

EQUIVARIANT EMBEDDINGS OF NORMAL BUNDLES OF FIXED POINT LOCI

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0. Introduction

Fixed point loci of compact Lie group actions on Kähler (or more generally, Riemannian) manifolds have been studied by many authors. Since every compact Lie transformation group can be regarded as a subgroup of the group of isometries, differential geometric methods such as the usage of geodesics and exponential maps play a crucial role in their studies. Although such arguments do not go through in the case of reductive algebraic group actions on algebraic varieties, similar views still allow us to obtain analogous results: Basic theorems on affine varieties under reductive group action were obtained by Luna (cf. [3]), whereas Bialynicki-Birula established, in [1], a powerful stratification theorem for \mathbf{C}^* -actions on non-singular projective varieties.

In the below, combining the methods of Luna and Bialynicki-Birula, we shall give an embedding theorem of normal bundles of m -codimensional fixed point loci in varieties under $SL(m; \mathbf{C})$ -action. More precisely, we shall prove (cf. (3.1)):

Theorem. *Let the algebraic group $G=SL(m; \mathbf{C})$, with $m \geq 2$, act regularly and non-trivially on an n -dimensional irreducible non-singular variety V . Assume that the fixed point set V^G of our G -action on V contains an irreducible component W of codimension $\leq m$ in V . Then W is a non-singular connected component of V^G of codimension m in V , and there exists a G -equivariant embedding $i: N(V:W) \hookrightarrow V$ of the normal bundle $N(V:W)$ of W in V onto a G -invariant Zariski-open neighbourhood U of W in V in a natural way. Furthermore, we have the following:*

- i) *If ψ is a G -equivariant automorphism of V such that $\psi(W)=W$, then $\psi(U)=U$.*
- ii) *If ψ is a G -equivariant automorphism of V such that $\psi|_W=id_W$, then there exists a non-vanishing regular function f on W such that the restriction of $i^{-1} \circ \psi \circ i$ to each fibre $N_q(V:W)$, $q \in W$, coincides with the scalar multiplication by $f(q)$ in the*

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vector space $N_q(V:W)$.

A couple of applications of this theorem are given in § 4:

Theorem 4.1.2. *Let the algebraic group $G=SL(m; \mathbf{C})$, with $m \geq 2$, act regularly and non-trivially on an m -dimensional irreducible normal complete variety V . Assume that V^G is non-empty and contains a simple point of V . Then $V \simeq \mathbf{P}^m(\mathbf{C})$.*

Theorem 4.2.4. *Let $m, n \in \mathbf{Z}_+$, and let the algebraic group $G=SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ act regularly and essentially effectively on an $(m+n+1)$ -dimensional non-singular complete variety V . Denoting by G' (resp. G'') the subgroup $SL(m+1; \mathbf{C}) \times \{e\}$ (resp. $\{e\} \times SL(n+1; \mathbf{C})$) of G , we assume that $V^{G'}$ contains a subvariety W with two properties:*

- (a) $W \simeq \mathbf{P}^n(\mathbf{C})$,
- (b) W is a single G'' -orbit.

Then V is isomorphic to either $\mathbf{P}^{m+n+1}(\mathbf{C})$ or the projective bundle $\mathbf{P}(\underbrace{\mathcal{O}_{\mathbf{P}^n}(d) \oplus \mathcal{O}_{\mathbf{P}^n}(d) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^n}(d) \oplus \mathcal{O}_{\mathbf{P}^n}}_{(m+1)\text{-copies}})$ over $\mathbf{P}^n(\mathbf{C})$ with some $d \in \mathbf{Z}$.

In separate papers (cf. [4], [5], [6]), we use these two theorems to classify essentially effective regular actions of $SL(3; \mathbf{C})$ and $SL(2; \mathbf{C}) \times SL(2; \mathbf{C})$ on algebraic threefolds.

NOTATIONS AND CONVENTIONS

- (0.1) \mathbf{Z} = the set of all integers, \mathbf{Z}_+ = the set of all positive integers, \mathbf{C} = the complex number field, $\mathbf{C}^* = \mathbf{C} - \{0\}$.
- (0.2) All varieties and algebraic groups are defined over \mathbf{C} , and all algebraic group actions are, throughout this paper, assumed to be regular.
- (0.3) For every variety and algebraic group, we only use Zariski topology. (Hence "closed" (resp. "open") means "Zariski closed" (resp. "Zariski open").)
- (0.4) Let an algebraic group G act on a variety V , and let p be a point on V . We denote by V^G (resp. G_p) the fixed point set of the G -action on V (resp. the isotropy subgroup of G at p). A subset W of V is said to be G -invariant if, for every $(g, p) \in G \times W$, $g \cdot p$ lies in W .
- (0.5) Let an algebraic group G act on varieties V and V' . A morphism $f: V \rightarrow V'$ (resp. an isomorphism $f: V \simeq V'$) is said to be a G -morphism (resp. G -isomorphism), if it is G -equivariant, i.e., $f(g \cdot p) = g \cdot f(p)$ holds for every $(g, p) \in G \times V$.
- (0.6) An action of an algebraic group G on a variety V is said to be essentially effective, if the group of all those elements in G which act identically on V is finite.

(0.7) For every vector bundle E over a variety V , we denote by $P(E)$ the associated projective bundle of E over V , which is, by definition, the quotient of E —(zero section) by the group C^* acting by complex scalar multiplication. $\mathcal{O}_{P(E)}(-1)$ is the line bundle over $P(E)$ associated with the C^* -bundle E —(zero section) over $P(E)$. For a C -vector space E , regarding it as a vector bundle over a singleton, we use the same notation ($P(E)$, $\mathcal{O}_{P(E)}(-1)$, ...) as above.

(0.8) Every set consisting of just one element is often identified with the element itself, and the notations $\{p\}$ and p are used interchangeably. On a variety, a “point” is always understood as a closed point. “Locally free sheaves” and “vector bundles” are used interchangeably.

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1. Extension theorems

In this section, we shall give a couple of extension theorems on morphisms between varieties. Although the proofs are very simple, these theorems play a crucial role in our later study of almost-homogeneous algebraic group actions.

(1.1) Dominant G -equivariant completions

DEFINITION 1.1.1. Let an algebraic group G act on an irreducible variety U . Then a variety V with a G -action is said to be a G -equivariant completion of U , (cf. Sumihiro [13]), if the following two conditions are satisfied:

- i) U is (embedded as) a G -invariant open dense subset of V .
- ii) V is a complete variety.

A G -equivariant completion V of U is said to be *dominant* if the following two conditions are satisfied:

- i) V is a normal variety.
- ii) $V-U$ is a disjoint union of (a finite number of) 1-codimensional G -orbits in V .

Theorem 1.1.2. *Let an algebraic group G act on irreducible varieties V and V' . Let U' be a non-empty open (dense) G -invariant subset of V' . Then every G -morphism $f: U' \rightarrow V$ extends uniquely to a G -morphism $f: V' \rightarrow V$ if the following three conditions are satisfied:*

- (1) V is a complete variety.
- (2) V' is a normal variety.
- (3) $V'-U'$ is a disjoint union of a finite number of 1-codimensional G -orbits

in V' .

Corollary 1.1.3. *Let an algebraic group G act on an irreducible variety U , and V' be a dominant G -equivariant completion of U . Then,*

i) *For every G -equivariant completion V of U , there exists a surjective birational G -morphism: $V' \rightarrow V$ which makes the following diagram commutative:*

$$\begin{array}{ccc} U & \hookrightarrow & V' \\ & \searrow & \downarrow \scriptstyle G \\ & & V \end{array}$$

ii) *In particular, every dominant G -equivariant completion V'' of U is G -isomorphic to V' , where the isomorphism between V' and V'' is a canonical extension of the identity automorphism of U .*

Since this corollary follows immediately from (1.1.2), we prove just (1.1.2).

Proof of (1.1.2). Let $f: U' \rightarrow V$ be a G -morphism satisfying the above three conditions (1), (2) and (3). Regarding this f as a rational mapping from V' to V , we denote by S the set of points of indeterminacy. Then by (1) and (2), $\text{codim}_{V'} S \geq 2$. On the other hand, since S is a G -invariant subset of $V' - U'$, the condition (2) shows that S is either empty or purely 1-codimensional in V' . Hence, S is empty. The rest of the proof is routine. Q.E.D.

(1.2) The following theorem obtained as a straightforward consequence of Zariski's Main Theorem (cf. for instance, Mumford [9]) turns out to be useful later.

Theorem 1.2.1. *Let U^* (resp. U, U') be a non-empty open subset of a complete irreducible variety V^* (resp. V, V'). Assume that there exist morphisms $\sigma: V^* \rightarrow V$ and $\sigma': V^* \rightarrow V'$ such that*

- (1) $\sigma|_{U^*}$ maps U^* isomorphically onto U .
- (2) $\sigma'|_{U^*}$ maps U^* isomorphically onto U' .
- (3) For every point $q \in V' - U'$, $\sigma(\sigma'^{-1}(q))$ is a finite set. Furthermore, we assume that:
- (4) V' is a normal variety.

Then the mapping $(\sigma|_{U^}) \circ (\sigma'|_{U^*})^{-1}: U' \rightarrow U$ extends uniquely to a surjective birational morphism $\tau: V' \rightarrow V$, and this τ satisfies $\sigma = \tau \circ \sigma'$.*

2. Algebraic groups with properties (R-m) and (L-m)

In this section, two properties (R-m) and (L-m) featuring the algebraic group $SL(m; \mathbf{C})$ will be discussed. A quick review of basic facts on reductive algebraic group actions will also be given.

(2.1) Properties (R-m) and (L-m).

DEFINITION 2.1.1. Fix an arbitrary positive integer m . A connected linear algebraic group G is said to have *Property (R-m)*, if the following two conditions are satisfied:

(1) G is reductive.

(2) For every non-trivial linear G -action on \mathbb{C}^m , the subset $\mathbb{C}^m - \{0\}$ of \mathbb{C}^m is a single G -orbit.

A connected linear algebraic group G is said to have *Property (L-m)*, if the following condition is satisfied, (cf. Bialynicki-Birula [1;p.491]).

(3) For each non-trivial linear G -action on \mathbb{C}^m , every G -equivariant biregular automorphism ψ of \mathbb{C}^m with $\psi(0)=0$ is necessarily a scalar multiplication of the vector space \mathbb{C}^m .

Proposition 2.1.2. *Let m be an integer with $m \geq 2$. Then the algebraic group $SL(m; \mathbb{C})$ has both Properties (R-m) and (L-m).*

Proof of (2.1.2). Since $SL(m; \mathbb{C})$ is a reductive algebraic group, it suffices to show (2) and (3) of Definition (2.1.1).

Proof of (2). Every non-trivial representation of $SL(m; \mathbb{C})$ on \mathbb{C}^m is equivalent to either the standard representation or its contragredient representation. In both cases, the subset $\mathbb{C}^m - \{0\}$ of \mathbb{C}^m forms a single orbit.

Proof of (3). Fix a non-trivial representation of $G=SL(m; \mathbb{C})$ on $V=\mathbb{C}^m$. Since we have the relation $G_{\psi(v)}=G_v$ for isotropy subgroups at v and $\psi(v)$, in view of the proof of (2) it follows that $\psi(v) \in \mathbb{C}^* \cdot v$ for every $v \in V$. Hence for every $v \in V - \{0\}$ and every $r \in \mathbb{C}^*$, there exist $g \in G$ and $s \in \mathbb{C}^*$ such that $r \cdot v = g \cdot v$ and $\psi(v) = s \cdot v$. Since $\psi(r \cdot v) = \psi(g \cdot v) = g \cdot \psi(v) = g \cdot (s \cdot v) = s \cdot (g \cdot v) = rs \cdot v = r \cdot \psi(v)$, there exists a regular function $f: \mathbb{P}^m(\mathbb{C}) (= \mathbb{P}(V)) \rightarrow \mathbb{C}^*$ such that $\psi(v) = f([v]) \cdot v$ for every $v \in V - \{0\}$, where $[v] \in \mathbb{P}(V)$ denotes the canonical image of $v \in V - \{0\}$. As f is constant, this completes the proof of (3).

(2.2) Reductive algebraic group actions

We shall first briefly discuss useful facts on fixed point loci in varieties under reductive algebraic group actions.

Proposition 2.2.1. *Let a connected reductive algebraic group G act on an irreducible non-singular variety V . Assume that V^G is non-empty. Then*

- i) *For every point p on V^G , there exists a G -invariant quasi-affine open neighbourhood V' of p in V , such that for some realization $V' \subseteq \mathbb{C}^N$ as a quasi-affine variety, the G -action on V' extends to a linear action on \mathbb{C}^N centered at $p=0$.*
- ii) *V^G is a non-singular variety.*

Proof of i) of (2.2.1). By Sumihiro [13; Lemma 8], there exists a G -invariant quasi-projective open neighbourhood $U(\subseteq V)$ of p on which G acts

linearly. Since every representation of the reductive group G is completely reducible, the linear G -action on $U \subseteq \mathbf{P}^N(\mathbf{C})$ is written, for suitable homogeneous coordinates $(x_0: x_1: \cdots: x_N)$ of $\mathbf{P}^N(\mathbf{C})$, in the form

$$G \rightarrow \mathrm{PGL}(N+1; \mathbf{C})$$

$$g \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{g} & \\ 0 & & & \end{pmatrix}$$

with $p=(1:0 \cdots :0)$. We then put $V'=U \cap \{x_0 \neq 0\}$. It is now easy to check that this V' has the required properties.

Proof of ii) of (2.2.1). Fix an arbitrary point p on V^G . By i) above, we may assume that V is a quasi-affine variety $\subseteq \mathbf{C}^N$ on which G acts linearly around the origin $p=0$. Since G is reductive, denoting by $T(V: p)$ the tangent space of V at p , we have a decomposition $\mathbf{C}^N = T(V: p) \oplus T(V: p)^\perp$ into G -invariant subspaces. Since the restriction pr_{1V} to V of the canonical projection $pr_1: \mathbf{C}^N \rightarrow T(V: p)$ induces an identity automorphism $(= (pr_{1V})_*: p)$ of $T(V: p)$ at p , replacing V by a smaller open neighbourhood of p , we may assume without loss of generality that

(#) the G -morphism $pr_{1V}: V \rightarrow T(V: p)$ is étale.

In particular, the connectedness of G implies $(pr_{1V})^{-1}(T(V: p)^G) = V^G$. Since $pr_{1V}(p)=0$ is a simple point of $T(V: p)^G$ (= a linear subspace of $T(V: p)$), the above (#) shows that p is also a simple point of V^G . Q.E.D.

We now quote the following fact on isotropy representations of reductive algebraic groups.

Theorem 2.2.2 (Bialynicki-Birula [1; Lemma 2.4]). *Let a connected reductive algebraic group G act effectively on an irreducible quasi-affine variety V . Let $p \in V^G$. Then the isotropy representation of G on the Zariski tangent space $T(V: p)$ is an effective G -action.*

Corollary 2.2.3. *Let a connected reductive algebraic group G act effectively (resp. essentially effectively, non-trivially) on an irreducible non-singular variety V . Let $p \in V^G$. Then the isotropy representation of G on $T(V: p)$ is an effective (resp. essentially effective, non-trivial) G -action.*

Proof of (2.2.3). This is straightforward from i) of (2.2.1) and the above (2.2.2).

Corollary 2.2.4. *Let a connected linear algebraic group G with Property (R-m) act non-trivially on an n -dimensional irreducible non-singular variety V .*

Then

$$\dim V^G \leq n - m.$$

Proof of (2.2.4). By ii) of (2.2.1), V^G is non-singular. Hence V^G is a disjoint union of its irreducible components V_1, V_2, \dots, V_k . We number these so that

$$\dim V^G = \dim V_1 \geq \dim V_2 \geq \dots \geq \dim V_k.$$

Fix a point p on V_1 . Now, for contradiction, we assume

$$\dim V^G (= \dim V_1) > n - m.$$

Pick a linear subspace L of $T(V_1: p)$ of dimension $n - m$. Since, by (2.2.3), G acts non-trivially on the m -dimensional vector space $T(V: p)/L$, and since G has Property $(R - m)$, it follows that $(T(V: p)/L) - \{0\}$ forms a single G -orbit. But then, this contradicts the triviality of the G -action on the non-zero subspace $T(V_1: p)/L$ of $T(V: p)/L$. Q.E.D.

3. Main Theorem

In this section, we shall prove the following. (This theorem combined with Proposition (2.1.2) immediately implies the theorem in the introduction.)

Theorem 3.1. *Let a connected linear algebraic group G with Property $(R - m)$ and $(L - m)$, with $m \geq 2$, act non-trivially on an n -dimensional irreducible non-singular variety V . Assume that V^G contains an irreducible component W of codimension $\leq m$ in V . Then W is a non-singular connected component of V^G of codimension m in V , and there exists a G -equivariant embedding $i: N(V: W) \hookrightarrow V$ onto a G -invariant open neighbourhood U of W in V in a natural way. Furthermore, we have the following:*

- i) *If ψ is a G -equivariant automorphism of V such that $\psi(W) = W$, then $\psi(U) = U$.*
- ii) *If ψ is a G -equivariant automorphism of V such that $\psi|_W = id_W$, then there exists a non-vanishing regular function f on W such that the restriction of $i^{-1} \circ \psi \circ i$ to each fibre $N_q(V: W)$, $q \in W$, coincides with the scalar multiplication by $f(q)$ in the vector space $N_q(V: W)$.*

We first prove the following local version of Theorem 3.1.

Proposition 3.2. *Let a connected linear algebraic group G with Property $(R - m)$ and $(L - m)$, with $m \geq 2$, act non-trivially on an n -dimensional irreducible non-singular variety V . Assume that V^G contains an irreducible component W of codimension $\leq m$ in V . Then W is a non-singular connected component of V^G of codimension m in V . Furthermore, for every point p on W , there exist an open*

neighbourhood W^p of p in W and a G -equivariant embedding $i^p: N(V: W^p) \hookrightarrow V$ onto a G -invariant open neighbourhood U^p of W^p in V with the following two properties:

(1) The differential $(i^p)_*$ of i^p induces the identity automorphism of $N(V: W^p)$ in a natural way.

(2) $i^p(N_q(V: W^p)) = \{q' \in V; \overline{G \cdot q'} \cap W^p = q\}$ for every $q \in W$, where $\overline{G \cdot q'}$ denotes the closure of $G \cdot q'$ in V .

Proof of (3.2). By (2.2.4) and ii) of (2.2.1), every irreducible component W of V^G with $\text{codim}_V W \leq m$ is a non-singular connected component of V^G with $\text{codim}_V W = m$. Hence the proof is reduced to showing the existence of such W^p and i^p as specified in the above. From now on, we fix an arbitrary point $p \in W$ once and for all.

Step 1. Replacing V by $V - (V^G - W)$, we may assume, from the beginning, (*) $W = V^G$.

In view of i) of (2.2.1), we may furthermore assume that V is a quasi-affine variety sitting in \mathbf{C}^N and passing through $p=0 \in \mathbf{C}^N$, and that the G -action on V is induced from a linear G -action on \mathbf{C}^N . Now let $L_V (= T(V: p))$ (resp. $L_W (= T(W: p))$) denote the G -invariant subspace of \mathbf{C}^N spanned by the vectors tangent to V (resp. W) at $p=0$. Since every representation of the reductive algebraic group G is completely reducible, there exists a G -invariant vector subspace L_V^\perp of \mathbf{C}^N (resp. $L_{V/W}$ of L_V) such that $\mathbf{C}^N = L_V \oplus L_V^\perp$ (resp. $L_V = L_W \oplus L_{V/W}$). In particular, the projection to the first factor

$$\rho: \mathbf{C}^N (= L_V \oplus L_V^\perp) \rightarrow L_V$$

is G -equivariant. Now, using Luna's argument [3], we shall define an étale morphism α as follows:

Let \bar{V} denote the closure of V in \mathbf{C}^N . Then the restriction $\rho|_{\bar{V}}: \bar{V} \rightarrow L_V$ of ρ to the G -invariant affine subvariety \bar{V} of \mathbf{C}^N is also G -equivariant. At the point p , this mapping induces an isomorphism of tangent spaces $(\rho|_{\bar{V}})_*: T(V: p) \simeq T(L_V: 0)$. Hence, by a theorem of Luna [3; p. 94] applied to the G -morphism $\rho|_{\bar{V}}$ and to the G -fixed point p , there exists an affine open neighbourhood V_1 of p in \bar{V} with the following properties:

(a) The restriction, (denoted by $\alpha: V_1 \rightarrow L_V$), of the morphism $\rho|_{\bar{V}}$ to V_1 is étale.

(b) The image $V'_1 \stackrel{\text{defn}}{=} \alpha(V_1)$ is an affine open subset of L_V .

(c) $V_1 = \pi^{-1}(\pi(V_1))$ and $V'_1 = \pi'^{-1}(\pi'(V'_1))$, where $\pi: \bar{V} \rightarrow \bar{V}/G$ and $\pi': L_V \rightarrow L_V/G$ denote the canonical quotient morphisms (= "good quotients" in Seshadri's sense, cf. Seshadri [12], Mumford [8; p. 27]).

(d) The canonically induced morphism $\bar{\alpha}: \pi(V_1)(=V_1/G) \rightarrow \pi'(V'_1)(=V'_1/G)$ is étale

Step 2. In this step, we shall construct U^p and W^p : Since G acts on $L_{V/W}(\simeq \mathbf{C}^m)$ non-trivially, (cf. (2.2.3)), and since G has Property (R-m), every G -orbit in L_V is either a G -fixed point on $L_W (=L_V)^G$ or written in the form

$$\{\text{a point on } L_W\} \times (L_{V/W} - \{0\}) (\subseteq L_W \oplus L_{V/W} (=L_V)).$$

Hence the quotient variety L_V/G (which parametrizes the closed G -orbits in L_V) is isomorphic to $L_W=(L_V)^G$. We shall show that a similar isomorphism also exists between V_1/G and $(V_1)^G$. First of all, since V'_1/G is an open subset of the non-singular variety $L_V/G (=L_W)$, (d) of Step 1 says that V_1/G is non-singular. Secondly, the unramified étale covering $\alpha_{1G,q}: G \cdot q \rightarrow \alpha(G \cdot q)$ is an isomorphism for every $q \in V_1 - (V_1)^G$, because the orbit $\alpha(G \cdot q) = G \cdot \alpha(q)$ is written in the form

$$\{\text{a point on } L_W\} \times (L_{V/W} - \{0\}) (\simeq \mathbf{C}^m - \{0\}),$$

which is simply connected by $m \geq 2$. Now, for such q , the orbit $G \cdot q (\simeq \mathbf{C}^m - \{0\})$ is a non-affine set and cannot be closed in the affine variety V_1 . Hence $(V_1)^G$ is the set of the closed orbits in V_1 . Then the restriction $\pi_{|(V_1)^G}: (V_1)^G \rightarrow V_1/G$ of the quotient morphism π (cf. (c) of Step 1) is bijective, and therefore, is an isomorphism. Let us denote this isomorphism by

$$j: (V_1)^G \simeq V_1/G.$$

Now, set $W^p = (V_1)^G \cap V$, which is $W \cap V_1$ (=an open neighbourhood of p in W) by the assumption (*) of Step 1. We then define: $U^p = \pi^{-1}(j(W^p))$. Since $\pi: \bar{V} \rightarrow \bar{V}/G$ is a "good quotient" in Seshadri's sense, U^p is written as

$$U^p = \{q \in V_1; \overline{G \cdot q} \cap W^p \neq \emptyset\},$$

where $\overline{G \cdot q}$ denotes the closure of $G \cdot q$ in V_1 . Since V is an open neighbourhood of W^p in \bar{V} , this expression of U^p shows that $U^p \subseteq V$. Then one can check that U^p is a G -invariant open neighbourhood of W^p in V with the property $(U^p)^G = W^p$.

Step 3. We finally construct an isomorphism $i^p: N(U^p: W^p) (=N(V: W^p)) \simeq U^p$ with the help of the following commutative diagram:

$$\begin{array}{ccc} \alpha_{|U^p}: U^p & \rightarrow & L_V \\ & \cup \cup \cup & \\ \alpha_{|W^p}: W^p & \rightarrow & L_W. \end{array}$$

The main point of the construction is to lift, using this diagram, the canonical

isomorphism $N(L_V: L_W) \simeq L_V$ to an isomorphism $i^p: N(U^p: W^p) \simeq U^p$.

Let $(\alpha_{|W^p})^*(T(L_V))$ denote the vector bundle over W^p induced from the tangent bundle $T(L_V)$ of L_V via the morphism $\alpha_{|W^p}: W^p \rightarrow L_W (\hookrightarrow L_V)$. Since $\alpha_{|U^p}$ is an étale G -morphism, (cf. (a) of Step 1), we obtain the canonical G -isomorphism

$$(**) \quad T(U^p)_{|W^p} \simeq (\alpha_{|W^p})^*(T(L_V)).$$

On the other hand since the morphism $\bar{\alpha}: V_1/G \rightarrow L_V/G$ is étale (cf. (d) of Step 1) and since W^p is an open (dense) subset of $(V_1)^G$, the commutativity of the diagram

$$\begin{array}{ccc} (V_1)^G & \simeq & V_1/G \\ \alpha \downarrow & \hookrightarrow & \downarrow \bar{\alpha} \\ L_W (= (L_V)^G) & \simeq & L_V/G \end{array}$$

implies that $\alpha_{|W^p}: W^p \rightarrow L_W$ is also étale. Then we obtain the isomorphism

$$(***) \quad T(W^p) \simeq (\alpha_{|W^p})^*(T(L_W)),$$

on both sides of which, G acts trivially. In view of (**) and (***) above, the following G -isomorphism is immediately defined:

$$\alpha': N(U^p: W^p) \simeq (\alpha_{|W^p})^*(N(L_V: L_W)).$$

We now construct a G -isomorphism $\gamma: U^p \simeq (\alpha_{|W^p})^*(N(L_V: L_W))$ as follows: Let $\psi: U^p \rightarrow N(L_V: L_W)$ be the composite of $\alpha_{|U^p}: U^p \rightarrow L_V$ with the natural isomorphism $L_V \simeq N(L_V: L_W)$. Consider also the G -morphism $j^{-1} \circ \pi_{|U^p}: U^p (= \pi^{-1}(j(W^p))) \rightarrow W^p (\subseteq (V_1)^G)$, (cf. Step 2). Since ψ is G -equivariant, we define a G -morphism γ by

$$\begin{aligned} \gamma: U^p &\rightarrow (\alpha_{|W^p})^*(N(L_V: L_W)) \\ q &\mapsto (\psi(q), j^{-1} \circ \pi(q)). \end{aligned}$$

γ is well-defined: Denoting by $\overline{G \cdot q}$ (resp. $\overline{G \cdot \alpha(q)}$) the closure of $G \cdot q$ (resp. $G \cdot \alpha(q)$) in V_1 (resp. L_V), we have $j^{-1} \circ \pi(q) \subseteq \overline{G \cdot q}$, (cf. Step 2). Hence $\alpha(j^{-1} \circ \pi(q)) \in \overline{\alpha(G \cdot q)} \subseteq \overline{G \cdot \alpha(q)}$. In view of the definition of ψ , this means that $\psi(q)$ lies on the fibre of $N(L_V: L_W)$ over $\alpha(j^{-1} \circ \pi(q))$, i.e., γ is well-defined.

γ is injective: Let $q, q' \in U^p$ be such that $\gamma(q) = \gamma(q')$. Then $\psi(q) = \psi(q')$ and $j^{-1} \circ \pi(q) = w = j^{-1} \circ \pi(q')$ for some $w \in W^p$. Since ψ is étale, the equality $W^p = (U^p)^G$ (cf. Step 2) implies $\psi(W^p) \cap \psi(U^p - W^p) = \emptyset$. On the other hand, from the definition of j , (cf. Step 2), $j^{-1} \circ \pi_{|W^p} = id_{W^p}$. Therefore we may assume $q, q' \in U^p - W^p$. Then both $\alpha_{|G \cdot q}: G \cdot q \rightarrow \alpha(G \cdot q)$ and $\alpha_{|G \cdot q'}: G \cdot q' \rightarrow \alpha(G \cdot q')$ are isomorphisms, (cf. Step 2). Since $\overline{G \cdot q} \cap \overline{G \cdot q'} \ni w$, it follows that $\overline{\alpha(G \cdot q)} \cap \overline{\alpha(G \cdot q')}$

$\ni \alpha(w)$. Hence $\alpha(G \cdot q) = \alpha(G \cdot q')$ in L_V . In view of the étaleness of α at the point w , the irreducibility of $\overline{\alpha(G \cdot q)} = \overline{\alpha(G \cdot q')}$ (through $\alpha(w)$) implies that $\overline{G \cdot q} \cup \overline{G \cdot q'}$ is also irreducible around w . Hence $G \cdot q = G \cdot q'$. Since $\alpha(q) = \alpha(q')$ (because $\psi(q) = \psi(q')$), and since $\alpha|_{G \cdot q}$ is an isomorphism, we have $q = q'$. Thus, γ is injective.

γ is surjective: Since γ is an injective G -morphism between equi-dimensional varieties, its image $\gamma(U^p)$ is a G -invariant open subset of $(\alpha|_{W^p})^*(N(L_V: L_W))$. Furthermore, $\gamma(U^p) \supseteq \gamma(W^p) = \text{zero section of } (\alpha|_{W^p})^*(N(L_V: L_W))$. Hence γ is surjective.

Thus, we have shown that γ is a G -isomorphism. We now set $i^p \stackrel{\text{defn}}{=} \gamma^{-1} \circ \alpha': N(U^p: W^p) (= N(V: W^p)) \simeq U^p (\subseteq V)$. Once we obtain this i^p , the properties (1) and (2) of Proposition (3.2) are easily verified: Since U^p is identified with $N(V: W^p)$ via the G -isomorphism i^p , we may assume $V = U^p = N(V: W^p)$ without loss of generality. Then (1) and (2) are straightforward. The proof of (3.2) is now complete.

Proof of (3.1). By (2.2.4) and ii) of (2.2.1), W is a non-singular connected component of V^G of codimension m in V . We now prove (3.1) in the following three steps:

Step 1. Construction of $i: N(V: W) \hookrightarrow V$. To each $p \in W$, we associate i^p , W^p , and U^p defined in Proposition (3.2). Let $\beta^p: U^p \rightarrow W^p$ denote the composite of $(i^p)^{-1}: U^p \rightarrow N(V: W^p)$ with the canonical projection: $N(V: W^p) \rightarrow W^p$. Then

$$(\#) \quad \begin{cases} \overline{G \cdot q'} \cap W^p = \beta^p(q') & \text{for every } q' \in U^p, \\ \overline{G \cdot q''} \cap W^p = \phi & \text{for every } q'' \in V - U^p. \end{cases}$$

We shall now glue $\{\beta^p; p \in W\}$ together: For arbitrary two points p' and p'' on W , we put $U_{p'; p''} = (\beta^{p'})^{-1}(W^{p'} \cap W^{p''}) \cap (\beta^{p''})^{-1}(W^{p'} \cap W^{p''})$, which is open dense in $U^{p'} \cap U^{p''}$. Since for every $q \in U_{p'; p''}$

$$\beta^{p'}(q) = \overline{G \cdot q} \cap (W^{p'} \cap W^{p''}) = \beta^{p''}(q),$$

it follows that $\beta^p|_{U_{p'; p''}} = \beta^{p''}|_{U_{p'; p''}}$, and hence $\beta^p|_{U^{p'} \cap U^{p''}} = \beta^{p''}|_{U^{p'} \cap U^{p''}}$. Thus, denoting by U the G -invariant open neighbourhood $\bigcup_{p \in W} U^p$ of W in V , we obtain a morphism $\beta: U \rightarrow W$ such that $\beta|_{U^p} = \beta^p$ for all $p \in W$.

In view of the equality $W = \bigcup_{p \in W} W^p$, the above (#) asserts that

$$\begin{cases} \overline{G \cdot q'} \cap W = \beta(q') = \text{a singleton} & \text{for every } q' \in U, \\ \overline{G \cdot q''} \cap W = \phi & \text{for every } q'' \in V - U. \end{cases}$$

Hence for every $p \in W$ and every $q \in W^p$, (cf. (2) of (3.2)),

$$(\#\#) \quad i^p(N_q(V: W)) (= \{q' \in V; \overline{G \cdot q'} \cap W^p = q\}) = \{q' \in V; \overline{G \cdot q'} \cap W = q\}.$$

From this equality, we now construct $i: N(V: W) \hookrightarrow V$ by gluing $\{i^p; p \in W\}$ together: Let p' and p'' be arbitrary points on W . Then for every $q \in W^{p'} \cap W^{p''}$, (cf. $(\#\#)$),

$$i^{p'}(N_q(V: W)) = \{q' \in V; \overline{G \cdot q'} \cap W = q\} = i^{p''}(N_q(V: W)),$$

and hence the G -morphism $\gamma^q \stackrel{\text{defn}}{=} (i^{p'}|_{N_q(V: W)})^{-1} \circ (i^{p''}|_{N_q(V: W)})$ is a well-defined biregular automorphism of $N_q(V: W)$. By (1) of (3.2), the differentials $(i^{p'})_*$ and $(i^{p''})_*$ coincide along the zero section of $N(V: (W^{p'} \cap W^{p''}))$. In particular

$$(\#\#\#) \quad \gamma^q(0) = 0, \quad \text{and} \quad (\gamma^q)_* \text{ at } 0 \text{ is } id_{T(N_q(V: W): 0)},$$

where 0 denotes the origin of $N_q(V: W)$. Since G acts on $N_q(V: W) (\simeq \mathbf{C}^m)$ non-trivially, (cf. (2.2.3)), and since G has Property (L-m), it follows that γ^q is a scalar multiplication of the vector space $N_q(V: W)$. In view of $(\#\#\#)$, we have $\gamma^q = id_{N_q(V: W)}$, i.e., $i^{p'}|_{N_q(V: W)} = i^{p''}|_{N_q(V: W)}$. Here $q \in W^{p'} \cap W^{p''}$ is arbitrary, and hence $i^{p'}|_{N(V: (W^{p'} \cap W^{p''}))} = i^{p''}|_{N(V: (W^{p'} \cap W^{p''}))}$. Thus, there exists a G -isomorphism $i: N(V: W) \xrightarrow{\simeq} U (\hookrightarrow V)$ such that $i|_{N(V: W^p)} = i^p$. (Then, by $(\#\#)$, $i(N_q(V: W)) = \{q' \in V; \overline{G \cdot q'} \cap W = q\}$ for every $q \in W$, where $\overline{G \cdot q'}$ denotes the closure of $G \cdot q'$ in V .)

Step 2. *Proof of i) of (3.1):* From the equality $i(N_q(V: W)) = \{q' \in V; \overline{G \cdot q'} \cap W = q\}$ just above, we see that $\psi(i(N_q(V: W))) = i(N_{\psi(q)}(V: W))$ for every $q \in W$. Hence $\psi(U) = \psi(i(N(V: W))) = \bigcup_{q \in W} \psi(i(N_q(V: W))) = \bigcup_{q \in W} i(N_{\psi(q)}(V: W)) = U$, ((because $\psi(W) = W$).

Step 3. *Proof of ii) of (3.1):* Fix an arbitrary point q on W . Since $\psi|_W = id_W$, we have $\psi(i(N_q(V: W))) = i(N_q(V: W))$, (cf. Step 2). Hence the G -morphism $i^{-1} \circ \psi \circ i|_{N_q(V: W)}$ is a biregular automorphism of $N_q(V: W)$. Since G acts on $N_q(V: W) (\simeq \mathbf{C}^m)$ non-trivially, (cf. (2.2.3)), and since $i^{-1} \circ \psi \circ i|_{N_q(V: W)}$ maps the origin of $N_q(V: W)$ onto itself, the Property (L-m) of G shows that:

$$i^{-1} \circ \psi \circ i|_{N_q(V: W)} = \text{scalar multiplication by } f(q) \text{ in } N_q(V: W),$$

where $f(q) \in \mathbf{C}^*$ is a non-zero constant which depends only on $q \in W$. Since $i^{-1} \circ \psi \circ i$ is a biregular automorphism of $N(V: W)$, the mapping $q \mapsto f(q)$ actually defines a non-vanishing regular function on W . This finishes the proof of ii) of (3.1). The proof of Theorem (3.1) is now complete.

REMARK 3.3. With a slight modification of the proof, one can show that both Theorem (3.1) and Proposition (3.2) are true also for $m=1$. Note that the algebraic torus \mathbf{C}^* has the Properties (R-1) and (L-1).

4. Applications of Theorem 3.1

In this section, we shall give a couple of applications of Theorem (3.1), (cf. (4.1.2) and (4.2.4)). In the below, we use the notation $(V; \gamma)$ which denotes a variety V endowed with an algebraic group action $\gamma: G \times V \rightarrow V$.

- DEFINITION 4.1.1. i) To every $SL(m; \mathbf{C})$ -action $\gamma: SL(m; \mathbf{C}) \times V \rightarrow V$ on a variety V , we associate a new action $\gamma^*: SL(m; \mathbf{C}) \times V \rightarrow V$ by putting $\gamma^*(g, x) = \gamma({}^t g^{-1}, x)$ for all $(g, x) \in SL(m; \mathbf{C}) \times V$.
- ii) More generally, consider an action $\gamma: G \times V \rightarrow V$ of an algebraic group G on a variety V . For every algebraic group automorphism h of G , we define an action $\gamma^h: G \times V \rightarrow V$ by putting $\gamma^h(g, x) = \gamma(h(g), x)$ for all $(g, x) \in G \times V$.
- iii) Let E be an m -dimensional \mathbf{C} -vector space. Then $P(E \oplus \mathbf{C})$ is a disjoint union of $P(E \oplus \{0\}) (= P(E))$ and its complement $\simeq E$. Since $SL(E) (= SL(m; \mathbf{C}))$ acts on $P(E)$ transitively, $P(E \oplus \mathbf{C}) (= P^m(\mathbf{C}))$ endowed with the canonical $SL(E)$ -action (we denote this action by α_m) is a dominant $SL(E)$ -equivariant completion of $E (= \mathbf{C}^m)$.

We first give the following application of Theorem (3.1). For similar results, see Oeljeklaus [11; Satz 3.2] and Nagano [10].

Theorem 4.1.2. *Let the algebraic group $G = SL(m; \mathbf{C})$, with $m \geq 2$, act non-trivially on an m -dimensional irreducible normal complete variety X . Assume that X^G is non-empty and contains a simple point of X . Then X is G -isomorphic to either $(P^m(\mathbf{C}); \alpha_m)$ or $(P^m(\mathbf{C}); (\alpha_m)^*)$. (For the definition of α_m and $(\alpha_m)^*$, see (i) and (iii) of (4.1.1).)*

Proof of (4.1.2). Let V be the set of all simple points of X . Since V is a G -invariant open dense subset of X , we can canonically define a non-trivial G -action on V by restriction. Now, by the assumption, the fixed point set V^G contains at least one point p . Applying Theorem (3.1) to our V, G , and $W = \{p\}$, we can embed the tangent space $T(V: p)$ into V as a G -invariant open subset of V . In terms of this embedding, X is a G -equivariant completion of $T(V: p)$.

Since, by (2.2.3), G acts on $T(V: p) (\simeq \mathbf{C}^m)$ linearly and non-trivially, a standard fact on representations of $G = SL(m; \mathbf{C})$ allows us to choose a suitable basis for $T(V: p)$ in such a way that, in terms of this basis, our G -action on $T(V: p)$ is expressed either as the standard $SL(m; \mathbf{C})$ -action on \mathbf{C}^m or as its contragredient. Hence, by iii) of (4.1.1), either $(P^m(\mathbf{C}); \alpha_m)$ or $(P^m(\mathbf{C}); (\alpha_m)^*)$ is a dominant G -equivariant completion of $T(V: p)$. We now apply i) of (1.1.3), and obtain a surjective birational G -morphism from either $(P^m(\mathbf{C}); \alpha_m)$ or $(P^m(\mathbf{C}); (\alpha_m)^*)$ onto X .

Now, every birational morphism from $P^m(\mathbf{C})$ onto a normal variety is an isomorphism, because by $b_2(P^m(\mathbf{C})) = 1$, no fibres of the morphism can have

positive dimensions. Thus, X is G -isomorphic to either $(\mathbf{P}^m(\mathbf{C}); \alpha_m)$ or $(\mathbf{P}^n(\mathbf{C}); (\alpha_m)^*)$.
 Q.E.D.

(4.2.1) *Definition of an $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -action $\sigma \times \tau$ on $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$.*

Recall that the standard $SL(n+1; \mathbf{C})$ -action on $\mathbf{P}^n(\mathbf{C})$ is naturally lifted to an action on $\mathcal{O}_{\mathbf{P}^n}(d)$, $d \in \mathbf{Z}$, as follows: Since $\mathcal{O}_{\mathbf{P}^n}(d) = (\mathcal{O}_{\mathbf{P}^n}(-1))^{\otimes -d}$, we may just consider the case $d = -1$. Then, by identifying $\mathcal{O}_{\mathbf{P}^n}(-1)$ with the blowing-up of \mathbf{C}^n at the origin 0 , we can naturally define an $SL(n+1; \mathbf{C})$ -action on $\mathcal{O}_{\mathbf{P}^n}(-1)$. This action actually lifts the $SL(n+1; \mathbf{C})$ -action on $\mathbf{P}^n(\mathbf{C})$, because the projection $\mathcal{O}_{\mathbf{P}^n}(-1) \rightarrow \mathbf{P}^n(\mathbf{C})$ canonically extends the $SL(n+1; \mathbf{C})$ -equivariant projection $\mathbf{C}^{n+1} - \{0\} \rightarrow \mathbf{P}^n(\mathbf{C})$.

Denoting by $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$ (resp. $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}$) the vector bundle over $\mathbf{P}^n(\mathbf{C})$ obtained as the direct sum of $m+1$ copies of $\mathcal{O}_{\mathbf{P}^n}(d)$ (resp. $\mathcal{O}_{\mathbf{P}^n}$), we now define actions σ and τ as follows: Letting $SL(m+1; \mathbf{C})$ act on $\mathbf{P}^n(\mathbf{C})$ trivially and on \mathbf{C}^{m+1} canonically, we can define a natural $SL(m+1; \mathbf{C})$ -action on $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}$ ($= \mathbf{P}^n(\mathbf{C}) \times \mathbf{C}^{m+1}$). Since $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d) = \mathcal{O}_{\mathbf{P}^n}(d) \otimes (\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n})$, this canonically induces an $SL(m+1; \mathbf{C})$ -action σ on $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$. On the other hand, we define τ to be the $SL(n+1; \mathbf{C})$ -action on $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$ canonically induced by the $SL(n+1; \mathbf{C})$ -action on $\mathcal{O}_{\mathbf{P}^n}(d)$.

Combining these actions σ and τ , we obtain an $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -action $\sigma \times \tau$ on $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$. The following properties are easily verified:

- i) The zero section of $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$ is exactly the fixed point set of the action σ .
- ii) The zero section of $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$ forms a single orbit under the action τ , and is $SL(n+1; \mathbf{C})$ -isomorphic to $\mathbf{P}^n(\mathbf{C})$.

(4.2.2) $\mathbf{P}((\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)) \oplus \mathcal{O}_{\mathbf{P}^n})$ as a dominant $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -equivariant completion of $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$.

Note that the projective bundle $X \stackrel{\text{defn}}{=} \mathbf{P}((\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)) \oplus \mathcal{O}_{\mathbf{P}^n})$ (cf. (0.7)) is a disjoint union of $X_\infty \stackrel{\text{defn}}{=} \mathbf{P}((\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)) \oplus \{0\})$ ($\simeq \mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$) and its complement $\simeq \bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$. Hence the $SL(m+1; \mathbf{C})$ -action σ (resp. $SL(n+1; \mathbf{C})$ -action τ) on $\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)$ ($= X - X_\infty$) defined in (4.2.1) canonically extends to an $SL(m+1; \mathbf{C})$ -action $\bar{\sigma}$ (resp. $SL(n+1; \mathbf{C})$ -action $\bar{\tau}$) on X . Combining these two actions $\bar{\sigma}$ and $\bar{\tau}$, we obtain an $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -action $\bar{\sigma} \times \bar{\tau}$ on X . The following are immediate:

- i) Under this action $\bar{\sigma} \times \bar{\tau}$, the subvariety X_∞ of X forms a single orbit, which is $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -isomorphic to $\mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$ with standard action.

ii) $(X; \bar{\sigma} \times \bar{\tau})$ is a dominant $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -equivariant completion of $(\bigoplus^{m+1} \mathcal{O}_{P^n}(d); \sigma \times \tau)$, (cf. (4.2.1)).

Furthermore, a routine computation shows that:

iii) The normal bundle $N(X: X_\infty)$ of X_∞ in X is just $pr_1^*(\mathcal{O}_{P^n}(1)) \otimes pr_2^*(\mathcal{O}_{P^n}(-d))$ under the identification of X_∞ with $P^m(\mathbf{C}) \times P^n(\mathbf{C})$, where $pr_1: P^m(\mathbf{C}) \times P^n(\mathbf{C}) \rightarrow P^m(\mathbf{C})$ (resp. $pr_2: P^m(\mathbf{C}) \times P^n(\mathbf{C}) \rightarrow P^n(\mathbf{C})$) denotes the projection to the first (resp. second) factor.

We now assume $d=1$. We shall show that there exists an $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -morphism from $X = P((\bigoplus^{m+1} \mathcal{O}_{P^n}(1)) \oplus \mathcal{O}_{P^n})$ onto $P^{m+n+1}(\mathbf{C})$ as follows: Let E (resp. F) be an $(m+1)$ -dimensional (resp. $(n+1)$ -dimensional) \mathbf{C} -vector space, and $s: F' \rightarrow F$ be the blowing-up of F at the origin 0 . Then $id_E \times s: E \times F' \rightarrow E \times F (= E \oplus F)$ canonically induces a morphism $s': E \times F' \rightarrow P(E \oplus F)$. Since the natural identification $F' = \mathcal{O}_{P^n}(-1)$ enables us to identify $E \times F'$ with $(\bigoplus^{m+1} \mathcal{O}_{P^n}) \oplus \mathcal{O}_{P^n}(-1)$, our s' induces the morphism $s'': P((\bigoplus^{m+1} \mathcal{O}_{P^n}) \oplus \mathcal{O}_{P^n}(-1)) \rightarrow P(E \oplus F)$. Combining this with the natural isomorphism $P((\bigoplus^{m+1} \mathcal{O}_{P^n}(1)) \oplus \mathcal{O}_{P^n}) \simeq P((\bigoplus^{m+1} \mathcal{O}_{P^n}) \oplus \mathcal{O}_{P^n}(-1))$, we finally obtain a morphism

$$\Sigma: X (= P((\bigoplus^{m+1} \mathcal{O}_{P^n}(1)) \oplus \mathcal{O}_{P^n})) \rightarrow P(E \oplus F) (= P^{m+n+1}(\mathbf{C})),$$

with the following properties:

- 1) The restriction $\Sigma|_{X-X_\infty}: X-X_\infty \rightarrow P(E \oplus F) - P(E \oplus \{0\})$ is an isomorphism.
- 2) Σ is an $SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -morphism from $(X; \bar{\sigma} \times \bar{\tau})$ onto $(P(E \oplus F); \nu)$, where ν is the natural $SL(E) \times SL(F)$ -action $(= SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ -action) on $P(E \oplus F) (= P^{m+n+1}(\mathbf{C}))$.
- 3) In terms of the isomorphisms $X_\infty \simeq P^m(\mathbf{C}) \times P^n(\mathbf{C})$ and $\Sigma(X_\infty) (= P(E \oplus \{0\})) \simeq P^m(\mathbf{C})$, the morphism $\Sigma|_{X_\infty}: P^m(\mathbf{C}) \times P^n(\mathbf{C}) (= X_\infty) \rightarrow P^m(\mathbf{C}) (= \Sigma(X_\infty))$ is just the projection to the first factor.

Before getting into the proof of Theorem (4.2.4), we shall first prove:

Proposition 4.2.3. *Let $m, n \in \mathbf{Z}_+$, and let E be a vector bundle of rank $m+1$ over $P^n(\mathbf{C})$. Assume that the algebraic group $G = SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ acts on E essentially effectively, satisfying the following conditions:*

- (a) *For every $p \in P^n(\mathbf{C})$, the subgroup $G' = SL(m+1; \mathbf{C}) \times \{e\}$ of G leaves the fibre E_p over p invariant in such a way that G' acts on E_p linearly.*
- (b) *The locus Z of the zero section of E forms a single G'' -orbit, where G'' denotes the subgroup $\{e\} \times SL(n+1; \mathbf{C})$ of G .*

Then, for some $d \in \mathbf{Z}$, E is G -isomorphic to $\bigoplus^{m+1} \mathcal{O}_{P^n}(d)$ endowed with one of the following G -actions: $\sigma \times \tau, \sigma^ \times \tau, \sigma \times \tau^*, \sigma^* \times \tau^*$, (cf. (4.2.1), (i) of (4.1.1)).*

Proof of (4.2.3). Let β (resp. γ) denote the G' -action (resp. G'' -action) on E . Since our G -action on E is written as $\beta \times \gamma$, it suffices to show the equalities $E = \bigoplus_{i=0}^{m+1} \mathcal{O}_{P^n}(d)$, $\beta = \sigma$ or σ^* , $\gamma = \tau$ or τ^* .

Step 1. By (2.2.3), G' acts on E_p ($p \in P^n(\mathbf{C})$) non-trivially. Hence, for each p , we may choose a \mathbf{C} -basis $\{e_{p;i}; i=0, 1, \dots, m\}$ of E_p in such a way that our G' -action β on E_p is, in terms of this basis, written either as the standard $SL(m+1; \mathbf{C})$ -action on \mathbf{C}^{m+1} or as its contragredient. We now fix a point $p_0 \in P^n(\mathbf{C})$. Then, putting $Y = \{p \in P^n(\mathbf{C}); \beta|_{E_p} \text{ is equivalent to } \beta|_{E_{p_0}}\}$ and $Y' = \{p \in P^n(\mathbf{C}); \beta|_{E_p} \text{ is equivalent to } \beta^*|_{E_{p_0}}\}$, we have: $Y \cup Y' = P^n(\mathbf{C})$. For each i , let $G'(i)$ be the isotropy subgroup of G' at the point $e_{p_0;i}$ and $G'(i)^*$ be the subgroup of G' ($=SL(m+1; \mathbf{C})$) defined by $G'(i)^* = \{g^{-1}; g \in G'(i)\}$. Then

$$\begin{cases} Y = \pi(P(E)^{G'(i)}) = \text{closed in } P^n(\mathbf{C}), \\ Y' = \pi(P(E)^{G'(i)^*}) = \text{closed in } P^n(\mathbf{C}), \end{cases} \quad (\text{cf. (0.4)}),$$

where $\pi: P(E) \rightarrow P^n(\mathbf{C})$ denotes the projective bundle associated with E , and $P(E)$ is endowed with a natural G' -action induced from the one on E . Since Y contains the point p_0 , the connectedness of $P^n(\mathbf{C})$ immediately implies $Y = P^n(\mathbf{C})$.

In view of the decomposition $E_{p_0} = \bigoplus_{i=0}^m (E_{p_0})^{G'(i)}$ (direct sum of 1-dimensional vector spaces), our equality $Y = P^n(\mathbf{C})$ now says that:

$$E = \bigoplus_{i=0}^m E^{G'(i)} \quad (\text{direct sum of line bundles}).$$

Since $G'(i)$, $i=0, 1, \dots, m$, are mutually conjugate in G' , there exists $g_i \in G'$ for each i such that $G'(i) = g_i \cdot G'(0) \cdot g_i^{-1}$. Hence $E^{G'(i)} = g_i \cdot E^{G'(0)}$, i.e., $E^{G'(i)} \simeq E^{G'(0)}$ as line bundles over $P^n(\mathbf{C})$. Since the line bundle $E^{G'(0)}$ over $P^n(\mathbf{C})$ is written in the form $\mathcal{O}_{P^n}(d)$ with some $d \in \mathbf{Z}$, it then follows that $E \simeq \bigoplus_{i=0}^{m+1} \mathcal{O}_{P^n}(d)$. From our choice of the \mathbf{C} -basis $\{e_{p;i}; 0 \leq i \leq m\}$ for E_p , one can easily check that $E^{G'(i)}$ is the line bundle generated by $e_{p;i}$ over each $p \in P^n(\mathbf{C})$. Then, our G' -action β on E coincides with either σ or σ^* under the identification $E = \bigoplus_{i=0}^{m+1} \mathcal{O}_{P^n}(d)$.

Step 2. By (b) above, for a suitable choice of the homogeneous coordinates of the base space $Z = P^n(\mathbf{C})$, the restriction $\gamma|_Z$ of our G'' -action γ is written either as the standard $SL(n+1; \mathbf{C})$ -action on $P^n(\mathbf{C})$ or as its contragredient. Hence, identifying E with $\bigoplus_{i=0}^{m+1} \mathcal{O}_{P^n}(d)$, we obtain: $\tau|_Z = \gamma|_Z$ or $\gamma^*|_Z$.

Case 1) $\tau|_Z = \gamma|_Z$. In this case, we shall show that τ and γ coincides on the whole space E . Now, fix an arbitrary $g \in G''$ and consider the G -equivariant automorphism ψ_g of E defined by

$$\psi_g(x) = \tau(g^{-1}, \gamma(g, x)) \quad \text{for all } x \in E.$$

Since $\psi_{g|_Z} = id_Z$ and $E^{G'} = Z$, applying ii) of Theorem (3.1) to $W = Z$ and $V = E = N(V: W)$, we have a non-zero constant $c_g \in \mathbf{C}^*$ such that the restriction $\psi_{g|_{E_p}}$ to each fibre E_p is exactly the scalar multiplication by c_g in the vector space E_p . We shall now show that $c_g \cdot c_h = c_{g \cdot h}$ for all $g, h \in G''$: First note that $c_f \cdot \tau(f, x) = \tau(f, c_f \cdot x) = \tau(f, \psi_f(x)) = \gamma(f, x)$ for every $f \in G''$ and every $x \in E = \bigoplus_{m+1} \mathcal{O}_{\mathbf{P}^m}(d)$. Then $(c_g \cdot c_h) \cdot \tau(g \cdot h, x) = (c_g \cdot c_h) \cdot \tau(g, \tau(h, x)) = c_g \cdot \tau(g, c_h \cdot \tau(h, x)) = \gamma(g, \gamma(h, x)) = \gamma(g \cdot h, x) = c_{g \cdot h} \cdot \tau(g \cdot h, x)$, i.e., $c_g \cdot c_h = c_{g \cdot h}$. Thus, the mapping $g \mapsto c_g$ defines an algebraic homomorphism of $G'' = SL(n+1; \mathbf{C})$ into \mathbf{C}^* . But then it must be trivial, i.e., $\psi_g = id_E$ for all $g \in G''$. Hence $\tau = \gamma$ on the whole space E .

Case 2) $\tau|_Z = \gamma^*|_Z$. Putting $\psi_g(x) = \tau(g^{-1}, \gamma^*(g, x))$, the same argument as above shows that $\gamma = \tau^*$ on E . The proof of (4.2.3) is now complete.

Q.E.D.

We finally show:

Theorem 4.2.4. *Let $m, n \in \mathbf{Z}_+$, and let V be an $(m+n+1)$ -dimensional non-singular complete variety endowed with an essentially effective action γ of the algebraic group $G = SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$. Denoting by G' (resp. G'') the subgroup $SL(m+1; \mathbf{C}) \times \{e\}$ (resp. $\{e\} \times SL(n+1; \mathbf{C})$) of G , we assume that the fixed point set $V^{G'}$ contains a subvariety W with two properties:*

- (a) $W = \mathbf{P}^n(\mathbf{C})$,
- (b) W is a single G'' -orbit.

Then, for some automorphism h of G , the space $(V; \gamma^h)$ (cf. (ii) of (4.1.1)) is G -isomorphic to one of the following:

$$\begin{aligned} &(\mathbf{P}^{m+n+1}(\mathbf{C}); \nu), && \text{(cf. (2) of (4.2.2)),} \\ &(\mathbf{P}((\bigoplus_{m+1} \mathcal{O}_{\mathbf{P}^m}(d)) \oplus \mathcal{O}_{\mathbf{P}^n}); \bar{\sigma} \times \bar{\tau}), \quad d=0, \pm 1, \pm 2, \dots, && \text{(cf. (4.2.2)).} \end{aligned}$$

REMARK 4.2.5. From the proof below, we shall see that one of the following four automorphisms of $G = SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ can be chosen as h in (4.2.4) above:

$$id_G, \quad \theta_{G'} \times id_{G''}, \quad id_{G'} \times \theta_{G''}, \quad \theta_{G'} \times \theta_{G''},$$

where $\theta_{G'}$ (resp. $\theta_{G''}$) denotes the automorphism of $G' = SL(m+1; \mathbf{C})$ (resp. $G'' = SL(n+1; \mathbf{C})$) defined by $g \mapsto {}^t g^{-1}$

REMARK 4.2.6. We shall use the following lemma in the proof of (4.2.4):

Lemma. *Let $G = SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ act on a variety A in such a way that there exists a surjective G -morphism $\xi: \mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) \rightarrow A$, where G*

acts naturally on $\mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$. We assume that ξ is neither an isomorphism nor a contraction to a point. Then one of the following two situations happens:

i) $A \simeq \mathbf{P}^m(\mathbf{C})$, and via this isomorphism, the above ξ is regarded as the canonical projection: $\mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) \rightarrow \mathbf{P}^m(\mathbf{C})$ to the first factor.

ii) $A \simeq \mathbf{P}^n(\mathbf{C})$, and via this isomorphism, the above ξ is regarded as the canonical projection: $\mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) \rightarrow \mathbf{P}^n(\mathbf{C})$ to the second factor.

Proof of Lemma. Let $H_m = \{(h_{ij}) \in SL(m+1; \mathbf{C}); h_{i1} = 0 \text{ for all } i \neq 1\}$ and let $H_n = \{(h_{ij}) \in SL(n+1; \mathbf{C}); h_{i1} = 0 \text{ for all } i \neq 1\}$. Then every closed subgroup K of $G = SL(m+1; \mathbf{C}) \times SL(n+1; \mathbf{C})$ satisfying $H_m \times H_n \subseteq K \subseteq G$ is either $H_m \times SL(n+1; \mathbf{C})$ or $SL(m+1; \mathbf{C}) \times H_n$. Our Lemma is now immediate from this observation.

Proof of Theorem 4.2.4. Let W' be the irreducible component of $V^{G'}$ containing W . Then, by Theorem (3.1) applied to V , W' , and G' , it follows that $W = W'$, and there exists a G -isomorphism $i: N(V: W) \simeq U(\hookrightarrow V)$ onto a G -invariant open subset U of V . Now, by Proposition (4.2.3) applied to $E = N(V: W)$, there exists $d \in \mathbf{Z}$ such that $N(V: W) = (\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d))$ with one of the G -actions $\sigma \times \tau$, $\sigma^* \times \tau$, $\sigma \times \tau^*$, $\sigma^* \times \tau^*$. Therefore, for some automorphism h (cf. (4.2.5)) of G , the space $(V; \gamma^h)$ is a G -equivariant completion of $(\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d); \sigma \times \tau)$.

We now put $X = \mathbf{P}((\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)) \oplus \mathcal{O}_{\mathbf{P}^n})$, $X_\infty = \mathbf{P}((\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d)) \oplus \{0\})$, (cf. (4.2.2)). Then, by ii) of (4.2.2), $(X; \bar{\sigma} \times \bar{\tau})$ is a dominant G -equivariant completion of $(\bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d); \sigma \times \tau)$, and hence i) of Corollary (1.1.3) shows that the G -isomorphism $i: \bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d) (= N(V: W)) \simeq U$ extends to a surjective birational G -morphism $f: (X; \bar{\sigma} \times \bar{\tau}) \rightarrow (V; \gamma^h)$. Recall that $X_\infty (= X - \bigoplus^{m+1} \mathcal{O}_{\mathbf{P}^n}(d))$ forms a single G -orbit which is G -isomorphic to $\mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$, (cf. (i) of (4.2.2)). Then by the above Lemma, the following four cases are possible:

Case 1) $f|_{X_\infty}: X_\infty \rightarrow f(X_\infty)$ is an isomorphism. In this case, f is a surjective birational morphism with finite fibres. By Zariski's Main Theorem, $f: (X; \bar{\sigma} \times \bar{\tau}) \simeq (V; \gamma^h)$.

Case 2) $f(X_\infty)$ is a singleton. In this case, the argument in Grauert [2; §3, Satz 9] shows that some negative tensor power of the normal bundle $N(X: X_\infty)$ admits a non-zero section, which contradicts iii) of (4.2.2). Hence, Case 2) cannot happen.

Case 3) $f(X_\infty) \simeq \mathbf{P}^n(\mathbf{C})$, and via this isomorphism, the restriction $f|_{X_\infty}$ is regarded as the projection $\mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) (= X_\infty) \rightarrow \mathbf{P}^n(\mathbf{C}) (= f(X_\infty))$ to the first factor: Since X_∞ in X is an exceptional subvariety of the first kind in the sense of Moisëzon [7], a theorem of Moisëzon [7; Chapter III, Corollary 2] says that $d=1$, (cf.

(iii) of (4.2.2)). We now compare the G -morphisms $f: (X; \bar{\sigma} \times \bar{\tau}) \rightarrow (V; \gamma^h)$ and $\Sigma: (X; \bar{\sigma} \times \bar{\tau}) \rightarrow (\mathbf{P}^{m+n+1}(\mathbf{C}); \nu)$, (cf. (2) of (4.2.2)). Then in view of 3) of (4.2.2), both $f(\Sigma^{-1}(q))$ and $\Sigma(f^{-1}(q'))$ are finite sets for every $q \in \Sigma(X_\infty)$ and $q' \in f(X_\infty)$. Applying Theorem (1.2.1), we conclude that the isomorphism $(\Sigma|_{X-X_\infty}) \circ (f|_{X-X_\infty})^{-1}$ of $V - f(X_\infty)$ with $\mathbf{P}^{m+n+1}(\mathbf{C}) - \Sigma(X_\infty)$ (cf. (1) of (4.2.2)) canonically extends to a G -isomorphism: $(V; \gamma^h) \simeq (\mathbf{P}^{m+n+1}(\mathbf{C}); \nu)$.

Case 4) $f(X_\infty) \simeq \mathbf{P}^n(\mathbf{C})$, and via this isomorphism, the restriction $f|_{X_\infty}$ is regarded as the projection $\mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C}) (= X_\infty) \rightarrow \mathbf{P}^n(\mathbf{C}) (= f(X_\infty))$ to the second factor: In this case, we apply the theorem of Moisëzon [7; Chapter III, Corollary 2] again to $X_\infty = \mathbf{P}^m(\mathbf{C}) \times \mathbf{P}^n(\mathbf{C})$, only to get $1 = -1$, which is a contradiction. Hence Case 4) cannot happen.

Thus, the above four cases 1)-4) complete the proof of Theorem (4.2.4).

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