Invent. math. 131, 25–83 (1998)

Inventiones mathematicae

Equivariant cohomology, Koszul duality, and the localization theorem

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Oblatum 21-VI-1996 & 31-I-1997

1 Introduction

(1.1) This paper concerns three aspects of the action of a compact group K on a space X. The first is concrete and the others are rather abstract.

(1) *Equivariantly formal spaces*. These have the property that their cohomology may be computed from the structure of the zero and one dimensional orbits of the action of a maximal torus in K.

(2) *Koszul duality*. This enables one to translate facts about equivariant cohomology into facts about its ordinary cohomology, and back.

(3) *Equivariant derived category*. Many of the results in this paper apply not only to equivariant cohomology, but also to equivariant intersection cohomology. The equivariant derived category provides a framework in both of these may be considered simultaneously, as examples of "equivariant sheaves".

We treat singular spaces on an equal footing with nonsingular ones. Along the way, we give a description of equivariant homology and equivariant intersection homology in terms of equivariant geometric cycles.

Most of the themes in this paper have been considered by other authors in some context. In Sect. 1.7 we sketch the precursors that we know about. For most of the constructions in this paper, we consider an action of a compact connected Lie group K on a space X, however for the purposes of the introduction we will take $K = (S^1)^n$ to be a torus.

* partially supported by N.S.F. grant DMS 9303550 and DMS 9304580
** partially supported by N.S.F. grant DMS 8913971
*** partially supported by N.S.F. grant DMS 9106522

(1.2) Equivariantly formal spaces. Suppose a compact torus K acts on a (possibly singular) space X. The equivariant cohomology of X is the cohomology $H_K^*(X;\mathbb{R}) = H^*(X_K;\mathbb{R})$ of the *Borel construction* $X_K = X \times_K EK$. Let $\pi : X_K \to BK$ denote the fibration of X_K over the classifying space BK with fiber $\pi^{-1}(b) = X$. We say that X is equivariantly formal if the spectral sequence

$$H^p(BK; H^q(X;\mathbb{R})) \Longrightarrow H_K^{p+q}(X;\mathbb{R}) \tag{1.2.1}$$

for this fibration collapses. (This condition is discussed at length in [B3] Sect. XII.) The class of equivariantly formal spaces is quite rich: it includes (1) symplectic manifolds with Hamiltonian K-actions, (2) any space with a K-invariant CW decomposition, and (3) any K-space whose (ordinary) cohomology vanishes in odd degrees (cf. Sect. 14.1).

Now suppose that X is a (possibly singular) complex projective algebraic variety with an algebraic action of a complex torus $T = (\mathbb{C}^*)^n$. Let $K = (S^1)^n \subset T$ denote the compact subtorus. The equivariant cohomology $H_K^*(X;\mathbb{R})$ is an algebra: it is a ring under the cup product and it is a module over the symmetric algebra $\mathbf{S} = H^*(BK;\mathbb{R}) \cong S(\mathfrak{k}^*)$ of polynomial functions on the Lie algebra $\mathfrak{k}$ of K. Suppose that T acts with only finitely many fixed points $x_1, x_2, \ldots, x_k$ and finitely many one-dimensional orbits $E_1, E_2, \ldots, E_\ell$. If X is equivariantly formal, then there is a concise and explicit formula for its equivariant cohomology algebra: Each 1-dimensional T-orbit E_j is a copy of $\mathbb{C}^*$ with two fixed points (say x_{j_0} and x_{j_∞}) in its closure. So $\bar{E}_j = E_j \cup \{x_{j_0}\} \cup \{x_{j_\infty}\}$ is an embedded Riemann sphere. The K action rotates this sphere according to some character $\Xi_j : K \to \mathbb{C}^*$. The kernel of Ξ_j may be identified,

$$\mathfrak{k}_j = \ker \Xi_j = \mathrm{Lie}(\mathrm{Stab}_K(e)) \subset \mathfrak{k}$$

with the Lie algebra of the stabilizer of any point $e \in E_j$. In Sect. 7.2 we prove

Theorem 1.2.2. *Suppose the algebraic variety X is equivariantly formal. Then the restriction mapping $H_K^*(X;\mathbb{R}) \to H_K^*(F;\mathbb{R}) \cong \bigoplus_{x_i \in F} S(\mathfrak{k}^*)$ is injective, and its image is the subalgebra*

$$H = \left\{ (f_1, f_2, \ldots, f_k) \in \bigoplus_{i=1}^{k} S(\mathfrak{k}^*) \,\middle|\, f_{j_0}|\mathfrak{k}_j = f_{j_\infty}|\mathfrak{k}_j \quad \text{for } 1 \leq j \leq \ell \right\} \tag{1.2.3}$$

consisting of polynomial functions $(f_1, f_2, \ldots, f_k)$ such that for each 1-dimensional orbit E_j, the functions f_{j_0} and f_{j_∞} agree on the subalgebra $\mathfrak{k}_j$.

Remarks. The K action admits a moment map $\mu : X \to \mathfrak{k}^*$ which takes each 1-dimensional orbit E_j to a straight line segment connecting the points $\mu(x_{j_0})$ and $\mu(x_{j_\infty})$. Let $\langle e_j \rangle \subset \mathfrak{k}^*$ denote the 1-dimensional subspace of $\mathfrak{k}^*$ which is parallel to this straight line segment. Then the subspace $\mathfrak{k}_j \subset \mathfrak{k}$ is the annihilator of $\langle e_j \rangle$. So the equivariant cohomology module $H_K^*(X)$ is completely determined by the "graph" $\mu(X_1) \subset \mathfrak{k}^*$ (where $X_1 \subset X$ is the union of the 0 and 1 dimensional T orbits in X.) This is made explicit in Sect. 7.5. (Actually, $\mu(X_1) \subset \mathfrak{k}^*$ may fail to be an embedded graph because the moment map images of distinct orbits may cross or even coincide.)

The ordinary cohomology of an equivariantly formal space X may be obtained from its equivariant cohomology by extension of scalars,

$$H^*(X; \mathbb{Q}) \cong \frac{H_K^*(X; \mathbb{Q})}{M \cdot H_K^*(X; \mathbb{Q})} \tag{1.2.4}$$

where M denotes the augmentation ideal in the polynomial algebra $\mathbf{S} = H_K^*(\mathrm{pt})$. So Theorem 1.2.2 also gives a formula for the ordinary cohomology (ring) in terms of the graph $\mu(X_1) \subset \mathfrak{k}^*$. Even if the ordinary cohomology groups $H^*(X; \mathbb{R})$ are known (say, from a Bialynicki-Birula decomposition), the formulas (1.2.3) and (1.2.4) have several advantages: they determine the cup product structure on cohomology, and they are functorial. If a finite group (a Weyl group, for example) acts on X in a way which commutes with the action of K, then it will take fixed points to fixed points and it will take 1-dimensional orbits to 1-dimensional orbits, so its action on $H_K^*(X; \mathbb{R})$ and on $H^*(X; \mathbb{R})$ are determined by these equations.

There are many situations in which an algebraic torus T acts with finitely many fixed points and finitely many 1-dimensional orbits on an algebraic variety X (e.g. toric varieties, or Schubert varieties [Ca]). But there exist formulas analogous to (1.2.3) which may be used in more general situations as well (cf. Sect. 6.3).

The space $X_1 \subset X$ is a kind of algebraic 1-skeleton of X. Theorem 1.2.2 is parallel to Witten's point of view on Morse theory: the cohomology of a Riemannian manifold with a generic Morse function is determined by the graph whose vertices are the critical points and whose edges are the gradient flow orbits which connected critical points whose Morse indices differ by 1.

Theorem 1.2.2 says that the equivariant cohomology of X coincides with the coordinate ring of the affine variety which is obtained from the disjoint union $\bigcup_{x_i \in F} \mathfrak{k}$ by making the following identifications: for each $j = 1, 2, \ldots, \ell$, identify the subspace $\mathfrak{k}_j$ in the copy of $\mathfrak{k}$ corresponding to the fixed point x_{j_0} with the subspace $\mathfrak{k}_j$ in the copy of $\mathfrak{k}$ corresponding to the fixed point x_{j_∞}.

(1.3) Cohomology operations. In order to apply this formula for equivariant cohomology, we need a way to identify equivariantly formal spaces. In Theorem 14.1 we list nine sufficient conditions for a space to be equivariantly formal, perhaps the most interesting of which is given in terms of cohomology operations.

If a torus $K = (S^1)^n$ acts on a reasonable space X, then for each monomial $a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ in n variables, there is a cohomology operation λ_a which lowers cohomology by degree $i = 2\Sigma a_i - 1$. If the monomial a has degree one, the operation λ_a is defined on all of $H^*(X; \mathbb{R})$: it is a primary operation. If the monomial a has degree greater than one, then λ_a is defined on elements for which the previous cohomology operations (λ_b for $b|a$) vanish, and it is defined up to an indeterminacy given by the images of these previous cohomology operations. In this case λ_a is a higher operation. In Sect. 13.4 we prove,

Theorem 1.3.1. *The space X is equivariantly formal if and only if all the cohomology operations λ_a vanish on the (ordinary) cohomology of X.*

The proof of this theorem consists of identifying these higher cohomology operations with the differentials of the spectral sequence (1.2.1).

To illustate the geometry behind the operation λ_a let us consider the case $K = S^1$. Denote by $\lambda_{(i)}$ the homology operation which raises degree by $2i - 1$ and which is adjoint to λ_{x^i}. A geometric k-chain ξ on X may be swept around by the circle orbits to produce a K-invariant $(k+1)$-chain $S\xi$. If ξ was a cycle then $S\xi$ is also. It is easy to see that the resulting homology class $[S\xi]$ depends only on the homology class $[\xi]$ of ξ. So we obtain a homomorphism $\lambda_{(1)} : H_k(X;\mathbb{R}) \to H_{k+1}(X;\mathbb{R})$. If $\lambda_{(1)}([\xi])) = 0$ then $S\xi$ is the boundary of some chain, call it $\partial^{-1}S\xi$. Then $S\partial^{-1}S\xi$ turns out to be a cycle, and the map sending $[\xi]$ to $[S\partial^{-1}S\xi]$ is $\lambda_{(2)}$. The indeterminacy comes from a choice of pre-image ∂^{-1}. Similarly, $\lambda_{(3)}$ sends $[\xi]$ to $[S\partial^{-1}S\partial^{-1}S\xi]$ and so on.

If $X = S^3$ is the three sphere with the free (Hopf) action of the circle, and if the cycle ξ is represented by a single point in X, then $S\xi$ is a single circle, which bounds a disk $\partial^{-1}S\xi$, whose sweep $S\partial^{-1}S\xi$ is X itself. So $\lambda_{(2)}([\xi]) = [X] \in H_3(X)$ is the fundamental class.

For a general torus K, all the homology operations on a class $[\xi]$ involve sweeping the cycle ξ around by subtori $K' \subset K$. Therefore we have, (cf Sect. 14.1)

Corollary 1.3.2. *Suppose the ordinary homology $H_*(X;\mathbb{R})$ is generated by classes which are representable by cycles ξ, each of which is invariant under the action of K. Then X is equivariantly formal.*

(1.4) Geometric cycles for equivariant homology and intersection homology. Let us say that an *equivariant geometric chain* is a geometric chain ξ together with a *free* action of K, and an equivariant mapping $\xi \to X$. In Sects. 4.2 and 4.6 we show that the equivariant homology groups $H_*^K(X)$ are given by the homology of the complex of equivariant geometric chains. Similarly the equivariant intersection homology $IH_*^K(X)$ is isomorphic to the homology of the subcomplex of such geometric chains which satisfy the allowability conditions for intersection homology. The proof is essentially a remark. However these descriptions will appeal to those who want to think about equivariant homology geometrically. They played a role in the development of the ideas in this paper, but are not needed for the proofs of the main results in this paper.

(1.5) Koszul duality. Suppose a compact torus K acts on a reasonable space X. Our object now is to treat the ordinary cohomology $H^*(X;\mathbb{R})$ and the equivariant cohomology $H_K^*(X;\mathbb{R})$ in a completely parallel manner. The equivariant cohomology is a module over the symmetric algebra $\mathbf{S}(\mathfrak{k}*)$ $\cong H^*(BK;\mathbb{R})$. The ordinary cohomology $H^*(X)$ is a module over the exterior algebra $\Lambda_\bullet = \bigwedge(\mathfrak{k})$ in the following way: There is a canonical isomorphism $\Lambda_\bullet \cong H_*(K;\mathbb{R})$. Let $[\xi] \in H^*(X)$ and let $\lambda \in H_*(K)$. Then $\lambda \cdot [\xi]$ is the slant product of λ with the cohomology class $\mu^*([\xi]) \in H^*(K \times X)$ (where μ denotes the K-action mapping $\mu : K \times X \to X$). For each 1-dimensional subtorus $S^1 \subset K$ the action of its fundamental class $x = [S^1] \in H_1(K)$ coincides with the primary cohomology operation λ_x described in Sect. 1.3.

There is a beautiful relation between modules over the symmetric algebra $\mathbf{S}(\mathfrak{k}^*)$ and modules over the exterior algebra $\Lambda_\bullet = \bigwedge(\mathfrak{k})$ given by the Koszul duality of Bernstein, Gelfand and Gelfand [BGG], which was further developed by Beilinson, Ginzburg, and Soergel [B][BB][G][BGS]. For any given K-space X, one might hope that the $\mathbf{S}(\mathfrak{k}^*)$ module $H_K^*(X)$ and the $\Lambda_\bullet$ module $H_*(X)$ determine each other by Koszul duality (as anticipated by V. Ginzburg [G]), but this turns out to be false: the homology $H_*(S^3)$ of the three-sphere, as a module over $\Lambda_\bullet = H_*(S^1)$, is the same whether S^1 acts trivially on S^3 or whether it acts nontrivially via the Hopf action. But the equivariant cohomology $H_K^*(S^3)$ is different for these two actions.

However the corresponding statement is true on the cochain level, up to quasi-isomorphism. It is possible to lift the action of the exterior algebra $\Lambda_\bullet$ (on the cohomology $H^*(X)$) to an appropriate model of the cochain complex $C^*(X;\mathbb{R})$, thus giving an element of the derived category $D_+(\Lambda_\bullet)$ of cochain complexes which are (differential graded) $\Lambda_\bullet$-modules, in such a way that elements of $\mathfrak{k}$ lower degree by one. This "enhanced" cochain complex $C^*(X) \in D_+(\Lambda_\bullet)$ is a finer invariant of the K-space X than the $\Lambda_\bullet$–module $H^*(X)$. For example, it contains the information of all the cohomology operations from Sect. 1.3 whereas the $\Lambda_\bullet$ action on $H^*(X)$ contains only the information of the primary cohomology operations.

Similarly, it is possible to lift the action of the symmetric algebra $\mathbf{S} = S(\mathfrak{k}*)$ (on the equivariant cohomology $H_K^*(X;\mathbb{R})$) to an action on an appropriate model for the equivariant cochain complex $C_K^*(X;\mathbb{R})$, in such a way that that elements $x \in \mathfrak{k}^* \subset S(\mathfrak{k}^*)$ raise degrees by two. This gives rise to an element of the derived category $D_+(\mathbf{S})$ of differential graded $\mathbf{S}$-modules. In Sects. 8.4 and 11.2 we show,

Theorem 1.5.1. *The Koszul duality functor* $h : D_+(\mathbf{S}) \to D_+(\Lambda_\bullet)$ *is an equivalence of categories, with an explicit quasi-inverse functor* $t : D_+(\Lambda_\bullet) \to D_+(\mathbf{S})$. *For any* K-space X, *the functor* h *takes* $C_K^*(X)$ *to* $C^*(X)$.

The functors h and t are modifications of the Koszul duality isomorphisms of Bernstein, Gelfand, and Gelfand (who consider derived categories in which $\Lambda_\bullet$ and $\mathbf{S}$ act without degree shifts). This theorem implies that knowledge of the element $C_K^*(X) \in D_+(\mathbf{S})$ determines the element $C^*(X) \in D_+(\Lambda_\bullet)$ and hence determines the ordinary cohomology $H^*(X)$

together with all its higher cohomology operations. Similarly, knowledge of $C^*(X) \in D_+(\Lambda_\bullet)$ determines $C_K^*(X)$ and therefore also the equivariant cohomology $H_K^*(X)$, even if X fails to be equivariantly formal.

However, equivariantly formal spaces have their most elegant and natural characterization in the language of derived categories. A chain complex $C^* \in D_+(\Lambda_\bullet)$ is called *split* if it is quasi-isomorphic to its cohomology, considered as a chain complex all of whose differentials are zero. In Sects. 13.4 and 9.3 we show,

Theorem 1.5.2. *A K-space X is equivariantly formal if and only if $C^*(X) \in D_+(\Lambda_\bullet)$ is split and the $\Lambda_\bullet$ action on $C^*(X) \cong H^*(X)$ is trivial. A K-space X is equivariantly formal if and only if $C_K^*(X) \in D_+(\mathbf{S})$ is split and the $\mathbf{S}$ action on $C_K^*(X) \cong H_K^*(X)$ is free.*

These two statements are Koszul dual to each other, in the sense that the Koszul duality functor h takes split elements of $D_+(\mathbf{S})$ with free $\mathbf{S}$ action to split elements of $D_+(\Lambda_\bullet)$ with trivial $\Lambda_\bullet$ action.

(1.6) Equivariant derived category. We are often interested not only in the (equivariant) cohomology of a K-space, but also in its (equivariant) intersection cohomology. These are both special cases of a much more general, object, namely (equivariant) cohomology of an *equivariant complex of sheaves*.

By an equivariant complex of sheaves, we mean an element of the equivariant derived category $D_K^+(X)$ (cf. [BB] [BL] [G] [J2]). The construction of [BL] is recalled in Sect. 5 below. The equivariant derived category enjoys a Grothendieck style formalism of push-forward and pull-back for equivariant mappings $X \to Y$. Every equivariant complex of sheaves $A \in D_K^+(X)$ has an associated equivariant cochain complex $C_K^*(X;A) \in D_+(\mathbf{S})$ (constructed in [BL] Sects. 12.3, 12.4) whose cohomology is the equivariant cohomology $H_K^*(X;A)$. In fact, Bernstein and Lunts construct an equivalence of categories $BL : D_K^+(\mathrm{pt}) \to D_+(\mathbf{S})$. The equivariant cochain complex is $C_K^*(X;A) = BL \circ c_*(A)$ where $c : X \to \mathrm{pt}$ is the constant mapping.

But the equivariant complex of sheaves $A \in D_K^+(X)$ also has an associated ordinary chain complex $C^*(X;A) \in D_+(\Lambda_\bullet)$, whose cohomology is the ordinary cohomology $H^*(X;A)$. In Sect. 11.2 we show,

Theorem 1.6.1. *For any $A \in D_K^+(X)$ the Koszul duality functor h takes the equivariant cochain complex $C_K^*(X;A) \in D_+(\mathbf{S})$ to the ordinary cochain complex $C^*(X;A) \in D_+(\Lambda_\bullet)$.*

Almost all of the theory about equivariantly formal spaces goes through in the context of equivariant complexes of sheaves. (The only exception is the cycle-theoretic result of Cor. 1.3.2). We call an equivariant complex of sheaves *equivariantly formal* if the spectral sequence for its equivariantly cohomology,

$$E_2^{pq} = H_K^p(\mathrm{pt}) \otimes H^q(X;A) \Longrightarrow H_K^{p+q}(X;A)$$

degenerates at E_2.

Theorem 1.6.2. *Suppose a compact torus $K = (S^1)^n$ acts on a reasonable space X. Let $A \in D_K^+(X)$ be an equivariant complex of sheaves. Then the following statements are equivalent:*

(1) *A is equivariantly formal*

(2) *$C^*(X;A) \in D_+(\Lambda_\bullet)$ is split and the $\Lambda_\bullet$ action is trivial*

(3) *$C_K^*(X;A) \in D_+(\mathbf{S})$ is split and the $\mathbf{S}$ action is free*

(4) *All the (primary and higher) cohomology operations λ_a vanish on $H^*(X;A)$*

(5) *The edge morphism $H_K^*(X;A) \to H^*(X;A)$ is surjective*

In this case, we also have

(6) *The ordinary cohomology is given by extension of scalars,*

$$H^*(X;A) \cong H_K^*(X;A) \otimes_{\mathbf{S}} \mathbb{R}$$

(7) *The restriction mapping $H_K^*(X;A) \to H_K^*(F;A)$ is injective, and its image is the kernel*

$$H_K^*(X;A) \cong \ker[H_K^*(F;A) \xrightarrow{\delta} H_K^*(X_1,F;A)]$$

(*Here $F \subset X$ denotes the fixed point set and $X_1 \subset X$ denotes the union of the* 1-*dimensional orbits of K*).

Part (7) is a refinement of the localization theorem (cf Sect. 6.2) which asserts that the mapping $H_K^*(X;A) \to H_K^*(F;A)$ is an isomorphism after localizing at an appropriate multiplicative set. It is the key step in the proof of (1.2.2). In theorem 14.1 we give additional sufficient conditions which guarantee that $A \in D_K^+(X)$ is equivariantly formal.

(1.7). Localization theorems and implications among the above conditions have been studied in various situations for the last 35 years. The following list is not meant to represent a historically accurate account of the subject, but it includes the references which we are most familiar with. In his fundamental paper [B3] (1960), Borel drew attention (Sect. XII Theorem 3.4) to the possible degeneration of the spectral sequence for equivariant cohomology, and its consequences. He also showed that the equivariant cohomology $H_K^*(X - F)$ is a torsion module over $\mathbf{S}$, although he did not use precisely this language. Localization is explored systematically by Segal [Se] (1968) and by Atiyah and Segal [AS2] (1968) in the context of fixed point theorems for equivariant K-theory. See also [AS1] (1965), Hsiang [H1a], [H1b] (1970) and Quillen [Q] (Theorem 4.2) (1971). The idea to restrict attention to the 1-dimensional orbits appears in Chang and Skjelbred [CS] (1974) (Lemma 2.3), whose results are also explained in [H2] (1975). Berline and Vergne [BV] (1985) describe the localization theorem in the context of the moment map, and this same point of view was taken by Atiyah and Bott in [AB] (1984); cf. Duistermaat and Heckman [DH] (1982). Related results occur tom Dieck [tD] Sect. III Proposition 1.18, and Littleman and Procesi [LP] (1989). Some of the above implications for intersection homology are considered by Joshua [J] (1987), Kirwan [Ki] (1988) and Brylinski [Br] (1992). In a recent preprint Evens and Mirković [EM] show that the "algebraic form" of the localization theorem may be extended to arbitrary sheaves in the equivariant derived category. For rationally nonsingular toric varieties X, Cappell and Shaneson [CP] consider the module (1.2.2) although they do not explicitly identify it as the equivariant cohomology. An equivalent formula appears in [Bri], cf. [BrV].

The construction of the equivariant derived category $D_K^+(X)$ is necessarily very delicate. It was achieved in the algebraic context by Beilinson and Ginzburg [BB] [G] and by Joshua [J2], and in the topological context by Bernstein and Lunts [BL]. The idea to relate equivariant cohomology to ordinary cohomology using Koszul duality was apparently first envisioned in print by Ginzburg [G], who indicated that this case motivated much of his beautiful later work on Koszul duality. The idea that the ordinary cohomology $H^*(X)$ is determined by the equivariant cochains $C_K^*(X)$ together with its $\mathbf{S}$-module structure is known in the literature on transformation groups, especially in the case of an action by a finite torus $(\mathbb{Z}/(2))^n$ (cf. Allday and Puppe [AP2] Sect. 4 (1984), [AP1] Theorem 1.2.6; Proposition 1.3.14 (1993)). For $K = S^1$, an element in $D_+(\Lambda_\bullet)$ is known as a *mixed complex*, its Koszul dual in $D_+(\mathbf{S})$ is known as the associated Connes' double complex, whose cohomology is then called the cyclic homology (cf. Sect. 13.7).

We wish to thank A. Beilinson, T. Braden, P. Deligne, V. Lunts, and an anonymous referee for useful discussions. The authors are grateful for support from the University of Utrecht during the spring of 1994, where much of this material was worked out. The first author would also like to thank the Institute for Advanced Study in Princeton for their support during the writing of this paper.

2. Partial table of contents

3. Subanalytic sets

References for this section are [Ha] and [Hi].

(3.1) Stratifications. Let $X \subset \mathbb{R}^N$ be a (closed) subanalytic set. If Z_1 and Z_2 are closed subanalytic subsets of X, then so is $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$. If $f : X \to Y$ is a proper subanalytic mapping, and if $Z_1 \subset X$ and $Z_2 \subset Y$ are subanalytic subsets, then so are $f(Z_1)$ and $f^{-1}(Z_2)$. Every subanalytic set X admits a "subanalytic" Whitney stratification in which the closure of each stratum is a subanalytic subset of X. If $Y \subset X$ is a closed subanalytic subset then a subanalytic Whitney stratification of X may be chosen so that Y is a union of strata.

If a compact Lie group K acts subanalytically on a subanalytic set X then X admits a subanalytic Whitney stratification by K-invariant strata. For any closed invariant subanalytic subset $Y \subset X$, an invariant subanalytic Whitney stratification of X exists such that Y is a union of strata.

A subanalytic set X is *compactifiable* if there is a compact subanalytic set $\overline{X}$ and a closed subanalytic subset $Y \subset \overline{X}$ such that $X = \overline{X} - Y$. A subanalytic Whitney stratification of a compactifiable subanalytic set X is the restriction of a subanalytic Whitney stratification of $\overline{X}$ such that $Y \subset \overline{X}$ is a union of strata.

(3.2) Triangulations and chains. A subanalytic triangulation T of a subanalytic set X is a simplicial complex K (possibly infinite) and a subanalytic homeomorphism $\phi : |K| \to X$. Any two subanalytic triangulations of X admit a common subanalytic refinement.

If T is a subanalytic triangulation of a subanalytic set X, define $C_*^T(X; \mathbb{Z})$ to be the chain complex of simplicial chains with respect to the triangulation T. These chain complexes form a directed system. Define the complex of subanalytic chains to be the inverse limit,

$$C_*(X;\mathbb{Z}) = \lim_{\leftarrow} C_*^T(X;\mathbb{Z}).$$

taken over all subanalytic triangulations of X. The homology of this complex is canonically isomorphic to the singular homology of X.

(3.3) Support. Any subanalytic chain $\xi \in C_n(X)$ is a simplicial chain with respect to some subanalytic triangulation T of X and hence may be written as a formal linear combination

$$\xi = \sum_{i=1}^{r} a_i \sigma_i$$

of n-dimensional simplices $\sigma_i \in T$. Define the support

$$|\xi| = \bigcup \{\sigma_i | a_i \neq 0\}$$

of ξ to be the union of all the n-dimensional simplices which occur with nonzero multiplicity in ξ. Then $|\xi| \subset X$ is a subanalytic subset which is independent of the choice of triangulation T which was used in its definition.

(3.4) Intersection chains. For any perversity $\bar{p}$ [GM1], [GM2] and any subanalytic Whitney stratification of a subanalytic set X, the complex of subanalytic intersection chains is the subcomplex of the complex of subanalytic chains, consisting of $(\bar{p}, i)$-allowable chains,

$$I^{\bar{p}}C_i(X) = \left\{ \xi \in C_i(X;\mathbb{Z}) \middle| \begin{array}{l} \dim(|\xi| \cap S_c) \leq i - c + p(c) \\ \dim(|\partial\xi| \cap S_c) \leq i - 1 - c + p(c) \end{array} \right\} \tag{3.4.1}$$

for each stratum $S_c \subset X$ of codimension c. The intersection homology $I^{\bar{p}}H_*(X;\mathbb{Z})$ is the homology of the complex $I^{\bar{p}}C_*(X;\mathbb{Z})$.

4. Equivariant chains

Throughout this section we suppose that $\mu_X : K \times X \to X$ denotes a subanalytic action of a compact Lie group K on a subanalytic set X. Let $k = \dim(K)$ denote the dimension of K as a smooth manifold.

(4.1) Definition. *An (abstract) subanalytic equivariant chain (ξ, f) of formal dimension i on X is a subanalytic $i + k$ dimensional chain $\xi \subset \mathbb{R}^N$ (contained in some Euclidean space), together with a free action of K on $|\xi|$ and a subanalytic K-equivariant mapping $f : \xi \to X$ (modulo the obvious identification with respect to the standard inclusion $\mathbb{R}^N \subset \mathbb{R}^{N+1} \subset \ldots$ of Euclidean spaces).*

Denote by $C_i^K(X;\mathbb{Z})$ the group of subanalytic equivariant chains with formal dimension i.

The boundary $(\partial\xi, f||\partial\xi|)$ of an equivariant subanalytic chain (ξ, f) is again an equivariant subanalytic chain, so $C_*^K(X;\mathbb{Z})$ forms a chain complex. In this section we will show that the homology of this complex is canonically isomorphic to the equivariant homology $H_*^K(X,\mathbb{Z})$.

Let $(\xi, f) \in C_i^K(X;\mathbb{Z})$ be a subanalytic equivariant chain on X. Since K is compact and the action of K on $|\xi|$ is free, a choice of orientation for K determines an orientation on the chain $|\xi|/K \to X/K$. For each subanalytic chain ξ the quotient mapping $|\xi| \to |\xi|/K$ is a principal K-bundle, and is hence classified by a unique homotopy class of K-equivariant mappings,

$$\begin{tikzcd} |\xi| \arrow[r, "e"] \arrow[d] & EK \arrow[d] \\ |\xi|/K \arrow[r] & BK \end{tikzcd}$$

(where $EK \to BK$ is a smooth subanalytic model for the classifying space of K; cf. Sects. 5.1, 10.7). Let $\psi : |\xi| \to X \times EK$ be the mapping $\psi(y) = (f(y), e(y))$. Then ψ is K-equivariant with respect to the diagonal action on $X \times EK$ so it passes to a mapping $\phi : |\xi|/K \to X \times_K EK$. If $\partial\xi = 0$ then ϕ induces a homomorphism $\phi_* : H_i(|\xi|/K) \to H_i(X \times_K EK)$.

(4.2) Theorem. *The mapping ϕ_* induces an isomorphism*

$$H_*(C_*^K(X);\mathbb{Z}) \cong H_*(X \times_K EK;\mathbb{Z}) = H_*^K(X;\mathbb{Z})$$

between the homology of the complex $C_*^K(X)$ of subanalytic equivariant chains, and the equivariant homology of the space X.

(*4.3*) *Proof.* The principal K-bundle $EK \to BK$ is a limit of smooth algebraic principal bundles $\pi_n : EK_n \to BK_n$ of increasing dimension (cf. Sect. 5.1, Sect. 10.7). In particular, $X \times_K EK_n$ has a subanalytic structure. Let $C_*(X \times_K EK_n)$ denote the complex of subanalytic chains on this space. Then we obtain a homomorphism

$$F : C_*(X \times_K EK_n) \to C_*^K(X)$$

as follows. Choose a subanalytic embedding $X \times EK_n \subset \mathbb{R}^N$ into some Euclidean space. For any subanalytic chain $\eta \in C_i(X \times_K EK_n)$ let $\xi = \pi_n^{-1}(\eta)$ denote the subanalytic chain on $X \times EK_n$ whose orientation is given by following the orientation of η with the orientation of K. Then $\dim(|\xi|) = i + k$ and K acts freely on $|\xi|$. Define $F(\eta)$ to be the subanalytic chain ξ together with the mapping to X which is given by the projection $|\xi| \subset X \times EK_n \to X$. It is easy to see that the induced homomorphism $F_* : H_*(X \times_K EK) \to H_*(C_*^K(X))$ is an inverse to ϕ_*. $\square$

(4.4) Equivariant intersection chains. Now fix a subanalytic K-invariant stratification of X. (Any subanalytic stratification of X admits a K-invariant refinement, which can even be chosen so that the fixed point set is a union of strata and so that the projection mapping $X \to X/K$ is a weakly stratified mapping.) Let $\bar{p}$ denote a perversity function [GM1]. The equivariant intersection homology $I^{\bar{p}}H_*^K(X)$ was introduced in [Br], [J], [Ki1]. The following geometric construction of equivariant intersection homology is due to T. Braden and R. MacPherson:

(4.5) Definition. *The complex $I^{\bar{p}}C_*^K(X)$ of subanalytic equivariant intersection chains is the subcomplex of $C_*^K(X;\mathbb{Z})$ consisting of $(\bar{p}, i)$-allowable subanalytic equivariant chains,*

$$I^{\bar{p}}C_i^K(X;\mathbb{Z}) = \left\{ (\xi, f) \in C_i^K(X;\mathbb{Z}) \middle| \begin{array}{c} \mathrm{cod}_{\xi} f^{-1}(S_c) \geq c - p(c) \\ \mathrm{cod}_{\partial\xi} f^{-1}(S_c) \geq c - p(c) \end{array} \right\}$$

where $\mathrm{cod}_{\xi} f^{-1}(S_c)$ denotes the codimension in $|\xi|$ of the pre-image of the stratum $S_c \subset X$ of codimension c.

(4.6) Theorem. *The mapping ϕ_* induces an isomorphism between the homology of the complex $I^{\bar{p}}C_*^K(X)$ and the equivariant intersection homology $H_*^K(X)$.*

(*4.7*) *Proof*. The proof is essentially the same as that for ordinary homology. It reduces to the fact that a K-invariant subanalytic stratification of X determines a stratification of the product $X \times EK_n$ with strata of the form $S \times EK_n$, since the finite approximations $EK_n \to BK_n$ may be chosen so as to be compact smooth subanalytic (even algebraic) manifolds. This stratification passes to the quotient $X \times_K EK_n$ and the equivariant intersection homology is given by the ordinary intersection homology [Ki1]

$$IH_i^K(X) = \lim_{n\to\infty} IH_i^K(X \times_K EK_n). \qquad \square$$

5. Equivariant sheaves

(5.1) Equivariant derived category. The relation between intersection cohomology and equivariant intersection cohomology is entirely analogous to the relation between ordinary cohomology and equivariant cohomology. Both equivariant cohomology and equivariant intersection cohomology are objects in the *equivariant derived category*, which was developed in the algebraic context by Beilinson and Ginzburg ([G] Sect. 7; see also [BB]) and Joshua [J2], and in the topological context by Bernstein and Lunts [BL].

Many of the properties of equivariant cohomology (e.g. the localization theorems) apply to any element $A \in D^b_K(X)$ and are best stated in terms of the language of the equivariant derived category. In this section we recall the construction [BL] and some of the basic properties of the equivariant derived category of sheaves of vectorspaces over the real numbers $\mathbb{R}$ (although these constructions work more generally for sheaves of modules over any ring of finite cohomological dimension.)

Throughout this paper, K will denote a compact Lie group. Let us fix once and for all a smooth subanalytic model $\pi : EK \to BK$ for the classifying space of K (cf. [BL] Sect. 12.4.1) This means that $EK = \bigcup_{n=1}^{\infty} EK_n$ and $BK = \bigcup_{n=1}^{\infty} BK_n$ where $\pi_n : EK_n \to BK_n$ is a smooth compact n-universal principal K-bundle (on which K acts from the left), and that both of the inclusions $EK_n \subset EK_{n+1} \subset \dots$ and $BK_n \subset BK_{n+1} \subset \dots$ are embeddings of closed submanifolds of increasing dimension. The weak topology on EK and BK is paracompact and the embeddings $EK_n \subset EK$ and $BK_n \subset BK$ are closed.

If X is a locally compact Hausdorff space, denote by $D^b(X)$ the bounded derived category of sheaves of $\mathbb{R}$-vectorspaces on X ([Ve1], [Ve2], [GM3], [B4], [Iv], [KS]). Suppose K acts (subanalytically) on X. Consider the diagram of topological spaces,

$$X \xleftarrow{p} X \times EK \xrightarrow{q} X \times_K EK \tag{5.1.1}$$

Definition. ([*BL*] *Sect.* 2.7.2, *Sect.* 2.1.3) *An object $A \in D^b_K(X)$ is a triple $(\mathbf{A_X}, \bar{\mathbf{A}}, \beta)$ where $\mathbf{A_X} \in D^b(X)$, $\bar{\mathbf{A}} \in D^b(X \times_K EK)$, and $\beta : p^*(\mathbf{A_X}) \to q^*(\bar{\mathbf{A}})$ is an isomorphism in $D^b(X \times EK)$. A morphism $\alpha : (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \to (\mathbf{B_X}, \bar{\mathbf{B}}, \gamma)$ is a pair $\alpha = (\alpha_X, \bar{\alpha})$ where $\alpha_X : \mathbf{A_X} \to \mathbf{B_X}$ and $\bar{\alpha} : \bar{\mathbf{A}} \to \bar{\mathbf{B}}$ such that the following diagram commutes in $D^b(X \times EK)$,*

$$\begin{array}{ccc} p^*(\mathbf{A_X}) & \xrightarrow{\beta} & q^*(\bar{\mathbf{A}}) \\ {\scriptstyle p^*(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle p^*(\bar{\alpha})} \\ p^*(\mathbf{B_X}) & \xrightarrow[\gamma]{} & q^*(\bar{\mathbf{B}}) \end{array} \tag{5.1.2}$$

(5.2) Constructible sheaves. A **compactifiable K-space** X is a locally closed union of strata of an equivariant Whitney stratification of some smooth compact manifold M on which K acts smoothly. In other words, $X = \overline{X} - Y$ where $\overline{X} \subset M$ is a compact Whitney stratified K-invariant subset, and $Y \subset \overline{X}$ is a closed union of strata. A complex of sheaves $\mathbf{A_X}$ on a compactifiable K-space X is said to be (cohomologically) *constructible* with respect to the given stratification, if its cohomology sheaves $\mathbf{H^i}(\mathbf{A_X})$ are finite dimensional and are locally constant on each stratum of X. (It follows that the cohomology $H^*(X; \mathbf{A_X})$ is finite dimensional).

Definition. ([BL] Sect. 2.8) *Let X be a compactifiable K-space. The constructible bounded equivariant derived category $D^b_{K,c}(X)$ is the full subcategory of $D^b_K(X)$ consisting of triples $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta)$ such that $\mathbf{A_X} \in D^b_c(X)$ is (cohomologically) constructible.*

(5.3) Forgetful functor. The forgetful functor $D^b_K(X) \to D^b(X)$ is given by

$$(\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \mapsto \mathbf{A_X}.$$

Such an element $(\mathbf{A_X}, \bar{\mathbf{A}}, \beta)$ is said to be an *equivariant lift* of the sheaf $\mathbf{A_X} \in D^b(X)$. The constant sheaf $\mathbb{R}_\mathbf{X}$ has a canonical lift $\mathbb{R}^K_X = (\mathbb{R}_\mathbf{X}, \mathbb{R}_{\mathbf{X}\times_\mathbf{K}\mathbf{EK}}, I)$ to the equivariant derived category. For any perversity $\bar{p}$ the sheaf $\mathbf{I}^{\bar{p}}\mathbf{C_X}$ of intersection cochains (with real coefficients) has a canonical lift $I^{\bar{p}}C^K_X = (\mathbf{I^{\bar{p}}C_X}, \mathbf{I^{\bar{p}}C_{X\times_K EK}}, \beta)$ to the equivariant derived category, which is given by the construction of [Br] Sect. 2.1, [J], [Ki1] Sect. 2.11, or by the sheaf-theoretic construction of [BL] Sect. 5.2 or by the equivariant geometric intersection chains of Sect. 4.5.

The equivariant derived category $D^b_K(X)$ is triangulated and supports the usual operations ($Rf_*, Rf_!, f^*, f^!, \overset{L}{\otimes}, R\mathrm{Hom}$, and Verdier duality) in a way which is compatible with the forgetful functor $D^b_K(X) \to D^b(X)$.

(5.4) Map to a point. Suppose a compact Lie group K acts on a locally compact Hausdorff space X. The constant map $c : X \to \mathrm{pt}$ gives rise to a functor $c^K_* : D^b_K(X) \to D^b_K(\mathrm{pt})$ which we now describe. Let $c' : X\times_K EK \to BK$ and $c'' : X \times EK \to EK$ denote the projections. Both squares in the following diagram are Cartesian.

$$\begin{array}{ccccc} X & \xleftarrow{p} & X \times EK & \xrightarrow{q} & X \times_K EK \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c''} & & \downarrow{\scriptstyle c'} \\ \mathrm{pt} & \xleftarrow[r]{} & EK & \xrightarrow[\pi]{} & BK \end{array} \tag{5.4.1}$$

Let $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D^b_K(X)$. Since π and q are fiber bundles with smooth compact fiber K, the adjunction morphism $\theta : \pi^* Rc'_*(\bar{\mathbf{A}}) \to Rc''_* q^*(\bar{\mathbf{A}})$ is a quasi-isomorphism. (cf. [GM4] (2.5), [BL] Sect. A1, or [B] Sect. V, 10.7).

Definition. The pushforward $c^K_*(A)$ is given by the triple

$$c^K_*(A) = (Rc_*(\mathbf{A_X}),\ Rc'_*(\bar{\mathbf{A}}),\ Rc''_*(\beta))$$

where $Rc''_*(\beta)$ is the composition

$$\begin{tikzcd} r^*(Rc_*(\mathbf{A_X})) \arrow[d, "\cong"'] & & \pi^*(Rc'_*(\bar{\mathbf{A}})) \\ Rc''_*p^*\mathbf{A_X} \arrow[rr, "\cong", "\beta"'] & & Rc''_*q^*(\bar{\mathbf{A}}) \arrow[u, "\cong", "\theta^{-1}"'] \end{tikzcd} \tag{5.4.2}$$

If X is a compactifiable K-space then the pushforward functor c_*^K also restricts to a functor on the constructible derived category,

$$c_*^K : D^b_{K,c}(X) \to D^b_{K,c}(\mathrm{pt}).$$

(5.5) Cohomology. There are two cohomological functors from $D^b_K(X)$ to real vectorspaces: the *equivariant cohomology* of $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D^b_K(X)$ is

$$H^*_K(X; A) = H^*(X \times_K EK; \bar{\mathbf{A}}) = H^*(BK; Rc'_*(\bar{\mathbf{A}})) \tag{5.5.1}$$

and the *ordinary cohomology* of A is

$$H^*(X; A) = H^*(X; \mathbf{A_X}) = H^*(\mathrm{pt}; Rc_*(\mathbf{A_X})). \tag{5.5.2}$$

These functors factor through c_*^K, i.e., $H^*_K(X; A) = H^*_K(\mathrm{pt}; c_*^K A)$. (cf. [BL] Sect. 13.1).

The equivariant cohomology $H^*_K(X; A)$ may be computed from the Leray spectral sequence for the fibration $c' : X \times_K EK \to BK$, with

$$E^{pq}_{(2)} = H^p(BK; R^q c'_*(\bar{\mathbf{A}})) \Rightarrow H^{p+q}(X \times_K EK; \bar{\mathbf{A}}) \tag{5.5.3}$$

The sheaves $R^q c'_*(\bar{\mathbf{A}})$ are constant and the isomorphism β may be used to construct a (non-canonical) isomorphism with the constant sheaf, $R^q c'_*(\bar{\mathbf{A}}) \cong H^q(X; \mathbf{A_X}) \otimes \mathbb{R}_{\mathbf{BK}}$. This gives the spectral sequence for equivariant cohomology,

$$H^p(BK) \otimes H^q(X; \mathbf{A_X}) \Rightarrow H^{p+q}_K(X; A) \tag{5.5.4}$$

(5.6) Free and trivial actions. Suppose a compact Lie group K acts freely on X. Then the equivariant cohomology is given by $H^*_K(X) \cong H^*(X/K)$. Similarly, if $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D^b_K(X)$ is an element of the equivariant derived category, then ([BL] Sect. 2.2.5) there exists $\mathbf{B} \in D^b(X/K)$ and a quasi-isomorphism, $\bar{\mathbf{A}} \cong \bar{\pi}^*(\mathbf{B})$ (where $\bar{\pi} : X \times_K EK \to X/K$ is the quotient mapping). Hence, the equivariant cohomology is given by $H^*_K(X; A) \cong H^*(X/K; \mathbf{B})$.

If K acts trivially on X then $H_K^*(X;\mathbb{R}) \cong H_K^*(\mathrm{pt};\mathbb{R}) \otimes H^*(X;\mathbb{R})$. However if $A \in D_K^b(X)$ is an element of the equivariant derived category (and if K acts trivially on X), it does not necessarily follow that $H_K^*(X;A) \cong H_K^*(\mathrm{pt};\mathbb{R}) \otimes H^*(X;\mathbf{A_X})$, and in fact, the spectral sequence (5.5.4) for equivariant cohomology may fail to degenerate.

(5.7) Exact sequence of a pair. Let $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D_K^b(X)$ be an element of the equivariant derived category of X. For any invariant subspace $j : Y \subset X$ the equivariant cohomology groups $H_K^*(Y;A)$ and $H_K^*(X,Y;A)$ are defined as follows. The inclusion j induces an inclusion $j_K : Y \times_K EK \to X \times_K EK$ so the triple $(\mathbf{A_X}|Y, \bar{\mathbf{A}}|(Y \times_K EK), \beta|(Y \times EK))$ defines an element $j^*(A) \in D_K^b(Y)$ of the equivariant derived category of Y, whose equivariant cohomology we denote by $H_K^*(Y;A) = H^*(Y \times_K EK; j_K^*(\bar{\mathbf{A}}))$. If Y is closed and invariant in X and if $i : X - Y \to X$ denotes the inclusion of the complement of Y, then we have a similar inclusion $i_K : X - Y \to (X - Y) \times_K EK$ and we define $H_K^*(X,Y;A) = H^*(X \times_K EK; (i_K)_! i_K^*(\bar{\mathbf{A}}))$ to be the cohomology with compact supports of the restriction $\bar{\mathbf{A}}|(X - Y) \times_K EK$. Standard results in sheaf theory now give,

Proposition. *If $Y \subset X$ is a closed invariant subspace and if $A \in D_K^b(X)$ then there is a long exact sequence in equivariant cohomology,*

$$\cdots \xrightarrow{\delta} H_K^i(X,Y;A) \to H_K^i(X;A) \to H_K^i(Y;A) \xrightarrow{\delta} H_K^{i+1}(X,Y;A) \to \cdots \quad (5.7.1)$$

If Y is a closed union of invariant strata in X then it admits a neighborhood basis in X consisting of "regular" neighborhoods U for which the homomorphism induced by inclusion,

$$H_K^*(U;A) \to H_K^*(Y;A)$$

is an isomorphism. $\square$

(5.8) Dualizing complex. The dualizing complex in $D_K^b(\mathrm{pt})$ is identified with the constant sheaf $(\mathbb{R}_{\mathrm{pt}}, \mathbb{R}_{\mathbf{BK}}, \mathrm{I})$ so the dualizing complex $\mathbf{D_X} \in D_K^b(X)$ is given by $c^!\mathbb{R} = (\mathbf{D_X}, (c')^!(\mathbb{R}_{\mathbf{BK}}))$. Although this is the (usual) dualizing complex on X, it is not the (usual) dualizing complex on $X \times_K EK$. In other words, although the cohomology $H^*(X, \mathbf{D_X})$ of the dualizing complex coincides with the ordinary homology $H_*(X)$, the equivariant cohomology $H_K^*(X, \mathbf{D_X})$ of the dualizing complex does not necessarily agree with the equivariant homology $H_*^K(X) = H_*(X \times_K EK)$.

6. Localization theorems for torus actions

(6.1) Notation. Throughout Sects. 6 and 7 we assume that a compact torus $K = (S^1)^n$ acts on a compactifiable $K-$ space X. (This means that X is a locally closed union of strata of an equivariant Whitney stratification of some smooth compact manifold M on which K acts smoothly. Any such stratification admits an invariant refinement so that the fixed point set is a union of strata.) The results in this section certainly apply to more general situations, however this assumption guarantees various technical conveniences: the torus K acts smoothly on each stratum of X and, although X may fail to be compact, only finitely many orbit types occur. (Recall that two points are in the same orbit type if their stabilizers are conjugate.) Since K is abelian, this means that only finitely many stabilizers occur.

Throughout this section we use complex coefficients, and denote by

$$\mathbf{S} = H_K^*(\mathrm{pt}; \mathbb{C}) \cong \mathbb{C}[\mathfrak{k}_{\mathbb{C}}^*]$$

the equivariant cohomology of a point which we have identified (using Chern Weil theory, cf. Sect. 17.2) with the polynomials on the complexified Lie algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$.

For any point $x \in X$ let K_x denote the stabilizer of x, K_x^0 its identity component, and $\mathfrak{k}_x^{\mathbb{C}} = \mathrm{Lie}(K_x^0) \otimes_{\mathbb{R}} \mathbb{C}$ its (complexified) Lie algebra. Denote by $F \subset X$ the fixed point set of K. Let $\mathscr{P}$ denote the finite set, partially ordered by inclusion, of Lie algebras of stabilizers of points $x \in X - F$. Each $\mathfrak{l} \in \mathscr{P}$ corresponds to a subtorus $L \subset K$ with fixed point set,

$$X^{\mathfrak{l}} = \{x \in X | \mathfrak{k}_x \supseteq \mathfrak{l}\}. \tag{6.1.1}$$

Denote by

$$PL = \ker(\mathbb{C}[\mathfrak{k}_{\mathbb{C}}^*] \to \mathbb{C}[\mathfrak{l}_{\mathbb{C}}^*]) \tag{6.1.2}$$

the prime ideal in $\mathbf{S} = \mathbb{C}[\mathfrak{k}_{\mathbb{C}}^*] = H_K^*(\mathrm{pt}; \mathbb{C})$ consisting of polynomials which vanish on $\mathfrak{l}_{\mathbb{C}}$. For any module M over the polynomial ring $\mathbf{S}$ let

$$\mathrm{spt}(M) = \bigcap \{V(f) | f \cdot M = 0\} \subset \mathfrak{k}_{\mathbb{C}} \tag{6.1.3}$$

denote the support of M, where $V(f) = \{x \in \mathfrak{k}_{\mathbb{C}} | f(x) = 0\}$.

(6.2) Localization theorem: algebraic part. *Suppose X is a compactifiable K-space, and $Z \subset X$ is a closed invariant subspace containing the fixed point set F. For any element $A \in D_K^b(X)$ in the equivariant derived category, we have*

(1) *The module $H_K^*(X, Z; A)$ is a torsion module over* $\mathbf{S}$, *and its support*

$$\mathrm{spt}(H_K^*(X, Z; A)) \subset \bigcup_{x \in X - Z} \mathfrak{k}_x^{\mathbb{C}} \tag{6.2.1}$$

is contained in the union of the Lie algebras of the (finitely many) stabilizers of points $x \in X - Z$.

(2) *If $f \in \mathbb{C}[\mathfrak{k}_{\mathbb{C}}^*]$ is any function such that $V(f) \supset \bigcup_{x \in X - Z} \mathfrak{k}_x^{\mathbb{C}}$ then the localized restriction mapping*

$$H_K^*(X;A)_f \to H_K^*(Z;A)_f$$

is an isomorphism.

(3) *For any subtorus $L \subset K$, the restriction homomorphism of localized modules*

$$H_K^*(X;A)_{PL} \to H_K^*(X^L;A)_{PL} \tag{6.2.2}$$

is an isomorphism.

Now suppose that X is a compactifiable $K = (S^1)^n$-space and let

$$X_1 = \{x \in X | \operatorname{corank}(K_x) \le 1\}$$

denote the set of points consisting of 0 and 1 dimensional orbits of K. Let δ denote the connecting homomorphism in the long exact sequence for the equivariant sheaf cohomology (5.7.1) of the pair (X_1, F). The following result is a sheaf theoretic version of the lemma of Chang and Skjelbred [CS]:

(6.3) Localization theorem: topological part. *Suppose the equivariant cohomology $H_K^*(X;A)$ is a free module over* $\mathbf{S}$. *Then the sequence*

$$0 \to H_K^*(X;A) \xrightarrow{\gamma} H_K^*(F;A) \xrightarrow{\delta} H_K^*(X_1, F;A) \tag{6.3.1}$$

is exact, and in particular the equivariant cohomology of X may be identified as the submodule of the equivariant cohomology of the fixed point set which is given by $\ker(\delta)$.

If $A = \mathbb{R}$ is the constant sheaf then δ is compatible with the cup product so (6.3.1) determines the cup product structure on $H_K^*(X;\mathbb{R})$. If $A = \mathbf{IC}^\bullet$ is the intersection cohomology sheaf then δ is a $H_K^*(X;\mathbb{R})$-module homomorphism, so (6.3.1) also determines the action of (equivariant) cohomology on the (equivariant) intersection cohomology. The proofs of Theorems 6.2 and 6.3 will appear in Sect. 15.

(6.4) Examples and counterexamples. In Theorem 14.1 we list nine situations in which it is possible to guarantee that the equivariant cohomology $H_K^*(X;A)$ is a free module over $\mathbf{S}$. However, even for projective algebraic varieties, it is not always the case that the equivariant cohomology is a free $\mathbf{S}$-module, and in fact the conclusion of Theorem 6.3 fails for the following example: Let $K = S^1$ act on $\mathbb{CP}^1 \cong S^2$ by rotation with fixed points at the North and South poles. Let X be three copies of $\mathbb{CP}^1$ joined at these fixed points so as to form a "ring". Then X is a projective algebraic variety but the equivariant cohomology $H_K^*(X)$ is not a free module, and the restriction map γ (of (6.3.1)) fails to be an injection.

7. Algebraic torus actions with finitely many 1-dimensional orbits

In many cases, Theorem 6.3 may be used to give an explicit formula for the equivariant cohomology module, in terms of generators and relations which in turn can often be indexed using data from a moment map. We carry this out in the case of an algebraic torus action on a projective algebraic variety having finitely 1-dimensional orbits and whose fixed point set consists of finitely many isolated fixed points.

Throughout this section we use complex coefficients. Let $T \cong (\mathbb{C}^*)^n$ be a complex algebraic torus with maximal compact subgroup $K \cong (S^1)^n$. The inclusion of Lie algebras $\mathfrak{k} \subset \mathfrak{t} \cong \mathfrak{k}_{\mathbb{C}}$ induces an isomorphism $\mathbf{S} = \mathbb{C}[\mathfrak{k}^*] \cong \mathbb{C}[\mathfrak{t}^*]$ between the symmetric algebra of complex valued polynomials on $\mathfrak{k}$ and the symmetric algebra of complex valued polynomials on $\mathfrak{t}$ We may also identify the equivariant cohomology functors with complex coefficients, $H_K^*(\bullet) \cong H_T^*(\bullet)$.

(7.1) Algebraic torus actions. Throughout Sect. 7 we assume that X is a complex projective algebraic variety on which the complex torus $T = (\mathbb{C}^*)^n$ acts algebraically with finitely many fixed points $F = \{x_1, x_2, \ldots, x_r\}$ (all isolated) and with finitely many 1-dimensional orbits, $\{E_1, E_2, \ldots, E_\ell\}$. For each 1-dimensional orbit E_j there is a linear action of T on $\mathbb{CP}^1$ and a T-equivariant isomorphism $h_j : \overline{E_j} \to \mathbb{CP}^1$. Hence the closure $\overline{E_j}$ is obtained from E_j by adding two fixed points,

$$\partial E_j = \{h_j^{-1}(0), h_j^{-1}(\infty)\} \subset F$$

which we denote by x_{a_j} and x_{b_j} respectively. (These labels depend on the choice of T-action on $\mathbb{CP}^1$ and on the isomorphism h_j. The inverse action of T on $\mathbb{CP}^1$ is compatible with a "reverse" isomorphism h_j' for which the labels a_j and b_j will be reversed.) Let $K_j \subset K$ denote the stabilizer of any point in E_j, and let $\mathfrak{k}_j = \mathrm{Lie}(K_j) \subset \mathfrak{k}$ denote its (complex) Lie algebra. For $j = 1, 2, \ldots, \ell$ define

$$\beta_j : \bigoplus_{i=1}^{r} \mathbb{C}[\mathfrak{k}^*] \to \mathbb{C}[\mathfrak{k}_j^*]$$

to be the mapping given by

$$\beta_j(f_1, f_2, \ldots, f_r) = f_{a_j}|\mathfrak{k}_j - f_{b_j}|\mathfrak{k}_j \tag{7.1.1}$$

where $x_{a_j} \cup x_{b_j} = \partial E_j$ are the two points in the boundary of the orbit E_j. (We have arbitrarily chosen to denote one of these points x_{a_j} and the other x_{b_j}. Reversing the labels will change the mapping β_j by a sign but will not change $\ker(\beta_j)$).

(7.2) Theorem. *Suppose the equivariant cohomology $H_K^*(X;\mathbb{C})$ is a free module over $\mathbf{S} = \mathbb{C}[\mathfrak{k}^*]$. Then the restriction map*

$$H_K^*(X) \to H_K^*(F) = \bigoplus_{i=1}^{r} \mathbb{C}[\mathfrak{k}^*]$$

is an injection, and its image is the intersection of kernels,

$$H_K^*(X) \cong \bigcap_{j=1}^{\ell} \ker(\beta_j).$$

(*7.3*) *Proof*. By the localization theorem 6.3 the equivariant cohomology $H_K^*(X)$ is given by the kernel of $\delta : H_K^*(F) \to H_K^*(X_1, F)$, which we now identify. The set X_1 consists of the closure of the union of the 1-dimensional T-orbits. Let E_j be a single such one dimensional orbit, with closure $\bar{E}_j$ containing fixed points $\partial E_j = x \cup y$. Let $T_j \subset T$ denote the stabilizer of any point in E_j. Then $H_K^*(E_j) \cong \mathbb{C}[\mathfrak{k}_j^*]$. Consider the Mayer-Vietoris exact sequence for the covering of $\bar{E}_j$ by two open equivariant subsets, $U_1 = \bar{E}_j - \{x\}$ and $U_2 = \bar{E}_j - \{y\}$. This sequence agrees with the long exact cohomology sequence for the pair $(\bar{E}_j, \partial E_j)$. Since $H_K^i(\bar{E}_j) = 0$ for i odd, the sequence splits into short exact sequences,

$$\begin{array}{ccccccccc}
0 & \to & H_K^i(\bar{E}_j) & \to & H_K^i(x) \oplus H_K^i(y) & \xrightarrow{\delta} & H_K^{i+1}(\bar{E}_j, x \cup y) & \to & 0 \\
 & & \uparrow\cong & & \uparrow & & \cong\uparrow & & \\
0 & \to & H_K^i(\bar{E}_j) & \xrightarrow{\alpha} & H_K^i(U_2) \oplus H_K^i(U_1) & \xrightarrow{\beta} & H_K^i(E_j) & \to & 0
\end{array}$$

where the map $\beta : \mathbb{C}[\mathfrak{k}^*] \oplus \mathbb{C}[\mathfrak{k}^*] \to \mathbb{C}[\mathfrak{k}_j^*]$ is given by $\beta(f,g) = f|\mathfrak{k}_j - g|\mathfrak{k}_j$. Applying this computation to each one-dimensional orbit gives the formula in Theorem 7.2. $\square$

(7.4) Moment map. Let $X \to \mathbb{CP}^N$ be an equivariant projective embedding. By averaging over the compact torus K, we may assume that the Kähler form on projective space is K-invariant. It follows that K acts by Hamiltonian vectorfields, and so it admits a moment mapping $\mu : \mathbb{CP}^N \to \mathfrak{k}^*$. For simplicity, let us assume that the moment map images $v_i = \mu(x_i)$ of the fixed points are distinct. Fix j (for $1 \le j \le \ell$) and consider the moment map image $e_j = \mu(E_j)$ of the 1-dimensional T-orbit E_j. It is a straight line segment connecting two of the vertices, say v_a and v_b, which correspond to the two fixed points x_a, x_b in the closure of E_j.

Let $\langle e_j \rangle \subset \mathfrak{k}^*$ denote the 1-dimensional subspace of $\mathfrak{k}^*$ which is parallel to the segment e_j. The symmetric algebra $\mathbf{S} = \mathbb{C}[\mathfrak{k}^*]$ may be identified with the algebra $\mathscr{D}(\mathfrak{k}^*)$ of linear differential operators with constant (complex) coefficients on $\mathfrak{k}^*$. Let $\phi_j : \mathscr{D}(\mathfrak{k}^*) \to \mathscr{D}(\mathfrak{k}^*/\langle e_j \rangle)$ denote the push forward mapping on differential operators. Define

$$\beta_j : \bigoplus_{i=1}^{r} \mathscr{D}(\mathfrak{k}^*) \to \mathscr{D}(\mathfrak{k}^*/\langle e_j \rangle) \tag{7.4.1}$$

by $\beta_j(D_1, D_2, \ldots, D_r) = \phi_a(D_a) - \phi_b(D_b)$. (Reversing the labelling of the endpoints v_a and v_b will change β_j by a sign but will not change its kernel.)

(7.5) Corollary. *Suppose the equivariant cohomology $H_K^*(X; \mathbb{C})$ is a free module over* $\mathbf{S}$. *Then, in terms of moment map data it is given by*

$$H_K^*(X; \mathbb{C}) = \ker\left(\bigoplus_{i=1}^{r} \beta_j : \bigoplus_{i=1}^{r} \mathscr{D}(\mathfrak{k}^*) \to \bigoplus_{j=1}^{\ell} \mathscr{D}(\mathfrak{k}^*/\langle e_j \rangle) \right). \tag{7.5.1}$$

(*7.6*) *Proof.* Since there is an exact sequence

$$0 \to \langle e_j \rangle \to \mathfrak{k}^* \to \mathfrak{k}_j^* \to 0$$

we may identify the symmetric algebra $\mathbb{C}[\mathfrak{k}_j^*]$ with the algebra $\mathscr{D}(\mathfrak{k}^*/\langle e_j \rangle)$. So the mapping (7.4.1) agrees with the mapping (7.1.1). $\square$

(7.7) Remarks. The module (7.5.1) appears in [CS] in the case that X is a rationally nonsingular toric variety, although they do not identify it with the equivariant cohomology. An equivalent formula appears in [Bri] cf. [BrV].

(7.8) Other groups and sheaves. If K is a maximal torus in a compact connected Lie group G and if the K action extends to a G action on X then, by a result of A. Borel, the G-equivariant cohomology is given by the invariants, $H_G^*(X) \cong (H_K^*(X))^W$ under the Weyl group $W = N_G(K)/K$. The formula (7.5.1) is compatible with the action of W: it permutes the fixed points $x_1, \ldots, x_k$ and it permutes the 1-dimensional orbits $E_1, \ldots, E_\ell$. So (7.5.1) may be used to determine the G-equivariant cohomology as well. There is a formula, analogous to that of theorem (7.2) for the K-equivariant cohomology of any element $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D_K^b(X)$ provided

(1) $H_K^*(X; A)$ is a free module over $\mathbb{C}[\mathfrak{k}^*]$
(2) $H_K^*(F; A) \cong H^*(F; \mathbf{A_X}|F) \otimes \mathbb{C}[\mathfrak{k}^*]$
(3) $H_K^*(E_j; A) \cong H^*(E_j; \mathbf{A_X}|E_j) \otimes \mathbb{C}[\mathfrak{k}^*]$

Note that if $A = I^m C_X^*$ is the (equivariant) middle intersection cohomology sheaf on X, then condition (1) holds whenever X is projective (cf. Theorem 14.1). Conditions (2) and (3) often hold (cf. Theorem 14.1, or [Br] Sect. 4.2.4, [BL] Sect. 15.14). For example, conditions (2) and (3) hold for Schubert varieties and for toric varieties because the stalk of the (ordinary) intersection cohomology vanishes in odd degrees (cf. Theorem 14.1).

8. Koszul duality

In this section (and Sect. 16) we give a modified version of the basic results from [BGG] (and [BGS]; see also [B]), rewritten so as to agree with the gradings on the complexes which occur in this paper. Let k be a field.

(8.1) Exterior and symmetric algebra of a graded vectorspace Let $P = \bigoplus_{j\in\mathbb{Z}} P_j$ denote a graded vectorspace over k, with homogeneous components of odd positive degrees only, and let $\Lambda_\bullet = \bigwedge P$ denote the exterior algebra on P, together with its grading by degree: If $\lambda = \lambda_1 \wedge \ldots \wedge \lambda_t$, then $\deg(\lambda) = \sum \deg(\lambda_i)$ for homogeneous elements $\lambda_i \in P$. We also denote by $|\lambda| = t$ the *weight* of λ. Then $(-1)^{\deg(\lambda)} = (-1)^{|\lambda|}$ and $(-1)^{\deg(\lambda)\deg(\mu)} = (-1)^{|\lambda||\mu|}$ if μ is another (bi-) homogeneous element of $\Lambda_\bullet$. For any homogeneous element $\lambda \in \Lambda_\bullet$ set

$$\bar{\lambda} = (-1)^{|\lambda|(|\lambda|-1)/2}\lambda$$

The bar operation defines an isomorphism between $\Lambda_\bullet$ and its opposite ring $\Lambda_\bullet^{\mathrm{op}}$, which is the identity on P. In other words, for every $x, y \in \Lambda_\bullet$ we have $\overline{x}\overline{y} = \overline{yx}$.

Let $\tilde{P}^*$ denote the dual vectorspace $P^* = \mathrm{Hom}_k(P, k)$, graded by homogeneous components of even degrees only, $(\tilde{P}^*)^m = (P^*)^{m-1}$. Let $\mathbf{S} = S(\tilde{P}^*)$ denote the symmetric algebra on $\tilde{P}^*$, with grading $\deg(s_1s_2\ldots s_r) = \sum_{i=1}^r \deg(s_i)$ for homogeneous elements $s_i \in \tilde{P}^*$.

(8.2) The derived category. We wish to consider the derived category of graded modules over $\Lambda_\bullet$ or $\mathbf{S}$, which we regard as differential graded algebras with zero differential. Let us recall the construction ([Il] Sect. VI.10; cf. [BL] Sect. 10):

A bounded below *differential graded* $\Lambda_\bullet$*-module* (N, d_N) is a graded module $N = \bigoplus_{i\geq i_0} N^i$ together with a differential $d_N : N^i \to N^{i+1}$ such that $d_N^2 = 0$ on which the algebra $\Lambda_\bullet$ acts such that $\Lambda_j \cdot N^i \subset N^{i-j}$ and such that $\lambda d_N n = (-1)^{\deg(\lambda)} d_N \lambda n$ for all $\lambda \in \Lambda_\bullet$ and for $n \in N$. Let $\mathbf{K}_+(\Lambda_\bullet)$ denote the category of bounded below, differential graded $\Lambda_\bullet$ modules and chain homotopy classes of maps. It is a triangulated category and is usually referred to as the *homotopy category* of $\Lambda_\bullet$ modules. The *derived category* $D_+(\Lambda_\bullet)$ is obtained by localizing the homotopy category $\mathbf{K}_+(\Lambda_\bullet)$ at the collection of quasi-isomorphisms. (cf [Ve1], [Ve2], [W], [B], [Il]).

Let $K_+^f(\Lambda_\bullet)$ denote the homotopy category whose objects are differential graded $\Lambda_\bullet$-modules N which are bounded from below, such that the cohomology $H^*(N)$ is a finitely generated $\Lambda_\bullet$-module; and homotopy classes of maps. Let $D_+^f(\Lambda_\bullet)$ denote the corresponding derived category obtained by inverting quasi-isomorphisms. The canonical functor $D_+^f(\Lambda_\bullet) \to D_+(\Lambda_\bullet)$ is fully faithful. In other words, $D_+^f(\Lambda_\bullet)$ is equivalent to the full subcategory of $D_+(\Lambda_\bullet)$ consisting of objects whose cohomology is finitely generated.

A (bounded below) differential graded $\mathbf{S}$-module (M, d_M) is a graded module $M = \bigoplus_{i \geq i_0} M^i$ together with a differential $d_M : M^i \to M^{i+1}$ such that $d_M^2 = 0$ on which $\mathbf{S}$ acts with $\mathbf{S}^j.M^i \subset M^{j+i}$, such that $s d_M m = d_M s m$ for all $s \in \mathbf{S}$ and for all $m \in M$. Let $\mathbf{K}_+(\mathbf{S})$ denote the *homotopy category* whose objects are (bounded below) differential graded $\mathbf{S}$-modules, and whose morphisms are homotopy classes of maps. The *derived category* $D_+(\mathbf{S})$ is obtained by localizing the homotopy category $\mathbf{K}_+(\mathbf{S})$ at the collection of quasi-isomorphisms.

Let $K_+^f(\mathbf{S})$ denote the homotopy category whose objects are differential graded $\mathbf{S}$-modules M which are bounded from below, such that the cohomology $H^*(M)$ is a finitely generated $\mathbf{S}$-module; and homotopy classes of maps. Let $D_+^f(\mathbf{S})$ denote the corresponding derived category obtained by inverting quasi-isomorphisms. The canonical functor $D_+^f(\mathbf{S}) \to D_+(\mathbf{S})$ is fully faithful.

Fix homogeneous dual bases $\{x_i\}$ and $\{\xi_i\}$ of P and $\tilde{P}^*$. This means that $\langle \xi_i, x_j \rangle = \delta_{ij}$ and that $\deg(\xi_i) = \deg(x_i) + 1$.

(8.3) Definition. [BGG] The first Koszul duality functor $h : \mathbf{K}_+(\mathbf{S}) \to \mathbf{K}_+(\Lambda_\bullet)$ assigns to any complex (M, d_M) of $\mathbf{S}$ modules the following complex of $\Lambda_\bullet$-modules:

$$h(M) = \mathrm{Hom}_k(\Lambda_\bullet, M) \tag{8.3.1}$$

with grading $h^p(M) = \bigoplus_{i+j=p} \mathrm{Hom}_k(\Lambda_i, M^j)$, with module structure $(x \cdot F)(\lambda) = F(\bar{x} \wedge \lambda)$ (for $x \in \Lambda_\bullet$ and $F \in h(M)$), and with differential

$$dF(\lambda) = -\sum_{i=1}^{r} \xi_i F(x_i \lambda) + (-1)^{\deg(\lambda)} d_M(F(\lambda)). \tag{8.3.2}$$

for homogeneous elements $\lambda \in \Lambda_\bullet$. The second Koszul duality functor $t : \mathbf{K}_+(\Lambda_\bullet) \to \mathbf{K}_+(\mathbf{S})$ assigns to any complex (N, d_N) of $\Lambda_\bullet$ modules the following complex of $\mathbf{S}$-modules:

$$t(N) = \mathbf{S} \otimes_k N \tag{8.3.3}$$

with module structure $\xi \cdot (s \otimes n) = \xi s \otimes n$ (for $\xi, s \in \mathbf{S}$ and $n \in N$), with grading $t^p(N) = \bigoplus_{i+j=p} \mathbf{S}^i \otimes N^j$ and with differential

$$d(s \otimes n) = \sum_{i=1}^{r} \xi_i s \otimes x_i n + s \otimes d_N n. \tag{8.3.4}$$

(8.4) Koszul duality theorem. [BGG] *The Koszul duality functors h and t pass to functors $h : D_+(\mathbf{S}) \to D_+(\Lambda_\bullet)$ and $t : D_+(\Lambda_\bullet) \to D_+(\mathbf{S})$, where they become quasi-inverse equivalences of categories. The Koszul duality functors h and t restrict to (quasi-inverse) equivalences of categories*

$$D_+^f(\Lambda_\bullet) \underset{\leftarrow}{\overset{\rightarrow}{}} D_+^f(\mathbf{S}). \tag{8.4.1}$$

(*8.5*) *Proof*. The proof is delayed until Sect. 16. The key point is that both $ht(k)$ and $th(k)$ are the Koszul complex.

(8.6) Remarks. The notions of Koszul duality were introduced in [BGG] and developed in [BGS] for a slightly different category (let us denote it by $D^{\mathrm{b,gr}}(\mathbf{S})$): it is the derived category whose objects are bounded complexes of graded $\mathbf{S}$-modules. We would like to thank A. Beilinson for pointing out to us that (even if $P = P_1$ is trivially graded), the canonical functor $D^{\mathrm{b,gr}}(\mathbf{S}) \to D(\mathbf{S})$ (which associates to each complex of graded modules the associated single complex) is *not* an equivalence of categories.

(8.7) Forgetful functor. Let $D_+(k)$ denote the (bounded below) derived category of the category of vectorspaces over k. The forgetful functor $F_{\mathbf{S}} : D_+(\mathbf{S}) \to D_+(k)$ assigns to any differential graded $\mathbf{S}$-module $(M, d_M) \in D_+(\mathbf{S})$ the underlying complex of vectorspaces.

Recall from [Il], or [BL] Sect. 10 that the functor $\otimes_{\mathbf{S}}$ passes to a derived functor $\overset{L}{\otimes}_{\mathbf{S}}$ on $D_+\mathbf{S}$ by

$$M_1 \overset{L}{\otimes}_{\mathbf{S}} M_2 = M_1 \otimes_{\mathbf{S}} B(M_2) \tag{8.7.1}$$

for any differential graded $\mathbf{S}$-modules M_1 and M_2, where $B(M_2)$ is the bar resolution of M_2. Extending scalars via the augmentation $\mathbf{S} \to k$ therefore defines another functor $D_+(\mathbf{S}) \to D_+(k)$ by

$$M \mapsto k \overset{L}{\otimes}_{\mathbf{S}} M \tag{8.7.2}$$

A similar construction defines the forgetful functor $F_\Lambda : D_+(\Lambda_\bullet) \to D_+(k)$ and extension of scalars

$$k \overset{L}{\otimes}_\Lambda \bullet : D_+(\Lambda_\bullet) \to D_+(k). \tag{8.7.3}$$

(8.8) Proposition. *The Koszul duality functor, extension of scalars functor, and the forgetful functors are related by natural isomorphisms in $D_+(k)$,*

$$k \overset{L}{\otimes}_{\mathbf{S}} M \cong F_\Lambda h(M) \tag{8.8.1}$$

and

$$k \overset{L}{\otimes}_{\mathbf{S}} N \cong F_{\mathbf{S}} t(N) \tag{8.8.2}$$

for any $M \in D_+(\mathbf{S})$ *and* $N \in D_+(\Lambda_\bullet)$.

(*8.9*) *Proof.* By the Koszul duality theorem, any $M \in D_+(\mathbf{S})$ is isomorphic to a complex of the form

$$M \cong t(N) = \mathbf{S} \otimes_k N$$

with differential given by (8.3.4). Therefore

$$k \overset{L}{\otimes}_{\mathbf{S}} M \cong k \overset{L}{\otimes}_{\mathbf{S}} \mathbf{S} \otimes_k N \cong F_\Lambda N \cong F_\Lambda h(M) \tag{8.9.1}$$

It is straightforward to check that these isomorphisms are canonical, and that the resulting differentials agree. A similar computation gives (8.8.2). □

9. Split complexes

In this section, as in Sect. 8, k denotes a field, P is a graded vectorspace over k with odd grading, $\Lambda_\bullet = \bigwedge(P)$ is its exterior algebra, and $\mathbf{S} = S(\tilde{P}^*)$ is the associated evenly graded symmetric algebra. We are primarily interested in the case that $k = \mathbb{R}$, $\Lambda_\bullet = H_*(K;\mathbb{R})$ and $\mathbf{S} = H^*(BK;\mathbb{R})$ where K is a compact connected Lie group.

(9.1) Spectral sequences for Koszul duality. Let $N \in D_+^f(\Lambda_\bullet)$ be a complex of $\Lambda_\bullet$-modules and let $t(N) = \mathbf{S} \otimes_k N$ denote its Koszul dual (8.3.3). This is the single complex associated to the double complex

$$T^{pq} = \mathbf{S}^{2p} \otimes_k N^{q-p} \tag{9.1.1}$$

with differential $d = d' + d''$ where $d'd'' = -d''d'$ and

$$d'(s \otimes n) = s \otimes d_N n \in T^{p,q+1} \tag{9.1.2}$$

and

$$d''(s \otimes n) = \sum_i \xi_i s \otimes x_i n \in T^{p+1,q} \tag{9.1.3}$$

for any $s \otimes n \in T^{pq}$. The double complex looks like this:

$$\begin{tikzcd}
\mathbf{S}^0 \otimes N^3 \arrow[r, "d''"] & \mathbf{S}^2 \otimes N^2 \arrow[r, "d''"] & \mathbf{S}^4 \otimes N^1 \arrow[r, "d''"] & \mathbf{S}^6 \otimes N^0 \\
\mathbf{S}^0 \otimes N^2 \arrow[u, "d'"] \arrow[r, "d''"] & \mathbf{S}^2 \otimes N^1 \arrow[u, "d'"] \arrow[r, "d''"] & \mathbf{S}^4 \otimes N^0 \arrow[u, "d'"] & \\
\mathbf{S}^0 \otimes N^1 \arrow[u, "d'"] \arrow[r, "d''"] & \mathbf{S}^2 \otimes N^0 \arrow[u, "d'"] & & \\
\mathbf{S}^0 \otimes N^0 \arrow[u, "d'"] & & &
\end{tikzcd}$$

The spectral sequence which is obtained by first taking cohomology with respect to the differential d' and then with respect to d'' has (cf. 5.5.4)

$$E_{(1)}^{pq}(N) = \mathbf{S}^{2p} \otimes_k H^{q-p}(N) \Longrightarrow H^{p+q}(t(N)) \tag{9.1.4}$$

Now let $M \in D_+^f(\mathbf{S})$ denote a complex of $\mathbf{S}$-modules, with Koszul dual $h(M) = \mathrm{Hom}(\Lambda_\bullet, M)$ (cf. 8.3.1). This is the single complex associated to the double complex $\mathrm{Hom}(\Lambda_p, M^q)$ which gives rise to a spectral sequence

$$E_{(1)}^{pq}(M) = \mathrm{Hom}_k(\Lambda_p, H^q(M)) \Longrightarrow H^{p+q}(h(M)). \tag{9.1.5}$$

(9.2) Split complexes. Let $N \in D_+^f(\Lambda_\bullet)$ be a complex of $\Lambda_\bullet$-modules. Let us say that N is *split and trivial* if it is isomorphic (in $D_+^f(\Lambda_\bullet)$) to its own cohomology, $N \cong \oplus_n H^n(N)[-n]$ together with the trivial action of $\Lambda_\bullet$. If $M \in D_+^f(\mathbf{S})$ is a complex of modules over $\mathbf{S}$, we will say that M is *split and free* if it is isomorphic (in $D_+^f(\mathbf{S})$) to its own cohomology, $M \cong \oplus_n H^n(M)[-n]$ and if this cohomology is a free module over $\mathbf{S}$.

(9.3) Proposition. *Let $N \in D_+^f(\Lambda_\bullet)$ be a complex of $\Lambda_\bullet$-modules. Then the following are equivalent*:

(1) *N is split and trivial*

(2) *The Koszul dual $M = t(N) \in D_+^f(\mathbf{S})$ is split and free*

(3) *The spectral sequence* (9.1.4) *collapses at $E_{(1)}$.*

(4) *For all $p \geq 0$ the edge morphism $H^p(M) \to E_{(1)}^{0p} = \mathbf{S}^0 \otimes H^p(N)$ is surjective.*

(*cf. Proposition* (13.4) *and* (13.8).) *In this case, the edge morphism induces an isomorphism of k-vectorspaces*

$$H^*(M)/\mathbf{S}^{>0}H^*(M) \cong H^*(N)$$

and any lift $H^*(N) \to H^*(M)$ of the edge morphism induces an isomorphism of graded $\mathbf{S}$ modules,

$$\mathbf{S} \otimes H^*(N) \cong H^*(M).$$

(*9.4*) *Proof*. Parts (1) and (2) are equivalent because explicit formulas (8.3.2) and (8.3.4) for the differentials are given. It is easy to see that (2) implies (3). Let us show that (3) implies (2). Suppose that $N \cong E(\mathbf{A}^\bullet)$ for some $\mathbf{A}^\bullet \in D_K^b(\mathrm{pt})$ which we consider to be a complex of sheaves whose cohomology sheaves are constant. Then $t(N) \cong G(\mathbf{A}^\bullet)$ and the spectral sequence (9.1.4) is isomorphic to the spectral sequence

$$E_{(2)}^{pq} = H^p(BK; \mathbf{H^q}(\mathbf{A})) \Longrightarrow H^{p+q}(BK; \mathbf{A}^\bullet) \cong H^{p+q}(G(\mathbf{A}^\bullet)) \tag{9.4.1}$$

for the cohomology of the complex of sheaves $\mathbf{A}^\bullet$. Now apply Deligne's degeneracy criterion [D1] for the functors

$$T_i(K) = \mathrm{Hom}_{D^b(BK)}(\mathbf{H^i}(\mathbf{A}), K) \cong H^0(R\mathrm{Hom}(\mathbf{H^i}(\mathbf{A}), K))$$

and take $K = \mathbf{A}^\bullet$ as in [D1]. Since the cohomology sheaves $\mathbf{H^i}(\mathbf{A})$ are constant on BK, the spectral sequence [D1] Sect.$(1.3)_i$ collapses if and only if the spectral sequence (9.1.4) collapses.

The edge morphism factors

$$H^p(M) \twoheadrightarrow E_{(\infty)}^{0p} \hookrightarrow E_{(1)}^{0p} = \mathbf{S}^0 \otimes H^p(N).$$

If the spectral sequence collapses then the second arrow is an isomorphism, so the edge morphism is surjective: thus part (3) implies part (4). On the other hand, the edge morphism is a surjection iff $E_{(\infty)}^{0p} = E_{(1)}^{0p}$ for all p, i.e., if all differentials leaving from the first column vanish. It follows by induction on r that $E_{(r)}$ is a free $\mathbf{S}$-module, generated by the first column, $H^*(N)$, and all the differentials $d_{(r)}^{pq}$ vanish (since they are $\mathbf{S}$-module homomorphisms.) Thus, the spectral sequence collapses.

The conclusion of the theorem is essentially the Leray-Hirsch theorem. It follows from parts (3) and (4): Choose any splitting of the edge morphism. This determines a homomorphism $\mathbf{S} \otimes H^*(N) \to H^*(M)$ of graded filtered $\mathbf{S}$ modules, which induces isomorphisms on the graded filtered pieces (since the spectral sequence collapses). Therefore it is an isomorphism. □

(9.5) Remark. The dual statement is also true: an object $M \in D_+^f(\mathbf{S})$ is split and trivial iff its Koszul dual $N = h(M) \in D_+^f(\Lambda_\bullet)$ is split and free.

10. Universal sheaves on BK

Throughout the rest of this paper, K denotes a compact connected Lie group, $\Lambda_\bullet = H_*(K{:}\mathbb{R})$ denotes its homology, and $\mathbf{S} = H^*(BK; \mathbb{R})$ denotes the cohomology of its classifying space (cf. Sect. 10.6).

(10.1) The goal. Suppose K acts subanalytically on a subanalytic space X. For any $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D_K^b(X)$ we would like to define an $\mathbf{S}$-module structure on the equivariant global sections $\Gamma(X \times_K EK; \bar{\mathbf{A}})$ and to define a $\Lambda_\bullet$-module structure on the (ordinary) global sections $\Gamma(X; \mathbf{A_X})$. Unfortunately it is not so clear how to do this. Instead, in Sect. 11 we replace these complexes of global sections by certain quasi-isomorphic complexes which admit the appropriate module structures. Our definition of these module structures involves some differential geometry which we review in this section.

In certain cases (for example, if $\mathbf{A_X}$ is the sheaf of subanalytic chains, or the sheaf of subanalytic intersection chains) there is a natural $\Lambda_\bullet$-module structure on the global sections $\Gamma(X; \mathbf{A_X})$, as described in Sect. 12. In Theorems 12.3 and 12.5 we will show that these two module structures agree.

We use $\mathbb{R}$ coefficients throughout Sects. 10–12.

(10.2) Lie algebra homology. Let $\mathfrak{k} = \mathrm{Lie}(K)$ denote the (real) Lie algebra of K and denote by

$$\partial_{\mathfrak{k}}(x_0 \wedge x_1 \wedge \ldots \wedge x_n) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_0 \ldots \wedge \hat{x}_i \ldots \wedge \hat{x}_j \ldots \wedge x_n$$

the Lie algebra differential on the exterior algebra $\bigwedge \mathfrak{k}$. Let $P \subset \bigwedge \mathfrak{k}$ denote the graded subspace of primitive elements in the exterior algebra of $\mathfrak{k}$,

$$P = \left\{ x \in \bigwedge \mathfrak{k} | \Delta_*(x) = x \otimes 1 + 1 \otimes x \right\} \tag{10.2.1}$$

where $\Delta_* : \bigwedge \mathfrak{k} \to \bigwedge \mathfrak{k} \otimes \bigwedge \mathfrak{k}$ is the map induced by the diagonal embedding $\Delta : \mathfrak{k} \to \mathfrak{k} \oplus \mathfrak{k}$.

Define $\Lambda_\bullet$ to be the exterior algebra $\Lambda_\bullet = \bigwedge P$ on the primitive elements. The elements $x \in P$ are cycles (i.e., $\partial_{\mathfrak{k}} x = 0$), they are K-invariant, they have odd degrees, and the inclusion $P \to (\bigwedge \mathfrak{k})^K$ induces an isomorphism

$$\Lambda_\bullet = \bigwedge P \cong \left(\bigwedge \mathfrak{k} \right)^K \cong H_*(\mathfrak{k}, \mathbb{R}) \tag{10.2.2}$$

between $\Lambda_\bullet$, the invariants in the exterior algebra of $\mathfrak{k}$ and the Lie algebra homology of $\mathfrak{k}$.

(10.3) Lie algebra cohomology. Let $\bigwedge \mathfrak{k}^*$ be the exterior algebra of $\mathfrak{k}^* = \mathrm{Hom}(\mathfrak{k}, \mathbb{R})$, with its Lie algebra differential

$$d_{\mathfrak{k}}\omega(v_0, v_1, \ldots, v_p) = \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_p). \tag{10.3.1}$$

Let $P^* \subset \bigwedge \mathfrak{k}^*$ denote the graded subspace of primitive elements,

$$P^* = \{\xi \in \bigwedge(\mathfrak{k}^*) | \mu_{\mathfrak{k}}^*(\xi) = 1 \otimes \xi + \xi \otimes 1\} \tag{10.3.2}$$

where $\mu_{\mathfrak{k}}^* : \bigwedge \mathfrak{k}^* \to \bigwedge \mathfrak{k}^* \otimes \bigwedge \mathfrak{k}^*$ is the map induced by the bracket $\mu_{\mathfrak{k}} : \mathfrak{k} \oplus \mathfrak{k} \to \mathfrak{k}$. The elements $\omega \in P^*$ are invariant under K, they have odd degrees, and they are cocyles, i.e. $d_{\mathfrak{k}}\omega = 0$.

Define the algebra $\Lambda^\bullet$ to be the exterior algebra $\Lambda^\bullet = \bigwedge P^*$ on the primitive elements. The inclusion $P^* \to (\bigwedge \mathfrak{k}^*)^K$ induces an algebra isomorphism

$$\Lambda^\bullet \to (\bigwedge \mathfrak{k}^*)^K \cong H^*(\mathfrak{k}, \mathbb{R}) \tag{10.3.3}$$

between the exterior algebra $\Lambda^\bullet$, the invariants in the exterior algebra of $\mathfrak{k}^*$, and the Lie algebra cohomology of $\mathfrak{k}$.

(10.4) Kronecker pairing. For any multivector $a \in \bigwedge^m \mathfrak{k}$, the interior product $i(a) : \bigwedge^n \mathfrak{k}^* \to \bigwedge^{n-m} \mathfrak{k}^*$ is given by $(i(a)(\omega))(b) = \omega(a \wedge b)$. The interior product restricts to a nondegenerate pairing

$$P \times P^* \xrightarrow{\langle , \rangle} \mathbb{R} \tag{10.4.1}$$

which identifies P and P^* as dual (graded) vectorspaces. (In other words, if $a \in P$ and $\mu \in P^*$ then $\langle a, u \rangle = i(a)(\mu) = 0$ unless $\deg(a) = \deg(\mu)$. cf. [GHV] III Sect. 5.2.1)

The algebra $\Lambda^\bullet$ has the structure of a (left) module over $\Lambda_\bullet$ by interior product, $\lambda \cdot \omega = i(\bar{\lambda})\omega$, while $\mathrm{Hom}(\Lambda_\bullet, \mathbb{R})$ has the structure of a (left) module by $(\lambda \cdot F)(v) = F(\bar{\lambda} v)$. These module structures are compatible with the canonical isomorphism $\Lambda^\bullet \cong \mathrm{Hom}(\Lambda_\bullet, \mathbb{R})$ which is induced by the pairing (10.4.1), in other words, $\langle \lambda \cdot \omega, a \rangle = \langle \omega, \lambda \cdot a \rangle$.

(10.5) Fundamental vectorfields and interior products. Suppose K acts (from the left) on a smooth manifold Y. To each $u \in \mathfrak{k}$ we associate the *fundamental vectorfield* $V_u = V_u^Y$ on Y by

$$V_u^Y(y) = \frac{d}{dt} \exp(tu) \cdot y|_{t=0}. \tag{10.5.1}$$

Then $V_{[u,v]} = -[V_u, V_v]$. If $\mu_g : Y \to Y$ denotes the action by $g \in K$ then $(\mu_g)_*(V_u) = V_{Ad_g(u)}$ and in particular the fundamental vectorfield V_u may fail to be invariant. Each multivector $u \in \bigwedge^r \mathfrak{k}$ determines a fundamental multivectorfield V_u^Y on Y: if $u = u_1 \wedge \ldots \wedge u_r$ then $V_u^Y = V_{u_1}^Y \wedge \ldots \wedge V_{u_r}^Y$. If $u \in (\bigwedge \mathfrak{k})^K = \Lambda_\bullet$ is an invariant multivector then the multivectorfield V_u^Y is left invariant.

Each multivectorfield V on Y defines an interior product $i(V) : \Omega^n(Y) \to \Omega^{n-\deg(V)}(Y)$ by $i(V)(\omega)(W) = \omega(V \wedge W)$ for any multivectorfield W of degree $n - \deg(V)$. Then $i(V \wedge W) = i(W) \circ i(V)$. The ring $\Lambda_\bullet$

acts on the smooth differential forms $\Omega^\bullet(Y)$ by interior product with fundamental vectorfields: If $u \in \Lambda_\bullet = (\bigwedge \mathfrak{k})^K$, let $V_{\bar{u}}^Y$ denote the fundamental multivectorfield on Y associated to $\bar{u} = (-1)^{|u|}u$ and set $u \cdot \omega = i(V_{\bar{u}}^Y)(\omega)$. Then $u \cdot v \cdot \omega = (u \wedge v) \cdot \omega$.

(10.6) Homology of *K*. Associating to each Lie algebra cochain $\lambda \in \bigwedge^q \mathfrak{k}^*$ the corresponding left-invariant differential form $L(\lambda) \in \Omega^q(K, \mathbb{R})$ determines a canonical isomorphism

$$L : \Lambda^\bullet \cong H^*(K, \mathbb{R}) \tag{10.6.1}$$

between the ring $\Lambda^\bullet$ and the (de Rham or the singular) cohomology of K, together with its cup product structure. Using the pairing (10.4.1), the adjoint of (10.6.1) is a canonical isomorphism of algebras,

$$\Lambda_\bullet \cong H_*(K; \mathbb{R}) \tag{10.6.2}$$

between $\Lambda_\bullet$ and the (singular) homology of the topological group K, together with the Pontrjagin product (i.e. the homomorphism which is induced on homology from the multiplication $K \times K \to K$).

(10.7) Differential forms on the classifying space. As in Sect. 5.1, fix a smooth model $\pi : EK \to BK$ for the classifying space of K (cf. [BL] Sect. 12.4.1). Let $\mathbf{\Omega}^\bullet_{\mathbf{EK_n}}$ denote the sheaf of smooth real valued differential forms on EK_n extended by 0 on EK. Define ([BL] Sect. 12.2.2) the de Rham complex

$$\mathbf{\Omega}^\bullet_{\mathbf{EK}} = \lim_{\leftarrow} \mathbf{\Omega}^\bullet_{\mathbf{EK_n}}$$

on EK. Then $\mathbf{\Omega}^\bullet_{\mathbf{EK}}$ is a soft resolution of the constant sheaf $\mathbb{R}_{\mathbf{EK}}$. In a similar manner, define the de Rham complex of sheaves

$$\mathbf{\Omega}^\bullet_{\mathbf{BK}} = \lim_{\leftarrow} \mathbf{\Omega}^\bullet_{\mathbf{BK_n}}$$

on BK. For each n let $\pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK_n}}$ denote the complex of sheaves on BK_n whose sections over an open set $U \subset BK_n$ consist of all K-invariant differential forms in $\pi^{-1}(U)$. By the usual averaging argument, the inclusion $\pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK_n}} \to \pi_* \mathbf{\Omega}^\bullet_{\mathbf{EK_n}}$ is a quasi-isomorphism, and the same is true of the corresponding inverse limits on BK,

$$\pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK}} \cong \pi_* \mathbf{\Omega}^\bullet_{\mathbf{EK}}.$$

(10.8) The universal $\Lambda_\bullet$ sheaf on *BK*. For each $u \in \mathfrak{k}$ (resp. $u \in \bigwedge \mathfrak{k}$) let V_u denote the fundamental vectorfield on (resp. the fundamental multivectorfield) on EK_n which is obtained by differentiating the action of $\exp(tu)$. For any invariant differential form $\omega \in \Omega^\bullet(EK_n)$, the interior product $i(V_u)(\omega)$ is also invariant.

For each n, define a $\Lambda_\bullet$ module structure on $\pi_*\mathbf{\Omega}^{\bullet K}_{\mathbf{EK_n}}$ as follows: For any open set $U \subset BK_n$, for any invariant differential form $\omega \in \Omega^\bullet(\pi_n^{-1}(U))$, and for any $\lambda \in \Lambda_\bullet$, set

$$\lambda \cdot \omega = i(V_{\bar{\lambda}})(\omega) \tag{10.8.1}$$

Then $d(\lambda \cdot \omega) = (-1)^{\deg(\lambda)} \lambda \cdot d\omega$.

Proposition. *The operation* (10.8.1) *determines on* $\pi_*\mathbf{\Omega}^{\bullet K}_{\mathbf{EK}}$ *the structure of a soft sheaf of differential graded* $\Lambda_\bullet$ *modules on* BK. □

(10.9) The universal S sheaf on BK. Let $\tilde{P}^*$ denote the vectorspace P^* with the modified grading, $(\tilde{P}^*)^m = (P^*)^{m-1}$. Then $\tilde{P}^*$ is graded by even degrees. Define $\mathbf{S} = S(\tilde{P}^*)$ to be the (graded) symmetric algebra (over $\mathbb{R}$) on the graded vectorspace $\tilde{P}^*$, with grading $\deg(x_1, x_2, \ldots, x_r) = \sum_i \deg(x_i)$. (cf Sect. 8.1)

A choice of compatible K-invariant connections in the smooth fiber bundles $\pi_n : EK_n \to BK_n$ together with a choice of transgression determines a collection of compatible Chern-Weil homomorphisms

$$\theta_n : \mathbf{S} \to \mathbf{\Omega}^\bullet_{\mathbf{BK_n}} \tag{10.9.1}$$

which induces an isomorphism $\mathbf{S} \cong H^*(BK; \mathbb{R})$ on cohomology. Define a $\mathbf{S}$-module structure on each sheaf $\mathbf{\Omega}^\bullet_{\mathbf{BK_n}}$ as follows: for any open set $U \subset BK_n$ and any differential form $\omega \in \Gamma(U, \mathbf{\Omega}^\bullet_{\mathbf{BK_n}})$ set

$$s \cdot \omega = (\theta_n(s)|U) \wedge \omega. \tag{10.9.2}$$

Proposition. *The operation* (10.9.2) *determines on* $\mathbf{\Omega}^\bullet_{\mathbf{BK}}$ *the structure of a complex of soft sheaves of* $\mathbf{S}$*-modules on* BK. □

11. Koszul duality and equivariant cohomology

As in Sect. 10, K denotes a compact connected Lie group acting subanalytically on a subanalytic space X. We use $\mathbb{R}$ coefficients. In this section we show that the equivariant cohomology and the ordinary cohomology of an equivariant sheaf $A \in D^b_K(X)$ are related by Koszul duality.

(11.1) The category $D^b_K(\mathbf{pt})$. The canonical functor $D^b_K(\mathrm{pt}) \to D^b(BK)$ (which is given by $(\mathbf{A}_{\mathrm{pt}}, \bar{\mathbf{A}}, \beta) \mapsto \bar{\mathbf{A}}$) defines an equivalence of categories between $D^b_K(\mathrm{pt})$ and the full subcategory of $D^b(BK)$ consisting of complexes of sheaves whose cohomology sheaves are constant ([BL] Sect. 2.7.2). We will often abuse notation by writing $\mathbf{A}^\bullet \in D^b_K(\mathrm{pt})$ to represent a complex of sheaves on BK whose cohomology sheaves are constant. Define the functor of "equivariant" global sections, $G : D^b_K(\mathrm{pt}) \to D_+(\mathbf{S})$ by

$$G(\mathbf{A}^\bullet) = \Gamma(BK; \mathbf{\Omega}^\bullet_{\mathbf{BK}} \otimes_{\mathbb{R}} \mathbf{A}^\bullet). \tag{11.1.1}$$

where $\mathbf{A}^\bullet$ is a complex of sheaves on BK with constant cohomology sheaves, and where $s \in \mathbf{S}$ acts on $\omega \otimes a$ to give $\theta(s) \wedge \omega \otimes a$ (cf. equation (10.9.2), where $\theta = \lim \theta_n$). Define the functor of "ordinary" global sections $E : D^b_K(\mathrm{pt}) \to D_+(\Lambda_\bullet)$ by

$$E(\mathbf{A}^\bullet) = \Gamma(BK; \pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK}} \otimes_{\mathbb{R}} \mathbf{A}^\bullet) \tag{11.1.2}$$

where $\lambda \in \Lambda_\bullet$ acts on $e \otimes a$ to give $\lambda \cdot e \otimes a$ as in (10.8.1).

(11.2) Theorem. *The functors G and E are equivalences of categories, and are related by Koszul duality: there are natural isomorphism of functors*

$$hG \cong E \tag{11.2.1}$$

and

$$G \cong tE \tag{11.2.2}$$

where h denotes the first Koszul duality functor and t denotes the second Koszul duality functor. These functors restrict to equivalences of the full subcategories,

$$\begin{tikzcd} & D^f_+(\mathbf{S}^\bullet) \arrow[dd, shift right, "h"'] \\ D^b_{K,c}(pt) \arrow[ur, "G"] \arrow[dr, "E"'] & \\ & D^f_+(\Lambda^\bullet) \arrow[uu, shift right, "t"'] \end{tikzcd}$$

If X is a compactifiable K-space, if $c : X \to \mathrm{pt}$ is the map to a point, and if $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D^b_K(X)$, then in the following diagram,

$$\begin{tikzcd} & & D^f_+(\mathbf{S}^\bullet) \arrow[r, "H"] \arrow[dd, shift right, "h"'] & \mathbf{S} - Mod \\ D^b_{K,c}(X) \arrow[r, "c^K_*"'] & D^b_{K,c}(pt) \arrow[ur, "G"] \arrow[dr, "E"'] & & \\ & & D^f_+(\Lambda^\bullet) \arrow[r, "H"'] \arrow[uu, shift right, "t"'] & \Lambda_\bullet - Mod \end{tikzcd}$$

the composition across the top is the equivariant cohomology,

$$HGc^K_*(A) = H^*_K(X; A) = H^*(X \times_K EK; \bar{\mathbf{A}})$$

while the composition across the bottom is the ordinary cohomology,

$$HEc_*^K(A) = H^*(X; A) = H^*(X; \mathbf{A_X}).$$

The proof will appear in Sect. 17.

(11.3) Note. The Koszul duality functor h does *not* commute with cohomology: even though $\Lambda_\bullet$ may act trivially on the cohomology $H^*(X, \mathbf{A}^\bullet)$, it does not necessarily follow that the equivariant cohomology $H_K^*(X, \mathbf{A}^\bullet)$ is a free module over $\mathbf{S}$. For example, take $\mathbf{A}^\bullet$ to be the constant sheaf and X to be the total space of the Hopf bundle, $S^{2n+1} \to \mathbb{CP}^n$. This is a principal $K = S^1$ bundle, and $\Lambda_\bullet = \bigwedge \mathbb{R}$ acts trivially on the ordinary cohomology $H^*(S^{2n+1}; \mathbb{R})$. However, for $n < \infty$, the equivariant cohomology $H_K^*(S^{2n+1}; \mathbb{R}) = H^*(\mathbb{CP}^n; \mathbb{R})$ is not a free module over $\mathbf{S} = \mathbb{R}[x]$.

If we are willing to forget the $\mathbf{S}$-module structure on the equivariant cohomology, or to forget the $\Lambda_\bullet$-module structure on the ordinary cohomology, then proposition (8.8) may be applied to give another description of the relationship between cohomology and equivariant cohomology.

(11.4) Corollary. There are natural isomorphisms of complex vectorspaces,

$$H^*(X; A) \cong H^*(Gc_*^K(A) \overset{L}{\otimes}_{\mathbf{S}} \mathbb{R}) \tag{11.4.1}$$

and

$$H_K^*(X; A) \cong H^*(Ec_*^K(A) \overset{L}{\otimes}_{\Lambda} \mathbb{R}). \tag{11.4.2}$$

In other words, the ordinary cohomology may be recovered from the equivariant cochains by tensoring over $\mathbf{S}$ with $\mathbb{R}$ then taking cohomology, and the equivariant cohomology may be recovered from the ordinary cochains by tensoring over $\Lambda_\bullet$ with $\mathbb{R}$ then taking cohomology. (Equation 11.4.1 appears in [BL] Corollary 13.12.2, however in a different language, in the case of the constant sheaf, it is fairly well known among the experts in transformation groups: see [AP1]).

12. The sweep action of $\Lambda_\bullet$ on chains

Suppose that $\mu : K \times X \to X$ denotes a subanalytic action of a compact Lie group K on a subanalytic set X. Let $\mathbb{R}_X^K = (\mathbb{R}_X, \mathbb{R}_{X \times_K EK}, I) \in D_K^b(X)$ denote the canonical lift of the constant sheaf to an element of the equivariant derived category. By theorem 11.2, the (ordinary) cohomology of X is the cohomology of the complex $E(c_*^K \mathbb{R}_X^K) \in D_+(\Lambda_\bullet)$ and it carries the structure of a module over $\Lambda_\bullet = H_*(K)$. On the other hand, the cohomology of X may also be realized as the cohomology of the complex of subanalytic chains

$C^*(X)$, and on this complex there is another action of $\Lambda_\bullet$, the *sweep* action, which is given by sweeping cycles in X around by cycles in K. The purpose of this section is to show that the resulting complex $C^*(X) \in D_+(\Lambda_\bullet)$ is naturally isomorphic to the complex $E(c_*^K \mathbb{R}_X^K)$. Similar remarks apply to the intersection cohomology of X.

(12.1) Sweeping chains. Denote the subanalytic chains on X with complex coefficients by $C_*(X)$ (cf. Sect. 3.2). If $S \in C_i(K)$ and $\xi \in C_j(X)$ are subanalytic chains, denote by $S \times C \in C_{i+j}(K \times X)$ the product chain whose orientation is given by the orientation of S followed by the orientation of ξ. Let $\mu_* : C_*(K \times X) \to C_*(X)$ denote the homomorphism induced on chains by the action μ_X and define the *sweep* $S\xi = \mu_*(S \times \xi) \in C_{i+j}(X)$ to be image chain. (If $\dim(\mu_X(S \times \xi)) < i + j$ then $S\xi = 0$.)

The sweep may be used to define an action of $\Lambda_\bullet$ on the subanalytic chains $C_*(X; \mathbb{R})$ as follows. Fix a basis $x_1, x_2, \ldots, x_r \in H_*(K)$ for the primitive homology $P \subset H_*(K; \mathbb{R})$. (Sect. 10.1) Let $S_1, S_2, \ldots, S_r \in C_*(K)$ be conjugation-invariant subanalytic cycle representatives of the homology classes $x_1, x_2, \ldots, x_r \in H_*(K)$. (This means that $kS_ik^{-1} = S_i$ for all $k \in K$.) For any decomposable element $u = x_{i_1}x_{i_2}\ldots x_{i_t} \in \Lambda_\bullet$ and for any chain $\xi \in C_*(X)$, define the chain $u \cdot \xi$ to be the iterated sweep,

$$u \cdot \xi = S_{i_1}S_{i_2}\ldots S_{i_t}\xi = \mu'(S_{i_1} \times S_{i_2} \times \ldots \times S_{i_t} \times \xi) \tag{12.1.1}$$

where $\mu' : K \times K \times \ldots \times K \times X \to X$ denotes the iterated multiplication. Then $\partial u \cdot \xi = (-1)^{|u|} u \cdot \partial\xi = (-1)^{\deg(u)} u \cdot \partial\xi$.

The sweep action of $\Lambda_\bullet$ on $C_*(X)$ dualizes to a (left) action of $\Lambda_\bullet$ on the complex

$$C^*(X; \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(C_*(X; \mathbb{R}), \mathbb{R}) \tag{12.1.2}$$

of subanalytic cochains by $(u \cdot h)(\xi) = h(\bar{u} \cdot \xi)$ for $u \in \Lambda_\bullet$, $\xi \in C_*(X)$, and $h \in C^*(X)$. Then $d(u \cdot h) = (-1)^{\deg(u)} u \cdot dh$ where $dh(\xi) = h(\partial\xi)$ denotes the differential in $C^*(X)$. In summary, *the subanalytic cochains may be realized as an element $C^*(X) \in D_+(\Lambda_\bullet)$ in the derived category of $\Lambda_\bullet$-modules by choosing cycle representatives for the primitive homology classes of K and allowing $\Lambda_\bullet$ to act by the sweep.* $\square$

(12.2) Remarks. Changing the representative cycles S_i will change the module structure on $C^*(X)$, but only up to homotopy, so the isomorphism class $C^*(X) \in D_+(\Lambda_\bullet)$ is independent of this choice. Particular conjugation-invariant subanalytic cycles S_i are described in [P], [Dyn], [S1], [S2]. The subanalytic assumption is only made for technical convenience. The sweep action of $\Lambda_\bullet$ may be defined on the complex of singular chains, using standard techniques.

Let $c : X \to \mathrm{pt}$ and $c_*^K(\mathbb{R}_X^K) \in D_K^b(\mathrm{pt})$ denote the pushforward (Sect. 5.4) of the constant sheaf $\mathbb{R}_X^K \in D_K^b(X)$. Let $E(c_*^K \mathbb{R}_X^K) \in D_+(\Lambda_\bullet)$ be the complex of (11.1.2).

(12.3) Theorem. *Integration of differential forms over subanalytic chains determines an isomorphism in* $D_+(\Lambda_\bullet)$,

$$I : E(c_*^K \mathbb{R}_X^K) \xrightarrow{\cong} C^*(X). \tag{12.3.1}$$

The proof will appear in Sect. 18.

(12.4) Intersection chains. Choose an equivariant subanalytic stratification of X. Fix a perversity function $\bar{p}$ ([GM1], [GM2]) and let $I^{\bar{p}}C_i(X)$ denote the subcomplex of $(\bar{p}, i)$-allowable subanalytic chains with real coefficients,

$$I^{\bar{p}}C_i(X) = \left\{ \xi \in C_i(X;\mathbb{R}) \middle| \begin{array}{l} \dim(|\xi| \cap S_c) \leq i - c + p(c) \\ \dim(|\partial\xi| \cap S_c) \leq i - 1 - c + p(c) \end{array} \right\} \tag{12.4.1}$$

for each stratum $S_c \subset X$ of codimension c. The sweep action of $\Lambda_\bullet$ preserves the perversity restriction, so it acts on the chain complex $I^{\bar{p}}C_*(X)$ and hence also on the complex of intersection cochains,

$$I^{\bar{p}}C^i(X) = \mathrm{Hom}_{\mathbb{R}}(I^{\bar{p}}C_i(X), \mathbb{R}).$$

Thus the subanalytic intersection cochains may also be realized as an element, $I^{\bar{p}}C^*(X) \in D_+(\Lambda_\bullet)$ of the derived category of $\Lambda_\bullet$-modules. Let $I^{\bar{p}}C_X^K \in D_K^b(X)$ denote the equivariant intersection complex on X. (cf Sect. 4.5)

(12.5) Theorem. *The isomorphism* (12.3.1) *induces an isomorphism in* $D_+(\Lambda_\bullet)$,

$$I : E(c_*^K I^{\bar{p}}C_X^K) \xrightarrow{\cong} I^{\bar{p}}C^*(X). \tag{12.5.1}$$

The proof will appear in Sect. 18.

13. Secondary cohomology operations

(13.1) The 1-dimensional case. Let $K = S^1$ denote a 1-dimensional compact torus, and $\Lambda_\bullet = H_*(K;\mathbb{R})$ its homology ring. Let $(N, d_N) \in D_+(\Lambda_\bullet)$ be a differential graded complex of $\Lambda_\bullet$ modules. Then the cohomology $H^*(N)$ of N is a graded $\Lambda_\bullet$ module. In this section we will describe a sequence of higher cohomology operations on $H^*(N)$ with the property that the original complex N is split and trivial iff (a) the cohomology $H^*(N)$ is a trivial $\Lambda_\bullet$ module, and (b) all the higher cohomology operations on $H^*(N)$ vanish.

Denote by $\lambda \in \Lambda_1$ the fundamental class $\lambda = [K] \in \Lambda_1 = H_1(K; \mathbb{R})$. We may think of the induced action of λ on $H^*(N)$ as a cohomology operation of degree –1, and denote it by

$$\lambda_{(1)} : H^i(N) \to H^{i-1}(N). \tag{13.1.1}$$

(13.2) Proposition. *For each integer $n \geq 1$ there is a higher cohomology operation $\lambda_{(n)}$ of degree $-2n+1$ such that*

(1) *the operation $\lambda_{(1)}$ is given by* (13.1.1)

(2) *the operation $\lambda_{(n)}$ is defined on the kernel of $\lambda_{(n-1)}$ and is well defined modulo the image of $\lambda_{(n-1)}$,*

(3) *If $a_0, a_2, \dots, a_{n-1}$ are homogeneous elements of N with* $\deg(a_j) = \deg(a_0) + 2j$, *and if $da_0 = 0$ and $da_j = \lambda a_{j-1}$ for $1 \leq j \leq n-1$ then $\lambda_{(n)}[a_0] = [\lambda a_{n-1}]$.*

Here, $[a_0]$ denotes the homology class represented by a_0. The chain λa_{n-1} is a cycle because $d\lambda a_{n-1} = -\lambda d a_{n-1} = \lambda\lambda a_{n-2} = 0$. By writing $b_j = da_j = \lambda a_{j-1}$, condition (3) may be interpreted as the existence of a string of homogeneous elements in N, starting at a_0 and ending at $b_n = \lambda_{(n)}(a_0)$, which for $n = 3$ looks like this:

$$b_3 \xleftarrow{\lambda} a_2 \xrightarrow{d} b_2 \xleftarrow{\lambda} a_1 \xrightarrow{d} b_1 \xleftarrow{\lambda} a_0 \tag{13.2.1}$$

(*13.3*) *Proof.* Since K is 1-dimensional, the polynomial algebra $\mathbf{S} \cong \mathbb{R}[\xi]$ may be additively identified with the complex $\bigoplus_{n\geq 0} \mathbb{R}[2n]$ (with zero differential). Thus the Koszul dual $t(N) = N \otimes_{\mathbb{R}} \mathbf{S}$ (with differential $d(a \otimes s) = da \otimes s + \lambda a \otimes \xi s$) may be identified as the single complex associated to the double complex (9.1.1),

$$M^{pq} = \begin{cases} N^{q-p} & \text{if } a \geq p \\ 0 & \text{if } q < p \end{cases} \tag{13.3.1}$$

with differential $d = d' + d''$ where $d''(a) = \lambda a \in M^{p+1,q}$, and $d'(a) = d_N a \in M^{p,q+1}$ for any $a \in M^{pq}$. The spectral sequence associated to this double complex has $E_1^{pq} = H^{q-p}(N)$ and differential $d_{(1)}([a]) = \lambda_{(1)}([a])$ for any $[a] \in H^*(N)$. It follows by induction that the secondary cohomology operation $\lambda_{(n)}$ acting on elements of degree q may be identified with the differential $d_{(n)} : E_{(n)}^{0q} \to E_{(n)}^{n,q-n+1}$. In particular, it is defined on the kernel of $d_{(n-1)}$ and is well defined modulo the image of $d_{(n-1)}$. In fact, the operation $\lambda_{(n)} = d_{(n)}$ is well defined on the homology of the previous operation,

$$\lambda_{(n)} : \frac{\ker(\lambda_{(n-1)})}{\mathrm{Im}(\lambda_{(n-1)})} \to \frac{\ker(\lambda_{(n-1)})}{\mathrm{Im}(\lambda_{(n-1)})} \cong E_{(n)} \tag{13.3.2}$$

(13.4) Proposition. *A differential graded $\Lambda_\bullet$-module $N \in D_+(\Lambda_\bullet)$ is split and trivial iff $\lambda \in \Lambda_1$ acts trivially on its cohomology $H^*(N)$ and all the higher cohomology operations $\lambda_{(n)}$ vanish.*

(13.5) *Proof*. The primary operation by λ and the higher operations by $\lambda_{(n)}$ (for $n \geq 2$) were identified with the differentials in the spectral sequence (9.1.4). These operations vanish iff the spectral sequence collapses, in which case N is split, by (9.3). □

(13.6) Remarks. The example in Sect. 11.4 describes a space X with a nonvanishing higher cohomology operation λ_n on its cohomology $H^*(X)$, together with the consequential failure of the equivariant cohomology $H^*_K(X)$ to be a free module over $\mathbf{S}$.

Assuming the hypotheses of Proposition (13.4), it is possible to construct explicit quasi-isomorphisms of differential graded $\Lambda_\bullet$ modules,

$$\bigoplus_{n\geq 0} H^n(N)[-n] \xleftarrow{\beta} \bigoplus_{n\geq 0} H^n(N)[-n] \otimes_{\mathbb{R}} K^\bullet \xrightarrow{\alpha} N$$

where $K^\bullet = \Lambda^\bullet \otimes \mathbf{S}$ denotes the Koszul complex. The quasi-isomorphism β is given by the augmentation $\epsilon : K^\bullet \to K^0 \cong \mathbb{R}$ while the map α is defined as follows: choose a collection of cycle representatives for the elements in a homogeneous basis of $H^*(N)$. For each such cycle $a \in N$ define $\alpha_a : K^\bullet[\deg(a)] \to N$ by induction, mapping the Koszul complex to an arbitrarily long string of elements of the sort described in (13.2.1).

(13.7) Cyclic homology. The double complex (13.3.1) (or (9.1.1) is sometimes referred to as Connes' double complex ([W] Sect. 9.8.2, [Hu]) which is associated to the mixed complex [Ka1] $N \in D_+(\Lambda_\bullet)$. In particular, we see that *the cyclic homology of a mixed complex N coincides with the cohomology of the Koszul dual complex $t(N)$.*

(13.8) The general case. If K denotes a compact connected Lie group with $\Lambda_\bullet = H_*(K;\mathbb{R})$ and $\mathbf{S} = H^*(BK;\mathbb{R})$ and if $N \in D^f_+(\Lambda_\bullet)$ then the differentials in the spectral sequence for Koszul duality (9.1.4) may be interpreted as a collection of higher cohomology operations, the vanishing of which is equivalent to the statement that N is split and trivial, or that $M = t(N)$ is split and free. Choose homogeneous generators $x_1, x_2, \ldots, x_r \in P$ for the primitive homology classes $P \subset \Lambda_\bullet$.

Proposition. *For each monomial* $a = x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}$ *there is a higher cohomology operation* λ_a *on the cohomology* $H^*(N)$, *of degree*

$$\deg(\lambda_a) = -\sum_{i=1}^{r} [n_i(\deg(x_i) + 1) - 1]$$

which is defined on the subgroup

$$\bigcap \{\ker(\lambda_b) \mid \deg(b) > 0 \text{ and } b|a\}$$

and takes well defined values in the quotient group

$$\frac{H^*(N)}{\sum \{\mathrm{Im}(\lambda_b) \mid \deg(b) > 0 \text{ and } b|a\}}.$$

The complex $N \in D_+(\Lambda_\bullet)$ *is split and trivial iff the action of* $\Lambda_\bullet$ *on the cohomology* $H^*(N)$ *is trivial and all the higher operations* λ_a *vanish.* □

(13.9) Conjecture. The triangulated category $D_+(\Lambda_\bullet)$ is equivalent to the category of graded $\Lambda_\bullet$-modules together with the collection $\{\lambda_a\}$ of secondary cohomology operations.

14. Sufficient conditions for a complex to split

Throughout this section we assume that a compact connected Lie group K acts subanalytically on a subanalytic space X, and we fix an element $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D_K^b(X)$. We give a number of conditions, any one of which suffices to guarantee that the equivariant cohomology $H_K^*(X; A) = H^*(X \times_K EK; \bar{\mathbf{A}})$ is a free module over $\mathbf{S} = H_K^*(\mathrm{pt}; \mathbb{R})$. This verifies the key technical assumption in the topological part of the localization theorem 6.3.

(14.1) Theorem. *Let* $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D_K^b(X)$. *Suppose that any one of the following conditions holds:*

(1) *The (ordinary) sheaf cohomology,* $H^*(X; A) = H^*(X; \mathbf{A_X})$ *vanishes in odd degrees.*

(2) *The action of* $\Lambda_\bullet$ *on the (ordinary) cohomology* $H^*(X; A) = H^*(X; \mathbf{A_X})$ *is trivial, and all the higher* $\Lambda_\bullet$ *operations vanish.*

(3) $A = \mathbb{R}_X^K = (\mathbb{R}_X, \mathbb{R}_{X \times_K EK}, I)$ *is the constant sheaf, and for all* i, *the (ordinary) homology groups* $H_i(X; \mathbb{R})$ *are generated by* K*-invariant subanalytic cycles* $\xi \in C_i(X; \mathbb{R})$.

(4) $A = I^{\bar{p}} C_X^K$ *is the (equivariant) intersection complex with respect to some perversity* $\bar{p}$, *and for all* i, *the (ordinary) intersection homology groups* $I^{\bar{p}} H_i(X; \mathbb{R})$ *are generated by* K*-invariant subanalytic* $(\bar{p}, i)$*-allowable cycles.*

(5) $A = \mathbb{R}_X^K = (\mathbb{R}_X, \mathbb{R}_{X\times_K EK}, I)$ *is the constant sheaf, and the space X has a cell decomposition by K-invariant subanalytic cells.*

(6) *The space X is a nonsingular complex projective algebraic variety, the group* $K \cong (S^1)^r$ *is the compact subtorus of an algebraic torus* $T \cong (\mathbb{C}^*)^r$, *the action of K is the restriction of an algebraic action of T on X, and the sheaf* $A = \mathbb{R}_X^K = (\mathbb{R}_X, \mathbb{R}_{X\times_K EK}, I)$ *is the constant sheaf.*

(7) *The space X is a complex projective algebraic variety, the group* $K = (S^1)^r$ *is the compact subtorus of an algebraic torus* $T \cong (\mathbb{C}^*)^r$, *the action of K is the restriction of an algebraic action of T on X, and the sheaf* $A = I^{\bar{m}}C_X^K$ *is the middle intersection complex.*

(8) *The space* $X = X(\mathbb{C})$ *is the complex points of a (possibly singular) complex algebraic variety, the group* $K = (S^1)^r$ *is the compact subtorus of an algebraic torus* $T \cong (\mathbb{C}^*)^r$, *the action of K on X is the restriction of an algebraic action of T on X, the sheaf* $A = (\mathbb{R}_X^K, \mathbb{R}_{X\times_K EK}, I)$ *is the constant sheaf, and moreover for every non-negative integer q, the cohomology group* $H^q(X, \mathbb{Q})$ *is pure of weight q.*

(9) *The space X is a compact symplectic manifold, K acts on X by Hamiltonian vectorfields, and* $A = \mathbb{R}_X^K$ *is the constant sheaf.*

Then the global "ordinary" sections $E(c_*^K(A)) \in D_+(\Lambda_\bullet)$ *is split and trivial, the equivariant global sections* $G(c_*^K(A)) \in D_+(\mathbf{S})$ *is split and free, and the equivariant cohomology*

$$H_K^*(X; A) = H^*(X \times_K EK; \bar{\mathbf{A}}) \cong H^*(X; \mathbf{A_X}) \otimes_{\mathbb{R}} \mathbf{S} \tag{14.1.1}$$

is a free module over $\mathbf{S}$.

(*14.2*) *Proof.* In case (1) the spectral sequence (9.4.1) collapses because the cohomology of BK also vanishes in odd degrees. So proposition (9.3) applies, and the equivariant cohomology is given by

$$H_K^*(X; A) \cong \mathbf{S} \otimes H^*(X; \mathbf{A_X}).$$

In case (2) the spectral sequence (9.4.1) collapses by proposition (13.8) so the same argument applies.

In case (3), choose a basis for the (ordinary) homology of X consisting of invariant subanalytic cycles. Consider the chain complex B_* with 0 differential and trivial $\Lambda_\bullet$ action which consists of the vectorspace (over $\mathbb{R}$) generated by these subanalytic cycles. The inclusion $B_* \to C_*(X; \mathbb{R})$ of B_* into the complex of subanalytic chains on X is a quasi-isomorphism, and it is a $\Lambda_\bullet$-equivariant mapping. Therefore the (dual) morphism of cochain complexes $B^* = \mathrm{Hom}_{\mathbb{R}}(B^*, \mathbb{R}) \leftarrow C^*(X)$ is an isomorphism in $D_+(\Lambda_\bullet)$. Combining this with theorem (12.3), we obtain isomorphisms

$$B^* \cong C^*(X) \cong E(c_*^K \mathbb{R}_X^K)$$

in $D_K^b(X)$. By theorem 11.2 the Koszul dual is given by

$$t(B^*) \cong tE(c_*^K \mathbb{R}_X^K) \cong Gc_* \mathbb{R}_X^K$$

But the complex B^* is split and trivial, so by proposition (9.3), its Koszul dual is split and free. This means the spectral sequence (9.1.4) or (9.4.1) degenerates, and the equivariant cohomology of X is given by

$$H_K^*(X; \mathbb{R}) \cong H^*(X; \mathbb{R}) \otimes_{\mathbb{R}} \mathbf{S}$$

which is a free module over $\mathbf{S}$. This completes the proof in case (3). The proof in case (4) is similar.

In case (5), let B_* denote the complex of cellular chains on X, with respect to an equivariant cell decomposition of X. The inclusion $B_* \subset C_*(X)$ of B_* into the complex of subanalytic chains, is a quasi-isomorphism, and it is $\Lambda_\bullet$-equivariant since the cells are K-invariant. But $\Lambda_\bullet$ acts trivially on B_*, so the dual cochain complex B^* is a split and trivial element of the derived category $D_+(\Lambda_\bullet)$. The same argument as in the preceding paragraph applies.

In case (7), (cf [Br], [Ki1], [G]) the space $X \times_T ET$ may be realized as a limit of projective algebraic varieties $X \times_T ET_n$ as follows: Take $ET_n = (\mathbb{C}^n - 0)^r$ with algebraic T action given by $(t_1, t_2, \dots, t_r) \cdot (x_1, x_2, \dots, x_r) = (t_1 x_1, t_2 x_2, \dots, t_r x_r)$ as in [Ki1]. Since T acts algebraically on X, the quotient $X \times_T ET_n$ is algebraic and projective and the mapping $\pi : X \times_T ET_n \to BT_n \cong (\mathbb{CP}^n)^r$ is an algebraic fiber bundle. The hyperplane class for $X \times_T ET_n$ induces a hard Lefschetz isomorphism for the middle intersection cohomology of each fiber $\pi^{-1}(y) \cong X$ ([D2], Theorem 6.2.13 or [BBD] Cor. 5.3.4). By the theorem of Blanchard and Deligne [Bla], [D1], this implies that the spectral sequence (9.4.1) collapses, so proposition (9.3) applies, and the equivariant intersection cohomology is given by

$$I^{\bar{m}} H_K^*(X; \mathbb{R}) \cong I^{\bar{m}} H^*(X) \otimes_{\mathbb{R}} \mathbf{S}.$$

This completes the proof of case (7).

The same proof also works whenever K is a maximal compact subgroup of a complex algebraic grup $K_{\mathbb{C}}$, which admits a model $BK = \lim BK_n$ for its classifying space such that each BK_n is a complex projective nonsingular algebraic variety, provided the action of K on X is the restriction of an algebraic action of $K_{\mathbb{C}}$ on X. More generally, if the sheaf $\mathbf{A_X}$ is "pure" then it is isomorphic to a direct sum of (shifts of) intersection cohomology sheaves, and the same argument implies that the spectral sequence (9.4.1) collapses. (cf. [Br] Theorem 4.2.3 or [BBD] Theorem 5.3.8).

Case (6) is a particular version of case (7).

In case (8) we use the same algebraic model $ET_n = (\mathbb{C}^n - \{0\})^r$, $BT_n = (\mathbb{CP}^n)^r$ as in case (7). By hypothesis, $H^q(X)$ is pure of weight q, and moreover, the cohomology $H^p(BT_n)$ is pure of weight p. Therefore the E_2 term of the spectral sequence for the fibration $X \times_T ET_n \to BT_n$ is

$$E_{(2)}^{pq} = H^p(BT_n) \otimes H^q(X)$$

which is pure of weight $p+q$. It follows from mixed Hodge theory ([D3], [D4], [D5]) that the differentials in this spectral sequence are strictly compatible with the weight, and hence they all vanish, so proposition (9.3) applies. □

In case (9) it follows from [Ki2] Sects. 5.8 that the spectral sequence (9.4.1) for equivariant cohomology collapses so Proposition (9.3) applies. □

15. Proof of Theorem 6.2

The proof of the localization theorem consists of combining the equivariant derived category techniques of Bernstein and Lunts [BL] with the localization arguments of Borel [B3], Quillen [Q], Hsiang [H2], and Chang and Skjelbred [CS] (cf [H2] Sects. III.1, IV.2). We have simplified the argument in [CS] by focusing (as in [AB]) on the support of various **S**-modules rather than on the primary decomposition of their annihilators. As in Sect. 6, we use complex coefficients in this section.

(15.1) Lemma. Let $Y = Kx \subset X$ denote the orbit of a single point $x \in X - F$ (where F denotes the fixed point set). Let $L = K_x^0$ denote the connected component of the stabilizer. For any equivariant sheaf $A \in D_K^b(X)$, the equivariant cohomology $H_K^*(Y;A)$ is a torsion module over $\mathbf{S} = H_K^*(\mathrm{pt};\mathbb{C})$ with

$$\mathrm{spt}(H_K^*(Y;A)) \subset \mathfrak{l}_{\mathbb{C}} = \mathrm{Lie}(L) \otimes_{\mathbb{R}} \mathbb{C}. \tag{15.1.1}$$

(*15.2*) *Proof.* Choose a splitting $K \cong L \times L'$ of the torus, which gives rise to splittings $EK \cong EL \times EL'$, $BK \cong BL \times BL'$, and $H_K^*(\mathrm{pt}) \cong H_L^*(\mathrm{pt}) \otimes H_{L'}^*(\mathrm{pt})$. Then $K \cong L \times L'$ acts (almost) freely on the space $EL \times Y$ by $(\ell,\ell').(e,y) = (\ell.e,\ell'.y)$ with quotient $EL \times_K Y \cong BL$. So the projection $EL \times Y \to Y$ is (in the language of [BL]) an infinite acyclic resolution, and we have a diagram

$$\begin{array}{ccccc} Y & \longleftarrow & EK \times Y & \longrightarrow & EK \times_K Y \\ \Big\downarrow & & \Big\downarrow f & & \Big\downarrow \bar{f} \\ Y & \underset{p}{\longleftarrow} & EL \times Y & \underset{q}{\longrightarrow} & BL \end{array} \tag{15.2.1}$$

Following [BL] Sects. 2.1.3, 2.7.2, we consider the category $D_K^b(Y, EL \times Y)$ of triples $(\mathbf{A_Y}, \beta, \bar{\mathbf{A}})$ where $\mathbf{A_Y} \in D^b(Y)$, $\bar{\mathbf{A}} \in D^b(BL)$, and $\beta : p^*(\mathbf{A_Y}) \to q^*(\bar{\mathbf{A}})$ is an isomorphism in $D^b(EL \times Y)$. The association $(\mathbf{A_Y}, \beta, \bar{\mathbf{A}}) \mapsto (\mathbf{A_Y}, f^*\beta, \bar{f}^*(\bar{\mathbf{A}}))$ defines an equivalence of categories $D^b(Y, EL \times Y) \to D_K^b(Y)$ as in

[BL] Sect. 2.9.3. Thus, we may assume that $A|Y = (\mathbf{A_Y}, \beta, \bar{\mathbf{A}}) \in D_K^b(Y, EL \times Y)$. Consider the effect of the isomorphism β on the stalk cohomology at a point $(e, y) \in EL \times Y$,

$$\mathbf{H}_y^*(\mathbf{A_Y}) \cong \mathbf{H}_{(e,y)}^*(p^*\mathbf{A_Y}) \xrightarrow{\beta} \mathbf{H}_{(e,y)}^*(q^*\bar{\mathbf{A}}) \cong \mathbf{H}_{q(e)}^*(\bar{\mathbf{A}}). \tag{15.2.2}$$

This shows that the cohomology sheaf $\mathbf{H}^*(\bar{\mathbf{A}})$ on BL is constant. Therefore the equivariant cohomology is given by $H_K^*(Y; A) \cong H^*(BL; \bar{\mathbf{A}})$. There is a spectral sequence of $\mathbf{S}$-modules for this group, with

$$E_{(2)} = H^*(BL) \otimes H_{q(e)}^*(\bar{\mathbf{A}}).$$

The support of this module is $\mathfrak{l}_{\mathbb{C}}$. It follows that $\operatorname{spt}(H_K^*(Y, A)) \subset \mathfrak{l}_{\mathbb{C}}$. $\square$

(15.3) Lemma. Let $Y \subset X$ be an invariant, compact subset on which K acts without fixed points. Then for any $A \in D_K^b(X)$ the support of the equivariant cohomology

$$\operatorname{spt}(H_K^*(Y; A)) \subset \bigcup_{y \in Y} \mathfrak{k}_y^{\mathbb{C}} \tag{15.3.1}$$

is contained in the union of the Lie algebras of the stabilizers of points $y \in Y$.

(*15.4*) *Proof*. Cover Y by finitely many regular neighborhoods of orbits. Apply Lemma (15.1) to each orbit and patch using Mayer-Vietoris.

(*15.5*) *Proof of theorem 6.2* (1) *and* (2). Let $U \subset X$ be an invariant regular neighborhood of Z, with invariant boundary ∂U. Then $H_K^*(X, Z; A) \cong H_K^*(X - U, \partial U; A)$. Apply Lemma 15.3 to $H_K^*(X - U; A)$ and to $H_K^*(\partial U; A)$. This proves part (1), and part (2) follows immediately. $\square$

(15.6) *Proof of theorem 6.2* (3). By the long exact cohomology sequence, it suffices to show that $H_K^*(X, X^L; A)_{PL} = 0$. By Theorem 6.2 (1), $\operatorname{spt}(H_K^*(X, X^L; A)) \not\supset \mathfrak{l}_{\mathbb{C}}$ since it is contained in a union of linear subspaces $\mathfrak{k}_y^{\mathbb{C}}$, none of which contains $\mathfrak{l}_{\mathbb{C}}$. Hence the localized module vanishes, $H_K^*(X, X^L; A)_{PL} = 0$. $\square$

(15.7) Let $\xi \in H_K^*(F)$ and let $I(\xi) = \operatorname{Ann}(\delta'(\xi))$ be the ideal in $\mathbf{S}$ which annihilates $\delta'(\xi)$ where δ' is the connecting homomorphism in the long exact sequence,

$$H_K^*(X; A) \xrightarrow{\gamma} H_K^*(F; A) \xrightarrow{\delta'} H_K^*(X, F; A) \tag{15.7.1}$$

For each $\mathfrak{m} \in \mathscr{P}$ let $\delta^{\mathfrak{m}}$ denote the connecting homomorphism in the exact sequence,

$$H_K^*(X^{\mathfrak{m}}; A) \xrightarrow{\gamma} H_K^*(F; A) \xrightarrow{\delta^{\mathfrak{m}}} H_K^*(X^{\mathfrak{m}}, F; A) \tag{15.7.2}$$

(15.8) Lemma. [CS] If $\delta'(\xi) \neq 0$ then the variety defined by $I(\xi)$ satisfies

$$V(I(\xi)) \subset \bigcup_{\substack{\mathfrak{m} \in \mathscr{P} \\ \delta^{\mathfrak{m}}(\xi) \neq 0}} \mathfrak{m}_{\mathbb{C}}. \tag{15.8.1}$$

(*15.9*) *Proof.* Since $\delta'(\xi) \in H_K^*(X, F; A)$ we have

$$V(\mathrm{Ann}(\delta'(\xi))) \subset \mathrm{spt}(H_K^*(X, F; A)) \subset \bigcup_{\mathfrak{l} \in \mathscr{P}} \mathfrak{l}_{\mathbb{C}} \tag{15.9.1}$$

by Theorem 6.2(1). Suppose $\mathfrak{m} \in \mathscr{P}$ and $\delta^{\mathfrak{m}}(\xi) = 0$. We have an exact sequence

$$H_K^*(X, X^{\mathfrak{m}}; A) \xrightarrow{j} H_K^*(X, F; A) \xrightarrow{v} H_K^*(X^{\mathfrak{m}}, F; A) \tag{15.9.2}$$

Then $\delta^{\mathfrak{m}}(\xi) = v\delta'(\xi) = 0$ so there exists $y \in H_K^*(X, X^{\mathfrak{m}}; A)$ with $\delta'(\xi) = j(y)$. Hence, $\mathrm{Ann}(y) \subset \mathrm{Ann}(\delta'(\xi))$ so

$$V(\mathrm{Ann}(\delta'(\xi))) \subset V(\mathrm{Ann}(y)) \subset \mathrm{spt}\, H_K^*(X, X^{\mathfrak{m}}; A) \subset \bigcup_{\substack{\mathfrak{l} \in \mathscr{P} \\ \mathfrak{l} \not\supset \mathfrak{m}}} \mathfrak{l}_{\mathbb{C}}$$

by 6.2(1). Since this holds for any such $\mathfrak{m}$, we conclude that

$$V(I(\xi)) \subset \bigcap_{\substack{\mathfrak{m} \in \mathscr{P} \\ \delta^{\mathfrak{m}}(\xi) = 0}} \left(\bigcup_{\substack{\mathfrak{l} \in \mathscr{P} \\ \mathfrak{l} \not\supset \mathfrak{m}}} \mathfrak{l}_{\mathbb{C}} \right).$$

The (finite) partially ordered set $\mathscr{P}$ is the union of the two disjoint subsets,

$$\mathscr{P}_+ = \{\mathfrak{m} \in \mathscr{P} | \delta^{\mathfrak{m}}(\xi) = 0\}$$
$$\mathscr{P}_- = \{\mathfrak{m} \in \mathscr{P} | \delta^{\mathfrak{m}}(\xi) \neq 0\}$$

Then $\mathscr{P}_+$ is upward saturated (and $\mathscr{P}_-$ is downward saturated): If $\mathfrak{m} \in \mathscr{P}_+$ then $\mathscr{P}_{\geq \mathfrak{m}} = \{\mathfrak{l} \in \mathscr{P} | \mathfrak{l} \geq \mathfrak{m}\} \subset \mathscr{P}_+$. It follows that

$$\mathrm{spt}(I(\xi)) \subset \bigcap_{\mathfrak{m} \in \mathscr{P}_+} \left(\bigcup_{\mathfrak{l} \in \mathscr{P} - \mathscr{P}_{\geq m}} \mathfrak{l}_{\mathbb{C}} \right) = \bigcup_{\mathfrak{m} \in \mathscr{P}_-} \mathfrak{m}_{\mathbb{C}}. \qquad \square$$

(*15.10*) *Proof of Theorem 6.3*. By Theorem 6.2(1), the kernel and cokernel of γ are torsion modules, however $H_K^*(X;A)$ is a free module, by assumption. Therefore $\ker(\gamma) = 0$. It is also clear that $\delta \circ \gamma = 0$. Now suppose $\delta(\xi) = 0$. We must show that $\delta'(\xi) = 0$, where δ' is the connecting homomorphism in the long exact sequence (15.7.1). Assume $\delta'(\xi) \neq 0$. Let $I(\xi) = \mathrm{Ann}(\delta'(\xi))$ as in (15.7), so $V(I(\xi)) \subset \bigcup\{\mathfrak{l}_{\mathbb{C}} \mid \mathfrak{l} \in \mathscr{P}$ and $\delta^{\mathfrak{l}}(\xi) \neq 0\}$ by (15.8). On the other hand, $I(\xi)$ is principal since $H_K^*(X;A)$ is free (see [CS] Sect. 2.2 or [H2] Sect. IV.2 Proposition 6). Therefore at least one of the tori L appearing in (15.8.1) has codimension 1; for this torus $X^L \subset X_1$. Then $\delta^{\mathfrak{l}}$ factors through δ,

$$H_K^*(F;A) \xrightarrow{\delta} H_K^*(X_1, F; A) \to \mathbf{H}_K^*(X^L, F; A)$$

which contradicts the assumption that $\delta(x) = 0$. □

16. Proof of Koszul duality theorem (8.4)

In this section we show how the proof of Koszul duality, as outlined in [BGS] may be modified so as to agree with the gradings and sign conventions used in Sect. 8.

(16.1) Step 1. The functor $h : K_+(\Lambda_\bullet) \to K_+(\mathbf{S})$ passes to a functor $t : D_+(\Lambda_\bullet) \to D_+(\mathbf{S})$ on the derived category: a morphism $f : N_1 \to N_2$ of complexes of $\Lambda_\bullet$-modules induces a map of spectral sequences (Sect. 9.1.4) with $E_{(1)}$ given by

$$E_{(1)}^{pq}(N_1) = \mathbf{S}^{2p} \otimes_k H^{q-p}(N_1) \to E_{(1)}^{pq}(N_2) = \mathbf{S}^{2p} \otimes_k H^{q-p}(N_2) \qquad (16.1.1)$$

If $f : N_1 \to N_2$ induces an isomorphism on cohomology then the map (16.1.1) determines an isomorphism of spectral sequences and hence determines an isomorphism $H^*(t(N_1)) \to H^*(t(N_2))$ on cohomology. So $t(f) : t(N_1) \to t(N_2)$ is a quasi-isomorphism. A similar argument applies to the functor $h : K_+(\mathbf{S}) \to K_+(\Lambda_\bullet)$ by substituting the spectral sequence (9.1.5) for the spectral sequence (9.1.4).

(16.2) Step 2. Construct an isomorphism of functors $I \to ht$ on $D_+(\Lambda_\bullet)$ as follows: For any $N \in D_+(\Lambda_\bullet)$ define an injection

$$\Phi : N \to ht(N) = \mathrm{Hom}_k(\Lambda_\bullet, \mathbf{S} \otimes_k N) \qquad (16.2.1)$$

by

$$\Phi_n(\lambda) = 1 \otimes \bar{\lambda} n \qquad (16.2.2)$$

We claim that Φ is an isomorphism in the category $D_+(\Lambda_\bullet)$, i.e.,

(a) Φ is a morphism of $\Lambda_\bullet$-modules
(b) Φ is a morphism of complexes
(c) Φ induces isomorphisms on cohomology.

(*16.3*) *Proof of* (*a*). Note that the $\Lambda_\bullet$-module structure on $ht(N)$ is given as follows: if $x \in \Lambda_\bullet$ and $F \in \mathrm{Hom}(\Lambda_\bullet, \mathbf{S} \otimes_k N)$ then $x.F \in \mathrm{Hom}(\Lambda_\bullet, \mathbf{S} \otimes_k N)$ is the homomorphism

$$(x.F)(\lambda) = F(\bar{x}\lambda) \tag{16.3.1}$$

So for all $\lambda \in \Lambda_\bullet$ we have

$$(x.\Phi_n)(\lambda) = \Phi_n(\bar{x}\lambda) = 1 \otimes \overline{\bar{x}\lambda} n = 1 \otimes \bar{\lambda} x n = \Phi_{xn}(\lambda) \tag{16.3.2}$$

(16.4) Proof of (b). For all $n \in N$ and for all $\lambda \in \Lambda_\bullet$, we have, by (8.3.2) and (8.3.4),

$$\begin{aligned}
(d\Phi_n)(\lambda) &= -\sum_i \xi_i \Phi_n(x_i\lambda) + (-1)^{\deg(\lambda)} d_{\mathbf{S}\otimes N}(\Phi_n(\lambda)) \\
&= -\sum_i \xi_i \otimes \overline{x_i\lambda} n + (-1)^{\deg(\lambda)} d_{\mathbf{S}\otimes N}(1 \otimes \bar{\lambda} n) \\
&= -\sum_i \xi_i \otimes \bar{\lambda}\bar{x}_i n + (-1)^{\deg(\lambda)} \sum_i \xi_i \otimes x_i \bar{\lambda} n \\
&\qquad + (-1)^{\deg(\lambda)} (1 \otimes d_N(\bar{\lambda} n)) \\
&= -\sum_i \xi_i \otimes \bar{\lambda} x_i n + \sum_i \xi_i \otimes \bar{\lambda} x_i n + 1 \otimes \bar{\lambda} d_N n \\
&= \Phi_{dn}(\lambda)
\end{aligned} \tag{16.4.1}$$

(16.5) Proof of (c). The augmentation $\epsilon : \mathbf{S} \to k$ extends to a map of complexes $\epsilon : \mathbf{S} \otimes_k N \to N$. Following [BGS], define $\Psi : \mathrm{Hom}_k(\Lambda_\bullet, \mathbf{S} \otimes N) \to N$ by assigning to any $F \in \mathrm{Hom}(\Lambda_a, \mathbf{S} \otimes N)$ the element

$$\Psi(F) = \begin{cases} \epsilon(F(1)) & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases} \tag{16.5.1}$$

Although Ψ is not a morphism of $\Lambda_\bullet$-modules, one easily checks that it is nevertheless a morphism of complexes (i.e. $d_N \Psi = \Psi d_{ht(N)}$) and that it is a splitting for the injection $\Phi : N \to ht(N)$. Thus, it suffices to verify that Ψ induces an isomorphism on cohomology.

The module $ht(N) = \mathrm{Hom}(\Lambda_\bullet, \mathbf{S} \otimes N)$ is actually a triple complex,

$$ht(N)^{abc} = \mathrm{Hom}_k(\Lambda_a, \mathbf{S}^b \otimes_k N^c) \cong \mathrm{Hom}_k(\Lambda_a, \mathbf{S}^b) \otimes_k N^c \tag{16.5.2}$$

with total degree $a+b+c$ and differential $d = d' + d'' + d'''$ which (by (8.3.2) and (8.3.4)) is given by

$$\begin{aligned} d'(f \otimes n)(\lambda) &= -\sum_i \xi_i f(x_i \lambda) \otimes n \\ d''(f \otimes n)(\lambda) &= (-1)^{\deg(\lambda)} \sum_i \xi_i f(\lambda) \otimes x_i n \\ d'''(f \otimes n)(\lambda) &= (-1)^{\deg(\lambda)} f(\lambda) \otimes d_N n \end{aligned} \tag{16.5.3}$$

for any $f \in \mathrm{Hom}(\Lambda_\bullet, \mathbf{S})$ and $n \in N$. Then $ht(N)$ may be regarded as the single complex which is associated to the double complex

$$T^{pq}(N) = \bigoplus_{\substack{q=2a+b+c \\ p=-a}} \mathrm{Hom}(\Lambda_a, \mathbf{S}^b) \otimes N^c \tag{16.5.4}$$

with differentials $\delta'_T = d' : T^{pq} \to T^{p+1,q}$ and $\delta''_T = d'' + d''' : T^{pq} \to T^{p,q+1}$. We may also regard N as the single complex associated to the double complex

$$N^{pq} = \begin{cases} N^q & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} \tag{16.5.5}$$

and with differential $\delta''_N : N^{0,q} \to N^{0,q+1}$ given by d_N (and all other differentials vanishing). With these choices, the morphism $\Psi : \mathrm{Hom}(\Lambda_\bullet, \mathbf{S} \otimes N) \to N$ is actually a morphism of double complexes, $T^{pq} \to N^{pq}$.

The horizontal differential $\delta' = d'$ is the tensor product $d' = \partial \otimes I_N$ where ∂ is the Koszul differential on the Koszul complex $\mathrm{Hom}_k(\Lambda_\bullet, \mathbf{S})$, which in turn is a resolution of the constants $k \cong \mathrm{Hom}(\Lambda_0, \mathbf{S}^0)$ in degree 0 (cf. [C2], or [Ka2] XVIII eq. (7.13) for an explicit trivializing homotopy). Therefore the $E_{(1)}$ term of the spectral sequence for T^{pq} becomes

$$E^{pq}_{(1)}(T) = \begin{cases} N^q & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases} \tag{16.5.6}$$

Furthermore the differential d'' maps $\mathrm{Hom}(\Lambda_0, \mathbf{S}^0 \otimes N^q)$ to $\mathrm{Hom}(\Lambda_0, \mathbf{S}^2 \otimes N^{q+1})$ and hence it vanishes when we pass to $\mathbf{E}_{(1)}(T)$, in other words, $\delta'_T = d''' = d_N$ on $E_{(1)}(T)$. Therefore, Ψ induces an isomorphism

$$(E^{pq}_{(1)}(T), \delta''_T) \to (N, d_N) \tag{16.5.7}$$

of spectral sequences, and hence also an isomorphism on cohomology.

(16.6) Step 3 Construct an isomorphism of functors $th \to I$ on $D_+(\mathbf{S})$ as follows: For all $M \in D_+(\mathbf{S})$ define the surjection

$$\Theta : \mathbf{S} \otimes_k \mathrm{Hom}_k(\Lambda_\bullet, M) \to M \tag{16.6.1}$$

by

$$\Theta(s \otimes F) = s.F(1) \in M \tag{16.6.2}$$

Then, as in (16.3) and (16.4), Θ is a morphism of complexes of $\mathbf{S}$-modules. In fact it is a quasi-isomorphism, as may be seen by applying the preceding spectral sequence argument (16.5) to the splitting $M \to \mathbf{S} \otimes_k \mathrm{Hom}_k(\Lambda_\bullet, M)$ which is given by $m \mapsto 1 \otimes f_m$ where

$$f_m(\lambda) = \begin{cases} \lambda m & \text{if } \lambda \in \Lambda_0 \cong k \\ 0 & \text{otherwise.} \end{cases} \tag{16.6.3}$$

(16.7) Step 4. Now let us check the finiteness properties which are described in theorem 8.4. Let $D_+^F(\mathbf{S})$ denote the derived category of complexes of $\mathbf{S}$-modules which are finitely generated. We have canonical functors

$$D_+^F(\mathbf{S}) \xrightarrow{\alpha} D_+^f(\mathbf{S}) \xrightarrow{\beta} D_+(\mathbf{S}). \tag{16.7.1}$$

Bernstein and Lunts show (Sect. 11.1.3) that the composition $\beta\alpha$ is fully faithful, and the same argument applies to β.

We claim that the functor α is an equivalence of categories. This may be seen from the following argument, for which we thank V. Lunts [L]: It suffices to show that every object $M \in D^f(\mathbf{S})$ is quasi-isomorphic (within $D_+(\mathbf{S})$) to a complex of finitely generated $\mathbf{S}$-modules. This follows by induction on the cohomological dimension of the $\mathbf{S}$-module $H^*(M)$: if $H^*(M)$ is a free $\mathbf{S}$-module, then $H^*(M) \cong M$ and we are done. Otherwise, there is a finitely generated, bounded below $\mathbf{S}$-module P, with 0 differentials, and a morphism $u : P \to M$ which induces a surjection on cohomology. Let $C(u) \in D_+(\mathbf{S})$ denote the cone of this morphism. Then the cohomological dimension of $H^*(C(u))$ is less than that of $H^*(M)$, and $H^*(C(u))$ is finitely generated. By induction, $C(u)$ is isomorphic to a complex $C'(u)$ of finitely generated $\mathbf{S}$-modules. Since the functor $D_+^F(\mathbf{S}) \to D_+(\mathbf{S})$ is fully faithful, the third morphism $C'(u) \to P$ of the above distinguished triangle is also in $D_+^F(\mathbf{S})$. But M is isomorphic to the cone of this morphism $C'(u) \to P$, i.e. M is isomorphic to a complex of finitely generated $\mathbf{S}$-modules. Similar remarks apply to the derived categories of $\Lambda_\bullet$-modules.

The functors h and t take complexes with finitely generated cohomology to complexes with finitely generated cohomology, because $h(\mathbf{S})$ is the Koszul complex whose cohomology is k, and t even takes finitely generated $\Lambda_\bullet$-complexes to finitely generated $\mathbf{S}$-complexes. $\square$

17. Proof of Theorem 11.2

We must display an isomorphism of functors $hG \cong E$. First let us lift the Koszul duality functor h to a functor on sheaves. As in Sect. 10.1, let P denote the r-dimensional vectorspace of primitive elements in $\bigwedge \mathfrak{k}$, let $\Lambda_\bullet = \bigwedge P$ and $\Lambda^\bullet = \bigwedge P^*$ where $P^* = \mathrm{Hom}_{\mathbb{R}}(P, \mathbb{R})$ denotes the dual space for P. The Kronecker pairing (Sect. 10.4) $\langle\ ,\ \rangle : \Lambda_\bullet \times \Lambda^\bullet \to \mathbb{R}$ identifies $\Lambda^\bullet \cong \mathrm{Hom}_{\mathbb{R}}(\Lambda_\bullet, \mathbb{R})$. Let $\tilde{P}^*$ denote the dual space with modified grading, and $\mathbf{S} = S(\tilde{P}^*)$. If $\mu \in P^*$ write $\tilde{m}u \in \tilde{P}^*$ for the corresponding element. Fix dual bases $\{x_i\}$ and $\{\xi_i\}$ (with $1 \leq i \leq r$) for P and $\tilde{P}^*$.

(17.1) Definition. *Let* $\mathbf{B}^\bullet$ *be a soft complex of sheaves of* $\mathbf{S}$*-modules on the classifying space* BK*. Define*

$$\mathbf{h}(\mathbf{B}^\bullet) = \mathbf{Hom}(\Lambda_\bullet, \mathbf{B}^\bullet) \tag{17.1.1}$$

to be the complex of sheaves of $\Lambda_\bullet$*-modules on* BK *whose sections over an open set* U *are* $\Gamma(U, \mathbf{h}(\mathbf{B}^\bullet)) = \mathrm{Hom}_{\mathbb{R}}(\Lambda_\bullet, \Gamma(U, \mathbf{B}^\bullet))$ *with differential*

$$dF(\lambda) = -\sum_{i=1}^{r} \xi_i F(x_i \lambda) + (-1)^{\deg(\lambda)} d_{\mathbf{B}}(F(\lambda)) \tag{17.1.2}$$

(for homogeneous elements $\lambda \in \Lambda_\bullet$*), and with* $\Lambda_\bullet$*-module structure* $(x.f)(\lambda) = F(\bar{x}\lambda)$ *for* $x \in \Lambda_\bullet$ *and* $F \in \mathrm{Hom}_{\mathbb{R}}(\Lambda_\bullet \Gamma(U, \mathbf{B}^\bullet))$*. It follows that* $h(\Gamma(BK; \mathbf{B}^\bullet)) = \Gamma(BK; \mathbf{h}(\mathbf{B}^\bullet))$.

In the next few sections we will use Chern-Weil theory to show that $\mathbf{h}$ transforms the universal $\mathbf{S}$-sheaf into the universal $\Lambda_\bullet$-sheaf: in Lemma 17.6 we will describe a quasi-isomorphism of sheaves of $\Lambda_\bullet$-modules on BK,

$$\mathbf{h}(\mathbf{\Omega}^\bullet_{\mathbf{BK}}) \to \pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK}}. \tag{17.1.3}$$

(17.2) Chern-Weil construction. Fix a transgression $\tau : P^* \to S(\tilde{\mathfrak{k}}^*)^K$. Then τ is homogeneous of degree 1. The composition $\tilde{P}^* \to P^* \xrightarrow{\tau} S(\tilde{\mathfrak{k}}^*)^K$ is homogeneous of degree 0 and extends in a unique way to a homomorphism of graded algebras,

$$T : \mathbf{S} = S(\tilde{P}^*) \to S(\tilde{\mathfrak{k}}^*)^K. \tag{17.2.1}$$

A fundamental result of Chevalley, Koszul and Cartan ([C2] Theorem 2) states that the homomorphism T is an isomorphism of graded algebras.

In Sect. 10.9 a (left invariant) connection was chosen in the principal K-bundle $EK \to BK$. Let $f : \mathfrak{k}^* \to \Omega^1(EK)$ be the associated connection 1-form, where $\Omega^\bullet(EK)$ denotes the complex of smooth complex-valued differential forms on EK. The mapping f has a unique extension to a homomorphism of graded algebras, $f : \bigwedge(\mathfrak{k}^*) \to \Omega^\bullet(EK)$ however it does not commute with the differentials. The curvature 2-form, $\Theta : \tilde{\mathfrak{k}}^* \to \Omega^2(EK, \mathbb{R})$ is given by

$\Theta(\tilde{\xi}) = df(\xi) - f(d_{\mathfrak{k}}\xi)$. It extends to a homomorphism of graded algebras, $\Theta : S(\tilde{\mathfrak{k}}^*) \to \Omega^\bullet(EK)$. If $\xi \in S(\tilde{\mathfrak{k}}^*)^K$ is an invariant polynomial on $\mathfrak{k}$ then $\Theta(\xi)$ lies in the subalgebra $\pi^*(\Omega^\bullet(BK))$ of "basic" elements, consisting of differential forms which are both invariant and are also annihilated by every invariant vertical vectorfield. This gives the Weil homomorphism $\Theta : S(\tilde{\mathfrak{k}}^*)^K \to \Omega^\bullet(BK)$. Composing with the mapping T gives the injective Chern-Weil homomorphism of graded algebras (cf. (10.9.1)),

$$\theta = \Theta \circ T : \mathbf{S} \to \Omega^\bullet(BK) \tag{17.2.2}$$

which induces an isomorphism on cohomology, $\mathbf{S} \cong H^*(BK, \mathbb{R})$. Set $\phi = \Theta\tau : P^* \to \Omega^\bullet(BK)$. In summary, we have a commutative diagram, (where [1] denotes a degree 1 mapping),

$$\begin{tikzcd}
P^* \arrow[r, "{[1]}"'] \arrow[d, equal] & \mathbf{S} = S(\tilde{P}^*) \arrow[d, "T"] & & \\
P^* \arrow[r, "\tau", "{[1]}"'] \arrow[d, equal] & S(\tilde{\mathfrak{k}}^*)^K \arrow[r, hook] \arrow[d, "\Theta"] & S(\tilde{\mathfrak{k}}^*) \arrow[d, "\Theta"] & \mathfrak{k}^* \arrow[l, "{[1]}"] \arrow[d, "df - fd", "{[1]}"'] \\
P^* \arrow[r, "\phi", "{[1]}"'] & \Omega^\bullet(BK) \arrow[r, hook, "\pi^*"] & \Omega^\bullet(EK) & \Omega^2(EK) \arrow[l]
\end{tikzcd}$$

(17.3). Define $\mathbf{E}^\bullet_{\mathbf{BK}} = \Lambda^\bullet \otimes \boldsymbol{\Omega}^\bullet_{\mathbf{BK}}$ to be the complex of sheaves of $\Lambda_\bullet$-modules on BK whose sections over an open set $U \subset BK$ are given by

$$\Gamma(U, \mathbf{E}^{\mathbf{k}}_{\mathbf{BK}}) = \bigoplus_{p+q=k} \Lambda^p \otimes \Omega^q(U, \mathbb{R}) \tag{17.3.1}$$

with differential,

$$\begin{aligned}
&d_E(\mu_0 \wedge \mu_1 \wedge \ldots \wedge \mu_{p-1} \otimes \omega) \\
&\quad = -\sum_{j=0}^{p-1} (-1)^j \mu_0 \wedge \ldots \wedge \hat{\mu}_j \wedge \ldots \wedge \mu_{p-1} \otimes \phi(\mu_j) \wedge \omega \\
&\qquad + (-1)^p \mu_0 \wedge \mu_1 \wedge \ldots \wedge \mu_{p-1} \otimes d\omega
\end{aligned} \tag{17.3.2}$$

for any $\mu_0, \mu_1, \ldots, \mu_{p-1} \in P^*$, and with module structure given by the interior multiplication, $\lambda \cdot \mu \otimes \omega = i(\bar{\lambda})(\mu) \otimes \omega$ for $\lambda \in \Lambda_\bullet$, $\mu \in \Lambda^\bullet$, and $\omega \in \Omega^\bullet(U)$.

(17.4) Lemma. Let

$$\alpha : \mathbf{E}^\bullet_{\mathbf{BK}} = \Lambda^\bullet \otimes \boldsymbol{\Omega}^\bullet_{\mathbf{BK}} \to \mathbf{h}(\boldsymbol{\Omega}^\bullet_{\mathbf{BK}})$$

be the isomorphism of sheaves, given by $\alpha(\mu \otimes \omega) = F_{\mu\otimes\omega}$ where $F_{\mu\otimes\omega}(\lambda) = \langle \lambda, \mu \rangle \omega$. Then α is an isomorphism of sheaves of $\Lambda_\bullet$-modules.

(17.5) *Proof*. It is easy to check that the actions of $\Lambda_\bullet$ are compatible, i.e. $\alpha(\lambda \cdot \mu \otimes \omega) = \lambda \cdot \alpha(\mu \otimes \omega)$. The main issue is to check that the differentials agree. For each $\mu_j \in P^*$ we have $\tilde{\mu}_j = \sum_{i=1}^n \langle x_i, \mu_j \rangle \xi_i \in \tilde{P}^*$. Since $\phi(\mu_j) = \Theta\tau(\mu_j) = \Theta T(\tilde{\mu}_j)$, the differential (17.3.2) may be rewritten as

$$d_E(\mu \otimes \omega) = -\sum_{j=1}^n x_j \cdot \mu \otimes \Theta T(\xi_j) \wedge \omega + (-1)^{|\mu|} \mu \otimes d\omega \tag{17.5.1}$$

where $\mu_0, \mu_1, \dots, \mu_{p-1} \in P^*$, with $\mu = \mu_0 \wedge \mu_1 \wedge \dots \wedge \mu_{p-1} \in \Lambda_\bullet$ and $(x_j \cdot \mu)\,(\lambda) = i(\bar{x}_j)(\mu)(\lambda)$. Now apply α and evaluate on any homogeneous element $\lambda \in \Lambda_\bullet$ to get

$$\begin{aligned}(\alpha d_E(\mu \otimes \omega))(\lambda) &= -\sum_{j=1}^n \langle \lambda, x_j \cdot \mu \rangle \Theta T(\xi_j) \wedge \omega + (-1)^{|\mu|} \langle \lambda, \mu \rangle d\omega \\ &= -\sum_{j=1}^n \theta(\xi_j) \wedge \langle x_j \lambda, \mu \rangle \omega + (-1)^{|\mu|} \langle \lambda, \mu \rangle d\omega \\ &= -\sum_{j=1}^n \xi_j . F_{\mu\otimes\omega}(x_j \lambda) + (-1)^{|\mu|} d(F_{\mu\otimes\omega}(\lambda)) \\ &= d_{h(\Omega)} \alpha(\mu \otimes \omega))(\lambda) \end{aligned} \tag{17.5.2}$$

by Sect. 10.4. □

(17.6) Lemma. *Let* $\psi : \mathbf{E}^\bullet_{\mathbf{BK}} \to \pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK}}$ be the mapping

$$\psi(\mu \otimes \omega) = f(\mu) \wedge \pi^*(\omega) \ ,$$

where $f : \Lambda^\bullet \to \Omega^\bullet(EK)^K$ *denotes the restriction of the connection form* f (Sect. 3.2) *to the invariant elements. Then* ψ *is a quasi-isomorphism of complexes of sheaves of* $\Lambda_\bullet$ *modules. Composing with* α *gives an isomorphism* (17.1.3) *in* $D_+(\Lambda_\bullet)$,

$$\mathbf{h}(\mathbf{\Omega}^\bullet_{\mathbf{BK}}) \xleftarrow{\alpha} \mathbf{E}^\bullet_{\mathbf{BK}} \xrightarrow{\psi} \pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{BK}}.$$

(17.7) *Proof*. If $\lambda \in \Lambda_\bullet$ then

$$\lambda \cdot \psi(\mu \otimes \omega) = i(V_{\bar{\lambda}}) f(\mu) \wedge \pi^*(\omega) = f(i(\bar{\lambda})u) \wedge \pi^*(\omega) = \psi(\lambda \cdot \mu \otimes \omega)$$

so ψ is a mapping of $\Lambda_\bullet$-modules. By direct computation from (17.3.2) we have $d\psi(\mu \otimes \omega) = (-1)^{\deg(\mu)} \psi d_E(\mu \otimes \omega)$. The mapping ψ induces an isomorphism on hypercohomology by [C2] Sect. 4, p. 61 as described in [GHV] III Sect. 9.3. However the sheaf theoretic statement consists of identifying the stalk of $\mathbf{E}^\bullet_{\mathbf{BK}}$ at $x \in BK$ with the fiber projection as in [GHV] III theorem X p. 390. The induced map on stalk cohomology, $\psi_* : H^*_x(\mathbf{E}^\bullet_{\mathbf{BK}}) \to H^*(\pi^{-1}(x))$ is the isomorphism (10.6.1) $\Lambda^\bullet \cong H^*(K)$. $\square$

(17.8) Proof of Theorem 11.2. For any $\mathbf{A}^\bullet \in D^b_K(\mathrm{pt})$ the quasi-isomorphisms of sheaves of $\Lambda_\bullet$-modules

$$\mathbf{h}(\mathbf{\Omega}^\bullet_{\mathbf{BK}} \otimes_{\mathbb{R}} \mathbf{A}^\bullet) \cong \mathbf{h}(\mathbf{\Omega}^\bullet_{\mathbf{BK}}) \otimes_{\mathbb{R}} \mathbf{A}^\bullet \xleftarrow{\alpha\otimes I} \mathbf{E}^\bullet_{\mathbf{BK}} \otimes_{\mathbb{R}} \mathbf{A}^\bullet \xrightarrow{\psi\otimes I} \pi_*\mathbf{\Omega}^{\bullet K}_{\mathbf{EK}} \otimes_{\mathbb{R}} \mathbf{A}^\bullet$$

induces an isomorphism on global sections,

$$\Gamma(BK; \mathbf{h}(\mathbf{\Omega}^\bullet_{\mathbf{BK}} \otimes \mathbf{A}^\bullet)) \leftarrow \Gamma(BK; \mathbf{E}^\bullet_{\mathbf{BK}} \otimes \mathbf{A}^\bullet) \to \Gamma(BK; \pi_*\mathbf{\Omega}^{\bullet K}_{\mathbf{EK}} \otimes \mathbf{A}^\bullet)$$

and hence induces an isomorphism in $D_+(\Lambda_\bullet)$ between

$$hG(\mathbf{A}^\bullet) = h(\Gamma(BK; \mathbf{\Omega}^\bullet_{\mathbf{BK}} \otimes \mathbf{A}^\bullet)) \cong \Gamma(BK; \mathbf{h}(\mathbf{\Omega}^\bullet_{\mathbf{BK}} \otimes \mathbf{A}^\bullet))$$

and

$$E(\mathbf{A}^\bullet) = \Gamma(BK; \pi_*\mathbf{\Omega}^{\bullet K}_{\mathbf{EK}} \otimes \mathbf{A}^\bullet)$$

as claimed. This completes the proof of the first part of Theorem 11.2.

By [BL] Sects. 12.3.5 and 12.7.2, the functor G is an equivalence of categories. In fact, a quasi-inverse for the functor G is given in [BL] Sects. 12.3.1 and 12.4.5 by the "localization functor" $\mathscr{L} : D_+(\mathbf{S}) \to D_+(X)$

$$M \mapsto M \overset{L}{\otimes}_{\mathbf{S}} \mathbf{\Omega}^\bullet_{\mathbf{BK}}. \tag{17.8.1}$$

The Koszul duality functor h is also an equivalence of categories by [BGG]. This proves that the functor E is an equivalence of categories.

By Theorem 8.4 the functors h and t are quasi-inverses. Therefore the second isomorphism (11.2.2) of functors $G \cong tE$ follows from the first isomorphism (11.2.1) $hG \cong E$.

Now consider the two cohomological statements. If $A = (\mathbf{A_X}, \bar{\mathbf{A}}, \beta) \in D^b_K(X)$, its equivariant cohomology is (5.5.1)

$$H^*_K(X; A) = H^*(BK; Rc'_*(\bar{\mathbf{A}})) \cong H^*(BK; Rc'_*\bar{\mathbf{A}} \otimes \mathbf{\Omega}^\bullet_{\mathbf{BK}}) = HGc^K_*(A).$$

The interesting part is the computation of the ordinary cohomology of A, which is given by the following sequence of functorial isomorphisms.

$$
\begin{aligned}
H^*(X;A) &\cong H^*(\mathrm{pt}; Rc_*(\mathbf{A_X})) && \text{(by 5.5.2)}\\
&\cong H^*(EK; r^*Rc_*(\mathbf{A}_X)) \underset{Rc''_*(\beta)}{\longrightarrow} H^*(EK; \pi^* Rc'_* \bar{\mathbf{A}}) && \text{(cf. 5.4.2)}\\
&\cong H^*(BK; R\pi_*\pi^* Rc'_*(\bar{\mathbf{A}}))\\
&\cong H^*(BK; R\pi_*\pi^*(\mathbb{R}_{\mathbf{BK}}) \otimes Rc'_*(\bar{\mathbf{A}})) \xrightarrow{\psi^{-1}} H^*(BK; \pi_*\mathbf{\Omega}^{\bullet K}_{\mathbf{EK}} \otimes Rc'_*(\bar{\mathbf{A}}))\\
&= HEc^K_*(A)
\end{aligned}
$$

where we have used the contratibility of EK in the second isomorphism, and where, by ψ^{-1}, we mean the quasi-isomorphism

$$R\pi_*\pi^*(\mathbb{R}_{\mathbf{BK}}) \to R\pi_*(\mathbb{R}_{\mathbf{EK}}) \to \pi_*(\Omega^{\bullet K}_{\mathbf{EK}}) \xrightarrow{\psi^{-1}} \mathbf{E}^\bullet.$$

of Lemma 17.6. This completes the proof. $\square$

18. Proof of theorems 12.3 and 12.5

Suppose a compact connected Lie group K acts on a subanalytic space X. In this section we will construct a quasi-isomorphism of complexes of $\Lambda_\bullet$-modules

$$E(c^K_* \mathbb{R}^K_X) \to C^*(X; \mathbb{R}) \tag{18.1.1}$$

between the $\Lambda_\bullet$-module of (ordinary) global sections, and the $\Lambda_\bullet$-module of subanalytic cochains together with the sweep action.

(18.1) First reduction. By replacing X with an equivariant subanalytic tubular neighborhood of X in some Euclidean space, we may assume that X is a smooth subanalytic manifold. The first step in constructing the quasi-isomorphism (18.1.1) is to replace the complex $E(c^K_* \mathbb{R}^K_X)$ by the complex $\Omega^\bullet(X; \mathbb{R})$ of smooth differential forms on X, together with the action of $\Lambda_\bullet$ which is given by interior multiplication with fundamental vectorfields (Sect. 10.5). Fix a smooth model $EK_n \to BK_n$ for the classifying space of K (Sect. 10.7). Throughout this section we refer to the notation of diagram (5.4.1).

By (11.1.2) the complex $E(c^K_* \mathbb{R}^K_X)$ may be realized as the global sections of the following sheaf on BK,

$$\mathbf{E}(c^K_* \mathbb{R}^K_X) = \pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK}} \otimes c'_* \mathbf{\Omega}^\bullet_{\mathbf{X} \times_{\mathbf{K}} \mathbf{EK}}$$

which in turn is a limit of sheaves on BK_n,

$$\mathbf{E}(c^K_* \mathbb{R}^K_X)_n = \pi_* \mathbf{\Omega}^{\bullet K}_{\mathbf{EK_n}} \otimes c'_* \mathbf{\Omega}^\bullet_{\mathbf{X} \times_{\mathbf{K}} \mathbf{EK_n}}$$

(18.2) Proposition. *There is a quasi-isomorphism of sheaves of $\Lambda_\bullet$-modules on BK_n,*

$$\mathbf{E}(c_*^K \mathbb{R}_X^K)_n = \pi_* \mathbf{\Omega}^{\bullet \mathbf{K}}_{\mathbf{EK_n}} \otimes c'_* \mathbf{\Omega}^\bullet_{\mathbf{X \times_K EK_n}} \cong (\pi_* c_* \mathbf{\Omega}^\bullet_{\mathbf{X \times EK_n}})^K$$

between the complex $\mathbf{E}(c_*^K \mathbb{R}_X^K)_n$ and the sheaf (on BK_n) of invariant differential forms on $X \times EK_n$, where $\Lambda_\bullet$ acts on $(\pi_* c_* \mathbf{\Omega}^\bullet_{\mathbf{X \times EK_n}})^{\mathbf{K}}$ by interior product with fundamental vectorfields which are obtained from the diagonal action of K on $X \times EK_n$.

In fact, such a quasi-isomorphism may be obtained by taking the sheaf of invariants under the following composition of quasi-isomorphisms,

$$\begin{aligned}
\pi_* \mathbf{\Omega}^\bullet_{\mathbf{EK_n}} \otimes c'_* \mathbf{\Omega}^\bullet_{\mathbf{X \times_K EK_n}} &\cong \pi_* (\mathbf{\Omega}^\bullet_{\mathbf{EK_n}} \otimes \pi^* c'_* \mathbf{\Omega}^\bullet_{\mathbf{X \times_K EK_n}}) \\
&\cong \pi_* (\mathbf{\Omega}^\bullet_{\mathbf{EK}} \otimes c_* q^* \mathbf{\Omega}^\bullet_{\mathbf{X \times_K EK_n}}) \\
&\cong \pi_* c_* (c^* \mathbf{\Omega}^\bullet_{\mathbf{EK_n}} \otimes q^* \mathbf{\Omega}^\bullet_{\mathbf{X \times_K EK_n}}) \\
&\cong \pi_* c_* \mathbf{\Omega}^\bullet_{\mathbf{X \times EK_n}}
\end{aligned}$$

(this last isomorphism reflects the fact that both sheaves are (quasi-) isomorphic to the constant sheaf on EK_n.) □

By taking global sections, we see that *the complex $E(c_*^K \mathbb{R}_X^K) \in D_+(\Lambda_\bullet)$ is quasi-isomorphic to the complex of smooth invariant differential forms on $X \times EK$ (relative to the diagonal action of K) together with the action of $\Lambda_\bullet$ which is given by the interior product with fundamental vectorfields.*

(18.3) Integration. Integration induces a mapping from the complex of differential forms to the complex of subanalytic cochains,

$$\Omega^\bullet(X \times EK_n) \xrightarrow{\int} C^*(X \times EK_n)$$

by $\omega \mapsto (\xi \mapsto \int_\xi \omega)$. The theorem of de Rham says that this mapping induces isomorphisms on cohomology. We claim that in fact it is a quasi-isomorphism of complexes of $\Lambda_\bullet$-modules, where $\Lambda_\bullet$ acts on the differential forms $\Omega^\bullet(X \times EK_n)$ by contraction with fundamental vectorfields, and $\Lambda_\bullet$ acts on subanalytic cochains by the sweep. It suffices to show:

(18.4) Proposition. *Suppose the compact Lie group K acts on a subanalytic manifold Y. Let $S \in C_i(K; \mathbb{R})$ be a conjugation-invariant subanalytic cycle, and $u \in \Lambda_\bullet = (\bigwedge \mathfrak{k})^K \cong H_*(K)$ be an invariant multivector, such that both S and u represent the same homology class in K. Let V_u^Y be the resulting fundamental vectorfield on Y. Then, for any subanalytic chain $\xi \in C_*(Y)$ and for any smooth differential form $\omega \in \Omega^\bullet(Y)$ we have*

$$\int_{\xi} i(V_u^Y)\omega = \int_{S\xi} \omega.$$

The proof will occupy the next few sections.

(18.5) Currents. For a smooth manifold Y let $\mathscr{D}_i'(Y)$ denote the vectorspace of i dimensional currents, i.e. continuous linear homomorphisms $T : \Omega_c^i(Y) \to \mathbb{R}$. As in [deR], denote the value of a current T on a test-form $\omega \in \Omega_c^i(Y)$ by $T[\phi]$. For each multivector $u \in \bigwedge^i \mathfrak{k}$ let V_u^K denote the left invariant multivectorfield on K whose value at the identity is u. Fix an orientation on K and let $d\mathrm{vol}_K \in \bigwedge^n \mathfrak{k} \cong \Omega^n(K)^K$ denote the unique left invariant differential form so that $\int_K d\mathrm{vol}_K = +1$. (where $n = \dim(K)$) Let $F : \bigwedge^i \mathfrak{k} \to \mathscr{D}_i'(K)$ be the mapping which assigns to any multivector $u \in \bigwedge^i \mathfrak{k}$ the current

$$F(u)[\phi] = \int_K \phi(V_u^K) d\mathrm{vol}_K.$$

We claim *the mapping F induces an isomorphism between $\bigwedge^i \mathfrak{k}$ and the left invariant currents $\mathscr{D}_i'(K)^K$*. In fact, F is the composition of isomorphisms,

$$\bigwedge^i \mathfrak{k} \xrightarrow{\alpha} \left(\bigwedge^{n-i} \mathfrak{k}\right)^* \xrightarrow{\beta} \Omega^{n-i}(K)^K \xrightarrow{\gamma} \mathscr{D}_i'(K)^K$$

where $\alpha(u)(a) = \langle a \wedge u, d\mathrm{vol}_K \rangle$, $\beta(\tau)$ is the left invariant differential form corresponding to $\tau \in \bigwedge \mathfrak{k}^*$, and $\gamma(\omega)$ is the current

$$\gamma(\omega)[\phi] = \int_K \omega \wedge \phi.$$

The mapping γ is an isomorphism since $\Omega^{n-i}(K)$ is dense in $\mathscr{D}_i'(K)$ and the invariants form a finite dimensional subspace.

Using the volume form $d\mathrm{vol}_K$ it is possible to average a current $T \in \mathscr{D}_i'(K)$ to obtain a left invariant current $\langle T \rangle \in \mathscr{D}_i'(K)^K$ whose value on a test form $\phi \in \Omega_c^i(K)$ is defined by

$$\langle T \rangle[\phi] = \frac{1}{\mathrm{vol}(K)} \int_K T[L_g^* \phi] d\mathrm{vol}_K(g).$$

(Here, $L_g : K \to K$ is the left multiplication, $L_g(x) = gx$.) Then $\langle \partial T \rangle = \partial \langle T \rangle$. If ϕ is a left invariant form, then $\langle T \rangle[\phi] = T[\phi]$. It follows that: *if $\partial T = 0$ then also $\partial \langle T \rangle = 0$ and the homology classes represented by T and $\langle T \rangle$ coincide.*

(18.6) Lemma. Suppose $S \in C_i(K; \mathbb{R})$ is an i-dimensional subanalytic cycle, which is invariant under conjugation. Let $u \in (\bigwedge^i \mathfrak{k})^K \cong H_i(K)$ be the invariant multivectorfield whose homology class coincides with that of S. Then, as currents,

$$\langle S \rangle = F(u) \in \mathscr{D}'_i(K)$$

(*18.7*) *Proof*. The mapping F restricts to an isomorphism between the bi-invariant currents on K and the invariant multivectors $(\bigwedge^i \mathfrak{k})^K \cong H_i(K)$. The current $\langle S \rangle$ is bi-invariant and the homology classes represented by $\langle S \rangle$ and by $F(u)$ coincide. □

(18.8) Integration over the fiber. Let π_1 and π_2 denote the projections of $K \times Y$ to the first and second factors respectively. Recall (e.g. [GHV] II Sect. 7.14), that integration over the fibers of π_2 is a mapping

$$\int_{\pi_2} : \Omega^i(K \times Y) \to \Omega^{i-\dim(K)}(Y) \tag{18.8.1}$$

such that

$$\int_{\pi_2} \pi_2^*(\omega) \wedge \eta = \omega \wedge \int_{\pi_2} \eta \tag{18.8.2}$$

for every $\omega \in \Omega^\bullet(Y)$ and $\eta \in \Omega^\bullet(K \times Y)$, and

$$i(V^Y) \int_{\pi_2} \omega = \int_{\pi_2} i(V^{K \times Y})\omega \tag{18.8.3}$$

whenever $V^{K \times Y}$ and V^Y are π_2-related vectorfields on $K \times Y$ and Y respectively.

(18.9) Proof of Theorem 12.3. Let $V_u^{K \times Y}$ denote the fundamental multivectorfield (cf. Sect. 10.5) on $K \times Y$ which arises from the following K action on $K \times Y$: $k \cdot (g, x) = (gk^{-1}, kx)$. Then for any $\alpha, \beta \in \Omega^\bullet(Y)$ and any $\gamma \in \Omega^\bullet(K)$ we have:

$$\begin{aligned} i(V_u^{K \times Y})\mu_Y^*(\alpha) &= 0 \\ i(V_u^{K \times Y})\pi_2^*(\beta) &= \pi_2^*(i(V_u^Y)\beta) \\ i(V_u^{K \times Y})\pi_1^*(\gamma) &= \pi_1^*(i(V_u^K)\gamma) \end{aligned} \tag{18.9.1}$$

where V_u^X is the corresponding fundamental multivectorfield on X, and where V_u^K is the fundamental multi-vectorfield on K which is determined by the action $k \cdot g = gk^{-1}$. (It follows that $V_u^K = W_{-u}$ is the left invariant multivectorfield on K whose value at the identity is $-u$.) For any *invariant* differential form $\omega \in \Omega^\bullet(Y)$ we have (c.f. [GHV] II Sect. 4.3),

$$\omega = \int_{\pi_2} \mu_Y^*(\omega) \wedge \pi_1^*(d\mathrm{vol}_K) \tag{18.9.2}$$

Therefore

$$\begin{aligned}
\int_\xi i(V_u^X)\omega &= \int_\xi i(V_u^X) \int_{\pi_2} \mu^*(\omega) \wedge \pi_1^* d\mathrm{vol}_K \\
&= (-1)^{n\deg(\omega)} \int_\xi \int_{\pi_2} i(V_u^{K\times X})(\pi_1^*(d\mathrm{vol}_k) \wedge \mu^*(\omega)) \\
&= (-1)^{n\deg(\omega)} \int_\xi \int_{\pi_2} \pi_1^*(i(V_u^K) d\mathrm{vol}_K) \wedge \mu^*(\omega)
\end{aligned}$$

where $n = \dim(K)$. Thus

$$\begin{aligned}
\int_\xi i(V_u^X)\omega &= (-1)^{n\deg(\omega)+\deg(u)} \int_\xi \int_{\pi_2} \pi_1^*(i(W_u) d\mathrm{vol}_K) \wedge \mu^*(\omega) \\
&= (-1)^{\deg(u)+n\deg(u)} \int_{K\times\xi} \pi_1^*(i(W_u) d\mathrm{vol}_K) \wedge \mu^*(\omega) \\
&= (-1)^{\deg(u)+n\deg(u)} \int_K i(W_u) d\mathrm{vol}_K \wedge \int_{\pi_1} (\mu^*\omega)|\xi \\
&= (+1) \int_K \left(\int_{\pi_1} (\mu^*\omega)|\xi \right) \wedge i(W_u) d\mathrm{vol}_K \\
&= \int_{S\times\xi} \mu^*(\omega) \qquad \text{(by 18.6)} \\
&= \int_{S\xi} \omega
\end{aligned}$$

as desired. □

(*18.10*) *Proof of Theorem 12.3*. Theorem 12.3 states that $E(c_*\mathbb{R}_X^K)$ and $C^*(X)$ are isomorphic in $D_+(\Lambda_\bullet)$. This follows by applying global sections to the isomorphism of Proposition 18.2 and composing this with the isomorphism of Proposition 18.4, then taking the limit as $n \to \infty$ to obtain

$$
\begin{aligned}
E(c_*^K \mathbb{R}_X^K) &= \Gamma(BK; \pi_* \mathbf{\Omega}_{\mathbf{EK}}^{\bullet K} \otimes c'_* \mathbf{\Omega}^{\bullet}_{\mathbf{X}\times_{\mathbf{K}}\mathbf{EK}}) \\
&\cong \Omega^\bullet(X \times EK)^K && \text{(by 18.2)} \\
&\cong C^*(X \times EK) && \text{(by 18.4)} \\
&\cong C^*(X)
\end{aligned}
$$

since EK is contractible. The proof of Theorem 12.5 is similar. By choosing a system of control data on X, the intersection cohomology may be realized as the cohomology of a certain complex of stratified differential forms on X. (See, for example, [Br].) This allows one to mimic the arguments in the preceding section, for intersection cohomology in place of ordinary cohomology.

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