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Equivariant homology and K-theory of affine Grassmannians and Toda lattices

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Dedicated to Vladimir Drinfeld on the occasion of his 50th birthday

Abstract

For an almost simple complex algebraic group G with affine Grassmannian $\mathrm{Gr}_G = G(\mathbb{C}((\mathsf{t})))/G(\mathbb{C}[[\mathsf{t}]])$, we consider the equivariant homology $H^{G(\mathbb{C}[[\mathsf{t}]])}(\mathrm{Gr}_G)$ and K-theory $K^{G(\mathbb{C}[[\mathsf{t}]])}(\mathrm{Gr}_G)$. They both have a commutative ring structure with respect to convolution. We identify the spectrum of homology ring with the universal group-algebra centralizer of the Langlands dual group $\check{G}$, and we relate the spectrum of K-homology ring to the universal group-group centralizer of G and of $\check{G}$. If we add the loop-rotation equivariance, we obtain a noncommutative deformation of the (K-)homology ring, and thus a Poisson structure on its spectrum. We relate this structure to the standard one on the universal centralizer. The commutative subring of $G(\mathbb{C}[[\mathsf{t}]])$-equivariant homology of the point gives rise to a polarization which is related to Kostant's Toda lattice integrable system. We also compute the equivariant K-ring of the affine Grassmannian Steinberg variety. The equivariant K-homology of Gr_G is equipped with a canonical basis formed by the classes of simple equivariant perverse coherent sheaves. Their convolution is again perverse and is related to the Feigin–Loktev fusion product of $G(\mathbb{C}[[\mathsf{t}]])$-modules.

1. Introduction

Let G be an almost simple complex algebraic group, and let Gr_G be its affine Grassmannian. Recall that if we set $\mathbf{O} = \mathbb{C}[[\mathsf{t}]]$ and $\mathbf{F} = \mathbb{C}((\mathsf{t}))$, then $\mathrm{Gr}_G = G(\mathbf{F})/G(\mathbf{O})$.

It is well known that the subgroup ΩK of polynomial loops into a maximal compact subgroup $K \subset G$ projects isomorphically to Gr_G; thus Gr_G acquires the structure of a topological group. An algebro-geometric counterpart of this structure is provided by the *convolution diagram* $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G$.

This allows one to define the *convolution* of two $G(\mathbf{O})$-equivariant geometric objects (such as sheaves, or constructible functions) on Gr_G. A famous example of such a structure is the category of $G(\mathbf{O})$-equivariant perverse sheaves on Gr ('Satake category' in the terminology of Beilinson and Drinfeld [BD00]); this is a semisimple abelian category, and convolution provides it with a symmetric monoidal structure. By results of [Gin95], [MV00] and [BD00], this category is identified with the category of (algebraic) representations of the Langlands dual group.

The starting point for the present work was the observation that a similar definition works in another setting, yielding a monoidal structure on the category of $G(\mathbf{O})$-equivariant *perverse coherent*

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sheaves on Gr (the 'coherent Satake category'). The latter is a nonsemisimple artinian abelian category, the heart of the middle perversity t-structure on the derived category of $G(\mathbf{O})$-equivariant coherent sheaves on Gr_G; existence of this t-structure is due to the fact that dimensions of all $G(\mathbf{O})$-orbits inside a given component of Gr_G are of the same parity (cf. [Bez00]). The resulting monoidal category turns out to be nonsymmetric, though its Grothendieck ring $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative. One of the results of this paper is a computation of this ring. Along with $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$, we compute its 'graded version', the ring $H^{G(\mathbf{O})}(\mathrm{Gr})$ of equivariant homology of Gr, where the algebra structure is again provided by convolution.[1] (The ring $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ was essentially computed by Peterson [Pet97] cf. also [Kos96].)

To describe the answer, let $\check{G}$ be the Langlands dual group to G, and let $\check{\mathfrak{g}}$ be its Lie algebra. Consider the *universal centralizers* $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ and $\mathfrak{Z}_{\check{G}}^{\check{G}}$: if we denote by $C_{\check{G},\check{\mathfrak{g}}} \subset \check{G}\times\check{\mathfrak{g}}$ (respectively $C_{\check{G},\check{G}} \subset \check{G}\times\check{G}$) the locally closed subvariety formed by all the pairs (g,x) such that $Ad_g(x)=x$ and x is regular (respectively all the pairs (g_1,g_2) such that $Ad_{g_1}g_2=g_2$ and g_2 is regular), then $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ (respectively $\mathfrak{Z}_{\check{G}}^{\check{G}}$) is the categorical quotient $C_{\check{G},\check{\mathfrak{g}}}//\check{G}$ (respectively $C_{\check{G},\check{G}}//\check{G}$) with respect to the diagonal adjoint action of $\check{G}$.

We identify $\mathrm{Spec}(H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G))$ with $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$. Also, we identify $\mathrm{Spec}(K^{G(\mathbf{O})}(\mathrm{Gr}_G))$ with a *variant* of $\mathfrak{Z}_{\check{G}}^{\check{G}}$ (the isomorphism $\mathrm{Spec}(K^{G(\mathbf{O})}(\mathrm{Gr}_G)) \simeq \mathfrak{Z}_{\check{G}}^{\check{G}}$ holds true if and only if G is of type E_8).

Notice that $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ inherits a canonical symplectic structure as a Hamiltonian reduction of the cotangent bundle $\mathsf{T}^*\check{G}$. Also, $\mathfrak{Z}_{\check{G}}^{\check{G}}$ inherits a canonical Poisson structure as a q-Hamiltonian reduction of the q-Hamiltonian $\check{G}$-space *internal fusion double* $\mathbf{D}(\check{G})$ (see [AMM98]); this Poisson structure is in fact symplectic if and only if $\check{G}$ is simply connected (that is, G is adjoint).

The corresponding Poisson structures on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$ come from a deformation of these commutative algebras to noncommutative algebras $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ (respectively $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$); here $\mathbb{G}_m$ acts on Gr_G by loop rotation. We conjecture that the noncommutative algebra $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ can also be obtained from the ring of differential operators on $\check{G}$ by quantum Hamiltonian reduction.

The space $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$ contains an open piece $\mathfrak{Z}(\check{G})$ which for $\check{G}$ adjoint (that is, for G simply connected) is a complexification of the Kostant phase space of the classical Toda lattice [Kos79, Theorem 2.6]. We remark in passing that the Toda lattice also appears in the (apparently related) computations by Givental, Kim and others of quantum cohomology of flag varieties (see e.g. [Kim99]).

Our computation should be compared with (and is to a large extent inspired by) [Gin95] where equivariant cohomology $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$ was computed[2] in terms of the $\check{G}$. (The precise relation between the two computations is spelled out in Remark 2.13.)

The second main object considered in this paper is another derived category of coherent sheaves with a convolution monoidal structure, namely the derived category $D^bCoh_{\Lambda_G}^{G(\mathbf{O})}(\mathsf{T}^*\mathrm{Gr})$ of $G(\mathbf{O})$-equivariant coherent sheaves on the cotangent bundle of Gr_G supported on the union Λ_G of conormal bundles to the $G(\mathbf{O})$-orbits (the definition of involved objects requires extra work since Gr_G is infinite dimensional). (In this case we do not find a t-structure compatible with convolution, so all we get is a monoidal triangulated category.) Notice that the singular support of a $G(\mathbf{O})$-equivariant D-module on Gr_G is an object of $Coh_{\Lambda_G}^{G(\mathbf{O})}(\mathsf{T}^*\mathrm{Gr})$; thus this category can be considered a 'classical limit' of the (derived) Satake category. We compute the Grothendieck ring of $D^bCoh_{\Lambda_G}^{G(\mathbf{O})}(\mathsf{T}^*\mathrm{Gr})$, identifying its spectrum with $(T\times\check{T})/W$, where $T\subset G$ and $\check{T}\subset\check{G}$

[1] The two rings are related via the Chern character homomorphism from $K^{G(\mathbf{O})}(\mathrm{Gr})$ to the completion of $H^{G(\mathbf{O})}(\mathrm{Gr})$.
[2] Another description for $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$ is provided by a general result of [KK86]; in fact, its extension from [KK90] also gives an answer for $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$, and a similar technique can be applied to compute $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$. However, this form of the answer does not make the relation to the (dual) group geometry explicit.

are Cartan subgroups. This is a singular variety birationally equivalent to $\mathrm{Spec}(K^{G(\mathbf{O})}(\mathrm{Gr}_G))$. Unlike the latter, the former remains unchanged if we replace G by $\check{G}$. This motivates a conjecture that the corresponding triangulated monoidal categories for G and $\check{G}$ are equivalent. The conjecture is compatible with a 'classical limit' of the geometric Langlands conjecture of Beilinson and Drinfeld (see § 7.9 for a more precise statement of the conjecture).

Finally, we remark that the convolution of $G(\mathbf{O})$-equivariant perverse coherent sheaves is closely related to the *fusion product* of $G(\mathbf{O})$-modules introduced by Feigin[3] [FL99] (see § 8). In fact, our desire to understand the category $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$, and the work of Feigin and Loktev [FL99], was one of the motivations for the present work.

2. Notation and statements of the results

2.1 Kostant slices. The group G is an almost simple algebraic group with the Lie algebra $\mathfrak{g}$. We choose a principal $\mathfrak{sl}_2$ triple (e,h,f) in $\mathfrak{g}$. Let $\phi:\ \mathfrak{sl}_2\to\mathfrak{g}$ (respectively $\Phi:\ SL_2\to G$) be the corresponding homomorphism. We denote by e_G (respectively f_G) the image $\Phi\big(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\big)$ (respectively $\Phi\big(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\big)$). We denote by $\mathfrak{z}(e)$ the centralizer of e in $\mathfrak{g}$, and by $Z(e)$ (respectively $Z^0(e)$) the centralizer of e (equivalently, of e_G) in G (respectively its neutral connected component). We denote by $\Sigma_\mathfrak{g}\subset\mathfrak{g}$ (respectively $\Sigma_G\subset G$) the *Kostant slice* $\mathfrak{z}(e)+f$ (respectively $Z^0(e)\cdot f_G$). It is known that $\Sigma_\mathfrak{g}\subset\mathfrak{g}^{\mathrm{reg}}$ (respectively $\Sigma_G\subset G^{\mathrm{reg}}$), and the projection to the categorical quotient $\Sigma_\mathfrak{g}\hookrightarrow\mathfrak{g}\twoheadrightarrow\mathfrak{g}//Ad_G=\mathfrak{t}/W$ induces an isomorphism $\Sigma_\mathfrak{g}\simeq\mathfrak{t}/W$. Similarly, if G is simply connected, the projection to the categorical quotient $\Sigma_G\hookrightarrow G\twoheadrightarrow G//Ad_G=T/W$ induces an isomorphism $\Sigma_G\simeq T/W$.

2.2 The universal centralizers. We consider the locally closed subvariety $C_{\mathfrak{g},\mathfrak{g}}\subset\mathfrak{g}\times\mathfrak{g}$ (respectively $C_{\mathfrak{g},G}\subset\mathfrak{g}\times G$, $C_{G,\mathfrak{g}}\subset G\times\mathfrak{g}$, $C_{G,G}\subset G\times G$) formed by all the pairs (x_1,x_2) such that $[x_1,x_2]=0$ and x_2 is regular (respectively all the pairs (x,g) such that $Ad_g(x)=x$ and g is regular; all the pairs (g,x) such that $Ad_g(x)=x$ and x is regular; all the pairs (g_1,g_2) such that $Ad_{g_1}(g_2)=g_2$ and g_2 is regular). The categorical quotients with respect to the diagonal adjoint action of G are denoted respectively $C_{\mathfrak{g},\mathfrak{g}}//G=\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}$, $C_{\mathfrak{g},G}//G=\mathfrak{Z}^\mathfrak{g}_G$, $C_{G,\mathfrak{g}}//G=\mathfrak{Z}^G_\mathfrak{g}$, and $C_{G,G}//G=\mathfrak{Z}^G_G$. The projections to the second (regular) factor are denoted by $\varpi:\ \mathfrak{Z}^\mathfrak{g}_\mathfrak{g}\to\mathfrak{g}^{\mathrm{reg}}//G=\mathfrak{t}/W$, $\varpi:\ \mathfrak{Z}^\mathfrak{g}_G\to G^{\mathrm{reg}}//G=T/W$, $\varpi:\ \mathfrak{Z}^G_\mathfrak{g}\to\mathfrak{g}^{\mathrm{reg}}//G=\mathfrak{t}/W$, and $\varpi:\ \mathfrak{Z}^G_G\to G^{\mathrm{reg}}//G=T/W$. In all the four cases ϖ is flat.

We consider the restrictions of our centralizer varieties to the Kostant slices: $C^\Sigma_{\mathfrak{g},\mathfrak{g}}=C_{\mathfrak{g},\mathfrak{g}}\cap(\mathfrak{g}\times\Sigma_\mathfrak{g})$, $C^\Sigma_{\mathfrak{g},G}=C_{\mathfrak{g},G}\cap(\mathfrak{g}\times\Sigma_G)$, $C^\Sigma_{G,\mathfrak{g}}=C_{G,\mathfrak{g}}\cap(G\times\Sigma_\mathfrak{g})$, and $C^\Sigma_{G,G}=C_{G,G}\cap(G\times\Sigma_G)$.

Then the locally closed embedding $C^\Sigma_{\mathfrak{g},\mathfrak{g}}\hookrightarrow C_{\mathfrak{g},\mathfrak{g}}\twoheadrightarrow\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}$ induces an isomorphism $C^\Sigma_{\mathfrak{g},\mathfrak{g}}\simeq\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}$. Similarly, we have isomorphisms $C^\Sigma_{G,\mathfrak{g}}\simeq\mathfrak{Z}^G_\mathfrak{g}$ and (for simply connected G) $C^\Sigma_{\mathfrak{g},G}\simeq\mathfrak{Z}^\mathfrak{g}_G$, and $C^\Sigma_{G,G}\simeq\mathfrak{Z}^G_G$.

Thus both $\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}\to\mathfrak{t}/W$ and $\mathfrak{Z}^\mathfrak{g}_G\to T/W$ (for simply connected G) are the sheaves of abelian Lie algebras, while both $\mathfrak{Z}^G_\mathfrak{g}\to\mathfrak{t}/W$ and $\mathfrak{Z}^G_G\to T/W$ (for simply connected G) are the sheaves of abelian Lie groups.

2.3 Isogenies. The center $Z(G)$ acts naturally on $\mathfrak{Z}^\mathfrak{g}_G$ (respectively $\mathfrak{Z}^G_\mathfrak{g}$) by $z(x,g)=(x,zg)$ (respectively $z(g,x)=(zg,x)$). The center $Z(G)$ acts on $\mathfrak{Z}^G_G$ on both sides: $z_1(g_1,g_2)z_2=(z_1g_1,z_2g_2)$. Let $\tilde{G}$ denote the universal cover of G. Then the fundamental group $\pi_1(G)$ is embedded into $Z(\tilde{G})$, and we have $\mathfrak{Z}^\mathfrak{g}_G=\pi_1(G)\backslash\mathfrak{Z}^\mathfrak{g}_{\tilde{G}}$, $\mathfrak{Z}^G_\mathfrak{g}=\pi_1(G)\backslash\mathfrak{Z}^{\tilde{G}}_\mathfrak{g}$, and $\mathfrak{Z}^G_G=\pi_1(G)\backslash\mathfrak{Z}^{\tilde{G}}_{\tilde{G}}/\pi_1(G)$.

[3] The relation between convolution and fusion was known to B. Feigin in 1997.

2.4 Symplectic structures. We fix an invariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$; hence $\mathfrak{t} \simeq \mathfrak{t}^*$. Then $\mathfrak{g} \times \mathfrak{g}$ gets identified with $\mathfrak{g} \times \mathfrak{g}^* = \mathsf{T}^*\mathfrak{g}$ (the cotangent bundle), and $G \times \mathfrak{g}$ gets identified with $G \times \mathfrak{g}^* = \mathsf{T}^*G$. After this, $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ (respectively $\mathfrak{Z}^{G}_{\mathfrak{g}}$) can be viewed as a Hamiltonian reduction of $\mathsf{T}^*\mathfrak{g}$ (respectively T^*G) with respect to the adjoint action of G; thus it inherits a canonical symplectic structure.

Identifying $\mathfrak{g} \times G$ with $\mathfrak{g}^* \times G = \mathsf{T}^*G$, we can view $\mathfrak{Z}^{\mathfrak{g}}_{G}$ as a Hamiltonian reduction of T^*G as well; thus it inherits a canonical Poisson structure. Note that $\mathfrak{Z}^{\mathfrak{g}}_{G}$ is smooth and symplectic if and only if G is simply connected. We have symplectic isomorphisms $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}} \simeq \mathsf{T}^*(\mathfrak{t}/W)$, and (when G is simply connected) $\mathfrak{Z}^{\mathfrak{g}}_{G} \simeq \mathsf{T}^*(T/W)$.

Note that $\mathfrak{Z}^{\mathfrak{g}}_{G}$ and $\mathfrak{Z}^{G}_{\mathfrak{g}}$ share a common open piece $\mathsf{Z}(G)$ formed by the classes of pairs (g, x) where both g and x are regular. The canonical symplectic structures agree on $\mathfrak{Z}^{\mathfrak{g}}_{G} \supset \mathsf{Z}(G) \subset \mathfrak{Z}^{G}_{\mathfrak{g}}$. Note also that for adjoint G the space $\mathsf{Z}(G)$ contains (a complexification of) the Kostant phase space $\mathfrak{Z}(G)$ of the classical Toda lattice [Kos79], and the embedding $\mathfrak{Z}(G) \hookrightarrow \mathfrak{Z}^{G}_{\mathfrak{g}}$ is given by Theorem 2.6 of [Kos79].

Alekseev, Malkin and Meinrenken introduced in [AMM98, Example 6.1] the q-Hamiltonian G-space *internal fusion double* $\mathbf{D}(G)$. Its q-Hamiltonian reduction is $\mathfrak{Z}^{G}_{G}$, so it inherits a canonical Poisson structure. For a simply connected G, the space $\mathfrak{Z}^{G}_{G}$ is smooth and symplectic.

2.5 Affine blow-ups. The set of roots of G (respectively $\check{G}$) is denoted by R (respectively $\check{R}$). We will view $\alpha \in R$ (respectively $\check{\alpha} \in \check{R}$) as a homomorphism $\mathfrak{t} \to \mathbb{C}$ (respectively $\check{\mathfrak{t}} \to \mathbb{C}$) or as a homomorphism $T \to \mathbb{C}^*$ (respectively $\check{T} \to \mathbb{C}^*$) depending on context. Also, for a root $\alpha \in R$ we denote by ${}^1\alpha$ (respectively ${}^2\alpha$) the linear function on $\mathfrak{t} \times \mathfrak{t}$ obtained as a composition of α with the projection on the first (respectively second) factor.

We consider the following affine blow-up of $\mathfrak{t} \times \mathfrak{t}$ at the diagonal walls: $\dot{\mathfrak{B}}^{\mathfrak{g}}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, {}^1\alpha/{}^2\alpha, \alpha \in R])$. We also set $\dot{\mathfrak{B}}^{G}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times T, {}^1\alpha/({}^2\alpha - 1), \alpha \in R])$; $\dot{\mathfrak{B}}^{G}_{G} = \operatorname{Spec}(\mathbb{C}[T \times T, ({}^1\alpha - 1)/({}^2\alpha - 1), \alpha \in R])$, $\dot{\mathfrak{B}}^{\check{G}}_{G} = \operatorname{Spec}(\mathbb{C}[\check{T} \times T, ({}^1\check{\alpha} - 1)/({}^2\alpha - 1), \alpha \in R])$; and let $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} = \dot{\mathfrak{B}}^{\mathfrak{g}}_{\mathfrak{g}}/W$, $\mathfrak{B}^{G}_{G} = \dot{\mathfrak{B}}^{G}_{G}/W$, $\mathfrak{B}^{\check{G}}_{G} = \dot{\mathfrak{B}}^{\check{G}}_{G}/W$ (thus $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, {}^1\alpha/{}^2\alpha, \alpha \in R]^W)$, etc.). We denote by ϖ the projection of $\mathfrak{B}$ to the second factor; thus we have $\varpi :\ \mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} \to \mathfrak{t}/W$, $\mathfrak{B}^{\mathfrak{g}}_{G} \to T/W$, $\mathfrak{B}^{G}_{\mathfrak{g}} \to \mathfrak{t}/W$, $\mathfrak{B}^{G}_{G} \to T/W$ and $\mathfrak{B}^{\check{G}}_{G} \to T/W$.

2.6 Poisson structures. We have the canonical trivializations of the tangent bundles $\mathsf{T}(\mathfrak{t} \times \mathfrak{t}) = (\mathfrak{t} \times \mathfrak{t}) \times (\mathfrak{t} \times \mathfrak{t})$, $\mathsf{T}(\mathfrak{t} \times T) = (\mathfrak{t} \times T) \times (\mathfrak{t} \times \mathfrak{t})$, $\mathsf{T}(T \times \mathfrak{t}) = (T \times \mathfrak{t}) \times (\mathfrak{t} \times \mathfrak{t})$, $\mathsf{T}(T \times T) = (T \times T) \times (\mathfrak{t} \times \mathfrak{t})$, and $\mathsf{T}(T \times \check{T}) = (T \times \check{T}) \times (\mathfrak{t} \times \check{\mathfrak{t}})$. Making use of the identification $\check{\mathfrak{t}} = \mathfrak{t}^* \simeq \mathfrak{t}$, we obtain the W-invariant symplectic structures on the above varieties. Thus the above affine blow-ups carry the rational Poisson structures (regular off the discriminants $\mathbf{D} \subset \mathfrak{B}$).

PROPOSITION 2.7. *The Poisson structure on $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} - \mathbf{D}$ (respectively $\mathfrak{B}^{\mathfrak{g}}_{G} - \mathbf{D}$, $\mathfrak{B}^{G}_{\mathfrak{g}} - \mathbf{D}$, $\mathfrak{B}^{G}_{G} - \mathbf{D}$, $\mathfrak{B}^{\check{G}}_{G} - \mathbf{D}$) extends to the global Poisson structure; it is a symplectic structure if the corresponding variety is smooth.*

PROPOSITION 2.8. *We are in the setup of § 2.5.*

(a) *The projection ϖ is flat if G is simply connected.*

(b) *There are natural identifications $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} \simeq \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$, $\mathfrak{B}^{\mathfrak{g}}_{G} \simeq \mathfrak{Z}^{\mathfrak{g}}_{G}$, $\mathfrak{B}^{G}_{\mathfrak{g}} \simeq \mathfrak{Z}^{G}_{\mathfrak{g}}$, and $\mathfrak{B}^{G}_{G} \simeq \mathfrak{Z}^{G}_{G}$ commuting with ϖ.*

(c) *If G is simply laced and adjoint, we have an identification $\mathfrak{B}^{G}_{\check{G}} \simeq Z(\check{G})\backslash\mathfrak{Z}^{\check{G}}_{\check{G}}$ commuting with ϖ.*

(d) *If G is simply laced and simply connected, we have an identification $\mathfrak{B}^{G}_{\check{G}} \simeq \mathfrak{Z}^{G}_{G}/Z(G)$ commuting with ϖ.*

(e) *The above identifications respect the Poisson structures.*

2.9 Flat group sheaves. We consider the functor $\mathfrak{F}_\mathfrak{g}^\mathfrak{g}$ on the category $\mathrm{Flat}_{\mathfrak{t}/W}$ of schemes flat over $\mathfrak{t}/W$ to the category of sets, sending a test scheme S to the set of W-invariant morphisms $(\mathrm{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, \mathfrak{t}))^W$. Similarly, we consider the functor $\mathfrak{F}_G^\mathfrak{g}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $(\mathrm{Mor}(S \times_{T/W} T, \mathfrak{t}))^W$. Also, we consider the functor $\mathfrak{F}_\mathfrak{g}^G$ on the category $\mathrm{Flat}_{\mathfrak{t}/W}$ sending a test scheme S to the set of W-invariant morphisms $(\mathrm{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, T))_0^W \subset (\mathrm{Mor}(S \times_{\mathfrak{t}/W} \mathfrak{t}, T))^W$ subject to the condition (cf. [DG02, 4.2])

$$\alpha(f(\alpha^{-1}(0))) = 1 \ \forall \alpha \in R. \tag{1}$$

(Note that the W-invariance condition automatically implies $\alpha(f(\alpha^{-1}(0))) = \pm 1 \ \forall \alpha \in R$.)

Furthermore, we consider the functor $\mathfrak{F}_G^G$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $(\mathrm{Mor}(S \times_{T/W} T, T)t)_0^W \subset (\mathrm{Mor}(S \times_{T/W} T, T))^W$ subject to the condition

$$\alpha(f(\alpha^{-1}(1))) = 1 \ \forall \alpha \in R. \tag{2}$$

(Note that the W-invariance condition automatically implies $\alpha(f(\alpha^{-1}(1))) = \pm 1 \ \forall \alpha \in R$.)

Finally, we consider the functor $\mathfrak{F}_G^{\check{G}}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $(\mathrm{Mor}(S \times_{T/W} T, \check{T}))_0^W \subset (\mathrm{Mor}(S \times_{T/W} T, \check{T}))^W$ subject to the condition

$$\check{\alpha}(f(\alpha^{-1}(1))) = 1 \ \forall \alpha \in R. \tag{3}$$

(Note that the W-invariance condition automatically implies $\check{\alpha}(f(\alpha^{-1}(1))) = \pm 1 \ \forall \alpha \in R$.)

The following proposition is a generalization of [DG02, 11.6].

Proposition 2.10. *Assume that G is simply connected. The functor $\mathfrak{F}_\mathfrak{g}^\mathfrak{g}$ (respectively $\mathfrak{F}_G^\mathfrak{g}$, $\mathfrak{F}_\mathfrak{g}^G$, $\mathfrak{F}_G^G$, $\mathfrak{F}_G^{\check{G}}$) is representable by the scheme $\mathfrak{B}_\mathfrak{g}^\mathfrak{g}$ (respectively $\mathfrak{B}_G^\mathfrak{g}$, $\mathfrak{B}_\mathfrak{g}^G$, $\mathfrak{B}_G^G$, $\mathfrak{B}_G^{\check{G}}$).*

2.11 Equivariant Borel–Moore homology. For the definition of convolution in equivariant Borel–Moore homology, we refer the reader to [CG97, 2.7, 8.3] or [Lus95, Chapter 2].

We have $H_\bullet^{G(\mathbf{O})}(pt) = H^\bullet_{G(\mathbf{O})}(pt) = \mathbb{C}[\mathfrak{t}/W]$, and $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = H^\bullet_{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = \mathbb{C}[\mathfrak{t}/W][\hbar]$, where $\hbar$ is the generator of $H^2_{\mathbb{G}_m}(pt)$. We will consider the $\mathbb{C}[\mathfrak{t}/W]$-algebra (respectively $\mathbb{C}[\mathfrak{t}/W][\hbar]$-algebra) (with respect to convolution) $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (respectively $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$). Note that setting $\hbar = 0$ in $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we obtain $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$; indeed for any group H, a space X with an $H \times \mathbb{G}_m$ action, and an $H \times \mathbb{G}_m$-equivariant complex $\mathcal{F}$ on X, we have a long exact sequence $\cdots \to H^{i-2}_{H\times\mathbb{G}_m}(X, \mathcal{F}) \xrightarrow{\hbar} H^i_{H\times\mathbb{G}_m}(X, \mathcal{F}) \to H^i_H(X, \mathcal{F}) \to H^{i-1}_{H\times\mathbb{G}_m}(X, \mathcal{F}) \to \dots$ coming from the principal $\mathbb{G}_m$-bundle $E(H \times \mathbb{G}_m) \times_H X \to E(H \times \mathbb{G}_m) \times_{H\times\mathbb{G}_m} X$; if the space of $H \times \mathbb{G}_m$-equivariant cohomology is $\hbar$-torsion free, then we get $H^\bullet_H(X, \mathcal{F}) = H^\bullet(X, \mathcal{F})|_{\hbar=0}$.

Theorem 2.12.

(a) *The algebra $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative.*

(b) *Its spectrum together with the projection onto $\mathfrak{t}/W = \check{\mathfrak{t}}/W$ is naturally isomorphic to $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}} \xrightarrow{\varpi} \check{\mathfrak{t}}/W$.*

(c) *The Poisson structure on $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ arising from the $\hbar$-deformation $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ corresponds under the above identification to the Poisson structure of § 2.4 on $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$.*

Remark 2.13. The equivariant cohomology ring $H^\bullet_{G(\mathbf{O})}(\mathrm{Gr}_G, \mathbb{C}) = H^\bullet_{G(\mathbf{O})}(\mathrm{Gr}_G)$ was computed by Ginzburg [Gin95]. More precisely, the projection to the second (regular) factor $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{\mathfrak{g}}} \to \check{\mathfrak{g}}^{\mathrm{reg}}//\check{G} = \check{\mathfrak{t}}/W$ makes $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{\mathfrak{g}}}$ a sheaf of abelian Lie algebras. Ginzburg identifies $H^\bullet_{G(\mathbf{O})}(\mathrm{Gr}_G)$ with the global sections of the relative universal enveloping algebra $U_{\check{\mathfrak{t}}/W}(\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{\mathfrak{g}}})$. One can easily check that this result is compatible with our Theorem 2.12, part (b) as follows. For a group scheme A over a base S one has a natural pairing $U(\mathfrak{a}) \times \mathcal{O}(A) \to \mathcal{O}(S)$ where $U(\mathfrak{a})$ is the enveloping (over $\mathcal{O}(S)$) of the Lie algebra of A; the pairing sends (ξ, f) to $\xi(f)$ restricted to the identity of A. On the other hand, for a compact (or ind-compact) H-space X we have a pairing $H^\bullet_H(X) \times H^H_\bullet(X) \to H^\bullet_H(pt)$ induced by the action of cohomology on homology, and the push-forward map in Borel–Moore homology $H^H_\bullet(X) \to H^\bullet_H(pt)$. The isomorphisms of [Gin95] and of Theorem 2.12 take the first pairing into the second one.

2.14 Equivariant K-theory. For the definition of convolution in equivariant K-theory we refer the reader to [CG97, ch. 5].

We have $K^{G(\mathbf{O})}(pt) = \mathbb{C}[T/W]$ and $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = \mathbb{C}[T/W][q^{\pm 1}]$. We will consider the $\mathbb{C}[T/W]$-algebra (respectively $\mathbb{C}[T/W][q^{\pm 1}]$-algebra) (with respect to convolution) $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (respectively $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$). Note that setting $q = 1$ in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we obtain $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$.

THEOREM 2.15.

(a) *The algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative.*

(b) *Its spectrum together with the projection onto T/W is naturally isomorphic to $\mathfrak{B}^{\check{G}}_G \xrightarrow{\varpi} T/W$.*

(c) *The Poisson structure on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ arising from the q-deformation $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ corresponds under the above identification to the Poisson structure of Proposition 2.7 on $\mathfrak{B}^{\check{G}}_G$ when the latter variety is smooth, i.e. G is simply connected.*

3. Calculations in rank 1

In this section $G = SL_2$, and $\check{G} = PGL_2$. The Weyl group $W = \mathbb{Z}/2\mathbb{Z}$, the Cartan torus $T = \mathbb{G}_m = \mathbb{C}^*$ with a coordinate z, and the only simple root $\alpha(z) = z^2$. The dual torus $\check{T} = \mathbb{G}_m = \mathbb{C}^*$ with a coordinate t, and $\check{\alpha}(t) = t$. The Cartan Lie algebra $\mathfrak{t} = \mathbb{C}$ with a coordinate $x = \alpha(x)$. We fix a $\sqrt{-1}$.

3.1 Calculating $\mathfrak{Z}^G_G$ and $\mathfrak{B}^G_G$. We choose the standard $\mathfrak{sl}_2$-triple

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the Kostant slice

$$\Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}, \ a \in \mathbb{C} \right\}.$$

One checks that a matrix $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} a-1 & a-2 \\ 1 & 1 \end{smallmatrix}\right)$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \sqrt{-1} \begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix} \quad \text{for } b, c \in \mathbb{C}.$$

Then the condition $\det = 1$ reads as

$$1 = abc - b^2 - c^2. \tag{4}$$

Thus, $\mathfrak{Z}^G_G$ is identified with a hypersurface $\mathcal{S}$ in $\mathbb{A}^3$ given by (4). The left (respectively right) multiplication by $-1 \in Z(SL_2)$ is an involution $\imath$ (respectively $\jmath$) on $\mathcal{S}$ given by $\imath(a, b, c) = (a, -b, -c)$ (respectively $\jmath(a, b, c) = (-a, b, -c)$). Hence, $\mathfrak{Z}^{\check{G}}_{\check{G}} = \imath \backslash \mathcal{S} / \jmath$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g\sqrt{-1} \begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$$

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and

$$g\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

for some $y, z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to simultaneous inversion. Then we have

$$a = z + z^{-1}, \quad b = \frac{-\sqrt{-1}}{2}\left(y + y^{-1} + \frac{(y-y^{-1})(z+z^{-1})}{z - z^{-1}}\right), \quad c = -\sqrt{-1}\frac{y - y^{-1}}{z - z^{-1}}. \tag{5}$$

We conclude that $\mathbb{C}[\mathcal{S}] = \mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y-y^{-1})/(z-z^{-1})]^W$ where the nontrivial element $w \in W$ acts by $w(y,z) = (y^{-1}, z^{-1})$. We can rewrite $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y-y^{-1})/(z-z^{-1})]^W$ as $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y^2-1)/(z^2-1)]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}^G_G]$. All in all, we have $\mathfrak{B}^G_G \simeq \mathcal{S} \simeq \mathfrak{Z}^G_G$. Since we can identify $\check{T}$ with $T/Z(G)$, the identifications $\mathfrak{B}^G_{\check{G}} \simeq \mathcal{S}/\jmath$, $\mathfrak{B}^{\check{G}}_G \simeq \imath\backslash\mathcal{S}$, $\mathfrak{B}^{\check{G}}_{\check{G}} \simeq \imath\backslash\mathcal{S}/\jmath \simeq \mathfrak{Z}^{\check{G}}_{\check{G}}$ follow immediately.

3.2 Calculating $\mathfrak{Z}^G_{\mathfrak{g}}$ and $\mathfrak{B}^G_{\mathfrak{g}}$. The Kostant slice

$$\Sigma_{\mathfrak{g}} = \left\{\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix},\ \delta \in \mathbb{C}\right\}.$$

One checks that a matrix $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} 0 & \delta \\ 1 & 0 \end{smallmatrix}\right)$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix} \quad \text{for } \xi, \eta \in \mathbb{C}.$$

Then the condition det $= 1$ reads as

$$1 = \xi^2 - \delta\eta^2. \tag{6}$$

Thus, $\mathfrak{Z}^G_{\mathfrak{g}}$ is identified with a hypersurface $\mathcal{S}'$ in $\mathbb{A}^3$ given by (6). The action of $-1 \in Z(SL_2)$ is an involution $\imath$ on $\mathcal{S}'$ given by $\imath(\delta, \xi, \eta) = (\delta, -\xi, -\eta)$. Hence, $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}} = \imath\backslash\mathcal{S}'$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g\begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$$

and

$$g\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$$

for some $y \in \mathbb{G}_m = \mathbb{C}^* = T,\ x \in \mathbb{C} = \mathfrak{t}$, defined up to $(y, x) \mapsto (y^{-1}, -x)$. Then we have

$$\delta = x^2, \quad \xi = \frac{y + y^{-1}}{2}, \quad \eta = \frac{y - y^{-1}}{2x}.$$

We conclude that $\mathbb{C}[\mathcal{S}'] = \mathbb{C}[y^{\pm 1}, x, (y - y^{-1})/x]^W$ where the nontrivial element $w \in W$ acts by $w(y, x) = (y^{-1}, -x)$. We can rewrite $\mathbb{C}[y^{\pm 1}, x, (y-y^{-1})/x]^W$ as $\mathbb{C}[y^{\pm 1}, x, (y^2-1)/x]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}^G_{\mathfrak{g}}]$. All in all, we have $\mathfrak{B}^G_{\mathfrak{g}} \simeq \mathcal{S}' \simeq \mathfrak{Z}^G_{\mathfrak{g}}$. Since we can identify $\check{T}$ with $T/Z(G)$, the identification $\mathfrak{B}^{\check{G}}_{\check{\mathfrak{g}}} \simeq \imath\backslash\mathcal{S}' \simeq \mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ follows immediately.

3.3 Calculating $\mathfrak{Z}^{\mathfrak{g}}_G$ and $\mathfrak{B}^{\mathfrak{g}}_G$. Recall the Kostant slice

$$\Sigma_G = \left\{\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix},\ a \in \mathbb{C}\right\}.$$

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One checks that a traceless matrix $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11}\end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} a-1 & a-2 \\ 1 & 1\end{smallmatrix}\right)$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11}\end{pmatrix} = \zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2\end{pmatrix} \quad \text{for } \zeta \in \mathbb{C}.$$

Thus, $\mathfrak{Z}_G^{\mathfrak{g}}$ is identified with $\mathbb{A}^2$ with coordinates a, ζ. The action of $-1 \in Z(SL_2)$ is an involution $\jmath$ on $\mathbb{A}^2$ given by $\jmath(a,\zeta) = (-a,-\zeta)$. Hence, $\mathfrak{Z}_{\check{G}}^{\check{\mathfrak{g}}} = \mathbb{A}^2/\jmath$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g\zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2\end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x\end{pmatrix}$$

and

$$g \begin{pmatrix} a-1 & a-2 \\ 1 & 1\end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}$$

for some $x \in \mathbb{C} = \mathfrak{t},\ z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to $(x,z) \mapsto (-x, z^{-1})$. Then we have

$$a = z + z^{-1}, \quad \zeta = \frac{x}{z - z^{-1}}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[x, z^{\pm 1}, x/(z-z^{-1})]^W$ where the nontrivial element $w \in W$ acts by $w(x,z) = (-x, z^{-1})$. We can rewrite $\mathbb{C}[x, z^{\pm 1}, x/(z-z^{-1})]^W$ as $\mathbb{C}[x, z^{\pm 1}, x/(z^2-1)]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_G^{\mathfrak{g}}]$. All in all, we have $\mathfrak{B}_G^{\mathfrak{g}} \simeq \mathbb{A}^2 \simeq \mathfrak{Z}_G^{\mathfrak{g}}$. Since we can identify $\check{T}$ with $T/Z(G)$, the identification $\mathfrak{B}_{\check{G}}^{\check{\mathfrak{g}}} \simeq \mathbb{A}^2/\jmath \simeq \mathfrak{Z}_{\check{G}}^{\check{\mathfrak{g}}}$ follows immediately.

3.4 Calculating $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ and $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}}$. Recall the Kostant slice

$$\Sigma_{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0\end{pmatrix},\ \delta \in \mathbb{C} \right\}.$$

One checks that a traceless matrix $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11}\end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} 0 & \delta \\ 1 & 0\end{smallmatrix}\right)$ if and only if

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11}\end{pmatrix} = \begin{pmatrix} 0 & \delta\theta \\ \theta & 0\end{pmatrix} \quad \text{for } \theta \in \mathbb{C}.$$

Thus, $\mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$ is identified with $\mathbb{A}^2$ with coordinates δ, θ.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that

$$g \begin{pmatrix} 0 & \delta\theta \\ \theta & 0\end{pmatrix} g^{-1} = \begin{pmatrix} u & 0 \\ 0 & -u\end{pmatrix}$$

and

$$g \begin{pmatrix} 0 & \delta \\ 1 & 0\end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x\end{pmatrix}$$

for some $u, x \in \mathbb{C} = \mathfrak{t}$, defined up to $(u,x) \mapsto (-u,-x)$. Then we have

$$\delta = x^2, \quad \theta = \frac{u}{x}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[u, x, u/x]^W$ where the nontrivial element $w \in W$ acts by $w(u,x) = (-u,-x)$. Hence we get an identification $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}} \simeq \mathbb{A}^2 \simeq \mathfrak{Z}_{\mathfrak{g}}^{\mathfrak{g}}$.

3.5 Calculating $\mathfrak{B}_{\mathfrak{g}}^{G}$ and $\mathfrak{F}_{\mathfrak{g}}^{G}$. Recall the setup of Proposition 2.10. We will prove that the functor $\mathfrak{F}_{\mathfrak{g}}^{G}$ is representable by the scheme $\mathfrak{B}_{\mathfrak{g}}^{G}$; the other parts of Proposition 2.10 are proved absolutely similarly, as well as the Proposition 2.10 for G replaced by $\check{G}$. For a scheme S flat over $\mathfrak{t}/W$

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we will denote by $S_{\mathfrak{t}}$ the cartesian product $S \times_{\mathfrak{t}/W} \mathfrak{t}$. Our usual coordinate x on $\mathfrak{t}$ gives rise to the same named function on $S_{\mathfrak{t}}$. The nontrivial element $w \in W$ acts by the involution of $S_{\mathfrak{t}}$. Finally, we denote by $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ the affine blow-up of $S_{\mathfrak{t}} \times T$, that is $S_{\mathfrak{t}} \times_{\mathfrak{t}} \mathfrak{B}^G_{\mathfrak{g}}$. Clearly, w acts as an involution of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$.

Note that the condition (1) is void in the case under consideration. Given a w-equivariant morphism $f: S_{\mathfrak{t}} \to T = \mathbb{G}_m$ we see that $f^2 - 1$ is divisible by x; hence f lifts uniquely to a section $\hat{f}$ of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, $\hat{f}$ is w-invariant. If we consider $\hat{f}$ as a closed subscheme of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$, then $\hat{f}/W$ is a closed subscheme of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}/W = S \times_{\mathfrak{t}/W} \mathfrak{B}^G_{\mathfrak{g}}$ which is the graph of a morphism $\tilde{f}: S \to \mathfrak{B}^G_{\mathfrak{g}}$.

Conversely, given a morphism $\tilde{f}: S \to \mathfrak{B}^G_{\mathfrak{g}}$ we consider its graph $\Gamma_{\tilde{f}}$ as a closed subscheme of $S \times_{\mathfrak{t}/W} \mathfrak{B}^G_{\mathfrak{g}}$, and then the cartesian product $\Gamma_{\tilde{f}} \times_{S \times_{\mathfrak{t}/W} \mathfrak{B}^G_{\mathfrak{g}}} (\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ is a section $\hat{f}$ of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, $\hat{f}$ gives rise to a w-equivariant function $f: S_{\mathfrak{t}} \to T$.

3.6 A basis in equivariant K-theory. We recall a few standard facts about the affine Grassmannians Gr_G and $\mathrm{Gr}_{\check{G}}$. The $G(\mathbf{O})$-orbits (equivalently, $\check{G}(\mathbf{O})$-orbits) on $\mathrm{Gr}_{\check{G}}$ are numbered by nonnegative integers and denoted by $\mathrm{Gr}_{\check{G},n}$, $n \in \mathbb{N}$. The orbits $\mathrm{Gr}_{\check{G},2n}$, $n \in \mathbb{N}$, form a connected component of $\mathrm{Gr}_{\check{G}}$ equal to Gr_G. The open embedding of an orbit into its closure will be denoted by $j_n: \mathrm{Gr}_{\check{G},n} \hookrightarrow \overline{\mathrm{Gr}}_{\check{G},n}$ or simply by j if no confusion is likely. We have $\dim \mathrm{Gr}_{\check{G},n} = n$; in particular, $\mathrm{Gr}_{\check{G},0}$ is a point.

We have $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(G)$ with a basis $\mathbf{v}(n)$, $n \in \mathbb{N}$, formed by the classes of irreducible G-modules $\mathcal{V}(n)$. Also, $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(\check{G}) \subset \mathrm{Rep}(G)$ has a basis $\mathbf{v}(2n)$, $n \in \mathbb{N}$.

For $m > 0$ the $G(\mathbf{O})$-equivariant line bundles in $\mathrm{Gr}_{\check{G},m}$ are numbered by integers and denoted by $\mathcal{L}(n)_m$. Among them, the $\check{G}(\mathbf{O})$-equivariant line bundles are exactly $\mathcal{L}(2n)_m$, $n \in \mathbb{Z}$. We define $\mathcal{V}(n)_m$ as $j_*\mathcal{L}(n)_m[m/2]$, that is, the (nonderived) direct image to the orbit closure placed in the homological degree $-m/2$. Note that, since the complement $\overline{\mathrm{Gr}}_{\check{G},m} - \mathrm{Gr}_{\check{G},m}$ has codimension 2, the above direct image is a coherent sheaf. The degree shift will become clear later. The class $[\mathcal{L}(n)_m]$ in $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ will be denoted by $\mathbf{v}(n)_m$. Thus, it is natural to denote $\mathbf{v}(n)$ above by $\mathbf{v}(n)_0$; we will keep both names.

The collection $\{\mathbf{v}(n)_m : n \in \mathbb{N} \text{ if } m = 0;\ n \in \mathbb{Z} \text{ if } m \in \mathbb{N} - 0\}$ forms a basis in $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$. Among this collection, all the $\mathbf{v}(n)_m$ with n even (respectively m even) form a basis in $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ (respectively $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$).

3.7 Convolution: commutativity. In this subsection G is an arbitrary semisimple group. We prove Theorem 2.15, part (a). We refer the reader to [Gai01] for the basics of the Beilinson–Drinfeld Grassmannian. Recall that $\mathrm{Gr}^{\mathrm{BD}}_G \xrightarrow{\pi} \mathbb{A}^1$ is a flat ind-scheme such that $\pi^{-1}(\mathbb{A}^1 - 0) = (\mathbb{A}^1 - 0) \times \mathrm{Gr}_G \times \mathrm{Gr}_G$, while $\pi^{-1}(0) = \mathrm{Gr}_G$. We also have the deformed convolution diagram $\mathrm{Gr}^{\mathrm{BD,conv}}_G \xrightarrow{\Pi} \mathrm{Gr}^{\mathrm{BD}}_G$ such that Π is an isomorphism over $\mathbb{A}^1 - 0$, while over $0 \in \mathbb{A}^1$ our Π is the usual convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \xrightarrow{\Pi_0} \mathrm{Gr}_G$.

Given two $G(\mathbf{O})$-equivariant complexes of coherent sheaves $\mathcal{A}, \mathcal{B}$ on Gr_G, we can form their 'deformed convolution' complex $\mathcal{A} \,\tilde{\star}\, \mathcal{B}$ on $\mathrm{Gr}^{\mathrm{BD,conv}}_G$ such that over $\mathbb{A}^1 - 0$ it is isomorphic to $\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}$, while over $0 \in \mathbb{A}^1$ it is isomorphic to the usual twisted product $\mathcal{A} \ltimes \mathcal{B}$ on the convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G$. In addition, if $\mathcal{A}, \mathcal{B}$ are coherent sheaves, then $\mathcal{A} \,\tilde{\star}\, \mathcal{B}$ is flat over $\mathbb{A}^1$. It implies that in the K-group the class $[\mathcal{A} \ltimes \mathcal{B}]$ is the *specialization* (see [CG97, 5.3]) of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\mathrm{Gr}^{\mathrm{BD,conv}}_G \xrightarrow{\pi \circ \Pi} \mathbb{A}^1$, and also the class $[\mathcal{A} \star \mathcal{B}] = [\Pi_{0*}(\mathcal{A} \ltimes \mathcal{B})]$ is the specialization of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\mathrm{Gr}^{\mathrm{BD}}_G \xrightarrow{\pi} \mathbb{A}^1$. Hence we have the desired commutativity.

3.8 Convolution: relations. We return to the setup of § 3.6. Note that $\mathrm{Gr}_{\check{G},1} \simeq \mathbb{P}^1$, and $\mathcal{V}(n)_1$ is the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. The twisted product $\mathcal{V}(n)_1 \ltimes \mathcal{V}(l)_1$ is the line bundle $\mathcal{O}(n,l)$ on the two-dimensional subvariety $\mathcal{H}_2 \subset \check{G}(\mathbf{F}) \times_{\check{G}(\mathbf{O})} \mathrm{Gr}_{\check{G}}$ isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$ over $\mathbb{P}^1$. The projection $\Pi_0 : \ \mathcal{H}_2 \to \mathrm{Gr}_{\check{G},2}$ is the contraction of the -2-section $\mathbb{P}^1 \hookrightarrow \mathcal{H}_2$.

Now it is easy to compute $\mathbf{v}(n)_1 \star \mathbf{v}(n)_1 = \mathbf{v}(2n)_2, \ \mathbf{v}(1)_1 \star \mathbf{v}(-1)_1 = \mathbf{v}(0)_2 + 1$. Taking into account the evident relation $\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 = \mathbf{v}(1)_1 + \mathbf{v}(-1)_1$ we arrive at

$$\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 \star \mathbf{v}(1)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1 + \mathbf{v}(0)_1 \star \mathbf{v}(0)_1 + 1. \tag{7}$$

A moment of reflection shows that $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is generated as algebra by $\mathbf{v}(1)_0, \ \mathbf{v}(0)_2 = \mathbf{v}(0)_1 \star \mathbf{v}(0)_1, \ \mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1, \ \mathbf{v}(1)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(0)_1$ (one has to use that $\mathbf{v}(k)_{2l} \star \mathbf{v}(n)_{2m} = \mathbf{v}(k+n)_{2l+2m}$ plus the terms supported on the smaller orbits). Similarly, $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ is generated as algebra by $\mathbf{v}(2)_0 = \mathbf{v}(1)_0 \star \mathbf{v}(1)_0 - 1, \ \mathbf{v}(0)_1, \ \mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1, \ \mathbf{v}(2)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_0 - \mathbf{v}(0)_1$.

Note that both algebras $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ lie in the vector space $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$, and their intersection is the common subalgebra $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)$. The tensor product algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G) \otimes_{K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)} K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ can be identified as a vector space with $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$, and then it is generated by the three basic elements $\mathbf{v}(1)_0, \mathbf{v}(0)_1$ and $\mathbf{v}(1)_1$ subject to the only relation (7).

The comparison of (7) and (4) shows that the assignments $a \mapsto \mathbf{v}(1)_0, \ b \mapsto \mathbf{v}(0)_1, \ c \mapsto \mathbf{v}(1)_1$ establish an isomorphism $\mathbb{C}[\mathcal{S}] \simeq K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$. It identifies the spectrum of $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ with $\imath \backslash \mathcal{S} \simeq \mathfrak{B}_G^{\check{G}}$, and the spectrum of $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ with $\mathcal{S}/\jmath \simeq \mathfrak{B}_{\check{G}}^G$.

3.9 Iwahori-equivariant *K*-theory. Let $I \subset G(\mathbf{O})$ be the Iwahori subgroup. The space $K^I(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G) = K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K(T(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\mathrm{Gr}_G) = K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K(G(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\mathrm{Gr}_G)$ commuting with the right action of $K^G(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T} \times T]$. The action of W on $K^T(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T} \times T]$.

Our aim in this subsection is to identify the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O})) \ltimes W - K^G(\mathrm{Gr}_G)$-bimodule $K^T(\mathrm{Gr}_G)$ with the $\mathbb{C}[\check{T} \times T] \ltimes W - \mathbb{C}[\mathfrak{B}_G^{\check{G}}]$-bimodule $\mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]$ (and similarly for G replaced by $\check{G}$). As in § 3.8, it suffices to identify the $K(T(\mathbf{O})\backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \ltimes W - K^G(\mathrm{Gr}_{\check{G}})$-bimodule $K^T(\mathrm{Gr}_{\check{G}})$ with the $\mathbb{C}[T \times T] \ltimes W - \mathbb{C}[\mathfrak{B}_G^G]$-bimodule $\mathbb{C}[\dot{\mathfrak{B}}_G^G]$.

Note that $K^G(\mathrm{Gr}_{\check{G}}) \subset K^T(\mathrm{Gr}_{\check{G}})$, and the $K^G(\mathrm{Gr}_{\check{G}})$-module $K^T(\mathrm{Gr}_{\check{G}})$ is free of rank 2 with the generators $1, z$, where z is the generator of $K^T(pt) = \mathbb{C}[T]$ (so that, for example, $\mathbf{v}(1)_0 = z + z^{-1}$). Furthermore, $\mathbb{C}[y^{\pm 1}, z^{\pm 1}] = \mathbb{C}[T \times T] = K(T(\mathbf{O})\backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \subset K^T(\mathrm{Gr}_{\check{G}})$, and one can check that

$$y + y^{-1} = \sqrt{-1}(2\mathbf{v}(0)_1 - \mathbf{v}(1)_0 \star \mathbf{v}(1)_1), \quad y - y^{-1} = \sqrt{-1}(z - z^{-1})\mathbf{v}(1)_1. \tag{8}$$

Comparing (8) with (5) we get the desired identification of the $K(T(\mathbf{O})\backslash \check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \ltimes W - K^G(\mathrm{Gr}_{\check{G}})$-bimodule $K^T(\mathrm{Gr}_{\check{G}})$ with the $\mathbb{C}[y^{\pm 1}, z^{\pm 1}] \ltimes W - \mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y - y^{-1})/(z - z^{-1})]^W$-bimodule $\mathbb{C}[y^{\pm 1}, z^{\pm 1}, (y - y^{-1})/(z - z^{-1})]$.

3.10 Borel–Moore homology. For an arbitrary semisimple G one proves the commutativity of $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (Theorem 2.12, part (a)) exactly as in § 3.7 using the Beilinson–Drinfeld Grassmannian and the *specialization* in Borel–Moore homology (see [CG97, 2.6.30]).

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For $\check{G} = PGL_2$, let us denote by $\delta \in H^4_{\check{G}(\mathbf{O})}(pt,\mathbb{Z}) = H_4^{\check{G}(\mathbf{O})}(pt,\mathbb{Z})$ the generator of the equivariant (co)homology. Furthermore, we denote by η (respectively ξ) the generator of $H_{-2}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1},\mathbb{Z})$ (respectively the generator of $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1},\mathbb{Z})$). Then it is easy to see that δ,ξ,η generate $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ (while $\delta,\xi^2,\eta^2,\xi\eta$ generate the subalgebra $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$), and we claim that

$$1 = \xi^2 - \delta\eta^2. \tag{9}$$

In effect, this is an equality in $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2})$. Since $\mathrm{Gr}_{\check{G},2}$ is rationally smooth, $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2}) = H^4_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2})$. Let us denote by $\mathbf{B}\mathrm{Gr}_{\check{G},2} \xrightarrow{p} \mathbf{B}\check{G}(\mathbf{O})$ the associated fiber bundle over the classifying space of $\check{G}(\mathbf{O})$ with the fiber $\mathrm{Gr}_{\check{G},2}$. Then $1 \in H^4_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2}) = H^4(\mathbf{B}\mathrm{Gr}_{\check{G},2})$ is the Poincaré dual class of the codimension-2 cycle $\mathbf{B}\check{G}(\mathbf{O}) = \mathbf{B}\mathrm{Gr}_{\check{G},0} \hookrightarrow \mathbf{B}\mathrm{Gr}_{\check{G},2}$, and $\delta\eta^2 = p^*\delta$.

Recall the convolution morphism $\Pi_0 : \mathcal{H}_2 \to \mathrm{Gr}_{\check{G},2}$ of § 3.8. This is a morphism of $\check{G}(\mathbf{O})$-varieties, and we denote by $\Pi_0 : \mathbf{B}\mathcal{H}_2 \to \mathbf{B}\mathrm{Gr}_{\check{G},2}$ the corresponding morphism of associated fiber bundles. Note that (additively) $H^\bullet(\mathbf{B}\mathcal{H}_2) = H^\bullet(\mathbf{B}\mathrm{Gr}_{\check{G},1}) \otimes_{H^\bullet(\mathbf{B}\check{G}(\mathbf{O}))} H^\bullet(\mathbf{B}\mathrm{Gr}_{\check{G},1})$. Recall that ξ is the generator of $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}) = H^2_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}) = H^2(\mathbf{B}\mathrm{Gr}_{\check{G},1})$. Finally, we have $\xi^2 = \Pi_{0*}(\xi\otimes\xi)$. Now (9) follows easily.

Comparing the sizes of $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ and $\mathbb{C}[\delta,\xi,\eta]/(\xi^2-\delta\eta^2-1)$ we conclude that they are isomorphic. The comparison with (6) establishes an isomorphism $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}}) \simeq \mathbb{C}[\mathcal{S}']$, and identifies the spectrum of $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ with $\mathcal{S}' \simeq \mathfrak{Z}^G_{\mathfrak{g}}$, and the spectrum of $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ with $\imath\backslash\mathcal{S}' \simeq \mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$.

4. Centralizers and blow-ups

The aim of this section is a proof of Proposition 2.8. We will consider $\mathfrak{B}^G_G$ and $\mathfrak{Z}^G_G$, the other cases being similar. Till further notice, G is assumed simply connected.

Lemma 4.1. *The projection $\varpi : \mathfrak{B}^G_G \to T/W$ is flat.*

Proof. It suffices to prove that the first projection of $\dot{\mathfrak{B}}^G_G$ to T is smooth (recall that $\dot{\mathfrak{B}}^G_G$ is defined as $\mathrm{Spec}(\mathbb{C}[T\times T,\ ({}^1\alpha-1)/({}^2\alpha-1),\ \alpha\in R]))$. In effect, then $\mathbb{C}[T\times T,\ ({}^1\alpha-1)/({}^2\alpha-1),\ \alpha\in R]$ is a flat $\mathbb{C}[T]$-module; hence it is a flat $\mathbb{C}[T]^W$-module (since $\mathbb{C}[T]$ is free over $\mathbb{C}[T]^W$; see [Ste75]). Finally, $\mathbb{C}[T\times T,\ ({}^1\alpha-1)/({}^2\alpha-1),\ \alpha\in R]^W$ is a direct summand of a flat $\mathbb{C}[T]^W$-module $\mathbb{C}[T\times T,\ ({}^1\alpha-1)/({}^2\alpha-1),\ \alpha\in R]$; hence it is flat.

The affine blow-up $\dot{\mathfrak{B}}^G_G$ is the result of the following successive blow-up of $T\times T$. We choose an ordering $\alpha_1,\dots,\alpha_\nu$ of the set of positive roots R^+. We define $\mathfrak{B}_1$ as the blow-up of $T\times T$ at the diagonal wall ${}^1\alpha_1 = {}^2\alpha_1 = 1$ with the proper preimage of the divisor ${}^1\alpha_1 = 1$ removed. We define $\mathfrak{B}_2$ as the blow-up of $\mathfrak{B}_1$ at the proper transform of the diagonal wall ${}^1\alpha_2 = {}^2\alpha_2 = 1$ with the proper preimage of the divisor ${}^1\alpha_2 = 1$ removed. Going on like this we construct $\mathfrak{B}_\nu$; evidently, it coincides with $\dot{\mathfrak{B}}^G_G$.

Note that at each step the center of the blow-up is smooth over the corresponding wall ${}^2\alpha_i = 1$ in T by the following claim. Thus the desired flatness assertion follows inductively from the

Claim. Let $p : X \to Y$ be a smooth morphism of smooth varieties; let $X' \subset X$ be a subvariety such that $Y' = f(X') \subset Y$ is a smooth hypersurface, and $p : X' \to Y'$ is also smooth. Then the blow-up $\mathrm{Bl}_{X'}X$ with the proper preimage of the divisor $p^{-1}(p(X'))$ removed is smooth over Y.

The smoothness is checked in the formal neighborhoods of points by direct calculation in coordinates. This completes the proof of the lemma. ☐

4.2 The simultaneous resolution. Recall that $\{(g,B):\ g\in B\}=\dot{G}\xrightarrow{p}G$ is the Grothendieck simultaneous resolution; here B is a Borel subgroup, and $p(g,B)=g$. We also have the projection $\varrho:\ \dot{G}\to T$ to the abstract Cartan, which we identify with T; namely, $\varrho(g,B)=g\pmod{\mathrm{rad}(B)}$. The preimage $p^{-1}(\Sigma_G)\subset\dot{G}$ is identified with T by ϱ. We denote by $\dot{\mathfrak{Z}}_G^G\subset G\times\dot{G}$ the subset of triples (g_1,g_2,B) such that $Ad_{g_1}(g_2)=g_2$ and $(g_2,B)\in p^{-1}(\Sigma_G)$. Note that necessarily $g_1\in B$ (as well as $g_2\in B$); hence we have the projections $\varrho_1,\varrho_2:\ \dot{\mathfrak{Z}}_G^G\to T$; namely, $\varrho_i(g_1,g_2,B)=g_i\pmod{\mathrm{rad}(B)}$. Equivalently, $\dot{\mathfrak{Z}}_G^G$ can be defined as the categorical quotient by the diagonal G-action of the variety of triples $\dot{C}_{G,G}=\{(g_1,g_2,B):\ Ad_{g_1}(g_2)=g_2,\ B\ni g_1,g_2,\text{ and }g_2\text{ is regular}\}$.

The natural projection $\dot{\mathfrak{Z}}_G^G\to\mathfrak{Z}_G^G$ (forgetting B) is a Galois W-covering. Finally, $\varrho_2:\ \dot{\mathfrak{Z}}_G^G\to T$ is flat.

4.3 The proof of Proposition 2.8. In order to identify $\mathfrak{Z}_G^G$ and $\mathfrak{B}_G^G$ it suffices to identify their Galois W-coverings $\dot{\mathfrak{Z}}_G^G\to T$ and $\dot{\mathfrak{B}}_G^G\to T$ in an equivariant way. Let $\mathbf{D}\subset T$ denote the discriminant, so that $T-\mathbf{D}=T^{\mathrm{reg}}$. Let $\Delta\in\mathbb{C}[T]^W$ denote the product $\prod_{\alpha\in R}(\alpha-1)$, so that $\mathbf{D}$ is the divisor cut out by Δ.

Evidently, both $\dot{\mathfrak{Z}}_G^G|_{T^{\mathrm{reg}}}$ and $\dot{\mathfrak{B}}_G^G|_{T^{\mathrm{reg}}}$ are isomorphic to $T\times T^{\mathrm{reg}}$. Hence both $\mathbb{C}[\dot{\mathfrak{Z}}_G^G]$ and $\mathbb{C}[\dot{\mathfrak{B}}_G^G]$ are flat $\mathbb{C}[T]$-modules embedded into $\mathbb{C}[T\times T](\Delta^{-1})$. We must prove that the identification of $\dot{\mathfrak{Z}}_G^G|_{T^{\mathrm{reg}}}$ and $\dot{\mathfrak{B}}_G^G|_{T^{\mathrm{reg}}}$ extends to the identification over the whole T. To this end it suffices to check that the identification extends over the codimension-1 points of T (indeed, for a flat quasi-coherent sheaf $\mathcal{F}$ on a normal irreducible scheme we have $\mathcal{F}\xrightarrow{\sim}j_*j^*\mathcal{F}$ if j is an open embedding with complement of codimension 2). Let $g\in T$ be a regular point of $\mathbf{D}$; that is, g is a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must construct an isomorphism between localizations $(\dot{\mathfrak{Z}}_G^G)_g$ and $(\dot{\mathfrak{B}}_G^G)_g$ which is compatible with the above isomorphism at the generic point. To this end note that $Z(g)$ is connected since G is simply connected. So $Z(g)$ is a Levi subgroup of G sharing with it the Cartan torus T. We have an evident projection $\dot{\mathfrak{B}}_{Z(g)}^{Z(g)}\leftarrow\dot{\mathfrak{B}}_G^G$ which becomes an isomorphism after localization: $(\dot{\mathfrak{B}}_{Z(g)}^{Z(g)})_g\simeq(\dot{\mathfrak{B}}_G^G)_g$.

Now let us construct a projection $\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)}\leftarrow\dot{\mathfrak{Z}}_G^G$ which becomes an isomorphism after localization: $(\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)})_g\simeq(\dot{\mathfrak{Z}}_G^G)_g$. We choose a subminimal parabolic subgroup $Z(g)\subset P\subset G$. We consider the variety of triples $\dot{C}_{P,P}\subset\dot{C}_{G,G}$ cut out by the requirement $g_1,g_2\in P,\ B\subset P$. The inclusion $\dot{C}_{P,P}\subset\dot{C}_{G,G}$ induces an isomorphism $\dot{\mathfrak{Z}}_P^P:=\dot{C}_{P,P}//P\xrightarrow{\sim}\dot{C}_{G,G}//G=\dot{\mathfrak{Z}}_G^G$. The projection $P\to Z(g)$ (quotient by the unipotent radical) induces the morphisms $\dot{C}_{P,P}\to\dot{C}_{Z(g),Z(g)}$ and $\dot{\mathfrak{Z}}_P^P\to\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)}$. Thus we have obtained the desired morphism $\dot{\mathfrak{Z}}_G^G\simeq\dot{\mathfrak{Z}}_P^P\to\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)}$ which becomes an isomorphism after localization: $(\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)})_g\simeq(\dot{\mathfrak{Z}}_G^G)_g$.

Finally, the desired identification $(\dot{\mathfrak{Z}}_{Z(g)}^{Z(g)})_g\simeq(\dot{\mathfrak{B}}_{Z(g)}^{Z(g)})_g$ follows from the calculations in § 3.1.

This completes the identification $\mathfrak{Z}_G^G\simeq\mathfrak{B}_G^G$ for a simply connected G. Evidently, this identification respects the left and right actions of the center $Z(G)$, so the isomorphism for an arbitrary G follows from that for its universal cover. The other isomorphisms in Proposition 2.8, part (b) are proved in a similar way.

To prove Proposition 2.8, parts (c) and (d) it suffices to note that the minimal level (viewed as a W-equivariant homomorphism $T\twoheadrightarrow\check{T}$) for a simply laced simply connected G identifies $\check{T}$ with $T/Z(G)$; also, $\check{G}=G/Z(G)$.

5. W-invariant sections and blow-ups

The aim of this section is a proof of Proposition 2.10. We concentrate on the last statement, the other being completely similar.

Let $T^{\mathrm{reg}} \subset T$, $T^{\mathrm{reg}}_\alpha \subset T$ be the open subschemes defined by $T^{\mathrm{reg}} = \{t \mid \alpha(t) \neq 1 \text{ for all roots } \alpha\}$; $T^{\mathrm{reg}}_\alpha = \{t \mid \beta(t) \neq 1 \text{ for all roots } \beta \neq \alpha\}$; and $\mathring{T} = \bigcup_\alpha T^{\mathrm{reg}}_\alpha$ (thus $T - \mathring{T}$ has codimension 2 in T (where the empty subscheme in a curve is considered to be of codimension 2)). Notice that since G is simply connected the action of W on T^{reg} is free.

We start with a lemma.

Lemma 5.1. *The map $\dot{\mathfrak{B}}^{\check{G}}_G \times_T \mathring{T} \to \mathfrak{B}^{\check{G}}_G/W \times_{T/W} \mathring{T}$ is an isomorphism.*

Proof. Let $X \to Y$ be a flat morphism of semiseparated (which means that the diagonal embedding is affine) schemes of finite type over a characteristic zero field, and let a finite group W act on X, Y so that the map is W-equivariant. Assume that Y is flat over Y/W. We then claim that the map $X \to X/W \times_{Y/W} Y$ is an isomorphism provided that for every Zariski point $y \in Y$ the action of $\mathrm{Stab}_W(y)$ on the scheme-theoretic fiber X_y is trivial (here X/W, Y/W stand for categorical quotients). To check this claim, we can assume X is affine: by semiseparatedness every W-invariant subset in X has a W-invariant affine neighborhood. Let us first assume also that Y/W is a point; then (by replacing Y by its connected component, and W by the stabilizer of that component) we can assume that Y is nilpotent. Then $\mathcal{O}_X$ is free over $\mathcal{O}_Y$, and the generators of $\mathcal{O}_X$ as an $\mathcal{O}_Y$-module can be chosen to be W-invariant (by semisimplicity of the W-action on $\mathcal{O}_X$, and triviality of the W-action on $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathsf{k}$); since $\mathcal{O}_Y^W = \mathsf{k}$ (where k is the base field) we see that $\mathcal{O}_X^W \otimes \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$ as claimed. Now for a general Y we see that the morphism in question is a morphism of flat schemes of finite type over Y/W, which induces an isomorphism on every fiber; and such a morphism is necessarily an isomorphism.

Now it remains to check that the above conditions hold for $X = \dot{\mathfrak{B}}^{\check{G}}_G \times_T \mathring{T}$, $Y = T$. For $y \in T^{\mathrm{reg}}$, the stabilizer of y is trivial, so there is nothing to check. Consider now $y \in T^{\mathrm{reg}}_\alpha$, $y \notin T^{\mathrm{reg}}$. Then the stabilizer of y is $\{1, s_\alpha\}$. The ring of functions on $\dot{\mathfrak{B}}^{\check{G}}_G$ is generated by ${}^1\check{\lambda}$, ${}^2\mu$, t_α where $\check{\lambda}$, μ run over weights of $\check{T}$, T respectively, $\alpha \in R^+$, and $t_\alpha({}^2\alpha - 1) = {}^1\check{\alpha} - 1$. We have $s_\alpha^*({}^1\check{\lambda}) = {}^1\check{\lambda} \cdot ({}^1\check{\alpha})^{\langle -\alpha, \check{\lambda}\rangle}$, $s_\alpha^*({}^2\mu) = {}^2\mu \cdot ({}^2\alpha)^{\langle -\mu, \check{\alpha}\rangle}$, and $s_\alpha^*(t_\alpha) = t_\alpha \cdot ({}^2\alpha/{}^1\check{\alpha})$. On the fiber we have ${}^2\alpha = 1$, hence ${}^1\check{\alpha} = 1$, so the action of s_α on the fiber is trivial. □

Proposition 2.10 clearly follows from the (ii) $\iff$ (iv) part of the next proposition.

Proposition 5.2. *Let $S \to T/W$ be a flat morphism, and set $\phi : S \times_{T/W} T^{\mathrm{reg}}/W \to (\check{T} \times T)/W$ to be a T^{reg}/W-morphism. Then the following are equivalent:*

- (i) *ϕ extends to a morphism $S \times_{T/W} \mathring{T}/W \to \mathfrak{B}^{\check{G}}_G \times_{T/W} \mathring{T}$;*
- (ii) *ϕ extends to a morphism $S \to \mathfrak{B}^{\check{G}}_G$;*
- (iii) *for every $\alpha \in R$ the morphism $\phi \times \mathrm{id}_{T^{\mathrm{reg}}} : S \times_{T/W} T^{\mathrm{reg}} \to \check{T} \times T^{\mathrm{reg}}$ extends to a morphism $S \times_{T/W} T^{\mathrm{reg}}_\alpha \to \check{T} \times T^{\mathrm{reg}}_\alpha$ such that (3) holds;*
- (iv) *$\phi \times \mathrm{id}_{T^{\mathrm{reg}}} : S \times_{T/W} T^{\mathrm{reg}} \to \check{T} \times T^{\mathrm{reg}}$ extends to a morphism $S \times_{T/W} T \to \check{T} \times T$, such that (3) holds for every $\alpha \in R$.*

Proof. It is enough to assume that S is affine. Indeed, a morphism from S extends if and only if its restriction to every affine open in S does, because compatibility on intersections follows from uniqueness of such an extension. This uniqueness follows from flatness: if S is flat affine, then tensoring the injection $\mathcal{O} \to j_*\mathcal{O}$ with $\mathcal{O}_S$ we get an embedding $\mathcal{O}_S \hookrightarrow j_*j^*\mathcal{O}_S$, where j stands for the embedding $T^{\mathrm{reg}}/W \to T/W$, or $T^{\mathrm{reg}} \to T$. So we will assume S affine from now on.

That (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) is obvious.

To check that (iii) $\Rightarrow$ (iv) we tensor (over $\mathcal{O}_{T/W}$) the exact sequence of $\mathcal{O}_T$-modules

$$0 \to \mathcal{O} \to \mathcal{O}_{T^{\mathrm{reg}}} \to \bigoplus_{\alpha} (\mathcal{O}_{T_\alpha^{\mathrm{reg}}}/\mathcal{O}_T) \tag{10}$$

with $\mathcal{O}_S$. The resulting exact sequence shows that a regular function on $S \times_{T/W} T^{\mathrm{reg}}$ extends to a regular function on $S \times_{T/W} T$ if and only if it extends to $S \times_T T_\alpha^{\mathrm{reg}}$ for all α. Applying this observation to $(\phi \times \mathrm{id})^*(f|_{\check{T}\times T^{\mathrm{reg}}})$ for each regular function f on $\check{T} \times T$, we see that (iii) implies extendability of $\phi \times \mathrm{id}$ to $S \times_{T/W} T$. It is also clear that (3) holds if it holds on $\mathring{T}$.

Verification of (i) $\Rightarrow$ (ii) is similar (with (10) replaced by the W-invariant part of (10)).

It remains to check (i) $\iff$ (iii). If (i) holds, i.e. ϕ extends to a map $S \times_{T/W} \mathring{T}/W \to \mathfrak{B}_G^{\check{G}} \times_{T/W} \mathring{T}$, then we can take the fiber product of this map with $\mathrm{id}_{\mathring{T}}$ over T/W. By Lemma 5.1 it yields a map $S \times_{T/W} \mathring{T} \to \dot{\mathfrak{B}}_G^{\check{G}} \times_T \mathring{T}$, which can be composed with the projection $\dot{\mathfrak{B}}_G^{\check{G}} \to \check{T} \times T$ to produce a map $S \times_{T/W} \mathring{T} \to \check{T} \times \mathring{T}$. It is clear that this map satisfies (3), because the image of the map $\dot{\mathfrak{B}}_G^{\check{G}} \to \check{T} \times T$ intersected with $\check{T} \times \mathrm{Ker}({}^2\alpha)$ is contained in $\mathrm{Ker}({}^1\check{\alpha}) \times T$.

Conversely, if (iii) holds, then restricting the given map $S \times_{T/W} \mathring{T} \to \check{T} \times \mathring{T}$ to $S \times_{T/W} (\mathrm{Ker}(\alpha) \cap \mathring{T})$ we get a map into $\mathrm{Ker}(\check{\alpha}) \times T$ (this is immediate from (3)). This means that the map lifts to a map into $\dot{\mathfrak{B}}_G^{\check{G}}$. Replacing both the source and the target by their quotients by W we get the map required in (i). □

6. K-theory and blow-ups

The aim of this section is a proof of Theorem 2.15. Recall that Theorem 2.15, part (a) was already proved in § 3.7. Here G is assumed simply connected till further notice.

6.1 Reminder on affine Grassmannians. Let $X = X_G$ be the lattice of characters of T, and let $Y = Y_G$ be the lattice of cocharacters of G. Note that $X_G = Y_{\check{G}}$, $Y_G = X_{\check{G}}$. Let $X^+ \subset X$ (respectively $Y^+ \subset Y$) be the cone of dominant weights (respectively dominant coweights). It is well known that the $G(\mathbf{O})$-orbits in Gr_G are numbered by the dominant coweights: $\mathrm{Gr}_G = \bigsqcup_{\check{\lambda} \in Y^+} \mathrm{Gr}_{G,\check{\lambda}}$. The adjacency relation of orbits corresponds to the standard partial order on coweights: $\overline{\mathrm{Gr}}_{G,\check{\lambda}} = \bigsqcup_{\check{\mu} \leqslant \check{\lambda}} \mathrm{Gr}_{G,\check{\mu}}$. The open embedding $\mathrm{Gr}_{G,\check{\lambda}} \hookrightarrow \overline{\mathrm{Gr}}_{G,\check{\lambda}}$ will be denoted by $j_{\check{\lambda}}$ or simply by j if no confusion is likely. The dimension $\dim(\mathrm{Gr}_{G,\check{\lambda}}) = \langle 2\rho, \check{\lambda} \rangle$ where $2\rho = \sum_{\alpha \in R^+} \alpha$, and $\langle\, , \rangle : \ X \times Y \to \mathbb{Z}$ is the canonical perfect pairing.

Recall that the T-fixed points in Gr_G are naturally numbered by Y; a point $\check{\mu}$ lies in an orbit $\mathrm{Gr}_{G,\check{\lambda}}$ if and only if $\check{\mu}$ lies in the W-orbit of $\check{\lambda}$. Each $G(\mathbf{O})$-orbit $\mathrm{Gr}_{G,\check{\lambda}}$ is partitioned into Iwahori orbits isomorphic to affine spaces and numbered by $\check{\mu} \in W\check{\lambda}$. Hence the basics of [CG97, Chapter 5] are applicable in our situation.

In particular, $K^T(\mathrm{Gr}_{G,\check{\lambda}})$ is a free $K^T(pt)$-module, and $K^{G(\mathbf{O})}(\mathrm{Gr}_{G,\check{\lambda}}) = K^G(\mathrm{Gr}_{G,\check{\lambda}})$ is a free $K^G(pt)$-module (recall that $K^T(pt) = \mathbb{C}[T]$, and $K^G(pt) = \mathbb{C}[T/W]$). Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\mathrm{Gr}_{G,\check{\lambda}}) \to K^T(\mathrm{Gr}_{G,\check{\lambda}})$ is an isomorphism, and $K^G(\mathrm{Gr}_{G,\check{\lambda}}) = K^T(\mathrm{Gr}_{G,\check{\lambda}})^W$ (cf. [CG97, 6.1.22]).

Since $K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G)$ (respectively $K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K^G(\mathrm{Gr}_G)$) is filtered by the support in $G(\mathbf{O})$-orbit closures, with the associated graded $\bigoplus_{\check{\lambda} \in Y^+} K^T(\mathrm{Gr}_{G,\check{\lambda}})$ (respectively $\bigoplus_{\check{\lambda} \in Y^+} K^G(\mathrm{Gr}_{G,\check{\lambda}})$), we arrive at the following lemma.

Lemma 6.2. *We have that $K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G)$ is a flat $K^T(pt)$-module, and $K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K^G(\mathrm{Gr}_G)$ is a flat $K^G(pt)$-module. Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\mathrm{Gr}_G) \to K^T(\mathrm{Gr}_G)$ is an isomorphism, and $K^G(\mathrm{Gr}_G) = (K^T(\mathrm{Gr}_G))^W$.*

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6.3 Localization. The space $K^T(\mathrm{Gr}_G) = K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K(T(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\mathrm{Gr}_G) = K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K(G(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\mathrm{Gr}_G)$ commuting with the right action of $K^G(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T}\times T]$. The action of W on $K^T(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T}\times T]$.

Let g be a general (regular) element of T. Then the fixed point set $(\mathrm{Gr}_G)^g = (\mathrm{Gr}_G)^T = Y$ coincides with the image of the embedding $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$. According to the Thomason localization theorem (see e.g. [CG97, 5.10]), after localization, $(K^T(\mathrm{Gr}_G))_g$ becomes a free rank-one $(K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O})))_g$-module. This means that after restriction to $T^{\mathrm{reg}} \subset T = \mathrm{Spec}(K^T(pt))$ we have an isomorphism $K^T(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}} \simeq \mathbb{C}[\check{T}\times T]|_{T^{\mathrm{reg}}}$ compatible with the natural W-actions. The localized algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W}$ is embedded into $(\mathrm{End}_{K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))|_{T^{\mathrm{reg}}}}(K^T(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}}))^W$. According to Lemma 6.2, $K^G(\mathrm{Gr}_G) = (K^T(\mathrm{Gr}_G))^W$; hence this embedding is an isomorphism, and we have $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W} \simeq \mathbb{C}[\check{T}\times T]^W|_{T^{\mathrm{reg}}/W}$.

Hence both $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ are flat $\mathbb{C}[T]^W$-modules embedded into $\mathbb{C}[\check{T}\times T](\Delta^{-1})$ (see § 4.3). We must prove that the identification of $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{\mathrm{reg}}/W}$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W}$ extends to the identification over the whole T/W. To this end it suffices to check that the identification extends over the codimension-1 points of T/W. Let $g\in T/W$ be a regular point of $\mathbf{D}$; that is, g is represented by a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must prove that the localizations $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$ and $(K^{G(\mathbf{O})}(\mathrm{Gr}_G))_g$ are isomorphic. To this end it suffices to identify $\mathbb{C}[\check{T}\times T,\ ({}^1\check{\alpha}-1)/({}^2\alpha-1),\ \alpha\in R]_g$ (which we denote by $\mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]_g$ for short) and $(K^T(\mathrm{Gr}_G))_g$. Note that the embedding of reductive groups $Z(g)\hookrightarrow G$ induces the isomorphism $\mathrm{Gr}_{Z(g)} = (\mathrm{Gr}_G)^g \hookrightarrow \mathrm{Gr}_G$. According to the Thomason localization theorem, we have an isomorphism of localizations $(K^T(\mathrm{Gr}_{Z(g)}))_g \simeq (K^T(\mathrm{Gr}_G))_g$. Finally, the isomorphism $K^T(\mathrm{Gr}_{Z(g)}) \simeq \mathbb{C}[\dot{\mathfrak{B}}_{Z(g)}^{\check{Z}(g)}]$ follows from the calculations in §§ 3.8 and 3.9, and together with the evident isomorphism of localizations $\mathbb{C}[\dot{\mathfrak{B}}_{Z(g)}^{\check{Z}(g)}]_g \simeq \mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]_g$ establishes the desired isomorphism $(K^T(\mathrm{Gr}_G))_g \simeq \mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]_g$.

This completes the proof of Theorem 2.15, part (b).

6.4 Comparison of Poisson structures. In order to compare the Poisson structures on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ it suffices to identify them on the open subset $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W} = \mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{\mathrm{reg}}/W} = \mathbb{C}[\check{T}\times T^{\mathrm{reg}}]^W$. The space

$$K^{T\times\mathbb{G}_m}(\mathrm{Gr}_G) = K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G) = K(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash G(\mathbf{F})\ltimes\mathbb{G}_m/G(\mathbf{O})\ltimes\mathbb{G}_m)$$

is equipped with the two commuting actions: $K(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash T(\mathbf{F})\ltimes\mathbb{G}_m/T(\mathbf{O})\ltimes\mathbb{G}_m)$ acts by convolutions on the left, and

$$K^{G\times\mathbb{G}_m}(\mathrm{Gr}_G) = K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G) = K(G(\mathbf{O})\ltimes\mathbb{G}_m\backslash G(\mathbf{F})\ltimes\mathbb{G}_m/G(\mathbf{O})\ltimes\mathbb{G}_m)$$

acts by convolutions on the right. Also, W acts on $K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ commuting with the right action of $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash T(\mathbf{F})\ltimes\mathbb{G}_m/T(\mathbf{O})\ltimes\mathbb{G}_m)$ is isomorphic to the group algebra $\mathbb{C}[\Gamma]$ of the following Heisenberg group Γ.

It is a $\mathbb{Z}$-central extension of $Y\times X$ with the multiplication (written multiplicatively)

$$(q^{n_1}, e^{\check{\lambda}_1}, e^{\mu_1})\cdot(q^{n_2}, e^{\check{\lambda}_2}, e^{\mu_2}) = (q^{n_1+n_2+\langle\mu_1,\check{\lambda}_2\rangle}, e^{\check{\lambda}_1+\check{\lambda}_2}, e^{\mu_1+\mu_2})$$

where $\langle\,,\rangle:\ X\times Y\to\mathbb{Z}$ is the canonical perfect pairing.

Finally, the action of the Weyl group W on $K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash T(\mathbf{F})\ltimes\mathbb{G}_m/T(\mathbf{O})\ltimes\mathbb{G}_m)$ and induces the natural (diagonal) action of W on $\mathbb{C}[\Gamma]$. From this we

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deduce, exactly as in § 6.3, that $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W} \simeq \mathbb{C}[\Gamma]|_{T^{\mathrm{reg}}/W}$. It follows that the Poisson structure on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{\mathrm{reg}}/W}$ coincides with the standard Poisson structure on $\mathbb{C}[\check{T} \times T^{\mathrm{reg}}]^W$.

This completes the proof of Theorem 2.15, part (c).

6.5 The case of nonsimply connected G. For general G, let $\tilde{G}$ denote its universal cover, and let $\tilde{T}$ stand for the Cartan of $\tilde{G}$. Note that the dual torus is $\check{T}/\pi_1(G)$. As in § 6.3, we have $K^G(\mathrm{Gr}_G) = (\mathrm{End}_{K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))}(K^T(\mathrm{Gr}_G)))^W$, so it suffices to identify the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O})) \ltimes W = \mathbb{C}[\check{T} \times T] \ltimes W$-module $K^T(\mathrm{Gr}_G)$ with $\mathbb{C}[\check{T} \times T,\ ({}^1\check{\alpha} - 1)/({}^2\alpha - 1),\ \alpha \in R] = \mathrm{Spec}\,\mathbb{C}[\dot{\mathfrak{Z}}^{\check{G}}_{G}]$. We do this by reduction to the known case of $\tilde{G}$.

Evidently, the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O})) \ltimes W = \mathbb{C}[\check{T} \times T] \ltimes W$-module $K^T(\mathrm{Gr}_G)$ equals $\mathbb{C}[\check{T} \times T] \ltimes W \otimes_{\mathbb{C}[(\check{T}/\pi_1(G))\times T]\ltimes W} K^T(\mathrm{Gr}_{\tilde{G}})$. On the other hand, it follows from § 6.3 that the $K(T(\mathbf{O})\backslash \tilde{T}(\mathbf{F})/\tilde{T}(\mathbf{O})) \ltimes W = \mathbb{C}[(\check{T}/\pi_1(G)) \times T] \ltimes W$-module $K^T(\mathrm{Gr}_{\tilde{G}})$ equals the invariants of $\pi_1(G)$ in $K^{\tilde{T}}(\mathrm{Gr}_{\tilde{G}})$, that is $\mathbb{C}[(\check{T}/\pi_1(G)) \times \tilde{T},\ ({}^1\check{\alpha} - 1)/({}^2\alpha - 1),\ \alpha \in R]^{\pi_1(G)} = \mathbb{C}[(\check{T}/\pi_1(G)) \times T,$ $({}^1\check{\alpha} - 1)/({}^2\alpha - 1),\ \alpha \in R]$.

This completes the proof of Theorem 2.15 for general G.

6.6 Borel–Moore homology and blow-ups. Theorem 2.12 is proved in absolute parallel fashion to the proof of Theorem 2.15.

7. Computation of $K_{G(\mathbf{O})}(\Lambda)$

7.1 The affine Grassmannian Steinberg variety. We denote by $\mathfrak{u} \subset \mathfrak{g}(\mathbf{O})$ (respectively $U \subset G(\mathbf{O})$) the nilpotent (respectively unipotent) radical. It has a filtration $\mathfrak{u} = \mathfrak{u}^{(0)} \supset \mathfrak{u}^{(1)} \supset \cdots$ by congruence subalgebras. The trivial (Tate) vector bundle $\underline{\mathfrak{g}(\mathbf{F})}$ with the fiber $\mathfrak{g}(\mathbf{F})$ over Gr_G has a structure of an ind-scheme. It contains a profinite dimensional vector subbundle $\underline{\mathfrak{u}}$ whose fiber over a point $g \in \mathrm{Gr}_G$ represented by a compact subalgebra in $\mathfrak{g}(\mathbf{F})$ is the pronilpotent radical of this subalgebra. The trivial vector bundle $\underline{\mathfrak{g}(\mathbf{F})} = \mathfrak{g}(\mathbf{F}) \times \mathrm{Gr}_G$ also contains a trivial vector subbundle $\mathfrak{u} \times \mathrm{Gr}_G$.

We will call $\underline{\mathfrak{u}}$ the *cotangent bundle* of Gr_G, and we will call the intersection $\Lambda := \underline{\mathfrak{u}} \cap (\mathfrak{u} \times \mathrm{Gr}_G)$ the *affine Grassmannian Steinberg variety*. It has a structure of an ind-scheme of ind-infinite type. Namely, if p stands for the natural projection $\Lambda \to \mathrm{Gr}_G$, then $\Lambda_{\leqslant\check{\lambda}} := p^{-1}(\overline{\mathrm{Gr}}_{G,\check{\lambda}})$ is a scheme of infinite type, and $\Lambda = \bigcup \Lambda_{\leqslant\check{\lambda}}$.

Note that for a fixed $\check{\lambda}$ and $l \gg 0$ the intersection of fibers of $\underline{\mathfrak{u}}$ over all points of $\overline{\mathrm{Gr}}_{G,\check{\lambda}}$ (as vector subspaces of $\mathfrak{g}(\mathbf{F})$) contains $\mathfrak{u}^{(l)}$. Thus $\mathfrak{u}^{(l)}$ acts freely (by fiberwise translations) on $\Lambda_{\leqslant\check{\lambda}}$, and the quotient is a scheme of finite type, to be denoted by $\Lambda^l_{\leqslant\check{\lambda}}$. For $k > l$ we have evident affine fibrations $p^k_l:\ \Lambda^k_{\leqslant\check{\lambda}} \to \Lambda^l_{\leqslant\check{\lambda}}$, and $\Lambda_{\leqslant\check{\lambda}}$ coincides with the inverse limit of this system.

Similarly, the total space of the vector bundle $\underline{\mathfrak{u}}$ (to be denoted by the same symbol) is a union of infinite type schemes $\underline{\mathfrak{u}}_{\leqslant\check{\lambda}}$, and for fixed $\check{\lambda}$ and $l \gg 0$, the scheme $\underline{\mathfrak{u}}_{\leqslant\check{\lambda}}$ is the inverse limit of affine fibrations $p^k_l:\ \underline{\mathfrak{u}}^k_{\leqslant\check{\lambda}} \to \underline{\mathfrak{u}}^l_{\leqslant\check{\lambda}}$ $(k > l)$. Note that the proalgebraic group $G(\mathbf{O})$ acts on all the above schemes, and the fibrations p^k_l are $G(\mathbf{O})$-equivariant.

A $G(\mathbf{O})$*-equivariant coherent sheaf* $\mathcal{F}$ *on* $\underline{\mathfrak{u}}$ is by definition supported on some $\underline{\mathfrak{u}}_{\leqslant\check{\lambda}}$. There, it is defined as a collection of $G(\mathbf{O})$-equivariant sheaves $\mathcal{F}^l$ on $\underline{\mathfrak{u}}^l_{\leqslant\check{\lambda}}$ for $l \gg 0$ together with isomorphisms $(p^k_l)^*\mathcal{F}^l \simeq \mathcal{F}^k$. We will consider the $G(\mathbf{O})$-equivariant coherent sheaves on $\underline{\mathfrak{u}}$ *supported on* Λ, and $D^b Coh^{G(\mathbf{O})}_{\Lambda}(\underline{\mathfrak{u}})$ stands for the derived category of such sheaves, and $K^{G(\mathbf{O})}(\Lambda)$ stands for the K-group of such sheaves.

7.2 Convolution in $D^bCoh^{G(\mathbf{O})}_{\Lambda}(\underline{\mathfrak{u}})$. We have a principal $G(\mathbf{O})$-bundle $G(\mathbf{F}) \to \mathrm{Gr}_G$. Given a $G(\mathbf{O})$-(ind)-scheme A we can form an associated bundle $\widetilde{A} = G(\mathbf{F}) \times_{G(\mathbf{O})} A \to \mathrm{Gr}_G$. Given a coherent $G(\mathbf{O})$-equivariant sheaf $\mathcal{F}$ on A we can form an associated sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{A}$ as $G(\mathbf{O})$-invariants in the direct image of $\mathcal{O}_{G(\mathbf{F})} \boxtimes \mathcal{F}$ from $G(\mathbf{F}) \times A$ to $G(\mathbf{F}) \times_{G(\mathbf{O})} A$. If $A = \mathrm{Gr}_G$, apart from the natural projection $p_1 : \widetilde{A} \to \mathrm{Gr}_G$, we have a multiplication map $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G$, to be denoted p_2. Then (p_1, p_2) identifies $\widetilde{\mathrm{Gr}_G}$ with $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Furthermore, $\widetilde{\underline{\mathfrak{u}}}$ is a vector bundle over $\widetilde{\mathrm{Gr}_G} = \mathrm{Gr}_G \times \mathrm{Gr}_G$ which is naturally identified with $p_2^*\underline{\mathfrak{u}}$. Thus we have an ind-proper morphism $p_2 : \ \widetilde{\underline{\mathfrak{u}}} \to \underline{\mathfrak{u}}$.

Note that both $\widetilde{\underline{\mathfrak{u}}} = p_2^*\underline{\mathfrak{u}}$ and $p_1^*\underline{\mathfrak{u}}$ are subbundles in the trivial (Tate) vector bundle $\underline{\mathfrak{g}(\mathbf{F})}$ over $\mathrm{Gr}_G \times \mathrm{Gr}_G$ with the fiber $\mathfrak{g}(\mathbf{F})$. Their intersection is naturally identified with $\widetilde{\Lambda}$. In particular, we have an embedding $\widetilde{\Lambda} \subset p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$, and an ind-proper morphism $p_2 : \ \widetilde{\Lambda} \to \underline{\mathfrak{u}}$.

Hence given $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on Λ we can consider the $G(\mathbf{O})$-equivariant complex $\mathcal{F} \star \mathcal{G} := (p_2)_*(p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}})$ (tensor product over the structure sheaf of the profinite dimensional vector bundle $p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$). Clearly, $\mathcal{F} \star \mathcal{G}$ is supported on Λ. Hence we get a convolution operation on $D^bCoh^{G(\mathbf{O})}_{\Lambda}(\underline{\mathfrak{u}})$ and on $K^{G(\mathbf{O})}(\Lambda)$ once we check that $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is bounded.

To this end, note that $\widetilde{\mathcal{G}}$ is flat over the first copy of Gr_G, and for some $\check{\lambda}$ the sheaf $\mathcal{F}$ is supported on $\Lambda_{\leqslant \check{\lambda}}$, so the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $(p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}})|_{\overline{\mathrm{Gr}}_{G,\check{\lambda}} \times \mathrm{Gr}_G} = \underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}} \subset \underline{\mathfrak{u}} \times \underline{\mathfrak{u}} = p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$. That is, $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is the direct image of $p_1^*\mathcal{F}|_{\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}} \overset{L}{\otimes}_{\mathcal{O}_{\underline{\mathfrak{u}}_{\check{\lambda}} \times \underline{\mathfrak{u}}}} \widetilde{\mathcal{G}}|_{\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}}$ under the closed embedding $\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}} \hookrightarrow \underline{\mathfrak{u}} \times \underline{\mathfrak{u}}$. On the other hand, $p_1^*\mathcal{F}$ is flat over the second copy of Gr_G, while the support of $\widetilde{\mathcal{G}}$ intersected with $\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}$ is contained in $\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}_{\leqslant \check{\mu}}$ for some $\check{\mu}$. Hence the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}_{\leqslant \check{\mu}}$. There exists $l \gg 0$ such that the diagonal fiberwise action of $\mathfrak{u}^{(l)}$ on $\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}_{\leqslant \check{\mu}}$ is free, and both $p_1^*\mathcal{F}$ and $\widetilde{\mathcal{G}}$ restricted to $\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}_{\leqslant \check{\mu}}$ are $\mathfrak{u}^{(l)}$-equivariant, that is, they are lifted from the sheaves on $(\underline{\mathfrak{u}}_{\leqslant \check{\lambda}} \times \underline{\mathfrak{u}}_{\leqslant \check{\mu}})/\mathfrak{u}^{(l)} =: V$; we abuse notation by keeping the same names for these sheaves. So the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed as the tensor product of coherent sheaves over the structure sheaf of the profinite dimensional vector bundle V over the finite dimensional scheme $\overline{\mathrm{Gr}}_{G,\check{\lambda}} \times \overline{\mathrm{Gr}}_{G,\check{\mu}}$.

Now there exists a vector subbundle $V' \subset V$ such that the quotient $\overline{V} := V/V'$ is a finite dimensional vector bundle, $p_1^*\mathcal{F}$ is lifted from $\overline{V}$, and the support of $\widetilde{\mathcal{G}}$ in V projects isomorphically onto its image in $\overline{V}$. Moreover, recall that $p_1^*\mathcal{F}$ is flat over $\overline{\mathrm{Gr}}_{G,\check{\mu}}$, while $\widetilde{\mathcal{G}}$ is flat over $\overline{\mathrm{Gr}}_{G,\check{\lambda}}$. Clearly, in this situation $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}} \in D^b(V)$. This explains why for $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on Λ the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is a bounded complex of coherent sheaves on $p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$ supported on $\widetilde{\Lambda}$. Hence the same is true for the bounded complexes of $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on $\underline{\mathfrak{u}}$ supported on Λ. Thus, $D^bCoh^{G(\mathbf{O})}_{\Lambda}(\underline{\mathfrak{u}})$ is closed with respect to convolution.

Theorem 7.3. *We have that $K^{G(\mathbf{O})}(\Lambda)$ is a commutative algebra isomorphic to $\mathbb{C}[\check{T} \times T]^W$.*

Remark 7.4. Since Λ_G is an affine Grassmannian analogue of the classical Steinberg variety, this result agrees well with the geometric realization of the Cherednik double affine Hecke algebra in [GG95] and [Vas02]. In effect, $K^{G(\mathbf{O})}(\Lambda_G)$ is the spherical subalgebra of the Cherednik algebra with both parameters trivial: $q = t = 1$.

7.5 Bialynicki–Birula stratifications. The proof of Theorem 7.3 uses the following lemma on K-theory of cellular spaces. Let M be a normal quasi-projective variety equipped with a torus H-action with finitely many fixed points. We assume that M is equipped with an H-invariant stratification $M = \bigsqcup_{\mu \in M^H} M_\mu$ such that each stratum M_μ contains exactly one H-fixed point μ, and M_μ is isomorphic to an affine space. For $\mu \in M^H$ we denote by $j_\mu : M_\mu \hookrightarrow M$ the locally closed embedding of the corresponding stratum. We denote by $i_\mu : \mu \hookrightarrow M_\mu$ the closed embedding of an H-fixed point in the corresponding stratum, or in the whole of M when no confusion is likely. We denote by $\mu \leqslant \nu$ the closure relation of strata. We denote by $M_{\leqslant\mu} \subset M$ the union $\bigcup_{\nu \leqslant \mu} M_\nu$.

Given an H-equivariant closed embedding of M into a smooth H-variety M' (for the existence see [Sum74]) we denote by T^*M the restriction of the cotangent bundle T^*M' to $M \subset M'$. We denote by $\imath : M \hookrightarrow \mathsf{T}^*M$ the embedding of the zero section. We also denote by i_μ the closed embedding of the conormal bundle $\mathsf{T}^*_\mu M' \hookrightarrow \mathsf{T}^*M$ when no confusion is likely. Finally, we denote by $\mathcal{L}'$ the union of conormal bundles $\bigcup_\mu \mathsf{T}^*_{M_\mu} M'$, and $\jmath$ stands for the closed embedding $\mathcal{L}' \hookrightarrow \mathsf{T}^*M$. We denote by $\mathcal{L}'_{\leqslant\mu} \subset \mathcal{L}'$ the union $\bigcup_{\nu \leqslant \mu} \mathsf{T}^*_{\mathcal{M}_\nu} M'$; it is a closed subvariety of $\mathcal{L}'$. It has a closed subvariety $\mathcal{L}'_{<\mu} := \bigcup_{\nu<\mu} \mathsf{T}^*_{M_\nu} M'$.

For $\mu \in M^H$ we have an embedding $i_{\mu*} : K^H(\mu) \hookrightarrow K^H(M)$. We have an embedding $\jmath_* : K^H(\mathcal{L}') \hookrightarrow K^H(\mathsf{T}^*M) \overset{\imath^*}{\simeq} K^H(M)$. Indeed, the exact sequences (see [CG97, Chapter 5])

$$0 \to K^H(\mathcal{L}'_{<\mu}) \to K^H(\mathcal{L}'_{\leqslant\mu}) \to K^H(\mathsf{T}^*_{M_\mu} M') \to 0,$$
$$0 \to K^H(\mathsf{T}^*M'|_{M_{<\mu}}) \to K^H(\mathsf{T}^*M'|_{M_{\leqslant\mu}}) \to K^H(\mathsf{T}^*M'|_{M_\mu})$$

give rise to the support filtrations on $K^H(\mathcal{L}')$ and $K^H(\mathsf{T}^*M)$ with associated graded groups $\bigoplus_{\mu\in M^H} K^H(\mathsf{T}^*_{M_\mu} M')$ and $\bigoplus_{\mu\in M^H} K^H(\mathsf{T}^*M'|_{M_\mu})$, respectively. Now $\jmath_*$ is strictly compatible with the support filtrations and clearly injective on the associated graded groups.

Note that the image $\jmath_*(K^H(\mathcal{L}')) \subset K^H(M)$ is independent of the choice of the closed embedding $M \hookrightarrow M'$. In effect, given another embedding $M \hookrightarrow \widetilde{M}$, we can consider the diagonal embedding $M \hookrightarrow M'' := M' \times \widetilde{M}$. Clearly, we have a projection $p : \mathsf{T}^*M''|_M \to \mathsf{T}^*M'|_M$ which realizes $\mathsf{T}^*M''|_M$ as a vector bundle over $\mathsf{T}^*M'|_M$. Moreover, if we denote by $\mathcal{L}''$ the union of conormal bundles $\bigcup_\mu \mathsf{T}^*_{M_\mu} M'' \subset \mathsf{T}^*M''|_M$ then $\mathcal{L}'' = p^{-1}\mathcal{L}'$. This shows that the images of $K^H(\mathcal{L}')$ and $K^H(\mathcal{L}'')$ in $K^H(M)$ coincide, and thus $\jmath_*(K^H(\mathcal{L}')) \subset K^H(M)$ is well defined.

Lemma 7.6. *In $K^H(M)$ we have an equality $\jmath_*(K^H(\mathcal{L}')) = \bigoplus_\mu i_{\mu*}(K^H(\mu))$.*

Proof. Let $K^H(D_M)$ stand for the K-group of weakly H-equivariant D-modules on M' supported on $M \subset M'$. Given such a D-module and passing to the associated graded sheaf with respect to a good filtration, we obtain an H-equivariant coherent sheaf on T^*M, and in this way one obtains a homomorphism $SS : K^H(D_M) \to K^H(\mathsf{T}^*M) \overset{\imath^*}{\simeq} K^H(M)$ (see e.g. [Gin86]). Let δ_μ stand for a δ-function D-module at the point $\mu \in M^H$ with its obvious H-equivariance. Then, evidently, $SS(\delta_\mu)$ generates $i_{\mu*}(K^H(\mu))$ as a module over $K^H(pt)$. Moreover, $\{SS(j_{\mu!}\mathcal{O}_{M_\mu}),\ \mu \in M^H\}$ forms a basis of $\jmath_*(K^H(\mathcal{L}'))$.

In effect, the closed embedding $\mathcal{L}'_{<\mu} \hookrightarrow \mathcal{L}'_{\leqslant\mu}$ gives rise to the exact sequence

$$0 \to K^H(\mathcal{L}'_{<\mu}) \to K^H(\mathcal{L}'_{\leqslant\mu}) \to K^H(\mathsf{T}^*_{M_\mu} M') \to 0$$

(see [CG97, Chapter 5]), and the image of $SS(j_{\mu!}\mathcal{O}_{M_\mu})$ in $K^H(\mathsf{T}^*_{M_\mu} M')$ clearly generates it.

So it is enough to check the equality in $K^H(\mathsf{T}^*M)$:

$$SS(\delta_\mu) = SS(j_{\mu!}\mathcal{O}_{M_\mu}) \cdot (-1)^{\dim M_\mu} \det(\mathsf{T}_\mu M_\mu), \tag{11}$$

where $\det(\mathsf{T}_\mu M_\mu)$ is the character of H (thus an invertible element of $K^H(pt) = \mathbb{C}[H]$) acting in the determinant of the tangent bundle of M_μ at μ.

To this end note that restriction to the H-fixed points gives rise to an embedding $\bigoplus_\nu i_\nu^* \imath^* : K^H(\mathsf{T}^*M) \hookrightarrow \bigoplus_\nu K^H(\nu)$. This is checked by induction in ν using the exact sequences

$$0 \to K^H(\mathsf{T}^*M'|_{M_{<\nu}}) \to K^H(\mathsf{T}^*M'|_{M_{\leqslant\nu}}) \to K^H(\mathsf{T}^*M'|_{M_\nu}) \to 0.$$

It is clear that for $\nu = \mu$ the restrictions $i_\mu^* \imath^*$ of the left- and right-hand sides of (11) coincide. We are going to check that for $\nu \neq \mu$ the restrictions $i_\nu^* \imath^*$ of the left- and right-hand sides of (11) both vanish. Evidently, $i_\nu^* \imath^* SS(\delta_\mu) = 0$.

Recall that i_ν also stands for the closed embedding $\mathsf{T}_\nu^* M' \hookrightarrow \mathsf{T}^*M$, so we just have to check that $i_\nu^* SS(j_{\mu!}\mathcal{O}_{M_\mu}) = 0 \in K^H(\mathsf{T}_\nu^* M')$. Note that the functor of global sections of H-equivariant coherent sheaves on the vector space $\mathsf{T}_\nu^* M'$ gives rise to an embedding $\Gamma : \ K^H(\mathsf{T}_\nu^* M') \hookrightarrow \mathbb{Z}^{X^*(H)}$ where $X^*(H)$ stands for the lattice of characters of H. Now for a D-module $\mathcal{F}$ we have $\Gamma(i_\nu^* SS\mathcal{F}) = \mathbf{i}_\nu^* \mathcal{F}$ where $\mathbf{i}_\nu^* \mathcal{F}$ stands for the fiber at $\nu \in M$ of the H-equivariant quasi-coherent $\mathcal{O}_{M'}$-module $\mathcal{F}$. Finally, for $\mathcal{F} = j_{\mu!}\mathcal{O}_{M_\mu}$ and $\nu \neq \mu$ we have $\mathbf{i}_\nu^* j_{\mu!}\mathcal{O}_{M_\mu} = 0$. This completes the proof of the lemma. □

7.7 Bialynicki–Birula stratification of Gr_G. We consider the stratification of Gr_G by the Iwahori orbits $\mathrm{Gr}_G = \bigsqcup_{\check\mu \in Y} \mathrm{Gr}_G^{\check\mu}$. This is a refinement of the stratification by the $G(\mathbf{O})$-orbits: $\mathrm{Gr}_{G,\check\lambda} = \bigsqcup_{\check\mu \in W\check\lambda} \mathrm{Gr}_G^{\check\mu}$. Let us denote by $\mathfrak{n} \supset \mathfrak{u}$ the nilpotent radical of the Iwahori subalgebra in $\mathfrak{g}(\mathbf{F})$. The union of conormal bundles to the Iwahori orbits is the following subvariety Λ_I of the cotangent bundle $\underline{\mathfrak{u}}$: by definition, $\Lambda_I := \underline{\mathfrak{u}} \cap (\mathfrak{n} \times \mathrm{Gr}_G)$. We have a closed embedding $\Lambda \subset \Lambda_I$.

Lemma 7.6 allows us to compute $K^T(\Lambda_I) = \bigoplus_{\check\mu \in Y} K^T(\check\mu) \subset K^T(\mathrm{Gr}_G)$, i.e. $K^T(\Lambda_I) \simeq \mathbb{C}[\check T \times T]$ (note that the natural W-action on $K^T(\mathrm{Gr}_G)$ induces the diagonal W-action on $\mathbb{C}[\check T \times T] \simeq K^T(\Lambda_I) \subset K^T(\mathrm{Gr}_G)$). Although Lemma 7.6 was formulated for finite dimensional varieties M, its proof goes through for Gr_G without changes: we only need to have the singular support map $SS : K^T(D_{\mathrm{Gr}_G}) \to K^T(\underline{\mathfrak{u}}) \simeq K^T(\mathrm{Gr}_G)$. For this see [KT95], [BD00, Chapter 15] and [GG95].

The embedding $\Lambda \hookrightarrow \Lambda_I$ gives rise to the embedding $K^T(\Lambda) \hookrightarrow K^T(\Lambda_I) \hookrightarrow K^T(\underline{\mathfrak{u}}) = K^T(\mathrm{Gr}_G)$. Note that W acts naturally on both $K^T(\Lambda)$ and $K^T(\mathrm{Gr}_G)$, and the embedding $K^T(\Lambda) \hookrightarrow K^T(\mathrm{Gr}_G)$ is W-equivariant. Also, $(K^T(\Lambda))^W = K^G(\Lambda) = K^{G(\mathbf{O})}(\Lambda)$. Hence, the image of the embedding $K^{G(\mathbf{O})}(\Lambda) \hookrightarrow K^T(\Lambda_I) \simeq \mathbb{C}[\check T \times T] \subset K^T(\mathrm{Gr}_G)$ lies in the invariants of the diagonal W-action on $\mathbb{C}[\check T \times T]$. Thus to prove Theorem 7.3 we must check that the image of this embedding contains $\mathbb{C}[\check T \times T]^W$.

We have projections $\pi : \ \Lambda \to \mathrm{Gr}_G$ and $\pi_I : \ \Lambda_I \to \mathrm{Gr}_G$. For $\check\lambda \in Y^+$ we denote by $\Lambda_{\check\lambda}$ (respectively $\Lambda_{\leqslant\check\lambda}$, $\Lambda_{<\check\lambda}$) the preimage $\pi^{-1}(\mathrm{Gr}_{G,\check\lambda})$ (respectively $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$, $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda} - \mathrm{Gr}_{G,\check\lambda})$). For $\check\lambda \in Y^+$ we denote by $\Lambda_{I,\check\lambda}$ (respectively $\Lambda_{I,\leqslant\check\lambda}$, $\Lambda_{I,<\check\lambda}$) the preimage $\pi_I^{-1}(\mathrm{Gr}_{G,\check\lambda})$ (respectively $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$, $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda} - \mathrm{Gr}_{G,\check\lambda})$). Clearly, $\Lambda_{<\check\lambda}$ (respectively $\Lambda_{I,<\check\lambda}$) is closed in $\Lambda_{\leqslant\check\lambda}$ (respectively $\Lambda_{I,\leqslant\check\lambda}$), with the open complement $\Lambda_{\check\lambda}$ (respectively $\Lambda_{I,\check\lambda}$). In K-groups we have exact sequences (see [CG97, Chapter 5])

$$0 \to K^T(\Lambda_{<\check\lambda}) \to K^T(\Lambda_{\leqslant\check\lambda}) \to K^T(\Lambda_{\check\lambda}) \to 0,$$
$$0 \to K^T(\Lambda_{I,<\check\lambda}) \to K^T(\Lambda_{I,\leqslant\check\lambda}) \to K^T(\Lambda_{I,\check\lambda}) \to 0.$$

Thus we obtain a support filtration on $K^T(\Lambda_I)$ (respectively $K^T(\Lambda)$) with associated graded $\bigoplus_{\check\lambda \in Y^+} K^T(\Lambda_{I,\check\lambda})$ (respectively $\bigoplus_{\check\lambda \in Y^+} K^T(\Lambda_{\check\lambda})$).

We have the embeddings $K^T(\Lambda_{\check\lambda}) \hookrightarrow K^T(\Lambda_{I,\check\lambda}) \hookrightarrow K^T(\underline{\mathfrak{u}}|_{\mathrm{Gr}_{\check\lambda}}) \simeq K^T(\mathrm{Gr}_{\check\lambda})$. The Weyl group W acts naturally on both $K^T(\Lambda_{\check\lambda})$ and $K^T(\mathrm{Gr}_{\check\lambda})$, and to prove Theorem 7.3 it suffices to check that the image of $(K^T(\Lambda_{\check\lambda}))^W$ in $K^T(\Lambda_{I,\check\lambda})$ contains (equivalently, coincides with) the intersection $K^T(\Lambda_{I,\check\lambda}) \cap (K^T(\mathrm{Gr}_{\check\lambda}))^W$.

To this end recall that $\mathrm{Gr}_{G,\check\lambda}$ can be G-equivariantly identified with the total space $\widetilde{\mathcal{B}}$ of a vector bundle over a certain partial flag variety $\mathcal{B}$ of the group G (the quotient $G/P_{\check\lambda}$ by a parabolic subgroup depending on $\check\lambda$). The Borel subgroup $B \subset G$ acts on $\mathcal{B}$ with finitely many orbits numbered by the cosets of parabolic Weyl subgroup $W^{\check\lambda} = W/W_{\check\lambda}$: we have $\mathcal{B} = \bigsqcup_{w\in W^{\check\lambda}} \mathcal{B}_w$. Let us denote by $\mathcal{L} \subset \mathsf{T}^*\mathcal{B}$ the union of conormal bundles $\mathcal{L} = \bigsqcup_{w\in W^{\check\lambda}} \mathsf{T}^*_{\mathcal{B}_w}\mathcal{B}$. Let us also denote by $\widetilde{\mathcal{B}}_w$ the preimage of $\mathcal{B}_w$ in $\widetilde{\mathcal{B}}$ (it coincides with a certain Iwahori orbit $\mathrm{Gr}^{\check\mu}_G \subset \mathrm{Gr}_{G,\check\lambda} = \widetilde{\mathcal{B}}$). We define $\widetilde{\mathcal{L}} := \bigsqcup_{w\in W^{\check\lambda}} \mathsf{T}^*_{\widetilde{\mathcal{B}}_w}\widetilde{\mathcal{B}} \subset \mathsf{T}^*\widetilde{\mathcal{B}}$. Then there exists a G-equivariant profinite dimensional vector bundle $\mathcal{V} \xrightarrow{p} \mathsf{T}^*\widetilde{\mathcal{B}}$ such that $\mathcal{V} \simeq \underline{\mathfrak{u}}|_{\mathrm{Gr}_{\check\lambda}}$, and under this isomorphism we have $\mathcal{V}|_{\widetilde{\mathcal{L}}} \simeq \Lambda_{I,\check\lambda}$, $\mathcal{V}|_{\widetilde{\mathcal{B}}\hookrightarrow \mathsf{T}^*\widetilde{\mathcal{B}}} \simeq \Lambda_{\check\lambda}$. Thus to prove Theorem 7.3 it is enough to check that the image of $(K^T(\widetilde{\mathcal{B}}))^W$ in $K^T(\mathsf{T}^*\widetilde{\mathcal{B}})$ contains the intersection $K^T(\widetilde{\mathcal{L}})\cap(K^T(\mathsf{T}^*\widetilde{\mathcal{B}}))^W$. Equivalently, we have to check that the image of $(K^T(\mathcal{B}))^W$ in $K^T(\mathsf{T}^*\mathcal{B})$ contains the intersection $K^T(\mathcal{L}) \cap (K^T(\mathsf{T}^*\mathcal{B}))^W$. This is the subject of the following lemma.

LEMMA 7.8. *Let $\imath: \mathcal{B} \hookrightarrow \mathsf{T}^*\mathcal{B}$ denote the embedding of the zero section, and let $\jmath: \mathcal{L} \hookrightarrow \mathsf{T}^*\mathcal{B}$ denote the natural closed embedding. Then $\imath_*(K^T(\mathcal{B}))^W$ coincides with $\mathrm{Im}(\jmath_*: K^T(\mathcal{L}) \hookrightarrow K^T(\mathsf{T}^*\mathcal{B}))\bigcap(K^T(\mathsf{T}^*\mathcal{B}))^W$.*

Proof. For $w \in W^{\check\lambda}$ we denote by $w \in \mathcal{B}_w \subset \mathcal{B}$ the corresponding T-fixed point. We denote by i_w the closed embedding $T^*_w\mathcal{B} \hookrightarrow \mathsf{T}^*\mathcal{B}$ (and also the closed embedding $w \hookrightarrow \mathcal{B}$, when confusion is unlikely), and we denote by $\mathfrak{i}_w$ the closed embedding $w \hookrightarrow \mathsf{T}^*\mathcal{B}$. According to Lemma 7.6, the image of $\jmath_*: K^T(\mathcal{L}) \hookrightarrow K^T(\mathsf{T}^*\mathcal{B})$ coincides with the image of $\bigoplus_{w\in W^{\check\lambda}} i_{w*}: \bigoplus_{w\in W^{\check\lambda}} K^T(\mathsf{T}^*_w\mathcal{B}) \to K^T(\mathsf{T}^*\mathcal{B})$. We have an embedding $\bigoplus_{w\in W^{\check\lambda}} \mathfrak{i}^*_w: K^T(\mathsf{T}^*\mathcal{B}) \hookrightarrow \bigoplus_{w\in W^{\check\lambda}} K^T(w)$, and similarly an embedding $\bigoplus_{w\in W^{\check\lambda}} i^*_w: K^T(\mathcal{B}) \hookrightarrow \bigoplus_{w\in W^{\check\lambda}} K^T(w)$.

Clearly, the W-invariants project injectively into any direct summand: $K^G(\mathcal{B}) = (K^T(\mathcal{B}))^W \overset{i^*_w}{\hookrightarrow} K^T(w)$ (respectively $K^G(\mathsf{T}^*\mathcal{B}) = (K^T(\mathsf{T}^*\mathcal{B}))^W \overset{\mathfrak{i}^*_w}{\hookrightarrow} K^T(w)$) for any $w \in W^{\check\lambda}$. Thus it suffices to check that for any $w \in W^{\check\lambda}$ we have a coincidence $\mathrm{Im}(\mathfrak{i}^*_w i_{w*}: K^T(\mathsf{T}^*_w\mathcal{B})^W \to K^T(w)) = \mathrm{Im}(\mathfrak{i}^*_w \imath_* \operatorname{Res}^G_T: K^G(\mathcal{B}) \to K^T(w))$. Note that if $w = e$ (the identity coset of $W_{\check\lambda}$ in W), then the image $i^*_e(K^T(\mathcal{B}))^W \subset K^T(e)$ (respectively $\mathfrak{i}^*_e(K^T(\mathsf{T}^*\mathcal{B}))^W \subset K^T(e)$) coincides with $(K^T(e))^{W_{\check\lambda}} = \mathbb{C}[T]^{W_{\check\lambda}}$. Moreover, under identification $K^T(\mathsf{T}^*_e\mathcal{B}) = K^T(e) = \mathbb{C}[T]$, we have $K^T(\mathsf{T}^*_e\mathcal{B})\cap(K^T(\mathsf{T}^*\mathcal{B}))^W = \mathbb{C}[T]^{W_{\check\lambda}}$.

Identifying both $K^T(\mathsf{T}^*_e\mathcal{B})$ and $K^T(e)$ with $\mathbb{C}[T]$, the map $\mathfrak{i}^*_e i_{e*}$ is a multiplication by the product $\Delta_1 = \prod_{k=1}^{\dim\mathcal{B}}(1-\chi_k)$ where the χ_k run through the characters of T in the tangent space $\mathsf{T}_e(\mathsf{T}^*_e\mathcal{B}) = \mathsf{T}^*_e\mathcal{B}$. Furthermore, identifying $K^G(\mathcal{B})$ with $\mathbb{C}[T]^{W_{\check\lambda}}$, and $K^T(e)$ with $\mathbb{C}[T]$, the map $\mathfrak{i}^*_e \imath_* \operatorname{Res}^G_T$ is a multiplication by the product $\Delta_2 = \prod_{k=1}^{\dim\mathcal{B}}(1-\chi'_k)$ where the χ'_k run through the characters of T in the tangent space $\mathsf{T}_e\mathcal{B}$. We can arrange the characters χ'_k so that we have $\chi'_k = \chi_k^{-1}$. Then we see that $\Delta_1 = \Delta_2 \prod_{k=1}^{\dim\mathcal{B}}(-\chi_k)$, so they differ by an invertible function. Hence the corresponding images coincide: $\Delta_1\mathbb{C}[T]^{W_{\check\lambda}} = \Delta_2\mathbb{C}[T]^{W_{\check\lambda}}$.

This completes the proof of the lemma along with Theorem 7.3.

7.9 Conjecture. In this subsection we describe (without striving for high precision) a conjectural picture motivating Theorem 7.3. We hope that the isomorphism $K^{G(\mathbf{O})}(\Lambda_G) = \mathbb{C}[\check T \times T]^W = \mathbb{C}[T \times \check T]^W = K^{\check G(\mathbf{O})}(\Lambda_{\check G})$ lifts to an equivalence of monoidal categories $F: D^b Coh^{G(\mathbf{O})}_{\Lambda_G}(\underline{\mathfrak{u}}_G) \simeq D^b Coh^{\check G(\mathbf{O})}_{\Lambda_{\check G}}(\underline{\mathfrak{u}}_{\check G})$. The conjectural equivalence F is related to the Langlands correspondence in the following way.

Recall that the conjectural (for $G = GL(n)$ mostly proven in [Gai02]) geometric Langlands correspondence is an equivalence of triangulated categories between the derived category of

D-modules on the stack Bun_G of G-bundles on a given smooth projective curve C, and the derived category of coherent sheaves on the stack of $\check{G}$ local systems on the same curve. One might expect its 'classical limit' to be an equivalence between the derived categories of coherent sheaves $L: \ D(\mathsf{T}^*\mathrm{Bun}_G) \simeq D(\mathsf{T}^*\mathrm{Bun}_{\check{G}})$ where $\mathsf{T}^*\mathrm{Bun}_G$ is the cotangent bundle to the moduli stack of G-bundles on C. Given a point $c \in C$, and identifying $\mathbf{O}$ with the algebra of functions on the formal neighborhood of c, one gets an action of $D^b Coh^{G(\mathbf{O})}_{\Lambda_G}(\underline{\mathfrak{u}}_G)$ on $D(\mathsf{T}^*\mathrm{Bun}_G)$. The 'classical limit' of the Hecke eigenproperty of geometric Langlands correspondence (see [BD00]) should be stated in terms of this action; it should say that the global equivalence L is compatible with our local equivalence F. □

8. Perverse sheaves and fusion

We refer the reader to [Bez00] for the definition of perverse equivariant coherent sheaves and related objects.

8.1 Recall the setup of § 6.1. Note that all the $G(\mathbf{O})$-orbits in a connected component of Gr_G have dimensions of the same parity. Thus it makes sense to consider the middle perversity function $p(\mathrm{Gr}_{G,\check{\lambda}}) = -\frac{1}{2}\dim(\mathrm{Gr}_{G,\check{\lambda}}) = -\langle\rho,\check{\lambda}\rangle$. It is obviously strictly monotone and comonotone, but at some connected components of Gr_G it takes values in half-integers. This means that we consider equivariant complexes formally placed in half-integer homological degrees. The theory of [Bez00] defines the artinian abelian category $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ of perverse $G(\mathbf{O})$-equivariant coherent sheaves (with respect to the above middle perversity). Let $D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$ denote the bounded derived category of $G(\mathbf{O})$-equivariant coherent sheaves on Gr_G (with the same convention that the complexes at 'odd' connected components are placed in half-integer homological degrees).

Given two complexes $\mathcal{F},\mathcal{G} \in D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$ we have their convolution $\mathcal{F}\star\mathcal{G} \in D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$. Recall that $\mathcal{F}\star\mathcal{G} = \Pi_{0*}(\mathcal{F}\ltimes\mathcal{G})$ where $\Pi_0: \ G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G \to \mathrm{Gr}_G$ is the convolution diagram, and $\mathcal{F}\ltimes\mathcal{G}$ is the twisted product of $\mathcal{F}$ and $\mathcal{G}$ on $G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G$.

Proposition 8.2. *The convolution preserves perverse sheaves: for $\mathcal{F},\mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ we have $\mathcal{F}\star\mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$.*

Proof. Denote the projection $G(\mathbf{F}) \to G(\mathbf{F})/G(\mathbf{O}) = \mathrm{Gr}_G$ by p, and consider a stratification $G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G = \bigsqcup_{\check{\lambda},\check{\mu}\in Y^+} p^{-1}(\mathrm{Gr}_{G,\check{\lambda}})\times_{G(\mathbf{O})}\mathrm{Gr}_{G,\check{\mu}}$. Clearly, $\mathcal{F}\ltimes\mathcal{G}$ is smooth (locally free) along this stratification, and perverse (with respect to the middle perversity). According to [MV00, 2.7], the map Π_0 is stratified semismall with respect to the above stratification. Now the perversity of $\Pi_{0*}(\mathcal{F}\ltimes\mathcal{G})$ follows in the same manner as in the constructible case (cf. [MV00]). □

8.3 The absence of commutativity constraint. According to Proposition 8.2, $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ acquires the structure of abelian artinian monoidal category. Moreover, according to Theorem 2.15, part (a), its K-ring is commutative. Nevertheless, $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ admits no commutativity constraint, as can be seen in the following example.

We recall the setup of § 3.6, and consider Gr_{PGL_2}. One can check that there are the nonsplit exact sequences in $\mathcal{P}^{PGL_2(\mathbf{O})}(\mathrm{Gr}_{PGL_2})$:

$$0 \to \mathcal{V}(0)_0 \to \mathcal{V}(0)_1\star\mathcal{V}(-2)_1 \to \mathcal{V}(-2)_2 \to 0,$$
$$0 \to \mathcal{V}(-2)_2 \to \mathcal{V}(-2)_1\star\mathcal{V}(0)_1 \to \mathcal{V}(0)_0 \to 0.$$

Thus $\mathcal{V}(0)_1\star\mathcal{V}(-2)_1$ and $\mathcal{V}(-2)_1\star\mathcal{V}(0)_1$ are nonisomorphic.

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8.4 $G(\mathbf{O}) \ltimes \mathbb{G}_m$-equivariant sheaves and fusion. The orbits of $G(\mathbf{O}) \ltimes \mathbb{G}_m$ on Gr_G coincide with the $G(\mathbf{O})$-orbits, so one can consider the abelian artinian monoidal category $\mathcal{P}^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ of $G(\mathbf{O}) \ltimes \mathbb{G}_m$-equivariant coherent perverse sheaves on Gr_G. For $\mathcal{F} \in \mathcal{P}^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we have $R\Gamma(\mathrm{Gr}_G, \mathcal{F}) \in D^b(G(\mathbf{O}) \ltimes \mathbb{G}_m - \mathrm{mod})$.

Feigin and Loktev define (under certain restrictions) in [FL99] the *fusion product* $V_1 \star \cdots \star V_k \in G(\mathbf{O}) \ltimes \mathbb{G}_m - \mathrm{mod}$ of $G(\mathbf{O}) \ltimes \mathbb{G}_m$-modules $V_1, \ldots, V_k$. We recall some of their results in the case $G = PGL_2$.

Let $V(n)$ be the $(n+1)$-dimensional $G(\mathbf{O}) \ltimes \mathbb{G}_m$-module factoring through $G(\mathbf{O}) \ltimes \mathbb{G}_m \twoheadrightarrow G \times \mathbb{G}_m \twoheadrightarrow G$. Recall the irreducible $PGL_2(\mathbf{O})$-equivariant perverse sheaf $\mathcal{V}(n)_m$ introduced in § 3.6. It can be lifted to the same named $PGL_2(\mathbf{O}) \ltimes \mathbb{G}_m$-equivariant perverse sheaf, where the action of $\mathbb{G}_m$ in the fiber over a $\mathbb{G}_m$-fixed point in the orbit $\mathrm{Gr}_{PGL_2,m}$ is set *trivial*. In particular, $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n)_1) = V(n)[\frac{1}{2}]$ for $n \geqslant 0$.

Now we can reformulate Theorem 2.5 of [FL99] as follows.

Proposition 8.5. *Let $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Then*

(a) *$R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \cdots \star \mathcal{V}(n_k)_1)$ is concentrated in degree $-k/2$;*

(b) *$R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \cdots \star \mathcal{V}(n_k)_1)[-k/2] \simeq V(n_k) \star \cdots \star V(n_1)$.*

8.6 Multiplication table. According to Proposition 8.5, the calculation of fusion product in $K(G(\mathbf{O}) \ltimes \mathbb{G}_m - \mathrm{mod})$ is closely related to the ring structure of $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. Let us formulate the recurrence relations in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ (compare [FL99, end of § 2.1]). So $\mathbf{v}(n)_m$ is the class of $\mathcal{V}(n)_m$ in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. We assume that $n \geqslant 0$. Then we obtain

$$q^{-l}\mathbf{v}(l+n)_1 0 \star \mathbf{v}(l)_1 = q^{-2l}\mathbf{v}(2l+n)_2 + q^2\mathbf{v}(n-2)_0 + q^4\mathbf{v}(n-4)_0 + \cdots \tag{12}$$

(the last summand being $q^n\mathbf{v}(0)_0$ if n is even, and $q^{n-1}\mathbf{v}(1)_0$ if n is odd),

$$q^{-l-2}\mathbf{v}(l-n)_1 0 \star \mathbf{v}(l)_1 = q^{-2l-2}\mathbf{v}(2l-n)_2 + q^{-2}\mathbf{v}(n-2)_0 + q^{-4}\mathbf{v}(n-4)_0 + \cdots \tag{13}$$

(the last summand being $q^{-n}\mathbf{v}(0)_0$ if n is even, and $q^{-n+1}\mathbf{v}(1)_0$ if n is odd),

$$\mathbf{v}(l+1)_1^{\star a} \star \mathbf{v}(l)_1^{\star b} = q^{\frac{1}{2}(a(1-a)+l(a+b)(1-a-b))}\mathbf{v}(a+l(a+b))_{a+b}. \tag{14}$$

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