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EQUIVARIANT h-COBORDISMS AND FINITENESS OBSTRUCTIONS

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ABSTRACT. We classify up to topological equivalence those equivariant h-cobordisms which admit a handle structure, giving a topologically invariant Whitehead group and an s-cobordism theorem, and giving the comparison to the Diff and PL classification via an equivariant version of the controlled Whitehead groups of Chapman and Quinn. We also construct stably triangulable, compact G-manifolds with boundary which realize arbitrary controlled equivariant finiteness obstructions. The controlled finiteness obstructions of their boundaries are generic for closed G-manifolds whose product with $\mathbf{R}$ is triangulable.

Here G is a finite group, and G-manifolds are assumed to be locally linear with codimension-3 gaps (proper inclusions of fixed-point components have codimension ≥ 3). By a G-h-cobordism on M we mean a proper equivariant h-cobordism which admits a handle structure (equivariant) in which no handles are attached to fixed-point components of M of dimension less than 5.

Let $\mathrm{Wh}_G^{\mathrm{PL}}(M)$ be the locally compact version as in [**Si**] of Illman's Whitehead group [**Il$_1$**], and let $\mathrm{Wh}_G^{\mathrm{PL},\rho}(M)$, be the subgroup consisting of pairs (Y, M) such that $Y_\alpha^H = M_\alpha^H \cup Y_\alpha^{>H}$ if either $M_\alpha^H = M_\alpha^{>H}$ or $\dim M_\alpha^H < 5$ for each component M_α^H of M^H. With our assumptions, G-h-cobordisms with an explicit handle structure are classified up to handle manipulation (or Cat isomorphism of Cat $=$ Diff or PL) by $\mathrm{Wh}_G^{\mathrm{PL},\rho}(M)$ (cf. [**BQ, R**]). This gives an s-cobordism theorem in Diff and PL, but as noted by [**Il$_2$**, **BH**, **R**] and [**DR**], any G-h-cobordism whose torsion lies in $\mathrm{Wh}_G^{\mathrm{PL}}(x) \subset \mathrm{Wh}_G^{\mathrm{PL}}(M)$, for $x \in M^G$, is topologically trivial.

We say that two G-pairs (Y, X) and (Z, X) are stably homeomorphic if there is a homeomorphism $Y \times Q_G \cong Z \times Q_G$ commuting up to proper G-homotopy with the inclusions of X, where Q_G is the product of infinitely many copies of the unit disc in the regular representation of G over $\mathbf{R}$. The

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stable homeomorphism classes of relative CW pairs (Y, X), with $X \subset Y$ a proper G-equivalence give a quotient group $\mathrm{Wh}_G^{h\text{-}\mathrm{Top}}(X)$ of $\mathrm{Wh}_G^{\mathrm{PL}}(X)$.

THEOREM 1 (EQUIVARIANT TOPOLOGICAL s-COBORDISM THEOREM). *Two G-h-cobordisms on M (with handle structure as above) are G-homeomorphic* rel M *if and only if their torsions agree in* $\mathrm{Wh}_G^{h\text{-}\mathrm{Top}}(M)$.

By a G-CE map $f: Y \to Z$ we mean a proper G-map for which $f^{-1}z \subset U$ is G_z-nullhomotopic for each G_z-neighborhood U of $f^{-1}z$ and each $z \in Z$. A G-CEPL map is one which is both G-CE and PL.

COROLLARY. $\mathrm{Wh}_G^{h\text{-}\mathrm{Top}}(M) = \mathrm{Wh}_G^{\mathrm{PL}}(M)/\sim$, *where* $\sim$ *is generated by* G-CE *maps* rel M.

Let $\mathrm{WH}_G^{\mathrm{CEPL}}(M) = \mathrm{Wh}_G^{\mathrm{PL}}(M)/\sim$, where $\sim$ is generated by G – CEPL maps rel M. Then $\mathrm{Wh}_G^{\mathrm{PL}}(M) \to \mathrm{Wh}_G^{h\text{-}\mathrm{Top}}(M)$ factors through $\mathrm{Wh}_G^{\mathrm{CEPL}}(M)$, and the kernel of $\mathrm{Wh}_G^{\mathrm{PL}}(M) \to \mathrm{Wh}_G^{\mathrm{CEPL}}(M)$ is $\sum_{f:G/H \to M} f_* \mathrm{Wh}_G^{\mathrm{PL}}(G/H)$, where the sum ranges over all maps of orbits into M. Thus all previously known examples of torsions of topologically trivial G-h-cobordisms die in $\mathrm{Wh}_G^{\mathrm{CEPL}}(M)$.

Let $\mathcal{E}$ = PL or CEPL. We compute the kernel of $\mathrm{Wh}_G^{\mathcal{E}}(M) \to \mathrm{Wh}_G^{h\text{-}\mathrm{Top}}(M)$ by an equivariant generalization of Chapman's controlled Whitehead groups [**C**]. For a G-map $p: X \to B$, with B metric, let $\mathrm{Wh}_G^{\mathrm{CEPL}}(X)_{p^{-1}\varepsilon}$ (ε a G-majorant of B) be the fully equivariant analogue of Chapman's group, and let $\mathrm{Wh}_G^{\mathrm{PL}}(X)_{p^{-1}\varepsilon}$ be obtained by strengthening the basic equivalence relation from diagrams of G-CEPL maps to those of G-simple maps (point inverses have the equivariant simple homotopy type of points). When $p = 1_X$ we denote these groups by $\mathrm{Wh}_G^{\mathcal{E}}(X)_\varepsilon$. Restricted groups $\mathrm{Wh}_G^{\mathcal{E},\rho}(M)_\varepsilon$ are defined as above.

THEOREM 2. *For* $\mathcal{E}$ = PL *or* CEPL *there is an exact sequence*

$$\varprojlim_{\varepsilon} \mathrm{Wh}_G^{\mathcal{E},\rho}(M)_\varepsilon \to \mathrm{Wh}_G^{\mathcal{E},\rho}(M) \to \mathrm{Wh}_G^{h\text{-}\mathrm{Top},\rho}(M) \to 0,$$

where $\mathrm{Wh}_G^{h\text{-}\mathrm{Top},\rho}(M)$ *is the image of* $\mathrm{Wh}_G^{\mathrm{PL},\rho}(M)$ *in* $\mathrm{Wh}_G^{h\text{-}\mathrm{Top}}(M)$.

We recover equivariant versions of all of the results of [**C**], with the obstructions groups for the Thin h-Cobordism Theorem and End Theorem given by the controlled PL Whitehead and $\tilde{K}_0$ groups. However, unlike the inequivariant case, $\varprojlim_{\varepsilon} \mathrm{Wh}_G^{\mathcal{E}}(X)_\varepsilon$ is rarely trivial.

Here, $\tilde{K}_{i_G}(X)_{p^{-1}\varepsilon} \subset \mathrm{Wh}_G^{\mathrm{PL}}(X \times T^{1-i})_{p^{-1}\varepsilon}$ (or, equivalently, CEPL) is the subgroup of elements invariant under all of the transfers from the S^1 factors of the $(1-i)$-torus T^{1-i} (with the trivial action) for $i \leq 0$. (The tilde is optional for $i < 0$.) The uncontrolled versions, $\tilde{K}_{i_G}(X)$, are isomorphic to those obtained from the Bass-Heller-Swan splitting when X is compact (cf. [**Ran**]). As in [$\mathbf{Q_1}$, $\mathbf{Q_2}$], all of these inverse systems are stable when p is an equivariant simplicial p-NDR [$\mathbf{Q_2}$] (e.g., $p = 1_X$, X a locally compact G-ANR, or p a G-simplicial map) and may be computed by a Leray spectral sequence with coefficients in the Whitehead and K-groups of p^{-1} (orbits) when p is simplicial. We obtain the following.

THEOREM 3. *Let X be a locally compact G-ANR. Then* $\varprojlim_\varepsilon K_{i_G}(X)_\varepsilon = 0$ *for* $i < -1$, $\varprojlim_\varepsilon K_{-1_G}(X)_\varepsilon \simeq H_0^{G,lf}(X : \mathcal{K}_{-1_G})$, *and there is an exact sequence*

$$H_3^{G,lf}(X; \mathcal{K}_{-1_G}) \to H_1^{G,lf}(X; \tilde{\mathcal{K}}_{0_G}) \to \varprojlim_\varepsilon \mathrm{Wh}_G^{\mathrm{CEPL}}(X)_\varepsilon \to H_2^{G,lf}(X; \mathcal{K}_{-1_G})$$
$$\to H_0^{G,lf}(X; \tilde{\mathcal{K}}_{0_G}) \to \varprojlim_\varepsilon \tilde{K}_{0_G}(X)_\varepsilon \to H_1^{G,lf}(X; \mathcal{K}_{-1_G}) \to 0.$$

Here, we take Bredon homology with locally finite chains with coefficient systems given by restriction of the functors $\tilde{K}_{i_G}$ to orbits.

COROLLARY. $\mathrm{Wh}_G^{h\text{-}\mathrm{Top}}(X)$ *and* $\varprojlim_\varepsilon \tilde{K}_{0_G}(X)_\varepsilon$ *are functors of the π_1-system of X for X compact.*

As a sample computation, note that if X has the π_1-system of the n-torus with trivial action, then $\mathrm{Wh}_G^{h-\mathrm{Top}}(X)$ is isomorphic to the Nil terms in

$$\mathrm{Wh}_G^{\mathrm{PL}}(X) \simeq \mathrm{Wh}_G^{\mathrm{PL}}(*) \oplus n\tilde{K}_{0_G}(*) \oplus \binom{n}{2} K_{-1_G}(*) \oplus \text{Nil terms}.$$

For any compact G-ANR X there is a well-defined controlled finiteness obstruction $\varprojlim_\varepsilon \sigma_\varepsilon(X) \in \varprojlim_\varepsilon \tilde{K}_{0_G}(X)_\varepsilon$ which maps to the ordinary equivariant finiteness obstruction [**A**] $\sigma(X)$ under the natural map $\varprojlim_\varepsilon \tilde{K}_{0_G}(X)_\varepsilon \to \tilde{K}_{0_G}(X)$. Examples of compact G-manifolds M with $\sigma(M) \neq 0$ have been given by [**Q$_1$**, **DR**] and others. These examples all have finiteness obstruction in $\tilde{K}_{0_G}(*) \subset \tilde{K}_{0_G}(M)$, $* \in M^G$.

THEOREM 4. *Let K be a finite G-complex and let* $x \in \varprojlim_\varepsilon \tilde{K}_{0_G}(K)_\varepsilon$. *Then there is a compact G-manifold M containing K as a π_1-equivalent retract, $r: M \to K$, such that* $r_*(\varprojlim_\varepsilon \sigma_\varepsilon(M)) = x$ *and $M \times \mathbf{R}$ is triangulable.*

For closed manifolds M, with $M \times \mathbf{R}$ triangulable, $\varprojlim_\varepsilon \sigma_\varepsilon(M)$ must have the form $\tau \pm \bar{\tau}$ for some $\tau \in \varprojlim_\varepsilon \tilde{K}_{0_G}(M)_\varepsilon$, where $\bar{\tau}$ is the suitably signed conjugate of τ. This is satisfied generally by $\partial(M \times I^n)$, $n \geq 3$, for the examples M of Theorem 4.

David Webb [**W**] has shown that $\varprojlim_\varepsilon \tilde{K}_{0_G}(X)_\varepsilon \to \tilde{K}_{0_G}(X)$ is not always injective, so there exist G-finite G-manifolds which do not admit a handle structure. This provides a negative answer to Question 4.4 of [**Sch**].

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