

## EQUIVARIANT COHOMOLOGY THEORIES ON G-CW COMPLEXES

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### Introduction

G.Bredon developed the equivariant (generalized) cohomology theories in [3], in which he had to restrict himself to the case of finite groups. One of the purposes of this note is to generalize his theory by replacing  $G$ -complexes with  $G$ -CW complexes. Then, for example, the followings are still true for the case in which  $G$  is an arbitrary topological group. The  $E_2$ -term of the Atiyah-Hirzebruch spectral sequence associated to a  $G$ -cohomology theory (in this note we frequently use ' $G$ -' instead of 'equivariant') is a classical  $G$ -cohomology theory, which is easy to calculate (§1~§4). The  $G$ -obstruction theory works in a classical  $G$ -cohomology theory (§5). Moreover, for a  $G$ -cohomology theory we get a representation theorem of E.Brown (§6) and the Maunder's spectral sequence (§7).

As an application we study the equivariant  $K^*$ -theory in the last section (§8). The Atiyah-Hirzebruch spectral sequence for  $K_G^*(X)$  collapses, if  $\dim X/G \leq 2$  or  $X$  satisfies some other conditions. The  $E_2$ -term depends only on the orbit type decomposition of the orbit space, if  $X$  is a regular  $O(n)$ -manifold or the like. These facts enable us to calculate the equivariant  $K^*$ -group of Hirzebruch-Mayer  $O(n)$ -manifolds and Jänich knot  $O(n)$ -manifolds. Our spectral sequence for a differentiable  $G$ -manifold is similar to that of G.Segal which is defined by the equivariant nerve of his [13], but ours is easier to calculate the  $E_2$ -term.

In this note  $G$  denotes a fixed topological group. Terminologies and notation follow those of [3], [9], [10] in general, though  $\sigma$  denotes a closed cell which is the closure of an (open) cell in the definition of a  $G$ -CW complex in [10]. And  $G\sigma$  denotes the  $G$ -orbit of  $\sigma$  and  $H_\sigma$  the unique isotropy subgroup at any interior point of  $\sigma$ . §0 is exposed for reference to the properties of  $G$ -CW complexes.

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## 0. Preliminaries about $G$ -CW complexes

We summarize here the properties of  $G$ -CW complexes and  $G$ -CW complexes with base point (the base point in  $G$ -CW complex is always assumed to be a vertex which is left fixed by each element of  $G$ ).

**Proposition 0.1.** ( $G$ -cellular approximation theorem) *Let  $f: X \rightarrow Y$  be a  $G$ -map between  $G$ -CW complexes (with base point). Then  $f$  is (base point preserving)  $G$ -homotopic to a  $G$ -map,  $f': X \rightarrow Y$  such that  $f'(X^n) \subset Y^n$  for any  $n$ .*

This is Theorem 4.4 of [10]. Moreover, if  $f$  is  $G$ -cellular on a  $G$ -subcomplex  $A$ , then we may require  $f' = f$  on  $A$ .

**Proposition 0.2.** ( $G$ -homotopy extension property) *Let  $f_0: X \rightarrow Y$  be a given  $G$ -map of a  $G$ -CW complex  $X$  into an arbitrary  $G$ -space  $Y$ . Let  $g_t: A \rightarrow Y$  be a  $G$ -homotopy of  $g_0 = f_0|_A$ , where  $A$  is a  $G$ -subcomplex of  $X$ . Then, there is a  $G$ -homotopy  $f_t: X \rightarrow Y$ , such that  $f_t|_A = g_t$ .*

This is (J) of [10].

For a pair of  $G$ -CW complexes  $(X, A)$ , collapsed  $A$  into a point,  $X/A$  forms a  $G$ -CW complex with a base point  $A/A$  (taken to be a disjoint point if  $A = \phi$ , in which case  $X^+$  denotes  $X/\phi$ ). Let  $i: A \rightarrow X$  be the inclusion. Consider the mapping cone  $C_i = X \cup CA = (X \times \{1\} \cup A \times I)/A \times \{0\}$  with the obvious  $G$ -action, trivial on  $I$ . Then, by the  $G$ -homotopy extension property, we can prove that the collapsing map,  $X \cup CA \rightarrow X \cup CA/CA = X/A$  is a  $G$ -homotopy equivalence. Therefore, we get

**Proposition 0.3.** *Let  $(X, A)$  be a pair of  $G$ -CW complexes (with base point) and let  $i: A \rightarrow X$  be the natural inclusion. Then, in the following cofiber sequence, the vertical maps are  $G$ -homotopy equivalences:*

$$\begin{array}{ccccccc} A & \xrightarrow{i} & X & \xrightarrow{j} & C_i & \xrightarrow{f} & C_j \rightarrow C_f \\ & & \searrow & & \downarrow \simeq G & \downarrow \simeq G & \downarrow \simeq G \\ & & & & X/A & \rightarrow & SA \rightarrow SX \end{array}$$

**Proposition 0.4.** (Theorem of J.H.C.Whitehead) *Let  $\varphi: (X, A) \rightarrow (Y, B)$  be a  $G$ -map between two pairs of  $G$ -CW complexes with base point. For each closed subgroup  $H$  which appears as an isotropy subgroup in  $X$  or  $Y$ , we assume that  $X^H, A^H, Y^H$  and  $B^H$  are arcwise connected, and the induced maps,*

$$\varphi_*: \pi_n(X^H, *) \rightarrow \pi_n(Y^H, *)$$

and

$$\varphi_*: \pi_n(A^H, *) \rightarrow \pi_n(B^H, *)$$

are bijective for  $1 \leq n \leq \max(\dim X, \dim Y)$ . Then,  $\varphi: (X, A) \rightarrow (Y, B)$  is a  $G$ -

*homotopy equivalence.*

This is a special case of \*) Theorem 5.3 of [10].

**Proposition 0.5.** *Let  $G$  be a compact Lie group. Then any compact differentiable  $G$ -manifold has a  $G$ -finite  $G$ -CW complex structure.*

This comes from Proposition 4.4 of [9].

**1. Definition of an equivariant cohomology theory on  $G$ -CW complexes**

On the category of pairs of  $G$ -finite  $G$ -CW complexes and  $G$ -homotopy classes of  $G$ -maps, a  *$G$ -cohomology theory* is defined to be a sequence of contravariant functors  $h_G^n(-\infty < n < \infty)$  into the category of abelian groups together with natural transformation  $\delta^n: h_G^n(A, \phi) \rightarrow h_G^{n+1}(X, A)$  such that the following axioms are satisfied (we put  $h_G^n(X) = h_G^n(X, \phi)$ ):

- (1) The inclusion  $(X, X \cap A) \rightarrow (X \cup A, A)$  induces an isomorphism,

$$h_G^n(X \cup A, A) \cong h_G^n(X, X \cap A).$$

- (2) If  $(X, A)$  is a pair of  $G$ -finite  $G$ -CW complexes, the sequence,

$$\dots \rightarrow h_G^n(X, A) \rightarrow h_G^n(X) \rightarrow h_G^n(A) \xrightarrow{\delta^n} h_G^{n+1}(X, A) \rightarrow \dots$$

is exact.

Standard argument can be used to prove the exactness of Mayer-Vietoris sequence and the long sequence of triples.

**Lemma 1.1.** *For a pair of  $G$ -finite  $G$ -CW complexes  $(X, A)$ , the collapsing map,  $(X, A) \rightarrow (X/A, A/A)$ , induces an isomorphism,*

$$h_G^n(X/A, A/A) \cong h_G^n(X, A)$$

Proof. By the proposition 0.3 the collapsing map,  $X \cup CA \rightarrow X \cup CA/CA = X/A$  is a  $G$ -homotopy equivalence. Moreover,  $CA \rightarrow *$  is an  $G$ -homotopy equivalence, and  $(X, A) \rightarrow (X \cup CA, CA)$  is an excision map. Hence, we get the commutative diagram (the homomorphisms are induced by the canonical  $G$ -maps),

$$\begin{array}{ccc} h_G^n(X \cup CA, *) & \cong & h_G^n(X \cup CA, CA) \\ \cong \downarrow & \swarrow & \uparrow \cong \\ h_G^n(X/A, A/A) & \rightarrow & h_G^n(X, A) \end{array}$$

q.e.d.

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\*) The footnote at p. 371 of [10] is inadequate. “\*)  $\pi_k(X, Y)$  vanishes’ should read “ $\pi_k(X, Y, y)$  vanishes for every point  $y$  of  $Y$ ” and also “ $\varphi_*: *) \pi_k(X) \rightarrow *) \pi_k(Y)$  is bijective or surjective’ should read “ $\varphi_*: \pi_k(X, x) \rightarrow \pi_k(Y, \varphi(x))$  is bijective or surjective for every point  $x$  of  $X$ ”. Then, the statements and proofs in [10] are true in the context except Theorem 5.2. In Theorem 5.2 we should add the assumption that each arcwise connected component of  $X$  or  $Y$  is  $n$ -simple for every  $n \geq 1$ .

For a  $G$ -CW complex with base point  $X$ ,  $SX = S \wedge X$  (with obvious  $G$ -action, trivial on the "circle factor"  $S$ ) denotes the reduced suspension of  $X$ . A *reduced  $G$ -cohomology theory* on the category of  $G$ -finite  $G$ -CW complexes with base point and base point preserving  $G$ -homotopy classes of base point preserving  $G$ -maps is a sequence of contravariant functors  $\tilde{h}_G^n (-\infty < n < \infty)$  into the category of abelian groups, together with natural transformations  $\sigma^n: \tilde{h}_G^n(X) \rightarrow \tilde{h}_G^{n+1}(SX)$  satisfying the following axioms:

- (1)'  $\sigma^n$  is an isomorphism for each  $n$  and  $X$ .
- (2)' The short sequence,

$$\tilde{h}^n(X/A) \rightarrow \tilde{h}_G^n(X) \rightarrow \tilde{h}_G^n(A)$$

is exact.

REMARK 1.2. By Proposition 0.3 and Axioms (1)', (2)' we get the long exact sequence for  $\tilde{h}_G^*(\cdot)$ .

Let  $h_G^*$  be a  $G$ -cohomology theory. Define  $\tilde{h}_G^*(X)$  by  $h_G^*(X, *)$ . Then  $h_G^*$  is a reduced  $G$ -cohomology theory by Lemma 1.1. Conversely let  $\tilde{h}_G^*$  be a reduced  $G$ -cohomology theory. Define  $h_G^*(X, A)$  by  $\tilde{h}_G^*(X/A)$ . Then  $\tilde{h}_G^*$  is a  $G$ -cohomology theory by Remark 1.2. This is a canonical one-to-one correspondence. Afterwards we identify  $h_G^n(X, A)$  and  $\tilde{h}_G^n(X/A)$ .

We enclose this section after giving some examples.

EXAMPLES 1.3. of  $G$ -COHOMOLOGY THEORIES:

- (i)  $h_G^n(X) = H^n(X/G; \mathbf{Z})$ .
- (ii)  $h_G^n(X) = K_G^n(X)$  when  $G$  is a compact Lie group.
- (iii)  $h_G^n(X) = h^n(X \times_G E_G)$  where  $E_G$  is a universal  $G$ -principal bundle and  $h^n$  a cohomology theory for spaces.

## 2. On classification of $G$ -maps between $G$ -cells of the same dimension up to $G$ -homotopy classes

Let  $H$  be a closed subgroup of  $G$ . Suppose that  $\bar{X}$  is a space and  $G/H \times \bar{X}$  is a  $G$ -space with the obvious  $G$ -action, trivial on  $\bar{X}$ . Let  $Y$  be a  $G$ -space and  $f: G/H \times \bar{X} \rightarrow Y$  be a  $G$ -map. Since  $f$  is  $G$ -equivariant, we get,  $f(H/H \times \bar{X}) \subset Y^H$  where  $Y^H$  is the  $H$ -pointwise fixed subspace of  $Y$ . Therefore, we may define a map,  $\tilde{f}: \bar{X} \rightarrow Y^H$ , by  $\tilde{f}(x) = f(H/H \times x)$ .

**Lemma 2.1.** *In the above situation, the correspondence,  $f \mapsto \tilde{f}$ , yields an isomorphism of sets,*

$$G\text{-maps } (G/H \times \bar{X}, Y) \cong \text{Maps } (\bar{X}, Y^H).$$

Moreover, the isomorphism induces another isomorphism,

$$[G/H \times \bar{X}; Y]_G \cong [\bar{X}; Y^H]$$

where  $[\cdot; \cdot]_G$  stands for the set of  $G$ -homotopy classes of  $G$ -maps.

Proof. Let  $\bar{f}: \bar{X} \rightarrow Y^H$  be a map. Define a map,  $f: G/H \times \bar{X} \rightarrow Y$ , by  $f(gH/H \times x) = g \cdot \bar{f}(x)$  for any  $g \in G$ , and any  $x \in \bar{X}$ . If  $gH/H = g'H/H$ , then  $g' = g \cdot h$  for some  $h \in H$ , so that  $g' \cdot \bar{f}(x) = g \cdot h \cdot \bar{f}(x) = g \cdot \bar{f}(x)$  (since  $\bar{f}(x)$  is fixed by  $H$ ), which shows that this definition is valid. By this definition  $f$  is certainly  $G$ -equivariant, and conversely if we assume that a map  $f: G/H \times \bar{X} \rightarrow Y$  is  $G$ -equivariant, we get  $f(gH/H \times x) = g \cdot f(H/H \times x)$ .

Therefore, the correspondence,  $\bar{f} \mapsto f$ , is the converse to the correspondence,  $f \mapsto \bar{f}$ . This proves the first isomorphism. The second isomorphism is induced, because the  $G$ -homotopy  $f_t (0 \leq t \leq 1)$  and homotopy  $\bar{f}_t (0 \leq t \leq 1)$  correspond each other in the same way.

q.e.d.

Assume that  $\bar{X}$  has a distinguished closed subspace  $\bar{A}$  and  $Y$  has a base point  $y_0$  (the base point is left fixed by  $G$ ).

**Lemma 2.1'.** *The correspondence,  $f \mapsto \bar{f}$ , yields an isomorphism,*

$$G\text{-maps } ((G/H \times \bar{X})/(G/H \times \bar{A}), Y/y_0)_0 \cong \text{Map } (\bar{X}/\bar{A}, Y^H/y_0)_0.$$

Moreover, the isomorphism induces another isomorphism,

$$[(G/H \times \bar{X})/(G/H \times \bar{A}); Y/y_0]_{G,0} \cong [\bar{X}/\bar{A}; Y^H/y_0]_0,$$

where  $[\cdot, \cdot]_{G,0}$  stands for the set of base point preserving  $G$ -homotopy classes of base point preserving  $G$ -maps.

Proof. The correspondence  $\bar{f} \mapsto f$ , is also defined in the same way as in Lemma 2.1.

q.e.d.

Therefore, we get

**Corollary 2.2.** *Let  $H$  and  $K$  be two closed subgroups of  $G$  and  $n \geq 0$  be a fixed integer. Then, "the restriction" yields the following isomorphisms,*

- (i)  $[G/H; G/K]_G \cong \pi_0((G/K)^H)$ ,
- (ii)  $[(G/H \times \Delta^n)/(G/H \times \partial\Delta^n); (G/K \times \Delta^n)/(G/K \times \partial\Delta^n)]_{G,0}$   
 $\cong \pi_n((G/K)^H \times \Delta^n)/((G/K)^H \times \partial\Delta^n, *)$ .

Here  $\pi_0(\cdot)$  stands for the set of arcwise connected components and  $*$  is the base point  $((G/K)^H \times \partial\Delta^n)/(G/K \times \partial\Delta^n)$ .

Now let  $Y$  be a space and  $n \geq 1$  be an integer.

**Lemma 2.3.**  *$Y \times \Delta^n / Y \times \partial\Delta^n$  is  $(n-1)$ -connected, and there are natural isomorphisms,*

$$\pi_n'(Y \times \Delta^n / Y \times \partial\Delta^n, *) \cong H_n(Y \times \Delta^n / Y \times \partial\Delta^n; \mathbf{Z}) \cong H_0(Y; \mathbf{Z})$$

Here  $\pi_n'(\cdot) = \pi_n(\cdot)$  for  $n \geq 2$  and  $\pi_1'(\cdot)$  is the abelianized group of  $\pi_1(\cdot)$  and  $H_n(\cdot; \mathbf{Z})$  is the singular homology group.

Proof. By the definition,  $Y \times \Delta^n / Y \times \partial \Delta^n$  is homeomorphic with the smash product  $Y^+ \wedge \Delta^n / \partial \Delta^n$ . Hence  $Y \times \Delta^n / Y \times \partial \Delta^n$  is  $(n-1)$ -connected. If we use the Hurwicz theorem, the rest is easily proved.

q.e.d.

Let  $\{Y_\lambda: \lambda \in \Lambda\}$  be the family of all the arcwise connected components of  $Y$ . Take an element  $y_\lambda \in Y_\lambda$  for each  $\lambda$ . Then each element of  $H_0(Y; \mathbf{Z})$  has  $\sum n_\lambda \cdot y_\lambda$  ( $n_\lambda = 0$  except the finite  $\lambda$ 's) as its representative. Also any map:  $(\Delta^n, \partial \Delta^n) \rightarrow (Y \times \Delta^n / Y \times \partial \Delta^n, *)$  determines  $n_\lambda$  uniquely.

Now let  $H$  and  $K$  be closed subgroups of  $G$ . Recall that for any element  $g \in N(H, K) = \{g \in G, Hg \subset gK\}$ ,  $\hat{g}: G/H \rightarrow G/K$  is defined by  $\hat{g}(aH) = agK$ , and this correspondence,  $g \mapsto \hat{g}$ , induces an isomorphism,

$$N(H, K)/K = (G/K)^H \xrightarrow{\cong} G\text{-maps}(G/H, G/K).$$

Suppose that  $\{g_\lambda \in G\}$  is the family of representatives of all arcwise connected components of  $N(H, K)/K = (G/K)^H$ . Then any base point preserving  $G$ -map,

$$f: (G/H \times \Delta^n) / (G/H \times \partial \Delta^n) \rightarrow (G/K \times \Delta^n) / (G/K \times \partial \Delta^n),$$

determines  $n_\lambda(f)$  such that  $f$  is equal to  $\sum n_\lambda(f) \cdot g_\lambda$  in  $\pi_n'(((G/K)^H \times \Delta^n) / ((G/K)^H \times \partial \Delta^n), *) \cong H_0((G/K)^H; \mathbf{Z})$ .

Let  $L$  be another closed subgroup of  $G$ . Suppose that  $g_\lambda \in N(H, K)$  and  $g_\mu \in N(K, L)$ , then we get

$$g_\lambda \cdot g_\mu \in N(H, L) \text{ (not } g_\mu \cdot g_\lambda!), \text{ and } (g_\lambda \cdot g_\mu)^\wedge = \hat{g}_\mu \circ \hat{g}_\lambda.$$

From this we get

**Proposition 2.4.** *Let  $H, K$  and  $L$  be closed subgroup of  $G$ . Suppose that  $\{g_\lambda \in G\}$ ,  $\{g_\mu \in G\}$  and  $\{g_\nu \in G\}$  are the families of representatives of all arcwise connected components of  $N(H, K)/K$ ,  $N(K, L)/L$  and  $N(H, L)/L$  respectively. Let  $f: (G/H \times \Delta^n) / (G/H \times \partial \Delta^n) \rightarrow (G/K \times \Delta^n) / (G/K \times \partial \Delta^n)$  and  $g: (G/K \times \Delta^n) / (G/K \times \partial \Delta^n) \rightarrow (G/L \times \Delta^n) / (G/L \times \partial \Delta^n)$ , be base point preserving  $G$ -maps. Then,*

$$n_\nu(g \circ f) = \sum n_\mu(g) n_\lambda(f).$$

Here the summation is taken over the pairs  $(\lambda, \mu)$  such that  $g_\lambda \cdot g_\mu$  and  $g_\nu$  are in the same arcwise connected component of  $N(H, L)/L$ .

### 3. Classical $G$ -cohomology theory on $G$ -CW complexes

We shall define a classical  $G$ -cohomology theory with coefficients in a (generic)  $G$ -coefficient system. In §4 the classical  $G$ -cohomology theory will be characterized as the  $G$ -cohomology theory which satisfies also the dimension

axiom.

DEFINITION 3.1. A (generic)  $G$ -coefficient system is a contravariant functor  $M_G$  of the category of the left coset spaces of  $G$  by closed subgroups,  $G/H$ , and  $G$ -homotopy classes of  $G$ -maps (equivariant with respect to left translation),  $G/H \rightarrow G/K$ , into the category of abelian groups.

REMARK. When  $G$  is a discrete group, any two distinct  $G$ -maps between  $G$ -coset spaces cannot be  $G$ -homotopic and hence this definition coincides with the generic equivariant coefficient system of Bredon in [3].

EXAMPLES 3.2. OF  $G$ -COEFFICIENT SYSTEMS:

- (i)  $M_G = h_G^G$ .
- (ii)  $M_G = \mathbf{Z}$  with a trivial  $G$ -action.
- (iii)  $M_G = \omega_n(Y)$  ( $n \geq 2$ ), where  $Y$  is a  $G$ -space with a base point  $y_0$  and  $\omega_n(Y)(G/H) = \pi_n(Y^H, y_0) \cong [(G/H \times \Delta^n)(G/H \times \partial \Delta^n), Y/y_0]_{G,0}$ .

Let  $M_G$  be a  $G$ -coefficient system. The  $n$ -dimensional  $G$ -cochain group of a pair of  $G$ -CW complexes  $(X, A)$  with coefficients in  $M_G$ , denoted by  $C_G^n(X, A; M_G)$ , is defined to be the group of all  $G$ -equivariant functions  $\varphi$  on the  $n$ -cells of  $(X, A)$  with  $\varphi(\sigma) \in M_G(G/H_\sigma)$  and  $M_G(\hat{g})\varphi(\sigma) = \varphi(g\sigma)$  for a right translation  $\hat{g}: G/H_{g\sigma} \ni aH_{g\sigma} = ag(H_\sigma)g^{-1} \mapsto agH_\sigma \in G/H_\sigma$ . (If  $\sigma$  is an  $n$ -cell of  $A$  or a  $p$ -cell ( $p \neq n$ ), then  $\varphi(\sigma) = 0$ .)

By the definition of the  $G$ -cochain group,  $C_G^n(X, A; M_G)$  is canonically isomorphic with  $C_G^n(X^n/X^{n-1} \cup A; M_G)$ . Moreover, since  $X^n/X^{n-1} \cup A = \vee(G\sigma/G\partial\sigma)$  where  $\sigma$  range over the representatives of all  $n$ -dimensional  $G$ -cells of  $(X, A)$ ,

$$C_G^n(X^n/X^{n-1} \cup A; M_G) = C_G^n(\vee(G\sigma/G\partial\sigma); M_G) = \prod C_G^n(G\sigma/G\partial\sigma; M_G).$$

Let  $f: (X, A) \rightarrow (Y, B)$  be a  $G$ -cellular map between pairs of  $G$ -CW complexes. Then, for every  $n$ ,  $f$  induces a  $G$ -map,

$$f^n: X^n/X^{n-1} \cup A \rightarrow Y^n/Y^{n-1} \cup B.$$

Suppose that  $\sigma$  and  $\tau$  are representatives of all  $G$ - $n$ -cells of  $(X, A)$  and  $(Y, B)$  respectively. Then we can define a  $G$ -map  $f_{\sigma\tau}$  (between  $G$ -cells of the same dimension  $n$ ) by  $f_{\sigma\tau} = c \circ f^n \circ i$  in the following diagram:

$$\begin{array}{ccc} X^n/X^{n-1} \cup A = \vee(G\sigma/G\partial\sigma) & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{i} \end{array} & G\sigma/G\partial\sigma = (G/H_\sigma \times \Delta^n)/(G/H_\sigma \times \partial\Delta^n) \\ \downarrow f^n & & \downarrow f_{\sigma\tau} \\ Y^n/Y^{n-1} \cup B = \vee(G\tau/G\partial\tau) & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{i} \end{array} & G\tau/G\partial\tau = (G/H_\tau \times \Delta^n)/(G/H_\tau \times \partial\Delta^n) \end{array}$$

where  $i$  is the inclusion and  $c$  is the collapsing of the other factors.

Let  $\{g_{\lambda(\sigma,\tau)} \in G\}$  be the family of representatives of all arcwise connected components of  $(G/H_\tau)^{H_\sigma}$  as in §2.

Define  $f^* = C_G^n(f; M_G): C_G^n(Y, B; M_G) \rightarrow C_G^n(X, A; M_G)$  by

$$(f^*\varphi)(\sigma) = \sum_{\tau} \sum_{\lambda(\sigma,\tau)} n_{\lambda(\sigma,\tau)}(f_{\sigma\tau}) M_G(\hat{g}_{\lambda(\sigma,\tau)}) \varphi(\tau)$$

where  $\tau$  ranges over the representatives of all  $G$ - $n$ -cells of  $(Y, B)$ . The sum is finite because  $n_{\lambda}(f_{\sigma\tau})=0$  except the finite  $\lambda$ 's.)

**Proposition 3.3.** *Let  $M_G$  be a  $G$ -coefficient system. Then,  $C_G^n(\cdot; M_G)$  is a contravariant functor from the category of pairs of  $G$ -CW complexes and  $G$ -cellular maps into the category of abelian groups.*

*Proof.* If we fix the representatives,  $(g \circ f)^* = f^* \circ g^*$  by Proposition 2.4. It is easily seen that  $f^*$  is determined independent of the representatives. Remark that  $f^*$  depends only on the  $G$ -homotopy class of the  $G$ -map  $f^n$ .

q.e.d.

Now recall that  $X^n/X^{n-1} \cup A$  has the same  $G$ -homotopy type with  $X^n \cup C(X^{n-1} \cup A)$  canonically. As a special case of Proposition 0.3, we have a Puppe sequence (the horizontal sequence),

$$\begin{array}{ccccccc} & & & & & S(X^{n-1}/X^{n-3} \cup A) & \\ & & & & \nearrow & \downarrow & \\ X^{n-1}/X^{n-2} \cup A & \rightarrow & X^n/X^{n-2} \cup A & \rightarrow & X^n/X^{n-1} \cup A & \xrightarrow{\partial} & S(X^{n-1}/X^{n-2} \cup A) \\ & & \downarrow & & \nearrow & & \downarrow S(\partial) \\ & & X^n/X^{n-3} \cup A & & & & S(X^{n-2}/X^{n-3} \cup A) \end{array}$$

Since both the vertical and oblique sequences are cofiberings, we get that  $S(\partial) \circ \partial$  is  $G$ -homotopic to the trivial map. On the other hand we have a canonical isomorphism,

$$\sigma: C_G^{n-1}(X^{n-1}/X^{n-2} \cup A; M_G) \xrightarrow{\cong} C_G^n(S(X^{n-1}/X^{n-2} \cup A); M_G).$$

Define the coboundary homomorphism

$$\delta: C_G^{n-1}(X, A; M_G) \rightarrow C_G^n(X, A; M_G)$$

by  $\delta = C_G^n(\partial) \circ \sigma$ . Then, because  $S(\partial) \circ \partial \simeq_G 0$ , we get  $\delta \circ \delta = 0$ .

**DEFINITION 3.4.** The *classical  $G$ -cohomology theory* on a pair of  $G$ -CW complexes  $(X, A)$  with the coefficients in a  $G$ -coefficient system  $M_G$ , denoted by  $H_G^*(X, A; M_G)$ , is defined by  $H_G^n(X, A; M_G) = H^n(C_G^*(X, A; M_G), \delta)$ .

**REMARK 3.5.** Let  $\sigma$  and  $\tau$  be  $n$ -cell and  $(n-1)$ -cell of  $(X, A)$ . We write  $[\sigma, g_{\lambda(\sigma,\tau)}\tau]$  for  $n_{\lambda(\sigma,\tau)}(\partial_{\sigma\tau})$  where  $\partial_{\sigma\tau}: G\sigma/G\partial\sigma \rightarrow S(G\tau/G\partial\tau)$ . Then, we get the formula,



$$(\delta\varphi)(\sigma) = \sum_{\tau} \sum_{\lambda(\sigma, \tau)} [\sigma, g_{\lambda(\sigma, \tau)} \tau] M_G(\hat{g}_{\lambda(\sigma, \tau)}) \varphi(\tau)$$

where  $\tau$  ranges over the representatives of all  $G$ -( $n-1$ )-cells and  $g_{\lambda(\sigma, \tau)}$  ranges over the representatives of all arcwise connected components of  $N(H_\sigma, H_\tau)/H_\tau$ .

**Theorem 3.6.** *The classical  $G$ -cohomology theory  $H_G^*(\cdot; M_G)$  is a  $G$ -cohomology theory in the sense of §1.*

*Proof.* We prove here only the  $G$ -homotopy axiom. The exision axiom and the exactness axiom is trivially satisfied. Let  $f: (X, A) \rightarrow (Y, B)$  be a  $G$ -map between pairs of  $G$ -CW complexes. By a  $G$ -cellular approximation theorem we may assume that  $f$  is  $G$ -cellular. The induced map  $f^*: C_G^n(Y, B; M_G) \rightarrow C_G^n(X, A; M)$  commutes with  $\delta$ , in fact,  $f^* \circ \delta = C_G^n(f) \circ C_G^n(\delta) \circ \sigma = C_G^n(\delta \circ f) \circ \sigma = C_G^n(S(f) \circ (\delta) \circ \sigma) = C_G^n(\delta) \circ C_G^n(f) \circ \sigma = C_G^n(\delta) \circ \sigma \circ C_G^{n-1}(f) = \delta \circ f^*$ . This gives an induced map  $f^*: H_G^*(Y, B; M_G) \rightarrow H_G^*(X, A; M_G)$ . If  $f$  is  $G$ -homotopic to  $g$ , we may assume that not only  $f$  and  $g$  are  $G$ -cellular but  $G$ -homotopy  $F: (X \times I, A \times I) \rightarrow (Y, B)$  with  $F|X \times \{0\} = f$ ,  $F|X \times \{1\} = g$  is also  $G$ -cellular. Then,  $F$  gives a homotopy connecting the chain maps,  $f^*$  and  $g^*: C_G^*(Y, B; M_G) \rightarrow C_G^*(X, A; M_G)$  and hence  $f^* = g^*: H_G^*(Y, B; M_G) \rightarrow H_G^*(X, A; M_G)$ . Therefore, even if  $f$  is not a  $G$ -cellular map the induced map  $f^*: H_G^*(Y, B; M_G) \rightarrow H_G^*(X, A; M_G)$  is well-defined and satisfies the  $G$ -homotopy axiom.

q.e.d.

#### 4. Spectral sequence of Atiyah-Hirzebruch type

Suppose that  $(X, A)$  is a fixed pair of  $G$ -finite  $G$ -CW complexes. Put  $H(p, q) = \Sigma h_G^n(X^{q-1}, X^{p-1} \cup A)$ . Then, the collection of  $H(p, q)$ 's satisfies the axioms (S.P. 1)-(S.P. 5) of Cartan-Eilenberg [5. p.334] and hence induces a spectral sequence resulting to  $h_G^*(X, A)$ . The  $E_1$ -term and the 1st differential of the spectral sequence are easily calculated as follows:

$$E_1^{p, q} = h_G^{p+q}(X^p, X^{p-1} \cup A)$$

$$d_1 = \delta: h_G^{p+q}(X^p, X^{p-1} \cup A) \rightarrow h_G^{p+q+1}(X^{p+1}, X^p \cup A)$$

where  $\delta$  is the coboundary homomorphism.

##### Lemma 4.1.

(i)  $h_G^{p+q}(X^p, X^{p-1} \cup A) = \tilde{h}_G^{p+q}(X^p/X^{p-1} \cup A)$  is decomposed into the direct product  $\prod \tilde{h}_G^{p+q}(G\sigma/G\partial\sigma)$ , where  $\sigma$  ranges over representatives of all  $p$ -dimensional  $G$ -cells of  $X/A$ .

(ii) And for each direct factor, there are isomorphisms,  $\tilde{h}_G^{p+q}(G\sigma/G\partial\sigma) = h_G^{p+q}(G\sigma, G\partial\sigma) \cong h_G^{p+q}(G/H_\sigma \times \Delta^p, G/H_\sigma \times \partial\Delta^p) \cong h_G^q(G/H_\sigma)$ .

*Proof of (i).* Since  $X^p/X^{p-1} \cup A = \vee (G\sigma/G\partial\sigma)$  is the one point union of finite  $(G\sigma/G\partial\sigma)$ 's we get the decomposition by the usual argument.

Proof of (ii). The 2nd isomorphism is induced by the  $G$ -characteristic map,  $Gf_\sigma: (G/H_\sigma \times \Delta^p, G/H_\sigma \times \partial\Delta^p) \rightarrow (G\sigma, G\partial\sigma)$ , which is a relative  $G$ -homeomorphism. Now we shall prove the last isomorphism. Put  $H=H_\sigma$ . Since the inclusion,  $G/H \times (\partial\Delta^p - \Delta^{p-1}) \rightarrow G/H \times \Delta^p$ , has a  $G$ -equivariant deformation retraction, we get the isomorphism,

$$h_G^{p+q}(G/H \times \Delta^p, G/H \times \partial\Delta^p) \xrightarrow{\cong} h_G^{p+q-1}(G/H \times \partial\Delta^p, G/H \times (\partial\Delta^p - \Delta^{p-1}))$$

in the exact sequence of a triple  $(G/H \times \Delta^p, G/H \times \partial\Delta^p, G/H \times (\partial\Delta^p - \Delta^{p-1}))$ . By the excision axiom, we get the isomorphism,

$$h_G^{p+q-1}(G/H \times \partial\Delta^p, G/H \times (\partial\Delta^p - \Delta^{p-1})) \xrightarrow{\cong} h_G^{p+q-1}(G/H \times \Delta^{p-1}, G/H \times \partial\Delta^{p-1}).$$

Combining the isomorphisms of these two types repeatedly, we get

$$\begin{aligned} h_G^{p+q}(G/H \times \Delta^p, G/H \times \partial\Delta^p) &\xrightarrow{\cong} h_G^{p+q-1}(G/H \times \Delta^{p-1}, G/H \times \partial\Delta^{p-1}) \\ &\dots \xrightarrow{\cong} h_G^q(G/H \times \Delta^0, G/H \times \partial\Delta^0) = h_G^q(G/H). \end{aligned}$$

q.e.d.

We shall consider the difference of taking another representative  $g\sigma$  instead of  $\sigma$ , as a representative of a  $p$ -dimensional  $G$ -cell  $G\sigma$ . Put  $H=H_\sigma$ . Then  $gHg^{-1} = H_{g\sigma}$ . Since we may identify  $agH$ -orbit of  $\sigma$  with  $agHg^{-1}$ -orbit of  $g\sigma$  in  $G\sigma$ , a canonical right translation  $\hat{g}: G/gHg^{-1} \ni agHg^{-1} \mapsto agH \in G/H$  induces a required isomorphism,  $h_G^q(\hat{g}): h_G^q(G/H_\sigma) \rightarrow h_G^q(G/H_{g\sigma})$ . This shows that  $h_G^{p+q}(X^p, X^{p-1} \cup A) \cong C_G^p(X^p, X^{p-1} \cup A; h_G^q)$ .

**Theorem 4.2.** *The  $E_2^{p,q}$ -term of the Atiyah-Hirzebruch spectral sequence for a  $G$ -cohomology theory,  $h_G^*$ , on  $G$ -finite  $G$ -CW complexes, is a classical  $G$ -cohomology theory with coefficients in  $h_G^q$ .*

Proof. By the result above we can identify  $E_1^{p,q} = h_G^{p+q}(X^p, X^{p-1} \cup A)$  with  $C_G^p(X, A; h_G^q)$ . And the coboundary homomorphisms are induced from  $\partial$  in the Puppe sequence in both cases.

q.e.d.

Assume that the  $G$ -cohomology theory  $h_G^*(\cdot)$  is defined also on (not  $G$ -finite)  $G$ -CW complexes, and satisfies the additivity axiom:

(3) The inclusions,  $i_\alpha: X_\alpha \rightarrow \coprod X_\alpha$ , induce an isomorphism,

$$\coprod h_G^n(i_\alpha): \coprod h_G^n(X_\alpha) \xrightarrow{\cong} h_G^n(\coprod X_\alpha)$$

Then, Lemma 4.1 and Theorem 4.2 are also valid for a pair of (not  $G$ -finite)  $G$ -CW complexes.

The classical  $G$ -cohomology theory is defined on  $G$ -CW complexes and satisfies the additivity axiom. Therefore, we get as usual

**Theorem 4.3.** *The classical  $G$ -cohomology theory is characterized to be*

the  $G$ -cohomology theory defined on  $G$ -CW complexes which satisfies also the additivity axiom and the dimension axiom.

Here we mean by dimension axiom,

$$(4) \quad h_G^n(G/H)=0 \text{ for } n \neq 0 \text{ and all closed subgroup } H \text{ of } G.$$

The additivity axiom and the dimension axiom are as follows, for the reduced  $G$ -cohomology theory.

(3)' The inclusions,  $i_\alpha: X_\alpha \rightarrow \vee X_\alpha$ , induce an isomorphism,

$$\prod \tilde{h}_G^n(i_\alpha): \prod \tilde{h}_G^n(X_\alpha) \xrightarrow{\cong} \tilde{h}_G^n(\vee X_\alpha).$$

(4)'  $\tilde{h}_G^n(G/H)^+=0$  for  $n \neq 0$  and all  $H$ .

### 5. $G$ -obstruction theory

Let  $Y$  be a  $G$ -space with a base point. Then in the classical  $G$ -cohomology group  $H_G^*(\cdot; \omega_n(Y))$ , we can make a  $G$ -obstruction theory similar to that of Bredon [3].

Let  $n \geq 1$  be a fixed integer and  $A$  be a  $G$ -subcomplex of a  $G$ -CW complex  $X$ . We shall assume, for simplicity, that the pointwise fixed subspace  $Y^H$  of  $Y$  by  $H$  is non-empty, arcwise connected and  $n$ -simple for each closed subgroup  $H$  of  $G$  which appears as an isotropy subgroup at a point of  $X$ .

Assume that we are given a  $G$ -map  $\varphi: X^n \cup A \rightarrow Y$ . Let  $\sigma$  be an  $(n+1)$ -cell of  $X$  and let  $f_\sigma: \partial\Delta^{n+1} \rightarrow X^n$  be the characteristic attaching map of  $\sigma$  and  $H_\sigma = H$ . Because the image of  $\partial\Delta^{n+1}$  by  $\varphi \circ f_\sigma$  is pointwise fixed by  $H$ , we get a map:  $\partial\Delta^{n+1} \rightarrow Y^H$ . We define  $c_\varphi(\sigma) \in \pi_n(Y^H, *) = \omega_n(Y)(G/H)$  to be the unique base point preserving homotopy class which is free homotopic to the above map ( $\pi_n(Y^H, *) \cong [S^n; Y^H]$  because  $Y^H$  is  $n$ -simple). Since  $\varphi$  is a  $G$ -map, we get  $c_\varphi(g\sigma) = g \cdot c_\varphi(\sigma) \in \pi_n(Y^{gHg^{-1}}, *) = \omega_n(Y)(G/gHg^{-1})$  and hence  $c_\varphi \in C_G^{n+1}(X, A; \omega_n(Y))$ .

**Lemma 5.1.**  $\delta c_\varphi = 0 \in C_G^{n+2}(X, A; \omega_n(Y))$ .

*Proof.* Let  $\tau$  be an  $(n+2)$ -cell of  $(X, A)$  and  $i: (G\tau, G\partial\tau) \rightarrow (X, A)$  be the inclusion. Then  $i^*\delta c_\varphi = \delta i^*c_\varphi$  and  $i^*c_\varphi \in C_G^{n+1}(G\tau, G\partial\tau; \omega_n(Y))$ . According to our definition of  $C_G^{n+1}(\cdot; \omega_n(Y))$  on  $G$ -CW complexes,  $C_G^{n+1}(G\tau, G\partial\tau; \omega_n(Y)) = 0$ . Therefore,  $i^*c_\varphi = 0$  and hence  $i^*\delta c_\varphi = 0$ , that is,  $c_\varphi(\tau) = 0$  for any  $(n+2)$ -cell  $\tau$  of  $(X, A)$ .

q.e.d.

Now identifying the  $G$ -homotopy classes of  $G$ -maps:  $G/H \times \partial\Delta^{n+1} \rightarrow Y$  and the homotopy classes of maps:  $\partial\Delta^{n+1} \rightarrow Y^H$ , we can reduce the proof of the following lemmas to the ordinary obstruction theory as Bredon did.

**Lemma 5.2.**  $c_\varphi = 0$  if and only if  $\varphi$  is extendable equivariantly on  $X^{n+1} \cup A$ .

**Lemma 5.3.** Let  $d \in C_G^n(X, A; \omega_n(Y))$ . Then, there is a  $G$ -map  $\theta: X^n \cup A \rightarrow Y$ , coinciding with  $\varphi$  on  $X^{n+1} \cup A$  such that  $d_{\theta, \varphi} = d$ .

Here the difference cochain  $d_{\theta, \varphi}$  is defined to be the class which corresponds to  $c_{\theta * \varphi}$  by the isomorphism,  $C_G^n(X, A; \omega_n(Y)) \rightarrow C_G^{n+1}(X \times I, A \times I \cup X \times \partial I; \omega_n(Y))$ .  $\theta * \varphi$  is a  $G$ -map:  $(X \times I)^n \cup A \times I \rightarrow Y$  which is  $\varphi$  on  $X^n \times \{0\} \cup X^{n-1} \times I$  and  $\theta$  on  $X^n \times \{1\}$ .

Combining these three lemmas, we get

**Theorem 5.4.** *Let  $\varphi: X^n \cup A \rightarrow Y$  be a  $G$ -map. Then  $\varphi|_{X^{n-1} \cup A}$  can be extended to  $G$ -map:  $X^{n+1} \cup A \rightarrow Y$  if and only if the  $G$ -cohomology class of  $c_\varphi$  in  $H_G^{n+1}(X, A; \omega_n(Y))$  vanishes.*

Also the argument of Bredon in 'primary obstructions' [3, II.5.2] is valid to this case. In particular, we get

**Proposition 5.5.** *Let  $n \geq 1$  be a fixed integer and let  $Y$  be a  $G$ -space with base point such that  $Y^H$  is non-empty, arcwise connected and  $n$ -simple for every closed subgroup  $H$  of  $G$ . Suppose that  $\omega_k(Y)$  vanishes for  $k \neq n$ , then a primary obstruction map,*

$$\alpha: [X; Y]_G \xrightarrow{\cong} H_G^n(X; \omega_n(Y))$$

is an isomorphism for any  $G$ -CW complex  $X$ .

**Proposition 5.5.'** *Under the assumption above, a primary obstruction map,*

$$\alpha': [X, Y]_{G,0} \xrightarrow{\cong} H_G^n(X; \omega_n(Y))$$

is an isomorphism for any  $G$ -CW complex  $X$  with base point.

## 6. Representation theorem of E. Brown

We shall prove the following representation theorem as an application of E. Brown's abstract homotopy theory [4].

**Theorem 6.1.** *If a reduced  $G$ -cohomology group  $\tilde{h}_G^n$  on  $G$ -CW complexes with base point satisfies the additivity axiom, then  $\tilde{h}_G^n$  is representable, that is, there is a  $G$ -space  $Y_n$  with base point and a natural transformation  $T: [\cdot; Y_n]_{G,0} \rightarrow \tilde{h}_G^n(\cdot)$  such that  $T$  is an isomorphism for any  $G$ -CW complex with base point, where  $[\cdot; \cdot]_{G,0}$  stands for the set of base point preserving  $G$ -homotopy classes of base point preserving  $G$ -maps.*

Let  $\mathcal{C}$  be the category of  $G$ -CW complexes with base point such that the  $H$ -stationary subspace is arcwise connected for each  $H$ , and base point preserving  $G$ -homotopy classes of base point preserving  $G$ -maps. In  $\mathcal{C}$  there is a (not unique) sequential direct limit by approximating  $G$ -maps by  $G$ -cellular maps and making their telescope. Also we get a (not unique) 'push out' as a double mapping cylinder in  $\mathcal{C}$ . If we choose one representative for each class of conjugate closed subgroups,  $\{(G/H \times \Delta^p)/(G/H \times \partial \Delta^p); H \text{ representative, } 0 < p < \infty\}$  is a

small subcategory of  $\mathcal{C}$ .

Let  $\mathcal{C}_0$  be a minimal subcategory which contains  $(G/H \times \Delta^p)/(G/H \times \partial \Delta^p)$ 's ( $0 < p < \infty$ ) and their 'push out'. Then  $\mathcal{C}_0$  is a small, full subcategory of  $\mathcal{C}$  and also a subcategory of  $G$ -finite  $G$ -CW complexes with base point and we get

**Proposition 6.2.** *A pair  $(\mathcal{C}, \mathcal{C}_0)$  is a homotopy category in the sense of E.Brown.*

Proof of Theorem 6.1. Since reduced  $G$ -cohomology theory has a Mayer-Vietoris exact sequence,  $\tilde{h}_G^n$  (restricted on  $\mathcal{C}$ ) with the additivity axiom is a homotopy functor in the sense of E.Brown. Moreover, we get  $\overline{\mathcal{C}}_0 = \mathcal{C}$  by an equivariant version of J.H.C.Whitehead's theorem. (See Proposition 0.4.). Therefore, by Theorem 2.8 of [4], we get a  $Y'_n \in \mathcal{C}$  unique up to  $G$ -homotopy equivalence and a natural transformation  $T: [\cdot; Y'_n]_{G,0} \rightarrow \tilde{h}_G^n(\cdot)$  such that  $T$  is an isomorphism for each  $X \in \mathcal{C}$ .

Define  $Y_n = \Omega Y'_{n+1}$ . For any  $G$ -CW complex  $X$  with base point,  $SX \in \mathcal{C}$ . Therefore, we get

$$\begin{array}{ccc} [X, Y_n]_{G,0} & & \tilde{h}_G^n(X) \\ \cong \downarrow & & \downarrow \cong \\ [SX, Y'_{n+1}]_{G,0} & \xrightarrow{\cong} & \tilde{h}_G^{n+1}(SX) \end{array}$$

q.e.d.

REMARK. Even when  $\tilde{h}_G^n$  is defined only on  $G$ -finite  $G$ -CW complexes, by the method of Adams [2], we get a reduced  $G$ -cohomology theory on  $G$ -CW complexes which satisfies the additivity axiom and coincides with  $\tilde{h}_G^n$  on  $G$ -finite  $G$ -CW complexes.

Let  $Y'_{n+1} \in \mathcal{C}$  be a representing space of  $\tilde{h}_G^n$  in the category of  $\mathcal{C}$ . Then, the isomorphism:  $h_G^{n+1}(X) \xrightarrow{\cong} h_G^{n+2}(SX)$  induces a  $G$ -map  $h'_{n+1}: Y'_{n+1} \rightarrow \Omega Y'_{n+2}$  which is a weak  $G$ -homotopy equivalence, that is,  $(h'_{n+1})_*: \pi_i(Y'_{n+1})^H \xrightarrow{\cong} \pi_i((\Omega Y'_{n+2})^H)$  for any  $i$  and any  $H$ . Hence, taking their loop spaces, we get also a weak  $G$ -homotopy equivalence,  $h_n: Y_n \rightarrow \Omega Y_{n+1}$ . Then,  $Y = \{Y_n, h_n; -\infty < n < \infty\}$  forms a weak  $\Omega$ -spectrum for  $\tilde{h}_G^*$ . This fact is used in §7 to make a spectral sequence of C.Mauder.

### 7. Killing the elements of the $G$ -homotopy groups and C.Mauder's spectral sequence

Let  $Y$  be a  $G$ -space with base point  $y_0$  such that  $Y^H$  is arcwise connected for each closed subgroup  $H$  of  $G$ . An element in the  $n$ -th homotopy group  $\pi_n(Y^H, y_0)$  of  $H$ -stationary subspace  $Y^H$  is called to be an element of  $G$ - $n$ -homotopy groups of  $Y$ . An element  $[f] \in \pi_n(Y^H, y_0)$  with  $f: S^n = \Delta^n / \partial \Delta^n \rightarrow Y^H$  is killed by attaching a  $G$ - $(n+1)$ -cell represented by an  $(n+1)$ -cell  $\sigma$  which has  $f$  as its charac-

teristic attaching map and  $H$  as its isotropy subgroup, that is,  $H_\sigma = H$ . If we fix  $n$  and kill all the elements of  $G$ - $n$ -homotopy groups, we get a relative  $G$ -CW complex  $\tilde{Y}$  such that  $\tilde{Y}^{-1} = Y$ . Then,  $i_*: \pi_n(Y^H, y_0) \rightarrow \pi_n(\tilde{Y}^H, y_0)$  is a zero map for any closed subgroup  $H$ , where  $i: Y^H \rightarrow \tilde{Y}^H$ . On the other hand, by the  $G$ -cellular approximation theorem we get  $\pi_k(\tilde{Y}^H, Y^H, y_0)$  vanishes for  $k < n$  and any  $H$ , that is,  $i_*: \pi_k(Y^H, y_0) \rightarrow \pi_k(\tilde{Y}^H, y_0)$  is an isomorphism for  $k < n$  and a surjection for  $k = n$ . Therefore,  $\pi_k(\tilde{Y}^H, y_0)$  is canonically isomorphic with  $\pi_k(Y^H, y_0)$  for  $k < n$  and vanishes for  $k = n$ . By this reason we call  $\tilde{Y}$  a  $G$ -space obtained of  $Y$  by killing the elements of  $G$ - $n$ -homotopy groups.

Let  $Y(1, p)$  be a  $G$ -space obtained of  $Y$  by killing the elements of  $G$ -homotopy groups of dimensions  $\geq (p+1)$  one after the other. Then,  $Y(1, p)$  is uniquely determined up to  $G$ -homotopy types rel.  $Y$  by the usual argument on (relative)  $G$ -CW complexes. For  $p \leq q$ ,  $Y(p, q)$  denotes the mapping track of  $i(p, q): Y(1, q) \rightarrow Y(1, p-1)$ . Moreover, let  $Y^{(r)}(p, q)$  denote the mapping track of  $i^{(r)}(p, q): Y(r, q) \rightarrow Y(r, p-1)$  for  $r < p \leq q$ . Then, it is easily seen that the natural  $G$ -map:  $Y^{(r)}(p, q) \rightarrow Y(p, q)$  has a  $G$ -homotopy inverse. Therefore, by taking mapping tracks repeatedly, we get a following  $G$ -fibering sequence of  $G$ -spaces. (The  $G$ -spaces are determined up to  $G$ -homotopy types.)

$$\Omega Y(r, t) \rightarrow \Omega Y(r, s) \xrightarrow{\delta} Y(s+1, t) \rightarrow Y(r, t) \rightarrow Y(r, s), \quad r \leq s < t.$$

Here, that  $X \rightarrow Y \rightarrow Z$  is a  $G$ -fibering stands for that  $X^H \rightarrow Y^H \rightarrow Z^H$  is a fibering for any  $H$ . In particular,  $\pi_k(Y(p, q)^H, y_0)$  is isomorphic with  $\pi_k(Y^H, y_0)$  for  $p \leq k \leq q$  and vanishes otherwise.

In §6 we have obtained a weak  $\Omega$ -spectrum for a  $G$ -cohomology theory  $\tilde{h}_G^*$ . Let  $X$  be a  $G$ -finite  $G$ -CW complex and put  $\bar{H}(p, q) = \sum_n [S(X^+); Y'_{n+1}(p+2, q)]_{G,0}$ . Then, by the  $G$ -fibering sequence above, we get a spectral sequence resulting to  $\tilde{h}_G^*(X) = \Sigma [S(X^+); Y'_{n+1}]_{G,0}$ . The  $E_2$ -term,  $\bar{E}_2^{p,q} = [S(X^+); Y'_{p+n+1}(p+1, p+1)]_{G,0}$  is isomorphic with  $H_G^{p+1}(S(X^+); \pi_{p+1}(Y'_{p+q+1})) = H_G^p(X; \tilde{h}_G^p)$  by Proposition 5.5'. Moreover, since  $[S((X^{p+1})^+); Y'_{p+q+1}(1, p+1)]_{G,0} \cong [S(X^+); Y'_{p+q+1}(1, p+1)]_{G,0}$  and  $[S(X^p/X^{p-1}); Y'_{p+q+1}]_{G,0} \cong [S(X^p/X^{p-1}); Y'_{p+q+1}(1, p+1)]_{G,0}$ , the Mauger's argument using exact couples [11] is also valid in this case. Hence, we get

**Theorem 7.1** *Let  $\tilde{h}_G^*$  be  $G$ -cohomology theory. Then, the spectral sequence above is isomorphic with the Atiyah-Hirzebruch spectral sequence except the  $E_1$ -term for any  $G$ -finite  $G$ -CW complex  $X$ .*

**Proposition 7.2.** *The  $r$ -th differential  $\bar{d}_r; \bar{E}_r^{p,q} \rightarrow \bar{E}_r^{p+r, q-r+1}$  in the Mauger's spectral sequence is induced from the 'higher cohomology operation' determined by the  $G$ -homotopy class of*

$$\begin{aligned} \delta_r &= \delta \circ h'_{p+q+1}: Y'_{p+q+1}(p+1, (p+r-1)) \xrightarrow{h'_{p+q+1}} \Omega Y'_{p+q+2}(p+2, p+r) \\ &\xrightarrow{\delta} Y'_{p+q+2}(p+r+1, p+r+1). \end{aligned}$$

Remark that  $[\delta_r] \in H_G^{p+r+1}(Y'_{p+q+1}(p+1, p+r-1), \omega_{p+q+1}(Y'_{p+q+2}))$ .

**Corollary 7.3.**  $E_r^{p,q} = \bar{E}_r^{p,q} (r \geq 2)$  together with the differentials  $d_r$  are  $G$ -homotopy type invariant.

This is also proved from Theorem 4.2 and comparison of spectral sequences.

**8. Applications to the equivariant  $K^*$ -theory**

In this section  $G$  denotes a compact Lie group. We shall applicate our results to  $K_G^*$ -theory.

**Theorem 8.1.** *Let  $X$  be a  $G$ -finite  $G$ -CW complex. There exists a spectral sequence  $E_r^{p,q} (r \geq 1, -\infty < p, q < \infty)$  with*

$$E_1^{p,q} \cong C_G^p(X, K_G^q)$$

$d_1$  being the coboundary homomorphism.

$$E_2^{p,q} \cong H_G^p(X, K_G^q),$$

$$E_\infty^{p,q} \cong G_p K_G^{p+q}(X) = K_{G,p}^{p+q}(X) / K_{G,p+1}^{p+q}(X)$$

where  $K_{G,p}^n(X) = \text{Kernel}(K_G^n(X) \rightarrow K_G^n(X^{p-1}))$ . The  $G$ -coefficient system,  $K_G^q(G/H)$  is isomorphic with  $K_G(G/H)$  for  $q$  even and vanishes for  $q$  odd (See [13]).

This is a special case of Theorem 4.2.

*A. Collapsing theorems*

If  $r$  is even, the  $r$ -th differential is a zero map, because  $d_r$  is a map of  $E_r^{p,q}$  into  $E_r^{p+r, q-r+1}$  where one of the domain or the image vanishes. Therefore, we get

**Theorem 8.2.** *If one of the following conditions is satisfied, then the above spectral sequence collapses :*

- (i)  $H_G^p(X; K_G)$  vanishes for every odd  $p$ .
- (ii)  $H_G^p(X; K_G)$  vanishes for every  $p \geq 3$ .

For the reduced  $K_G^*$ -theory, we get

**Theorem 8.2'.** *If  $X$  has a base point, then the spectral sequence,*

$$\bar{H}_G^p(X; K_G^q) \Rightarrow \bar{K}_G^{p+q}(X)$$

*collapses if:*

- (i)  $\bar{H}_G^p(X; K_G)$  vanishes for every odd  $p$  or for every even  $p$ , or
- (ii)  $\bar{H}_G^p(X; K_G)$  vanishes except  $p=r, r+1, r+2$  for some  $r$ .

*B. On  $E_2$ -term*

We consider the classical  $G$ -cohomology theory with coefficients in  $K_G$ .

$K_G(G/H)$  is canonically isomorphic with  $R(H)$ , where  $R(H)$  is the Grothendieck group of the isomorphic classes of complex representations of  $H$ .

Remark that  $K_G(\hat{g}); K_G(G/G) \rightarrow K_G(G/G)$  is an identity isomorphism for any  $g \in G$ , because any inner automorphism of  $G$  induces an identity isomorphism on  $R(G)$ . Therefore, if we assume that the restriction maps  $i^*: R(G) \rightarrow R(H)$  is surjective, then  $K_G(\hat{g}) = K_G(\hat{g}'): K_G(G/H) \rightarrow K_G(G/H')$  for any elements  $g, g'$  of  $N(H', H)$ . Hence, by Remark 3.5 we shall get

**Proposition 8.3.** *Let  $X$  be a  $G$ -finite  $G$ -CW complex whose isotropy subgroups satisfy the condition:*

(\*) *the restriction map:  $R(G) \rightarrow R(H)$  is a surjection for any closed subgroup  $H$  which appears as an isotropy subgroup at a point of  $X$ .*

*Then,  $H_G^p(X; K_G)$  can be calculated by considering only the orbit type decomposition of the orbit space.*

Proof. As we remark above, by the condition (\*),  $K_G(\hat{g}): K_G(G/H) \rightarrow K_G(G/H')$  is independent of the choice of  $g \in N(H', H)$  for any isotropy subgroups  $H, H'$ . So, we may write this map by  $K_G(H \rightarrow H')$ .

Then, we get the formula:

$$(\delta\varphi)(\sigma) = \sum_{\tau} \sum_{\lambda(\sigma, \tau)} [\sigma, g_{\lambda(\sigma, \tau)}\tau] K_G(H_{\sigma} \leftarrow H_{\tau}) \varphi(\tau).$$

On the other hand, it is easy to see that

$$\sum_{\lambda(\sigma, \tau)} [\sigma, g_{\lambda(\sigma, \tau)}\tau] = [\sigma/G, \tau/G] \in Z$$

where  $\sigma/G$  and  $\tau/G$  are the induced cells on  $X/G$ .

q.e.d.

REMARK 8.4. We call an  $O(n)$ -manifold to be a regular  $O(n)$ -manifold if each isotropy subgroup is conjugate to  $O(k)$  ( $k \leq n$ ). Then any regular  $O(n)$ -manifold satisfies the condition (\*) above, because the restriction map  $\rho_n: R(O(n)) \rightarrow R(O(n-1))$  is a surjection. This fact is easily checked by the classical representation theory as in [14], but we refer the reader to [12].

### C. A conclusion

Combining these results with Proposition 0.5, we get

**Proposition 8.5.** *For a compact regular  $O(n)$  manifold  $X$ , if  $\dim X/G \leq 2$ , then,  $K_G^0(X)/K_{G,2}^0(X)$ ,  $K_{G,2}^0(X)$  and  $K_G^1(X)$  depend only on the orbit type decomposition of the orbit space.*

### D. Examples

Now we shall calculate  $K_G^*(X)$  for some regular  $O(n)$ -manifolds.

(i) *Hirzebruch-Mayer  $O(n)$ -manifold  $W^{2n-1}(d)$  for  $n \geq 2$  [7]: The orbit*



space is a 2-disk  $D^2$  the orbit type of whose interior is  $(O(n-2))$  and the boundary  $(O(n-1))$ .

Define a presheaf  $\mathfrak{F}$  on the orbit space  $D^2$  by  $\Gamma(U, \mathfrak{F}) = \Gamma(U, U \times R(O(n-2)))$  if  $U \subset \text{Int } D^2$  and by  $\Gamma(U, \mathfrak{F}) = \Gamma(U, U \times R(O(n-1)))$  if  $U \cap \partial D^2 \neq \emptyset$ . Then, by Proposition 8.3,  $H_G^*(W^{2n-1}(d); K_G) \cong H^*(D^2, \mathfrak{F})$ . Remark that  $\mathfrak{F}$  forms a sheaf. Define  $\mathfrak{G}$  and  $\mathfrak{H}$  by  $\mathfrak{G} = \text{constant sheaf Ker } \rho_{n-1}$  on  $\partial D^2$  which is considered to be a sheaf over  $D^2$  and  $\mathfrak{H} = \text{constant sheaf } R(O(n-2))$  on whole  $D^2$ . Then, since  $\rho_{n-1}: R(O(n-1)) \rightarrow R(O(n-2))$  is surjective, we get an exact sequence of sheaves,

$$0 \rightarrow \mathfrak{G} \rightarrow \mathfrak{F} \rightarrow \mathfrak{H} \rightarrow 0.$$

The following notation is simpler and reasonable to denote this exact sequence.

$$\begin{array}{c} S^1 \\ \cap : 0 \\ D^2 \end{array} \rightarrow \left( \begin{array}{ccc} \text{Ker } \rho_{n-1} & \rightarrow & R(O(n-1)) \\ 0 & \rightarrow & R(O(n-2)) \end{array} \right) \rightarrow \left( \begin{array}{ccc} R(O(n-1)) & \rightarrow & R(O(n-2)) \\ R(O(n-2)) & \rightarrow & R(O(n-2)) \end{array} \right) \rightarrow 0$$

From the associated long exact sequence, we get

$$H_G^0 \cong R(O(n-1)), H_G^1 \cong \text{Ker } \rho_{n-1} \text{ and } H_G^2 \cong \text{Coker } \rho_{n-1} = 0.$$

Therefore,

$$K_G^0 \cong R(O(n-1)) \text{ and } K_G^1 \cong \text{Ker } \rho_{n-1}.$$

(ii) *Jänich knot  $O(n)$ -manifold for  $n \geq 3$*  [8]: Let  $S^1 \subset S^3$  be a knot. The orbit space is a 4-disk  $D^4$  where the orbit type of each difference domain of  $D^4 \supset S^3 \supset S^1$  is  $(O(n-2))$ ,  $(O(n-1))$ ,  $(O(n))$  respectively.

As in (i), we consider the following exact sequence of sheaves.

$$\begin{array}{c} S^1 \\ \cap \\ S^3 : 0 \\ \cap \\ D^4 \end{array} \rightarrow \left( \begin{array}{ccc} \text{Ker } \rho_n & \rightarrow & R(O(n)) \\ 0 & \rightarrow & \mathfrak{F}' = \left( \begin{array}{ccc} R(O(n-1)) & \rightarrow & R(O(n-1)) \\ R(O(n-2)) & \rightarrow & R(O(n-2)) \end{array} \right) \end{array} \right) \rightarrow 0$$

Then,  $H_G^* = H_G^*(X; K) \cong H^*(D^4; \mathfrak{F}')$  is calculated as follows:

$$H_G^0 \cong R(O(n)), H_G^1 \cong \text{Ker } \rho_n, H_G^2 = 0, H_G^3 \cong \text{Ker } \rho_{n-1} \text{ and } H_G^4 = 0.$$

In particular, if we consider that the  $O(n)$ -manifold has a base point, then  $\tilde{H}_G^0 = 0$  and  $\tilde{H}_G^*$  satisfies the condition (ii) of Theorem 8.2'. Therefore, we get

$$\tilde{K}_G^0 = 0, \text{ that is, } K_G^0 \cong R(O(n))$$

and

$$0 \rightarrow \text{Ker } \rho_{n-1} \rightarrow K_G^1 \rightarrow \text{Ker } \rho_n \rightarrow 0$$

is exact.

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