a space X is of *finite type* if:

(1) $\pi_1 X$ is a finitely generated group.

(2) $H_j X$ is a finitely generated group for all j.

(3) $H_j \tilde{X}$ is a finitely generated $\pi_1 X$-module for all j.

(4) $H_j \tilde{X}_\Gamma$ is a finitely generated $\pi_1 X$-module for all j. $\tilde{X}_\Gamma$ denotes the cover of X having fundamental group the maximal $\pi_1 X$-perfect subgroup of $\pi_1 X$. In other words, $\pi_1 X$ acts on its normal subgroups by conjugation, and Γ is the largest of those which are perfect with respect to this action.

For example, a locally finite space is of finite type (a locally finite space is one which has a finite number of cells in each dimension). Also, a space that satisfies the first three conditions and has a nilpotent fundamental group is of finite type.

PROPOSITION 4.1. *Let A be a space of finite type and suppose that there is a homology equivalence $A \to X$ in which X is nilpotent. Then X is of finite type.*

PROOF. By a result of Stallings **[16]** (see also Dror **[6]**) the map $\pi_1 A \to \pi_1 X$ is a surjection, so $\pi_1 X$ is also finitely generated. It remains to show that each $H_j \tilde{X}$ is finitely generated over $\pi_1 X$. It is well known for nilpotent spaces X that $H_* X$ is finitely generated (over Z) iff $\pi_* X$ is finitely generated (over Z). Thus we conclude that $\pi_* X = \pi_* \tilde{X}$ is finitely generated over Z. Therefore $H_* \tilde{X}$ is finitely generated.

Proposition 4.1 assures us that, in discussing pre-nilpotent spaces, we do not have to leave the class of spaces of finite type.

Our first theorem is the easy half of the characterization of pre-nilpotent spaces.

THEOREM 4.2. *If A is a pre-nilpotent space of finite type then $\pi_1 A$ acts pre-nilpotently on $H_* \tilde{A}_\Gamma$.*

PROOF. Find a homology equivalence to a nilpotent X, $A \to X$. Assume that the map is a cofibration. By Stallings' result in **[16]**, the kernel of $\pi_1 A \to \pi_1 X$ is $\Gamma \pi_1 A = \Gamma_j \pi_1 A$ for some j. The inverse image of A in $\tilde{X}$ is therefore $\tilde{A}_\Gamma$.

Consider the spectral sequence $E^2_{st} = H_s(\pi_1 X, H_t(\tilde{X}, \tilde{A}_\Gamma)) \Rightarrow H_{s+t}(X, A)$. Because the pair (X, A) has no homology, $E^2_{0,2} = 0$. $H_2(\tilde{X}, \tilde{A}_\Gamma)$ is therefore a perfect $\pi_1 X$-module, and we have enough finite generation to use Theorem 1.1. The entire second row in the E^2 term is trivial, so we deduce that $H_3(\tilde{X}, \tilde{A}_\Gamma)$ is perfect, and so on inductively. These modules are also perfect over $\pi_1 A$ because of the surjection on fundamental groups. Examination of $\cdots \to H_{n+1}(\tilde{X}, \tilde{A}_\Gamma) \to H_n\tilde{A}_\Gamma \to H_n\tilde{X} \to \cdots$ completes the proof.

THEOREM 4.3 (CONVERSE OF 4.2). *Let A be a space of finite type in which $\pi_1 A$ acts pre-nilpotently on $H_*\tilde{A}_\Gamma$. Then $\exists$ a nilpotent space X and a homology equivalence $A \to X$.*

PROOF. In order to see the simplicity of the basic idea we will assume that $\Gamma\pi_1 A = 0$ and thus that $\pi_1 A \equiv \pi$ is nilpotent. In an appendix we will reduce the general case to this situation. We are given, then, that π acts pre-nilpotently on $H_*\tilde{A}$.

Begin with the short exact sequence $0 \to \Gamma H_2\tilde{A} \to H_2\tilde{A} \to H_2\tilde{A}/\Gamma H_2\tilde{A} \to 0$, in which the last module is nilpotent by assumption. Use Theorem 1.6 to form a "resolution by frees and perfects" of $\Gamma H_2\tilde{A}$. Using this resolution, attach three- and four-cells to $\tilde{A}$ as in Proposition 3.3, to form a free π-complex $\tilde{A}(2)$. The cofiber $\tilde{A}(2)/\tilde{A}$ has homology only in dimensions three and four, where it is perfect. Thus, $A \to A(2)$ is a homology equivalence. Also observe that $H_2\tilde{A}(2)$ is nilpotent and that all higher homology modules of $\tilde{A}(2)$ are pre-nilpotent by Corollary 1.4.

We now proceed inductively, patching up the π-action on each homology group in order. We assume a homology equivalence $A \to A(n-1)$ where π acts nilpotently on $H_i\tilde{A}(n-1)$ for $2 \leqslant i \leqslant n-1$ and pre-nilpotently on the higher homology. From a resolution of frees and perfects of $\Gamma H_n\tilde{A}(n-1)$ we use Proposition 3.3 to attach cells of dimension $n+1$ and $n+2$ in such a way that π acts freely on the resulting complex $\tilde{A}(n)$, $H_n\tilde{A}(n)$ is a nilpotent π-module, the higher homology remains pre-nilpotent, and $A(n-1) \to A(n)$ is a homology equivalence. The only detail to be checked is the isomorphism hypothesis of Proposition 3.3, i.e., that $H_n\tilde{A}(n-1)$ can be split back to $\pi_n\tilde{A}(n-1)$. But we have already verified this in Lemma 3.5. The theorem is proved by setting $X = \varinjlim A(n)$.

If π is a finite p-group the cell construction is particularly elegant. In that case, each module $\Gamma H_n\tilde{A}(n-1)$ is cohomologically trivial (Rim [13]). Thus, the resolution by frees and perfects is in fact a free resolution. No new perfect modules are introduced into the higher homology as the "bad part" of $H_n\tilde{A}(n-1)$ is neatly excised.

We now give the reduction of the general case to the nilpotent fundamental group case used in the proof of Theorem 4.3.

APPENDIX. We are in the general case, where $\pi_1 A$ acts pre-nilpotently on $H_*\tilde{A}_\Gamma$. Notice that $H_1\tilde{A}_\Gamma$ is in fact perfect. This is because $\pi_1 A$ acts perfectly on Γ, therefore on $\Gamma/[\Gamma, \Gamma] = H_1\Gamma = H_1\tilde{A}_\Gamma$. Let $\pi = \pi_1 A/\Gamma$.

The idea is essentially the same as before: to attach cells to $\tilde{A}_\Gamma$ according to a free-perfect resolution of $H_1\tilde{A}_\Gamma$. The low dimension complicates the situation.

LEMMA 4.4. *Let $\tilde{B}$ be a (free) π-complex with perfect fundamental group ($\neq \pi$). Then two- and three-cells can be attached to $\tilde{B}$ to create a simply connected (free) π-complex $\tilde{D}$ with the inclusion $\tilde{B} \to \tilde{D}$ a homology equivalence.*

PROOF. Attach two-cells equivariantly to kill $\pi_1\tilde{B}$ and create a π-complex $\tilde{C}$. We have the sequence $0 \to H_2\tilde{B} \to H_2\tilde{C} \to H_2(\tilde{C}, \tilde{B}) \to 0$. $H_2(\tilde{C}, \tilde{B})$ is a free π-module, finitely generated if $\pi_1 B$ is finitely generated, so the sequence splits. Since $\pi_2\tilde{C} = H_2\tilde{C}$ we may attach three-cells equivariantly to kill the submodule $H_2(\tilde{C}, \tilde{B}) \subset H_2\tilde{C}$, thereby creating $\tilde{D}$. The composite $H_2\tilde{B} \to H_2\tilde{C} \to H_2\tilde{D}$ is an isomorphism.

Now, take a π-resolution of $H_1\tilde{A}_\Gamma$ by frees and perfects

$$F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} H_1\tilde{A}_\Gamma \to 0.$$

Since $\pi_1\tilde{A}_\Gamma \to H_1\tilde{A}_\Gamma$ is surjective we may attach two-cells to kill $H_1\tilde{A}_\Gamma$ via ∂_0. This complex, $\tilde{B}$ say, has perfect fundamental group. Apply Lemma 4.4 to form $\tilde{D}$. Since $\pi_2\tilde{D} \xrightarrow{\approx} H_2\tilde{D}$, we can apply Proposition 3.3 and finish the construction (using ∂_1), creating $\tilde{E}$ with $\tilde{E}/\tilde{A}_\Gamma$ having perfect homology in dimensions two and three. Modding out by the π-action yields $A \to E$, a homology isomorphism with $\pi_1 E = \pi$. Therefore, E fits the assumptions of Theorem 4.3.

5. Homotopy dimension. In [4] Brown and Kahn showed that the homotopy dimension of a nilpotent space is the same as its simple cohomological dimension, if the fundamental group is finitely generated. Paraphrased somewhat, they proved

THEOREM 5.1. *Suppose that X is a nilpotent complex with a finitely generated fundamental group, and that there is an integer $n \geq 3$ such that $H_n X$ is free Abelian and all higher homology vanishes. Then X is homotopy equivalent to an n-dimensional complex.*

As an application of the techniques of §§1–3 we will present here a short simple proof of this theorem. The proof, however, depends on the following characterization of projective π-modules:

5.2. A finitely generated π-module M is projective iff it is torsion free and $H_i(\alpha, M) = 0, \forall\, i \geq 1, \forall\, \alpha$ normal in π.

By Rim's results [13] the characterization 5.2 is valid for every finite group π. Since 5.1 is known to be true, it is reasonable to conjecture that 5.2 is satisfied for any finitely generated nilpotent group.

We will prove

THEOREM 5.1′. *Theorem* 5.1 *is true if* 5.2 *holds for π and if $H_n(\tilde{X}, \tilde{X}^{n-1})$ is finitely generated.*

PROOF. Let C denote the chain complex of the universal cover $\tilde{X}$, regarded as a complex of free π-modules. If B_n is the module of n-dimensional boundaries, it suffices to show that C_n/B_n is projective and that $H_i C = 0 \ \forall\, i \geq n + 1$. (See Proposition 1.1 of [4]: also [18] and [19].) The point is that one may then take a free complement R of C_n/B_n, wedge on $n - 1$ cells to $\tilde{X}^{n-1}$ to realize R, and reattach the n cells to realize $C_n/B_n \oplus R$. The resulting complex is easily shown to be homotopy equivalent to $\tilde{X}$.

Note that $C_n/B_n = H_n(\tilde{X}, \tilde{X}^{n-1})$.

Let α be any normal subgroup of π. We use $\tilde{X}_\alpha$ to denote the cover of X with fundamental group α (when $\alpha = 0$ we suppress the subscript). By a standard spectral sequence argument, π/α acts nilpotently on $H_*\tilde{X}_\alpha$.

One checks easily that the cofiber of $X^{n-1} \to X$ is $K'(F, n)$, where F is free Abelian. Look at the spectral sequence

$$E^2_{st} = H_s(\pi/\alpha, H_t(\tilde{X}_\alpha, \tilde{X}^{n-1}_\alpha)) \Rightarrow H_{s+t}(X, X^{n-1}).$$

Set $M_\alpha = H_n(\tilde{X}_\alpha, \tilde{X}^{n-1}_\alpha)$. As in Theorem 2.5, $H_1(\pi/\alpha, M_\alpha) = 0$. $M_\alpha \to M_\alpha/IM_\alpha$ is Z-split. Therefore, $H_s(\pi/\alpha, M_\alpha) = 0 \ \forall\, s \geq 1$ (Proposition 1.5), which means that the entire row $E^2_{sn} = 0$. This forces $H_0(\pi/\alpha, H_{n+1}(\tilde{X}_\alpha, \tilde{X}^{n-1}_\alpha))$ to be trivial, so $H_{n+1}\tilde{X}_\alpha$ is both perfect and nilpotent. Consequently the entire row $E^2_{s,n+1} = 0$. The obvious inductive argument shows that $H_{n+i}\tilde{X}_\alpha = 0 \ \forall\, i \geq 1$. In particular, $H_{n+i}\tilde{X} = 0 \ \forall\, i \geq 1$.

We have established that $\tilde{X}^{n-1}_\alpha \to \tilde{X}_\alpha$ is pseudo-induced. By Theorem 2.5, $H_i(\alpha, M) = 0 \ \forall\, i \geq 1$.

The only thing left is to show that M is torsion free.

Let G be any Abelian group; consider it a π-trivial module. Examine the spectral sequence $E^2_{st} = H_s(\pi, H_t(\tilde{X}, \tilde{X}^{n-1}; G)) \Rightarrow H_{s+t}(X, X^{n-1}; G)$. Note that $H_n(\tilde{X}, \tilde{X}^{n-1}; G) = M \otimes G$ and $H_{n+1}(\tilde{X}, \tilde{X}^{n-1}; G) = M * G$. Since F is free Abelian, $H_{n+i}(X, X^{n-1}; G) = 0 \ \forall\, i \geq 1$. Hence $H_1(\pi, M \otimes G) = 0$ (Theorem 2.5 again).

Now, the sequence $0 \to IM \to M \to M/IM \to 0$ is Z-split so $0 \to IM \otimes G \to M \otimes G \to M/IM \otimes G \to 0$ is also Z-split. But G is π-trivial, so $IM \otimes G = I(M \otimes G)$. Therefore, $M \otimes G \to M \otimes G/I(M \otimes G)$ is Z-split, and thus $H_i(\pi, M \otimes G) = 0 \ \forall\, i \geq 1$ (Proposition 1.5).

Once again we may conclude that $E^2_{0,n+1} = 0$, so that $H_{n+1}(\tilde{X}, \tilde{X}^{n-1}; G)$ is perfect. But $H_{n+1}(\tilde{X}; G) \xrightarrow{\approx} H_{n+1}(\tilde{X}, \tilde{X}^{n-1}; G)$ and $H_{n+1}(\tilde{X}; G)$ is nilpotent since G is nilpotent. Therefore, $M * G = 0$ for every Abelian group G, and the proof is complete.

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