Equivariant K-theory of Hilbert schemes via shuffle algebra

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Abstract In this paper we construct the action of *Ding-Iohara* and *shuffle* algebras on the sum of localized equivariant K-groups of Hilbert schemes of points on $\mathbb{C}^2$. We show that commutative elements K_i of shuffle algebra act through vertex operators over the positive part $\{\mathfrak{h}_i\}_{i>0}$ of the Heisenberg algebra in these K-groups. Hence we get an action of Heisenberg algebra itself. Finally, we normalize the basis of the structure sheaves of fixed points in such a way that it corresponds to the basis of Macdonald polynomials in the Fock space $\mathbb{C}[\mathfrak{h}_1, \mathfrak{h}_2, \ldots]$.

1. Introduction

For any surface X, let $X^{[n]}$ denote the Hilbert scheme of n points on X. The Heisenberg algebra $\{\mathfrak{h}_i\}_{i\in\mathbb{Z}\setminus 0}$ is known (see [8]) to act through natural correspondences in the sum of cohomology rings $\bigoplus_n H^*(X^{[n]})$.

From now on we deal only with the case $X = \mathbb{C}^2$. Then one can consider localized equivariant cohomologies instead of usual cohomologies. Let $R = \bigoplus_n H^{2n}_{\mathbb{T}}(X^{[n]}) \otimes_{H_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(H_{\mathbb{T}}(\mathrm{pt}))$. As shown in [6], R is isomorphic to the Fock space $\Lambda_F := \mathbb{C}(\hbar, \hbar')[\mathfrak{h}_1, \mathfrak{h}_2, \ldots]$, and after certain normalization, there is an isomorphism $\Delta : R \to \Lambda_F$ sending the basis of fixed points to Jack polynomials and $\{\mathfrak{h}_i\}_{i>0}$ to operators of multiplication by p_i.

In this paper we construct the action of $1+\sum_{i>0} \widetilde{K}_i z^i := \exp\bigl(\sum_{i>0}((-1)^{i-1}/i)\mathfrak{h}_i z^i\bigr)$ on the sum of localized equivariant K-groups $M = \bigoplus_n K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt}))$ in geometric terms. This determines the action of the Heisenberg algebra itself. We find an isomorphism $\Theta : M \to \Lambda_F$ which takes the normalized fixed point basis $\{\langle\lambda\rangle\}$ to Macdonald polynomials $\{P_\lambda\}$. Isomorphism Θ takes operators $\widetilde{K}_i$ to operators of multiplication by e_i acting in the Macdonald polynomials basis through Pieri formulas.

For achieving this result and for its own sake, we construct representations of two algebras: A and S (called *Ding-Iohara* and *shuffle* algebras correspondingly) on M. In fact, the subalgebra $\mathcal{S}$ of *shuffle* algebra S, generated by S_1, is of particular interest to us. It is a trigonometric analogue of Feigin-Odesskii algebra, studied in [5].

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The same operators appear in [10], where the action of the Hall algebra of an elliptic curve is constructed on R. We strongly recommend [10] to the reader as its results are more complete than ours and contain a completely different viewpoint.

In Section 2 we define Ding-Iohara and shuffle algebras and remind some properties of them. In Section 3 we construct the action of Ding-Iohara algebra A on M. In Section 4 we verify that the constructed operators do give representation of A. In Section 5 we define the action of shuffle algebra on M. In Section 6 we present operators $\widetilde{K_i}$, normalization of the fixed point basis $\{\langle\lambda\rangle\}$, and an isomorphism $\Theta : M \to \Lambda_F$ with the above-mentioned properties. Finally, in Section 7 we consider a vector $v = \sum_{n\geq 0} [O_{X^{[n]}}]$. We prove an analogue of Proposition 2.31 from [1] for it; that is, we show that this vector is an eigenvector for the negative half of our Heisenberg algebra.

2. Ding-Iohara and shuffle algebras

Let us fix any parameters q_1, q_2, q_3. Now we define the *Ding-Iohara algebra* A. This is an associative algebra generated by $e_i, f_i, \psi_j^{\pm}$ $(i \in \mathbb{Z}, j \in \mathbb{Z}_+)$ with the following defining relations:

$$\begin{aligned} &e(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w) \\ &\quad = -e(w)e(z)(w-q_1z)(w-q_2z)(w-q_3z), \end{aligned} \tag{1}$$

$$\begin{aligned} &f(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) \\ &\quad = -f(w)f(z)(z-q_1w)(z-q_2w)(z-q_3w), \end{aligned} \tag{2}$$

$$[e(z), f(w)] = \frac{\delta(z/w)}{(1-q_1)(1-q_2)(1-q_3)}\big(\psi^+(w) - \psi^-(z)\big), \tag{3}$$

$$\begin{aligned} &\psi^{\pm}(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w) \\ &\quad = -e(w)\psi^{\pm}(z)(w-q_1z)(w-q_2z)(w-q_3z), \end{aligned} \tag{4}$$

$$\begin{aligned} &\psi^{\pm}(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) \\ &\quad = -f(w)\psi^{\pm}(z)(z-q_1w)(z-q_2w)(z-q_3w), \end{aligned} \tag{5}$$

where the generating series are defined as follows:

$$e(z) = \sum_{i=-\infty}^{\infty} e_i z^{-i}, \qquad f(z) = \sum_{i=-\infty}^{\infty} f_i z^{-i},$$

$$\psi^{\pm}(z) = \sum_{j\geq 0} \psi_j^{\pm} z^{\mp j}, \qquad \delta(z) = \sum_{i=-\infty}^{\infty} z^i.$$

REMARK

These relations are very similar to the relations of quantum affine algebras (except for Serre relations).

We denote by A_+ (A_-) the subalgebra of A generated by $e_i(f_i)$ correspondingly.

Following [5], we define a shuffle algebra S depending on q_1,q_2,q_3. Fix a function

$$\lambda(x,y)=\frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3}.$$

The algebra S is an associative graded algebra $S=\bigoplus_{n\geq 0}S_n$. Each graded component S_n consists of rational functions of the form $F(x_1,\dots,x_n)=f(x_1,\dots,x_n)/\bigl(\prod_{1\leq i<j\leq n}(x_i-x_j)^2\bigr)$, where $f(x_1,\dots,x_n)$ is a symmetric Laurent polynomial. For $F\in S_m$ and $G\in S_n$, the product $F*G\in S_{n+m}$ is defined by the formula

$$(F*G)(x_1,\dots,x_{n+m})$$
$$=\operatorname{Sym}\Bigl(F(x_1,\dots,x_m)G(x_{m+1},\dots,x_{m+n})\prod_{1\leq i\leq m<j\leq m+n}\lambda(x_i,x_j)\Bigr).$$

Here the symbol Sym stands for the symmetrization. This endows S with a structure of an associative algebra.

Now we formulate some known properties of shuffle algebras.

THEOREM 2.1

For general parameters q_1,q_2,q_3, there is a natural isomorphism $\Xi:A_+\to S$, which takes $e_a\in A_+$ into $x^a\in S$. In particular, the whole algebra S is generated by S_1.

THEOREM 2.2

In the case where q_1,q_2 are generic and $q_1q_2q_3=1$, the subalgebra $\mathcal{S}$ generated by S_1 consists of rational functions of the form $F(x_1,\dots,x_n)=f(x_1,\dots,x_n)/\bigl(\prod_{1\leq i<j\leq n}(x_i-x_j)^2\bigr)$, where $f(x_1,\dots,x_n)$ is a symmetric Laurent polynomial satisfying $f(x_1,\dots,x_n)=0$ if $x_1/x_2=q_1, x_2/x_3=q_j$ for $j=2,3$.

REMARK 2.1

For any parameters the connection between Ding-Iohara algebra A and shuffle algebra S can be established in the following way (here algebras A and S are considered with the same parameters q_1,q_2,q_3). Let I_e be the kernel of the map Ξ from Theorem 2.1. (This theorem claims that I_e is trivial for generic parameters.) Denote by I_f the transposed ideal of A_-; that is, I_f is obtained from I_e by the change $e_i\longmapsto f_{-i}$. Then the factor of A by ideals I_f,I_e is just what we are most interested in. It may be viewed as a double of the shuffle algebra S.

In fact, the description of the ideal I_e is conjectured in [2] and is recently proved in [9].

THEOREM 2.3

For each $n \geq 1$, define elements $K_n \in S_n$ by

$$K_2(z_1,z_2) = \frac{(z_1 - q_1 z_2)(z_2 - q_1 z_1)}{(z_1 - z_2)^2}, \qquad K_n(z_1,\dots,z_n) = \prod_{1 \leq i < j \leq n} K_2(z_i,z_j).$$

In particular, $K_1(z) = 1$. If $q_1 q_2 q_3 = 1$, elements $K_n \in S_n$ commute.

The subalgebra generated by K_i is studied in [4]; in particular, Theorem 2.3 is proved here.

This work was motivated by [3] and [11].

3. Construction of operators

3.1. Correspondences

We recall that throughout this paper $X = \mathbb{C}^2$. In this case the Hilbert scheme of n points $X^{[n]}$ as a set is identified with the set of all ideals in $\mathbb{C}[x,y]$ of codimension n. Let us recall correspondences used by H. Nakajima to construct a representation of the Heisenberg algebra on $\bigoplus_n \mathbb{H}^*(X^{[n]})$. This action is constructed through the correspondences $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$. Though in the future we will need only $P[1], P[-1]$, let us mention the general definition of $P[i]$ for any i. For $i > 0$, the correspondence $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$ consists of all pairs of ideals (J_1, J_2) of $\mathbb{C}[x,y]$ of codimension $n, n+i$ correspondingly, such that $J_2 \subset J_1$ and the factor J_1/J_2 is supported at a single point. For $i = 1$ this condition is automatic. For $i < 0$, $P[i]$ is transposed to $P[-i]$. Let L be a tautological line bundle on $P[1]$ whose fiber at any point $(J_1,J_2) \in P[1]$ equals J_1/J_2. There are natural projections $\mathbf{p}, \mathbf{q}$ from $P[1]$ to $X^{[n]}$ and $X^{[n+1]}$ correspondingly.

3.2. Fixed points

There is a natural action of $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ on each $X^{[n]}$ induced from the one on X given by the formula $(t_1,t_2)(x,y) = (t_1 \cdot x, t_2 \cdot y)$. The set $(X^{[n]})^{\mathbb{T}}$ of $\mathbb{T}$-fixed points in $X^{[n]}$ is finite, and all these fixed points are parameterized by Young diagrams of size n. Namely, for each diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ we have an ideal $(t_1^{\lambda_1}, t_1^{\lambda_2} t_2, \dots, t_1^{\lambda_k} t_2^{k-1}, t_2^k) =: J_\lambda \in (X^{[n]})^{\mathbb{T}}$.

3.3. Equivariant K-groups

We denote by $'M$ the direct sum of equivariant (complexified) K-groups: $'M = \bigoplus_n K^{\mathbb{T}}(X^{[n]})$. It is a module over $K^{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathbb{T}] = \mathbb{C}[t_1,t_2]$. We define $M = {}'M \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt})) = {}'M \otimes_{\mathbb{C}[t_1,t_2]} \mathbb{C}(t_1,t_2)$.

We have an evident grading

$$M = \bigoplus_n M_n, \quad M_n = K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}\big(K^{\mathbb{T}}(\mathrm{pt})\big).$$

According to the Thomason localization theorem, restriction to the $\mathbb{T}$-fixed point set induces an isomorphism

$$K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}\big(K^{\mathbb{T}}(\mathrm{pt})\big) \to K^{\mathbb{T}}\big((X^{[n]})^{\mathbb{T}}\big) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}\big(K^{\mathbb{T}}(\mathrm{pt})\big).$$

The structure sheaves $\{\lambda\}$ of the $\mathbb{T}$-fixed points λ (see Section 3.2) form a basis in $\bigoplus_n K^{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt}))$. The embedding of a point λ into $X^{[n]}$ is a proper morphism, so the direct image in the equivariant K-theory is well defined, and we denote by $[\lambda] \in M_n$ the direct image of the structure sheaf $\{\lambda\}$. The set $[\lambda]$ forms a basis of M.

3.4. Representation of Ding-Iohara algebra on M

Let us now consider the tautological vector bundle $\mathfrak{F}$ on $X^{[n]}$, whose fiber at the point corresponding to an ideal J equals $\mathbb{C}[x,y]/J$. We introduce generating series $\mathbf{a}(z), \mathbf{c}(z)$ as follows:

$$\mathbf{a}(z) := \Lambda_{-1/z}(\mathfrak{F}) = \sum_{i \geq 0} \Lambda^i(\mathfrak{F}) \Big(\frac{-1}{z}\Big)^i,$$

$$\mathbf{c}(z) := \mathbf{a}(zt_1)\mathbf{a}(zt_2)\mathbf{a}(zt_1^{-1}t_2^{-1})\mathbf{a}(zt_1^{-1})^{-1}\mathbf{a}(zt_2^{-1})^{-1}\mathbf{a}(zt_1t_2)^{-1}.$$

We also define the operators

$$e_i = \mathbf{q}_*(L^{\otimes i} \otimes \mathbf{p}^*) : M_n \to M_{n+1}, \tag{6}$$

$$f_i = \mathbf{p}_*(L^{\otimes (i-1)} \otimes \mathbf{q}^*) : M_n \to M_{n-1}. \tag{7}$$

So e_i is a composition of pulling back by $P[1] \to M_n$, tensoring by $L^{\otimes i}$, and finally pushing forward along $P[1] \to M_{n+1}$, while f_{i+1} is obtained by the inverse order of these operations.

We consider the following generating series of operators acting on M:

$$e(z) = \sum_{r=-\infty}^{\infty} e_r z^{-r} : M_n \to M_{n+1}[[z, z^{-1}]], \tag{8}$$

$$f(z) = \sum_{r=-\infty}^{\infty} f_r z^{-r} : M_n \to M_{n-1}[[z, z^{-1}]], \tag{9}$$

$$\psi^+(z)|_{M_n} = \sum_{r=0}^{\infty} \psi_r^+ z^{-r} := \Big(-\frac{1 - t_1^{-1}t_2^{-1}z^{-1}}{1 - z^{-1}} \mathbf{c}(z)\Big)^+ \in M_n[[z^{-1}]], \tag{10}$$

$$\psi^-(z)|_{M_n} = \sum_{r=0}^{\infty} \psi_r^- z^{r} := \Big(-\frac{1 - t_1^{-1}t_2^{-1}z^{-1}}{1 - z^{-1}} \mathbf{c}(z)\Big)^- \in M_n[[z]], \tag{11}$$

where $()^{\pm}$ denotes the expansion at $z = \infty, 0$, respectively.

Formulas (10) and (11) should be understood as follows: $\psi^{\pm}(z)$ acts by multiplication in K-theory by $\big(-((1 - t_1^{-1}t_2^{-1}z^{-1})/(1 - z^{-1}))\mathbf{c}(z)\big)^{\pm}$, and $\psi_r^{\pm}$ are defined as the coefficients of these series.

THEOREM 3.1

The operators $e_i, f_i, \psi_j^{\pm}$ satisfy relations (1)–(5) with parameters $q_1 = t_1, q_2 = t_2, q_3 = t_1^{-1}t_2^{-1}$; that is, they give a representation of algebra A on the sum of localized equivariant K-groups of Hilbert schemes of points on $\mathbb{C}^2$.

This theorem is proven in Section 4.

Now we compute the matrix coefficients of the operators e_i, f_i and the eigenvalues of $\psi^{\pm}(z)$ in the fixed point basis. Let us take any diagram $\lambda=(\lambda_1,\dots,\lambda_k)$ and its box $\square_{i,j}$ with the coordinates (i,j) (i.e., it stands in the ith row and jth column), where $1\le i\le k, 1\le j\le \lambda_i$. We introduce functions $l(\square)$ and $a(\square)$, called *legs* and *arms*, correspondingly:

$$l(\square):=\lambda_i-j,\qquad a(\square):=\max\{k\mid \lambda_k\ge j\}-i.$$

We also denote by $\Sigma_1(\square)$ all boxes of λ with the coordinates $(i,k<j)$ and by $\Sigma_2(\square)$ all boxes of λ with the coordinates $(k<i,j)$. Sometimes we write $\lambda+j$ for the diagram $\lambda+\square_{j,\lambda_j+1}$ if it makes sense (i.e., if it is still a diagram). Finally, we call box $\square_{i,j}$ a *corner* if $j=\lambda_i>\lambda_{i+1}$ and a *hole* if $j=\lambda_i+1\le\lambda_{i-1}$.

LEMMA 3.1

(a) *The matrix coefficients of the operators e_i, f_i in the fixed point basis $[\lambda]$ of M are as follows:*

$$e_{i[\lambda,\lambda+k]}=(1-t_1)^{-1}(1-t_2)^{-1}(t_1^{\lambda_k}t_2^{k-1})^i$$
$$\times\prod_{s\in\Sigma_1(\square_{k,\lambda_k+1})}\frac{1-t_1^{-l(s)+1}t_2^{a(s)+1}}{1-t_1^{-l(s)}t_2^{a(s)+1}}\prod_{s\in\Sigma_2(\square_{k,\lambda_k+1})}\frac{1-t_1^{l(s)+1}t_2^{-a(s)+1}}{1-t_1^{l(s)+1}t_2^{-a(s)}},$$
$$f_{i[\lambda,\lambda-k]}=(t_1^{\lambda_k-1}t_2^{k-1})^{i-1}$$
$$\times\prod_{s\in\Sigma_1(\square_{k,\lambda_k})}\frac{1-t_1^{l(s)+1}t_2^{-a(s)}}{1-t_1^{l(s)}t_2^{-a(s)}}\prod_{s\in\Sigma_2(\square_{k,\lambda_k})}\frac{1-t_1^{-l(s)}t_2^{a(s)+1}}{1-t_1^{-l(s)}t_2^{a(s)}}.$$

All the other matrix coefficients of e_i, f_i vanish.

(b) *The eigenvalue of $\psi^{\pm}(z)$ on $[\lambda]$ equals*

$$\Big(-\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}}\prod_{\square\in\lambda}\frac{(1-t_1^{-1}\chi(\square)z^{-1})(1-t_2^{-1}\chi(\square)z^{-1})(1-t_1t_2\chi(\square)z^{-1})}{(1-t_1\chi(\square)z^{-1})(1-t_2\chi(\square)z^{-1})(1-t_1^{-1}t_2^{-1}\chi(\square)z^{-1})}\Big)^{\pm},$$

where $\chi(\square_{i,j})=t_1^{j-1}t_2^{i-1}$.

Proof

(a) For $(\lambda,\lambda')\in P[1]$, let $\rho: J_{\lambda'}\hookrightarrow J_\lambda, \pi: k[x,y]/J_{\lambda'}\twoheadrightarrow k[x,y]/J_\lambda$ be the natural maps. The tangent space $\mathfrak{T}_{(J_\lambda,J_{\lambda'})}(P[1])$ is a kernel of the map $\mathrm{Hom}(J_{\lambda'},k[x,y]/J_{\lambda'})\oplus\mathrm{Hom}(J_\lambda,k[x,y]/J_\lambda)\twoheadrightarrow\mathrm{Hom}(J_{\lambda'},k[x,y]/J_\lambda)$, which sends $(\alpha,\beta)\longmapsto\pi\circ\alpha-\beta\circ\rho$. Further, we write simply λ instead of J_λ.

Let us denote by $\chi_{(\lambda,\lambda')}$ the character of $\mathbb{T}$ in the tangent space $\mathfrak{T}_{(\lambda,\lambda')}(P[1])$ and by $\chi(L)_{(\lambda,\lambda')}$ the character of $\mathbb{T}$ in the fiber of L at the point (λ,λ'). We write $S\chi_{(\lambda)}$ (resp., $S\chi_{(\lambda,\lambda')}$) for the character of $\mathbb{T}$ in the symmetric algebra $\mathrm{Sym}^{\bullet}\mathfrak{T}_{(\lambda)}X^{[n]}$ (resp., $\mathrm{Sym}^{\bullet}\mathfrak{T}_{(\lambda,\lambda')}P[1]$).

According to the Bott-Lefschetz fixed point formula, the matrix coefficient $\mathbf{p}_*(L^{\otimes i}\otimes \mathbf{q}^*)_{[\lambda',\lambda]}$ of $\mathbf{p}_*(L^{\otimes i}\otimes \mathbf{q}^*): M_{n+1}\to M_n$ with respect to the basis elements $[\lambda]\in K^{\mathbb{T}}(X^{[n]}),[\lambda']\in K^{\mathbb{T}}(X^{[n+1]})$ equals $\chi(L)^i_{(\lambda,\lambda')}S\chi_{(\lambda,\lambda')}/S\chi_{(\lambda')}$. Similarly, the matrix coefficient $\mathbf{q}_*(L^{\otimes i}\otimes \mathbf{p}^*)_{(\lambda,\lambda')}$ of $\mathbf{q}_*(L^{\otimes i}\otimes \mathbf{p}^*): M_n\to M_{n+1}$ with respect to the basis elements $[\lambda]\in K^{\mathbb{T}}(X^{[n]})$, $[\lambda']\in K^{\mathbb{T}}(X^{[n+1]})$ equals $\chi(L)^i_{(\lambda,\lambda')}S\chi_{(\lambda,\lambda')}/S\chi_{(\lambda)}$.

Now it is straightforward to check the formulas.

(b) This follows from the exactness of $\Lambda_z(F):=\sum_{i\geq 0}\Lambda^i(F)z^i$ on the category of coherent sheaves and the fact that $\{\chi(\square)\mid \square\in\lambda\}-$ is a set of characters of $\mathbb{T}$ at the fiber $\mathfrak{F}|_\lambda$. □

Sometimes we use other expressions for the matrix coefficients of operators e_i, f_i.

PROPOSITION 3.1

$$e_{r[\lambda-i,\lambda]}=\frac{(t_1^{\lambda_i-1}t_2^{i-1})^r}{(1-t_1^{\lambda_1-\lambda_i+1}t_2^{1-i})(1-t_1t_2)}\prod_{j=1}^{\infty}\frac{1-t_1^{\lambda_j-\lambda_i+1}t_2^{j-i+1}}{1-t_1^{\lambda_{j+1}-\lambda_i+1}t_2^{j-i+1}},$$

$$f_{r[\lambda+i,\lambda]}=\frac{(t_1^{\lambda_i}t_2^{i-1})^{r-1}(1-t_1^{\lambda_i-\lambda_1+1}t_2^{i})}{1-t_1t_2}\prod_{j=1}^{\infty}\frac{1-t_1^{\lambda_i-\lambda_{j+1}+1}t_2^{i-j}}{1-t_1^{\lambda_i-\lambda_j+1}t_2^{i-j}}.$$

Proof

It is straightforward to get these formulas from Lemma 3.1. □

4. Proof of Theorem 3.1

DEFINITION 4.1

We denote by $\sigma_1,\sigma_2,\sigma_3$ the elementary symmetric polynomials in q_1,q_2,q_3, that is, $\sigma_1:=q_1+q_2+q_3=t_1+t_2+t_1^{-1}t_2^{-1}$, $\sigma_2:=q_1q_2+q_1q_3+q_2q_3=q_1^{-1}+q_2^{-1}+q_3^{-1}=t_1^{-1}+t_2^{-1}+t_1t_2$, $\sigma_3:=q_1q_2q_3=1$.

CONVENTION 4.1

In this section we check (1)–(5) explicitly in the fixed point basis. While comparing expressions of the left-hand side and right-hand side, we denote by P_i the mutual factor.

First, let us check (1).

Proof

For any integers i,j we have to prove the following equation:

$$e_{i+3}e_j-\sigma_1e_{i+2}e_{j+1}+\sigma_2e_{i+1}e_{j+2}-\sigma_3e_ie_{j+3}$$
$$=\sigma_3e_je_{i+3}-\sigma_2e_{j+1}e_{i+2}+\sigma_1e_{j+2}e_{i+1}-e_{j+3}e_i.$$

Let us compare the matrix elements of the left-hand side and right-hand side on any pair of Young diagrams $[\lambda, \lambda' = \lambda + \square_{i_1,j_1} + \square_{i_2,j_2}]$.

(a) Suppose $i_1 = i_2$; that is, the added two boxes lie in the same row:

$$(e_{i+3}e_j - \sigma_1 e_{i+2}e_{j+1} + \sigma_2 e_{i+1}e_{j+2} - \sigma_3 e_i e_{j+3})_{[\lambda,\lambda']}$$
$$= (\ldots)(1 - \sigma_1 t_1^{-1} + \sigma_2 t_1^{-2} - \sigma_3 t_1^{-3}) = 0$$

since t_1^{-1} is a root of $1 - \sigma_1 t + \sigma_2 t^2 - \sigma_3 t^3$.

Similarly $(\sigma_3 e_j e_{i+3} - \sigma_2 e_{j+1}e_{i+2} + \sigma_1 e_{j+2}e_{i+1} - e_{j+3}e_i)_{[\lambda,\lambda']} = 0$.

(b) Suppose $j_1 = j_2$; that is, the added two boxes lie in the same column.

This case is entirely similar since t_2^{-1} is also a root of $1 - \sigma_1 t + \sigma_2 t^2 - \sigma_3 t^3$.

(c) Suppose $i_1 < i_2, j_1 > j_2$.

The only difference occurs in the box $\square_{i_1,j_2}$.

Let us denote $a := j_1 - j_2, b := i_2 - i_1, \chi_1 := t_1^{j_1-1}t_2^{i_1-1}, \chi_2 := t_1^{j_2-1}t_2^{i_2-1}$. Then

$$(e_{i+3}e_j - \sigma_1 e_{i+2}e_{j+1} + \sigma_2 e_{i+1}e_{j+2} - \sigma_3 e_i e_{j+3})_{[\lambda,\lambda']}$$
$$= P_1(1 - t_1^{-a}t_2^{b})^{-1}(1 - t_1^{-a+1}t_2^{b})(1 - t_1^{a+1}t_2^{-b})^{-1}(1 - t_1^{a+1}t_2^{-b+1})\chi_1^j\chi_2^{i+3}$$
$$\times \left(1 - \sigma_1\left(\frac{\chi_1}{\chi_2}\right) + \sigma_2\left(\frac{\chi_1}{\chi_2}\right)^2 - \sigma_3\left(\frac{\chi_1}{\chi_2}\right)^3\right)$$
$$+ P_1(1 - t_1^{a}t_2^{-b})^{-1}(1 - t_1^{a}t_2^{-b+1})(1 - t_1^{-a}t_2^{b+1})^{-1}(1 - t_1^{-a+1}t_2^{b+1})\chi_1^{i+3}\chi_2^j$$
$$\times \left(1 - \sigma_1\left(\frac{\chi_2}{\chi_1}\right) + \sigma_2\left(\frac{\chi_2}{\chi_1}\right)^2 - \sigma_3\left(\frac{\chi_2}{\chi_1}\right)^3\right),$$

$$(\sigma_3 e_j e_{i+3} - \sigma_2 e_{j+1}e_{i+2} + \sigma_1 e_{j+2}e_{i+1} - e_{j+3}e_i)_{[\lambda,\lambda']}$$
$$= P_1(1 - t_1^{a}t_2^{-b})^{-1}(1 - t_1^{a}t_2^{-b+1})(1 - t_1^{-a}t_2^{b+1})^{-1}(1 - t_1^{-a+1}t_2^{b+1})\chi_1^j\chi_2^{i+3}$$
$$\times \left(\sigma_3 - \sigma_2\left(\frac{\chi_1}{\chi_2}\right) + \sigma_1\left(\frac{\chi_1}{\chi_2}\right)^2 - \left(\frac{\chi_1}{\chi_2}\right)^3\right)$$
$$+ P_1(1 - t_1^{-a}t_2^{b})^{-1}(1 - t_1^{-a+1}t_2^{b})(1 - t_1^{a+1}t_2^{-b})^{-1}(1 - t_1^{a+1}t_2^{-b+1})\chi_1^{i+3}\chi_2^j$$
$$\times \left(\sigma_3 - \sigma_2\left(\frac{\chi_2}{\chi_1}\right) + \sigma_1\left(\frac{\chi_2}{\chi_1}\right)^2 - \left(\frac{\chi_2}{\chi_1}\right)^3\right).$$

Denote $u := t_1^a, v := t_2^b$. Then $\chi_2/\chi_1 = u^{-1}v$. So the first summand of the left-hand side equals

$$P_1\chi_1^j\chi_2^{i+3}(u - v)^{-1}(u - t_1 v)(v - t_1 u)^{-1}(v - t_1 t_2 u)$$
$$\times (v - t_1 u)(v - t_2 u)(v - t_1^{-1}t_2^{-1}u)v^{-3}$$
$$= P_1\chi_1^j\chi_2^{i+3}(u - v)^{-1}(u - t_1 v)(v - t_1 t_2 u)(v - t_2 u)(v - t_1^{-1}t_2^{-1}u)v^{-3},$$

while the first summand of the right-hand side equals

$$P_1\chi_1^j\chi_2^{i+3}(v - u)^{-1}(v - t_2 u)(u - t_2 v)^{-1}(u - t_1 t_2 v)$$
$$\times (u - t_1 v)(u - t_2 v)(u - t_1^{-1}t_2^{-1}v)(-u^3 v^{-3})u^{-3}$$
$$= P_1\chi_1^j\chi_2^{i+3}(u - v)^{-1}(v - t_2 u)(u - t_1 t_2 v)(u - t_1 v)(u - t_1^{-1}t_2^{-1}v)v^{-3}.$$

So we get the same expressions for the first summands of the left-hand side and right-hand side. In the same way we check the equality of the second summands. This completes the proof in this case.

(d) Suppose $i_1 > i_2, j_1 < j_2$. This follows from (c). □

Equation (2) is entirely similar to the one above, so we omit it.

Now we compute $[e(z), f(w)]$. We prove the following proposition at first.

PROPOSITION 4.1

The coefficients of the the series $[e(z), f(w)]$ are diagonalizable in the fixed point basis $[\lambda]$.

Proof

We have to check for diagrams $[\lambda, \lambda' = \lambda + \square_{i_1,j_1} - \square_{i_2,j_2}]$ $((i_1, j_1) \neq (i_2, j_2))$ that the equality $(e_i f_j)_{[\lambda,\lambda']} = (f_j e_i)_{[\lambda,\lambda']}$ holds.

Let us consider the case $i_1 < i_2, j_1 > j_2$. (The case $i_1 > i_2, j_1 < j_2$ is completely analogous.) We define $a := j_1 - j_2, b := i_2 - i_1$. Then

$$\begin{aligned}(e_i f_j)_{[\lambda,\lambda']} &= P_2(1 - t_1^{1-a}t_2^b)^{-1}(1 - t_1^{1-a}t_2^{b+1})(1 - t_1^{-a}t_2^b)^{-1}(1 - t_1^{1-a}t_2^b)\\ &= P_2(1 - t_1^{1-a}t_2^{b+1})(1 - t_1^{-a}t_2^b)^{-1},\\ (f_j e_i)_{[\lambda,\lambda']} &= P_2(1 - t_1^{-a}t_2^{b+1})^{-1}(1 - t_1^{1-a}t_2^{b+1})(1 - t_1^{-a}t_2^b)^{-1}(1 - t_1^{-a}t_2^{b+1})\\ &= P_2(1 - t_1^{1-a}t_2^{b+1})(1 - t_1^{-a}t_2^b)^{-1}.\end{aligned}$$

So $(e_i f_j)_{[\lambda,\lambda']} = (f_j e_i)_{[\lambda,\lambda']}$. □

Now we introduce the operators $\Phi^+(z) = \sum_{i=0}^{\infty} \phi_i^+ z^{-i}, \Phi^-(z) = \sum_{i=0}^{\infty} \phi_i^- z^i$ diagonalizable in the fixed point basis and satisfying the equation

$$[e(z), f(w)] = \frac{\delta(z/w)}{(1 - t_1)(1 - t_2)(1 - t_1^{-1}t_2^{-1})}\big(\Phi^+(w) - \Phi^-(z)\big).$$

We show that $\phi_i^\pm$ are determined uniquely by the conditions $\phi_0^+ = -1, \phi_0^- = -1/t_1t_2$. Next we check

$$\begin{aligned}&\phi^\pm(z)e(w)(z - q_1w)(z - q_2w)(z - q_3w)\\ &\quad = -e(w)\phi^\pm(z)(w - q_1z)(w - q_2z)(w - q_3z),\end{aligned} \tag{12}$$

$$\begin{aligned}&\phi^\pm(z)f(w)(w - q_1z)(w - q_2z)(w - q_3z)\\ &\quad = -f(w)\phi^\pm(z)(z - q_1w)(z - q_2w)(z - q_3w).\end{aligned} \tag{13}$$

Finally, by showing that $\psi_i^\pm = \phi_i^\pm$, we get equalities (4) and (5) from equalities (12) and (13). And so Theorem 3.1 will be proved.

From Proposition 4.1 and the formulas of Lemma 3.1(a), one gets that $[e(z), f(w)]$ is diagonalizable in the fixed point basis and, moreover, its eigenvalue on

$[\lambda]$ equals

$$\sum_{a,b\in\mathbb{Z}} z^{-a}w^{-b}\gamma_{a+b},$$

where

$$\gamma_i = (1-t_1)^{-1}(1-t_2)^{-1}\sum_{\square-\text{corner}}\Big(\prod_{s\in\Sigma_1(\square)}\Big[\frac{(1-t_1^{l(s)+1}t_2^{-a(s)})(1-t_1^{-l(s)+1}t_2^{a(s)+1})}{(1-t_1^{l(s)}t_2^{-a(s)})(1-t_1^{-l(s)}t_2^{a(s)+1})}\Big]$$

$$\times\prod_{s\in\Sigma_2(\square)}\Big[\frac{(1-t_1^{-l(s)}t_2^{a(s)+1})(1-t_1^{l(s)+1}t_2^{-a(s)+1})}{(1-t_1^{-l(s)}t_2^{a(s)})(1-t_1^{l(s)+1}t_2^{-a(s)})}\Big]\Big)\chi^{i-1}(\square)$$

$$-(1-t_1)^{-1}(1-t_2)^{-1}\sum_{\square-\text{hole}}\Big(\prod_{s\in\Sigma_1(\square)}\Big[\frac{(1-t_1^{l(s)+1}t_2^{-a(s)})(1-t_1^{-l(s)+1}t_2^{a(s)+1})}{(1-t_1^{l(s)}t_2^{-a(s)})(1-t_1^{-l(s)}t_2^{a(s)+1})}\Big]$$

$$\times\prod_{s\in\Sigma_2(\square)}\Big[\frac{(1-t_1^{-l(s)}t_2^{a(s)+1})(1-t_1^{l(s)+1}t_2^{-a(s)+1})}{(1-t_1^{-l(s)}t_2^{a(s)})(1-t_1^{l(s)+1}t_2^{-a(s)})}\Big]\Big)\chi^{i-1}(\square).$$

Since we want an equality

$$[e(z),f(w)] = \frac{\delta\big(\frac{z}{w}\big)(\Phi^+(w)-\Phi^-(z))}{(1-t_1)(1-t_2)(1-t_1^{-1}t_2^{-1})}$$
$$= \frac{\big(\sum_{a+b>0} z^{-a}w^{-b}\phi^+_{a+b} - \sum_{a+b<0} z^{-a}w^{-b}\phi^-_{-a-b} + \sum_{a+b=0} z^{-a}w^{-b}(\phi^+_0-\phi^-_0)\big)}{(1-t_1)(1-t_2)(1-t_1^{-1}t_2^{-1})}$$

to hold, we determine $\phi^+_{s>0}, -\phi^-_{s>0}, \phi^+_{s=0}-\phi^-_{s=0}$ uniquely as they are equal to the corresponding $(1-t_1)(1-t_2)(1-t_1^{-1}t_2^{-1})\gamma_s$. So to determine all $\phi^\pm_i$, we need only to specialize the values ϕ^+_0, ϕ^-_0.

The next lemma is crucial.

LEMMA 4.1

We have $[e_0,f_0]|_\lambda = -\frac{1}{(1-t_1)(1-t_2)}$, $[e_0,f_1]|_\lambda = -\frac{1}{(1-t_1)(1-t_2)} + \sum_{\square\in\lambda}\chi(\square)$.

COROLLARY 4.1

The operator $[e_0, f_1-f_0]$ *is the operator of multiplication by* $\det(\mathfrak{F})$.

Proof of Lemma 4.1

In the proof below we use another expression for $\gamma_s=[e_0,f_s]$, which is obtained by using Proposition 3.1 instead of Lemma 3.1(a). With this purpose for any Young diagram $\lambda=(\lambda_1,\lambda_2,\ldots)$, we define $\chi_i := t_1^{\lambda_i-1}t_2^{i-1}$. Let us notice that due to the finiteness of λ: $\chi_i = t_1^{-1}t_2^{i-1}$ for all $i\gg 1$, which will be called the stabilizing condition,

$$\gamma_s = (1-t_1)^{-2}\sum_{i=1}^{l(\lambda)+1}\chi_i^{s-1}\prod_{i\neq j}^{\infty}\frac{(1-t_1^{\lambda_i-\lambda_j}t_2^{i-j+1})(1-t_1^{\lambda_j-\lambda_i+1}t_2^{j-i+1})}{(1-t_1^{\lambda_i-\lambda_j}t_2^{i-j})(1-t_1^{\lambda_j-\lambda_i+1}t_2^{j-i})}$$

(14)

$$- (1-t_1)^{-2}\sum_{i=1}^{l(\lambda)+1}(t_1\chi_i)^{s-1}\prod_{i\neq j}^{\infty}\frac{(1-t_1^{\lambda_j-\lambda_i}t_2^{j-i+1})(1-t_1^{\lambda_i-\lambda_j+1}t_2^{i-j+1})}{(1-t_1^{\lambda_j-\lambda_i}t_2^{j-i})(1-t_1^{\lambda_i-\lambda_j+1}t_2^{i-j})}.$$

(a) First, we prove $[e_0, f_0]|_\lambda = -1/((1-t_1)(1-t_2))$ for any λ. This is obvious for an empty diagram (and straightforward for any diagram λ consisting of 1 row). So it is enough to prove that $[e_0, f_0]$ does not depend on the diagram. Let $\lambda_{k-1}=0$. (We do not need $\lambda_{k-2}\neq 0$.) Then for $i\geq k-1$, we have $\chi_i = t_1^{-1}t_2^{i-1}$. Hence according to formula (14) and the stabilizing condition:

$$\gamma_0 = (1-t_1)^{-2}\Big(\sum_{i=1}^{k}\chi_i^{-1}\frac{\chi_i(1-t_1t_2^{2-k}\chi_i)}{\chi_i-t_2^{k-1}}\prod_{1\leq j\neq i}^{k-1}\frac{(\chi_j-t_2\chi_i)(\chi_i-t_1t_2\chi_j)}{(\chi_j-\chi_i)(\chi_i-t_1\chi_j)}$$
$$-\sum_{i=1}^{k}(t_1\chi_i)^{-1}\frac{\chi_i(1-t_1^2t_2^{2-k}\chi_i)}{\chi_i-t_1^{-1}t_2^{k-1}}\prod_{1\leq j\neq i}^{k-1}\frac{(\chi_i-t_2\chi_j)(\chi_j-t_1t_2\chi_i)}{(\chi_i-\chi_j)(\chi_j-t_1\chi_i)}\Big).$$

So we have a rational expression in χ_i $(1\leq i\leq k-2)$. Moreover, the degree of the numerator is not greater than that of the denominator. The possible poles of this function can occur only at $\chi_i=\chi_j, \chi_i=t_1\chi_j$ or at $\chi_i = t_2^{k-1}, t_1^{-1}t_2^{k-1}$. However, in cases $\chi_i = t_2^{k-1}, t_1^{-1}t_2^{k-1}$ there are no poles (since the poles coming from the denominator are compensated by zeros of $\chi_i - t_1t_2\chi_{k-1}$ or $\chi_i - t_2\chi_{k-1}$ coming from the numerator). All the poles $\chi_i=\chi_j, \chi_i=t_1\chi_j$ are simple. But it is straightforward to see that the residues at these points are in fact zero. Hence this rational function γ_0 is constant. This completes the proof of $[e_0,f_0]= -1/((1-t_1)(1-t_2))$.

(b) Let us check $[e_0, f_1]|_\lambda = -1/((1-t_1)(1-t_2)) + \sum_{\square\in\lambda}\chi(\square)$ for any λ. By the definition of χ_i, we have $\sum_{\square\in\lambda}\chi(\square) = \sum_{i=1}^{\infty}(t_2^{i-1}-t_1\chi_i)/(1-t_1)$. So we have to prove $[e_0,f_1] = -1/((1-t_1)(1-t_2)) + \sum_{i=1}^{\infty}(t_2^{i-1}-t_1\chi_i)/(1-t_1)$. It is obvious for an empty diagram, and it is a straightforward computation for any diagram λ consisting of 1 row. So it is enough to prove that $[e_0, f_1] - \sum_{i=1}^{\infty}(t_2^{i-1}-t_1\chi_i)/(1-t_1)$ does not depend on the diagram.

Let $\lambda_{k-1}=0$. Then for $i\geq k-1$ we have $\chi_i = t_1^{-1}t_2^{i-1}$. Hence according to formula (14),

$$\gamma_1 = (1-t_1)^{-2}\Big(\sum_{i=1}^{k}\frac{\chi_i(1-t_1t_2^{2-k}\chi_i)}{\chi_i-t_2^{k-1}}\prod_{1\leq j\neq i}^{k-1}\frac{(\chi_j-t_2\chi_i)(\chi_i-t_1t_2\chi_j)}{(\chi_j-\chi_i)(\chi_i-t_1\chi_j)}$$
$$-\sum_{i=1}^{k}\frac{\chi_i(1-t_1^2t_2^{2-k}\chi_i)}{\chi_i-t_1^{-1}t_2^{k-1}}\prod_{1\leq j\neq i}^{k-1}\frac{(\chi_i-t_2\chi_j)(\chi_j-t_1t_2\chi_i)}{(\chi_i-\chi_j)(\chi_j-t_1\chi_i)}\Big).$$

So we have a rational expression in χ_i $(1\leq i\leq k-2)$. Moreover, the degree of the numerator is not greater than that of the denominator plus 1. The possible poles of this function can occur only at $\chi_i=\chi_j, \chi_i=t_1\chi_j$ or at $\chi_i = t_2^{k-1}, t_1^{-1}t_2^{k-1}$. In cases $\chi_i = t_2^{k-1}, t_1^{-1}t_2^{k-1}$ the poles do not really occur (see the argument in part (a)). All the poles $\chi_i=\chi_j, \chi_i=t_1\chi_j$ are simple with vanishing residue. So γ_1 is a linear function in χ_i $(1\leq i\leq k-2)$. Finally, one checks that the principal part of γ_1 equals $\sum_{i=1}^{k-2}(-t_1/(1-t_1))\chi_i$. We conclude that $[e_0,f_1]-\sum_{i=1}^{\infty}(t_2^{i-1}-t_1\chi_i)/(1-t_1)$ is constant. This completes the proof of $[e_0,f_1]=-1/((1-t_1)(1-t_2))+\sum_{\square\in\lambda}\chi(\square)$. $\square$

COROLLARY 4.2

It follows from Lemma 4.1 that $\gamma_0\equiv -1/((1-t_1)(1-t_2))$. So we can define

$$\phi_0^+ := -1, \phi_0^- := -\frac{1}{t_1t_2}.$$

All the operators $\phi_s^{\pm}$ are diagonalizable in the fixed point basis according to Proposition 4.1.

Now we compute the matrix elements of the left-hand side and right-hand side of (12) in the fixed point basis. It is enough to check for any Young diagrams $\lambda, \lambda'=\lambda+i_1$, the following equality:

$$(\phi_{i+3}^+e_j-\sigma_1\phi_{i+2}^+e_{j+1}+\sigma_2\phi_{i+1}^+e_{j+2}-\sigma_3\phi_i^+e_{j+3})_{[\lambda,\lambda']} = (\sigma_3e_j\phi_{i+3}^+-\sigma_2e_{j+1}\phi_{i+2}^++\sigma_1e_{j+2}\phi_{i+1}^+-e_{j+3}\phi_i^+)_{[\lambda,\lambda']}. \tag{15}$$

Let us denote $\chi_1:=t_1^{j_1}t_2^{i_1-1}$, where $j_1:=\lambda_{i_1}+1$. Taking into account the equality $e_{j+k[\lambda,\lambda']}=\chi_1^ke_{j[\lambda,\lambda']}$ and the diagonalizability of ϕ_i, we reduce the above equation to the following:

$$(\phi_{i+3}^+-\sigma_1\chi_1\phi_{i+2}^++\sigma_2\chi_1^2\phi_{i+1}^+-\sigma_3\chi_1^3\phi_i^+)|_{\lambda'} = (\sigma_3\phi_{i+3}^+-\sigma_2\chi_1\phi_{i+2}^++\sigma_1\chi_1^2\phi_{i+1}^+-\chi_1^3\phi_i^+)|_{\lambda}, \tag{16}$$

where $\phi_j^+=0$ whenever $j<0$.

First, we prove the analogous equation for γ_i:

$$(\gamma_{i+3}-\sigma_1\chi_1\gamma_{i+2}+\sigma_2\chi_1^2\gamma_{i+1}-\sigma_3\chi_1^3\gamma_i)|_{\lambda'} = (\sigma_3\gamma_{i+3}-\sigma_2\chi_1\gamma_{i+2}+\sigma_1\chi_1^2\gamma_{i+1}-\chi_1^3\gamma_i)|_{\lambda}, \tag{17}$$

Proof of equation (17)

First case. The summand in the expression for γ_i corresponds to the *corner* $\square_{i_2,j_2}$ which appears in both sides of (17).

(a) Suppose $i_1<i_2, j_1>j_2$. Let us denote $a:=j_1-j_2, b:=i_2-i_1, u:=t_1^a, v:=t_2^b, \chi_2:=t_1^{j_2-1}t_2^{i_2-1}$. So $\chi_1/\chi_2=uv^{-1}$. Then

$$(\gamma_{i+3}-\sigma_1\chi_1\gamma_{i+2}+\sigma_2\chi_1^2\gamma_{i+1}-\sigma_3\chi_1^3\gamma_i)|_{\lambda'} = P_3\frac{(1-t_1^{-a}t_2^{b+1})(1-t_1^{a+1}t_2^{-b+1})}{(1-t_1^{-a}t_2^{b})(1-t_1^{a+1}t_2^{-b})}$$
$$\times\left(1-\sigma_1\left(\frac{\chi_1}{\chi_2}\right)+\sigma_2\left(\frac{\chi_1}{\chi_2}\right)^2-\sigma_3\left(\frac{\chi_1}{\chi_2}\right)^3\right)$$
$$=P_3(u-v)^{-1}(u-t_2v)(v-t_1t_2u)(v-t_2u)(v-t_1^{-1}t_2^{-1}u)v^{-3},$$

$$(\sigma_3\gamma_{i+3}-\sigma_2\chi_1\gamma_{i+2}+\sigma_1\chi_1^2\gamma_{i+1}-\chi_1^3\gamma_i)|_{\lambda} = P_3\frac{(1-t_1^{-a+1}t_2^{b+1})(1-t_1^{a}t_2^{-b+1})}{(1-t_1^{-a+1}t_2^{b})(1-t_1^{a}t_2^{-b})}$$

$$\times\left(\sigma_3-\sigma_2\Big(\frac{\chi_1}{\chi_2}\Big)+\sigma_1\Big(\frac{\chi_1}{\chi_2}\Big)^2-\Big(\frac{\chi_1}{\chi_2}\Big)^3\right)$$
$$=P_3(u-t_1t_2v)(u-v)^{-1}(v-t_2u)(u-t_2v)(u-t_1^{-1}t_2^{-1}v)v^{-3}.$$

We obtained the same expressions.

(b) Suppose $i_1>i_2, j_1<j_2$. This case is completely analogous to (a).

Second case. The summand in the expression for γ_i corresponds to the *hole* $\Box_{i_2,j_2}$ which appears in both sides of (17). In this case everything is analogous as the expression for the summands in γ corresponding to a *corner* and a *hole* differ only by the sign.

Third case. Let us finally consider the summands occurring only in one side of (17).

In this case the summands corresponding to deleting $\Box_{i_1,j_1}$ in the left-hand side of (17) and to inserting $\Box_{i_1,j_1}$ in the right-hand side of (17) are equal. All other summands are zero. (We use the argument that t_1^{-1}, t_2^{-1} are roots of the polynomial $1-\sigma_1 t+\sigma_2 t^2-\sigma_3 t^3$ again.) $\Box$

Now we are ready to verify equation (16).

Proof of equation (16)

If $i>0$, then (16) follows directly from (17). So let us consider the remaining cases: $i=-3,-2,-1,0$ (in the case $i<-3$ all summands are zero). According to (17) and the relation between γ_i and $\phi_i^{\pm}$, we have to check only the following equalities: $\phi_0^+|_{\lambda'}=\phi_0^+|_{\lambda}, \phi_0^-|_{\lambda'}=\phi_0^-|_{\lambda}, \phi_1^+|_{\lambda'}=(\phi_1^+ +(\sigma_1-\sigma_2)\chi_1\phi_0^+)|_{\lambda}, \phi_1^-|_{\lambda'}=(\phi_1^- +(\sigma_2-\sigma_1)\chi_1^{-1}\phi_0^-)|_{\lambda}$.

The first two are obvious since ϕ_0^+,ϕ_0^- are constant. It follows from Lemma 4.1 that $\gamma_1|_{\mu}=[e_0,f_1]|_{\mu}=-(1/(1-t_1)(1-t_2))+\sum_{\Box\in\mu}\chi(\Box)$. So

$$\phi_1^+|_{\lambda'}-\phi_1^+|_{\lambda}=(1-t_1)(1-t_2)(1-t_1^{-1}t_2^{-1})(\gamma_1^+|_{\lambda'}-\gamma_1^+|_{\lambda})=(\sigma_1-\sigma_2)\chi_1\phi_0^+$$

since $(\sigma_1-\sigma_2)=-(1-t_1)(1-t_2)(1-t_1^{-1}t_2^{-1})$ and $\phi_0^+=-1$.

The equation $\phi_1^-|_{\lambda'}=(\phi_1^-+(\sigma_2-\sigma_1)\chi_1^{-1}\phi_0^-)|_{\lambda}$ is proved in the same way. $\Box$

The proof of (13) is entirely similar to that of (12), and so we omit it.

Finally, let us prove $\Phi^+(z)=\psi^+(z)$. From (16) we get

$$\Phi^+(z)(1-\sigma_1\chi_1 z^{-1}+\sigma_2\chi_1^2 z^{-2}-\sigma_3\chi_1^3 z^{-3})|_{\lambda'}$$
$$=\Phi^+(z)(\sigma_3-\sigma_2\chi_1 z^{-1}+\sigma_1\chi_1^2 z^{-2}-\chi_1^3 z^{-3})|_{\lambda}.$$

Thus

$$\Phi^+(z)|_{\lambda'}=\Phi^+(z)|_{\lambda}\cdot\frac{(1-t_1^{-1}\chi_1 z^{-1})(1-t_2^{-1}\chi_1 z^{-1})(1-t_1t_2\chi_1 z^{-1})}{(1-t_1\chi_1 z^{-1})(1-t_2\chi_1 z^{-1})(1-t_1^{-1}t_2^{-1}\chi_1 z^{-1})}.$$

By induction $\Phi^+(z)|_\lambda = A\cdot \mathbf{c}(z)$, where A is a coefficient of proportionality, which is equal to $A=\Phi^+(z)|_{\text{empty}}$ (that is, here empty means an empty diagram)

$$A=\phi_0^+ - (1-t_1^{-1}t_2^{-1})\sum_{i<0} z^i = -1 - \frac{(1-t_1^{-1}t_2^{-1})z^{-1}}{1-z^{-1}} = -\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}}.$$

So $\Phi^+(z)=\psi^+(z)$, and analogously one gets $\Phi^-(z)=\psi^-(z)$. □

Theorem 3.1 is proved. □

5. The action of the shuffle algebra on M

In the previous section we constructed the action of the Ding-Iohara algebra A on M. Unfortunately, the parameters q_1, q_2, q_3 were not generic (we had $q_1q_2q_3=1$), so Theorem 2.1 does not give the representation of S automatically. However, if we write the formulas in the same way we get the representation of S on M.

Namely, we define the action of S on M in the following way. For any $F\in S_n$, we say that for any Young diagrams $\lambda, \lambda'=\lambda+i_1+\cdots+i_n$ $(i_1\le i_2\le\cdots\le i_n)$ the matrix element

$$F|_{[\lambda,\lambda']} := \frac{F(\chi_1,\dots,\chi_n)}{\prod_{1\le a<b\le n}\lambda(\chi_{i_a},\chi_{i_b})}\prod_{k=1}^{n} e_{0[\lambda+i_1+\cdots+i_{k-1},\lambda+i_1+\cdots+i_k]}, \tag{18}$$

where χ_{i_k} is the character of the kth added box to λ. All other matrix elements are zero.

Now we prove the following theorem.

THEOREM 5.1

Formula (18) gives a representation of the shuffle algebra S on M.

First, we note that the following proposition holds.

PROPOSITION 5.1

If $\lambda, \lambda+j_1, \lambda+j_1+j_2, \dots, \lambda'=\lambda+j_1+\cdots+j_n$ are Young diagrams, then $F|_{[\lambda,\lambda']} = (F(\chi_1,\dots,\chi_n)/\prod_{1\le a<b\le n}\lambda(\chi_{j_a},\chi_{j_b}))\prod_{k=1}^n e_{0[\lambda+j_1+\cdots+j_{k-1},\lambda+j_1+\cdots+j_k]}$, where χ_{j_k} is the character of the kth added box to λ. (We are adding boxes in the following order: j_1, then j_2, and so on.) So the formula for the matrix elements does not depend on the order of adding the boxes.

Proof

As the symmetric group is generated by transpositions, it is enough to check the statement only for them. But the case of transpositions follows from relation (1). This completes the proof of proposition. □

Now we prove the theorem.

Proof of Theorem 5.1

Let $F\in S_m, G\in S_n$, and let $\lambda, \lambda'=\lambda+j_1+\cdots+j_{m+n}$ be the Young diagrams.

Then by Proposition 5.1,

$$(F \circ G)|_{[\lambda,\lambda']} = \mathrm{Sym}\Big(\frac{G(\chi_1,\ldots,\chi_n)}{\prod_{1\le a<b\le n}\lambda(\chi_{j_a},\chi_{j_b})}\prod_{k=1}^{n} e_{0[\lambda+j_1+\cdots+j_{k-1},\lambda+j_1+\cdots+j_k]}$$

$$\times \frac{F(\chi_{n+1},\ldots,\chi_{n+m})}{\prod_{n+1\le a<b\le n+m}\lambda(\chi_{j_a},\chi_{j_b})}\prod_{k=n+1}^{n+m} e_{0[\lambda+j_1+\cdots+j_{k-1},\lambda+j_1+\cdots+j_k]}\Big)$$

$$= \mathrm{Sym}\Big(\frac{G(\chi_1,\ldots,\chi_n)F(\chi_{n+1},\ldots,\chi_{n+m})\prod_{k=1}^{n+m} e_{0[\lambda+j_1+\cdots+j_{k-1},\lambda+j_1+\cdots+j_k]}}{\prod_{1\le a<b\le n}\lambda(\chi_{j_a},\chi_{j_b})\prod_{n+1\le a<b\le n+m}\lambda(\chi_{j_a},\chi_{j_b})}\Big).$$

On the other hand,

$$(G * F)(\chi_1,\ldots,\chi_{n+m}) = \mathrm{Sym}\Big(G(\chi_1,\ldots,\chi_n)F(\chi_{n+1},\ldots,\chi_{n+m})\prod_{1\le a\le n<b\le n+m}\lambda(\chi_{i_a},\chi_{i_b})\Big).$$

Thus applying Proposition 5.1, we get

$$(F \circ G)|_{[\lambda,\lambda']} = (G * F)|_{[\lambda,\lambda']}.$$

This completes the proof of the theorem. □

Let us recall the definition of operators $K_i \in S_i$ $(i \in \mathbb{N})$ from Theorem 2.3:

$$K_2(z_1,z_2) = \frac{(z_1 - q_1 z_2)(z_2 - q_1 z_1)}{(z_1 - z_2)^2}, \qquad K_n(z_1,\ldots,z_n) = \prod_{1\le i<j\le n} K_2(z_i,z_j),$$

with the specialization $q_1 = t_1$. Then we have the following.

COROLLARY 5.1

If $i_1 < i_2 < \cdots < i_n$ and $\lambda + i_1 + \cdots + i_n$ is a Young diagram the matrix element

$$K_n|_{[\lambda,\lambda'=\lambda+i_1+\cdots+i_n]} = \prod_{1\le a<b\le n}\frac{(\chi_a-\chi_b)(\chi_b-t_1\chi_a)}{(\chi_a-t_2\chi_b)(\chi_a-t_1^{-1}t_2^{-1}\chi_b)}\prod_{1\le r\le n} e_{0[\lambda+i_1+\cdots+i_{r-1},\lambda+i_1+\cdots+i_r]},$$

where $\chi_a = t_1^{\lambda_{i_a}} t_2^{i_a-1}$.

All other matrix elements are zero.

REMARK 5.1

We have constructed the actions of Ding-Iohara and shuffle algebras on M. While the action of the Ding-Iohara algebra is purely geometric (it is given by operators $e_i, f_i, \psi_i^{\pm}$), the action of the shuffle algebra unfortunately is algebraic. Nevertheless, according to the criteria of Theorem 2.2, elements K_i belong to the subalgebra generated by S_1, and so they are geometrically represented since $S_1 \subset \Xi(A_+)$ (see Theorem 2.1).

6. Macdonald polynomials. Heisenberg algebra and vertex operators over it

6.1. Macdonald polynomials

In this subsection, we review basic facts about Macdonald polynomials. Our basic reference is Macdonald's book [7].

Recall that the algebra Λ_F of symmetric functions over $F=\mathbb{Q}(q,t)$ is freely generated by the power-sum symmetric functions p_k, where $k\in\mathbb{N}$; that is,

$$\Lambda = F[p_1, p_2, \ldots].$$

For any diagram $\lambda=(\lambda_1,\ldots,\lambda_k)=(1^{m_1}2^{m_2}\cdots)$, we define

$$p_\lambda := p_{\lambda_1}\cdots p_{\lambda_k}, \qquad z_\lambda := \prod_{r\geq 1} r^{m_r} m_r!.$$

Consider the Macdonald inner product $(\cdot,\cdot)_{q,t}$, s.t.

$$(p_\lambda, p_\mu)_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{1\leq i\leq k} (1-q^{\lambda_i})/(1-t^{\lambda_i}).$$

DEFINITION 6.1

Macdonald polynomials P_λ are characterized by two conditions:

(a) $P_\lambda = m_\lambda +$ lower terms;
(b) $(P_\lambda, P_\mu)_{q,t}=0$ if $\lambda\neq\mu$.

Here by *lower terms* we mean m_μ for $\mu\prec\lambda$.

Let e_r be the rth elementary symmetric function. The following result, called the Pieri formula, is proved in [7, Section VI.6].

LEMMA 6.1

We have $P_\mu e_r = \sum_\lambda \psi_{\lambda/\mu} P_\lambda$, where the sum is taken over λ such that λ/μ is a vertical r-strip. Here

$$\psi_{\lambda/\mu} = \prod \frac{(1-q^{\mu_i-\mu_j}t^{j-i-1})(1-q^{\lambda_i-\lambda_j}t^{j-i+1})}{(1-q^{\mu_i-\mu_j}t^{j-i})(1-q^{\lambda_i-\lambda_j}t^{j-i})}, \tag{19}$$

where the product is taken over all pairs (i,j) such that $i<j$ and $\lambda_i=\mu_i, \lambda_j=\mu_j+1$.

In particular,

$$\psi_{\mu+j/\mu} = \prod_{i=1}^{j-1} \frac{(1-q^{\mu_i-\mu_j}t^{j-i-1})(1-q^{\mu_i-\mu_j-1}t^{j-i+1})}{(1-q^{\mu_i-\mu_j}t^{j-i})(1-q^{\mu_i-\mu_j-1}t^{j-i})}. \tag{20}$$

6.2. Fixed points via Macdonald polynomials

Now we prove that the basis $[\lambda]$ of M can be normalized in such a way that normalized $K_i\in S_i$ acts as e_i in the basis of Macdonald polynomials in Λ_F. The normalization is found by comparing the matrix elements of K_1 with the matrix elements of e_1 in the basis of Macdonald polynomials.

We define the normalized vectors $\langle\lambda\rangle := c_\lambda \cdot [\lambda]$, where

$$c_\lambda := \left(-(1-t_2)^{-1}t_2\right)^{-|\lambda|} t_1^{\sum_i (\lambda_i(\lambda_i-1))/2} \prod_{\Box\in\lambda} \left(1-t_1^{l(\Box)}t_2^{-a(\Box)-1}\right)^{-1}. \tag{21}$$

First, we check the following lemma.

LEMMA 6.2

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$*. Then* $(1-t_1)(1-t_2)K_{1\langle\lambda,\lambda+j\rangle} = \psi_{\lambda+j/\lambda}|_{q:=t_1,t:=t_2^{-1}}$.

REMARK 6.1

This lemma means that $(1-t_1)(1-t_2)K_1 = (1-t_1)(1-t_2)\Xi(e_0)$ acts in a normalized basis like an operator of multiplication by e_1 in the basis of Macdonald polynomials. Moreover, this condition defines a normalization uniquely up to a mutual factor.

Proof of Lemma 6.2

$$\frac{c_{\lambda+j}}{c_\lambda} = -t_1^{\lambda_j}t_2^{-1}(1-t_2) \prod_{\Box\in\Sigma_1(\Box_{j,\lambda_j+1})} \frac{1-t_1^{l(\Box)}t_2^{-a(\Box)-1}}{1-t_1^{l(\Box)+1}t_2^{-a(\Box)-1}} \times \prod_{\Box\in\Sigma_2(\Box_{j,\lambda_j+1})} \frac{1-t_1^{l(\Box)}t_2^{-a(\Box)-1}}{1-t_1^{l(\Box)}t_2^{-a(\Box)-2}} (1-t_2^{-1})^{-1}.$$

Now we compute the products above:

$$\prod_{\Box\in\Sigma_2(\Box_{j,\lambda_j+1})} \frac{1-t_1^{l(\Box)}t_2^{-a(\Box)-1}}{1-t_1^{l(\Box)}t_2^{-a(\Box)-2}} = \prod_{i<j} \frac{1-t_1^{\lambda_i-\lambda_j-1}t_2^{i-j}}{1-t_1^{\lambda_i-\lambda_j-1}t_2^{i-j-1}},$$

$$\prod_{\Box\in\Sigma_1(\Box_{j,\lambda_j+1})} \frac{1-t_1^{l(\Box)}t_2^{-a(\Box)-1}}{1-t_1^{l(\Box)+1}t_2^{-a(\Box)-1}} = \frac{(1-t_2^{-1})}{(1-t_1^{\lambda_j}t_2^{j-k-1})} \prod_{i>j}^{k} \frac{1-t_1^{\lambda_j-\lambda_i}t_2^{j-i-1}}{1-t_1^{\lambda_j-\lambda_i}t_2^{j-i}}$$

$$= -(1-t_2^{-1}) \prod_{i>j}^{\infty} \frac{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j+1}}{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j}} t_2 t_1^{-\lambda_j}.$$

Thus

$$\frac{c_{\lambda+j}}{c_\lambda} = (1-t_2) \prod_{i>j}^{\infty} \frac{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j+1}}{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j}} \prod_{i<j} \frac{1-t_1^{\lambda_i-\lambda_j-1}t_2^{i-j}}{1-t_1^{\lambda_i-\lambda_j-1}t_2^{i-j-1}}.$$

On the other hand, it follows from Proposition 3.1 that

$$e_{0[\lambda,\lambda+j]} = (1-t_1)^{-1} \prod_{1\le i\ne j}^{\infty} \frac{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j+1}}{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j}}.$$

After the specializing of formula (20) for parameters $q := t_1, t := t_2^{-1}$, we have

$$\psi_{\lambda+j/\lambda} = \prod_{i=1}^{j-1} \frac{(1 - t_1^{\lambda_i-\lambda_j} t_2^{i-j+1})(1 - t_1^{\lambda_i-\lambda_j-1} t_2^{i-j-1})}{(1 - t_1^{\lambda_i-\lambda_j} t_2^{i-j})(1 - t_1^{\lambda_i-\lambda_j-1} t_2^{i-j})}.$$

Now it is straightforward to check that

$$\psi_{\lambda+j/\lambda} = (1-t_1)(1-t_2)e_{0[\lambda,\lambda+j]} \cdot \frac{c_\lambda}{c_{\lambda+j}}.$$

□

We denote $d_n := (-t_1)^{n-1}/((1-t_1)(1-t_2))$.

THEOREM 6.1

For any Young diagrams $\mu \subset \lambda$ such that λ/μ is a vertical n-strip with the boxes located in the rows $j_1 < \cdots < j_n$, we have: $\frac{1}{d_1 \cdots d_n} K_{n\langle\mu,\lambda\rangle} = \psi_{\lambda/\mu}|_{q:=t_1, t:=t_2^{-1}}$.

Proof

According to Corollary 5.1,

$$K_{n[\mu,\lambda]} = \prod_{1\le a<b\le n} \frac{(\chi_a - \chi_b)(\chi_b - t_1\chi_a)}{(\chi_a - t_2\chi_b)(\chi_a - t_1^{-1}t_2^{-1}\chi_b)} \prod_{1\le r\le n} e_{0[\lambda-j_r-\cdots-j_n, \lambda-j_{r+1}-\cdots-j_n]},$$

where $\chi_a = t_1^{\lambda_{j_a}-1} t_2^{j_a-1}$. Hence in the normalized basis,

$$K_{n\langle\mu,\lambda\rangle} = \prod_{1\le a<b\le n} \frac{(\chi_a - \chi_b)(\chi_b - t_1\chi_a)}{(\chi_a - t_2\chi_b)(\chi_a - t_1^{-1}t_2^{-1}\chi_b)} \prod_{1\le r\le n} e_{0\langle\lambda-j_r-\cdots-j_n, \lambda-j_{r+1}-\cdots-j_n\rangle}.$$

After specializing $q := t_1, t := t_2^{-1}$ the coefficients in the formula (19) look as follows:

$$\psi_{\lambda/\mu} = \prod \frac{(1 - t_1^{\mu_i-\mu_j} t_2^{i-j+1})(1 - t_1^{\lambda_i-\lambda_j} t_2^{i-j-1})}{(1 - t_1^{\mu_i-\mu_j} t_2^{i-j})(1 - t_1^{\lambda_i-\lambda_j} t_2^{i-j})}, \tag{22}$$

where the product is taken over all pairs (i,j) such that $j = j_b$ for some $1 \le b \le n$ and $j > i \ne j_a$ for any a.

Applying Lemma 6.2 we get that the right-hand side of (22) is equal to

$$\prod_{j_1,\ldots,j_{b-1}\ne i<j_b} \frac{(1 - t_1^{\mu_i-\mu_{j_b}} t_2^{i-j_b+1})(1 - t_1^{\lambda_i-\lambda_{j_b}} t_2^{i-j_b-1})}{(1 - t_1^{\mu_i-\mu_{j_b}} t_2^{i-j_b})(1 - t_1^{\lambda_i-\lambda_{j_b}} t_2^{i-j_b})}$$

$$= \prod_{a<b} \frac{(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}+1} t_2^{j_a-j_b})(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}} t_2^{j_a-j_b})}{(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}+1} t_2^{j_a-j_b+1})(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}} t_2^{j_a-j_b-1})} \cdot \frac{e_{0\langle\lambda-j_b-\cdots-j_n, \lambda-j_{b+1}-\cdots-j_n\rangle}}{d_1}.$$

Finally,

$$\prod_{a<b} \frac{(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}+1} t_2^{j_a-j_b})(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}} t_2^{j_a-j_b})}{(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}+1} t_2^{j_a-j_b+1})(1 - t_1^{\lambda_{j_a}-\lambda_{j_b}} t_2^{j_a-j_b-1})}$$

$$= \prod_{a<b} \frac{(\chi_b - t_1\chi_a)(\chi_b - \chi_a)}{(\chi_b - t_1t_2\chi_a)(\chi_b - t_2^{-1}\chi_a)} = \prod_{1\le a<b\le n} \frac{(\chi_a - \chi_b)(\chi_b - t_1\chi_a)}{(\chi_a - t_2\chi_b)(\chi_a - t_1^{-1}t_2^{-1}\chi_b)}(-t_1)^{1-b}.$$

This completes the proof of Theorem 6.1. □

6.3. Heisenberg action on M through vertex operators

The results of Section 6.2 provide us with an isomorphism $\Theta : M \to \Lambda_F$ which takes the normalized fixed point basis $\langle\lambda\rangle$ to Macdonald polynomial P_λ and which sends operators $\widetilde{K_i} := (1/d_1 \cdots d_i)K_i$ to operators of multiplication by e_i.

On the other hand, there is a well-known identity of generalized functions:

$$1 + \sum_{i>0} e_i z^i = \exp\Big(\sum_{i>0} \frac{(-1)^{i-1}}{i} p_i z^i\Big).$$

Hence the operators $\widetilde{K_i}$ acting on M, which may be viewed as a Fock space over p_i, are vertex operators over half of the Heisenberg algebra $\{\mathfrak{h}_i\}_{i>0}$. The isomorphism Θ takes $\mathfrak{h}_i$ to operators of multiplication by p_i (for $i > 0$). As a result, an action of the positive part of the Heisenberg algebra is obtained. Obviously starting from f_i instead of e_i we get in the analogous way the vertex operators over the negative half of the Heisenberg algebra. This provides Heisenberg algebra action on M.

REMARK 6.2

The disadvantage of our approach is that we do not know explicit formulas for K_i in terms of $x^{j_1} * x^{j_2} * \cdots * x^{j_i}$.

6.4. Specialization: $q = t^\alpha, t \to 1$

In [6] the authors studied the action of the Heisenberg algebra on the sum of localized equivariant cohomologies $R := \bigoplus_n H^{2n}_{\mathbb{T}}(X^{[n]}) \otimes_{H_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(H_{\mathbb{T}}(\mathrm{pt}))$. They proved that under certain normalizations of the fixed point basis there is an isomorphism $\Delta : M \to \Lambda_F$, which sends the basis of fixed points to Jack polynomials, and $\{\mathfrak{h}_i\}_{i>0}$ are sent to operators of multiplication by p_i. It is also known (see [7]) that Jack polynomials $J^{(\alpha)}_\lambda$ can be obtained from the Macdonald polynomials $P^{(q,t)}_\lambda$ by specializing $q := t^\alpha$, $t \to 1$. Considering the above-mentioned specialization of our normalization (21) we get the same formulas for normalization as those of [6] multiplied by some scalar* (see formulas (2.12), (2.14) of [6], and note that $l(\square)$, $a(\square)$ are interchanged with our notations). So as the formulas in the fixed point basis in $H^\bullet$ are additive analogues of the formulas for $K^\bullet$, our approach gives the same action of the Heisenberg algebra on R as the approach using higher correspondences $P[i]_{i\in\mathbb{Z}}$.

*This scalar comes from the fact that we used slightly different correspondences from Nakajima's construction. While we use the whole $P[1]$ (resp., $P[-1]$), Nakajima used only part of it, consisting of those $(J_1, J_2) \in P[1]$, such that the quotient is supported at a single point (which is automatic) with zero y-coordinate (resp., x-coordinate).

7. Whittaker vector

Let us consider the element $v=\sum_{n\geq 0}[O_{X^{[n]}}]$ from a completion of our space M. In the sequel we call this element the *Whittaker vector*. This term is justified by the following theorem and [1, Section 2.30].

THEOREM 7.1
Consider $\widetilde{K_{-n}}:=(1/d_1\cdots d_n)K_{-n}$ *in the analogy with* $\widetilde{K_n}$ *from Section 6.3. Then for any* $n\geq 1$ *we have* $\widetilde{K_{-n}}(v)=\widetilde{C_n}\cdot v$ *with* $\widetilde{C_n}=(1-t_2)^n/((1-t_2)(1-t_2^2)\cdots(1-t_2^n))$.

Before proving this theorem we will need a technical result.

PROPOSITION 7.1
In the basis $[\lambda]$ *the vector* v *is decomposed as follows:*

$$v=\bigoplus_{\lambda}a_\lambda\cdot[\lambda],a_\lambda=\prod_{\square\in\lambda}\left((1-t_1^{l(\square)+1}t_2^{-a(\square)})(1-t_1^{-l(\square)}t_2^{a(\square)+1})\right)^{-1}.$$

Proof
This follows from the Bott-Lefschetz fixed point formula. □

Proof of Theorem 7.1
The theorem is proved in two steps.

Step 1: Case $n=1$.

In this case the statement follows from $K_{-1}(v)=C_1\cdot v$ with $C_1=1/((1-t_1)(1-t_2))$ (indeed, then $\widetilde{C_1}=d_1^{-1}\cdot C_1=1$). Since $K_{-1}=f_0$, to prove $K_{-1}(v)=C_1\cdot v$ it is enough to check for any Young diagram λ the following identity: $C_1\cdot a_\lambda=\sum_{j\leq k+1}f_{0[\lambda+j,\lambda]}\cdot a_{\lambda+j}$, where k is a height of a diagram λ. Using Lemma 3.1 this is equivalent to

(23)
$$C_1=\sum_{j\leq k+1}\frac{a_{\lambda+j}}{\chi\cdot a_\lambda}\prod_{s\in\Sigma_1(\square_{j,\lambda_j+1})}\frac{1-t_1^{l(s)+1}t_2^{-a(s)}}{1-t_1^{l(s)}t_2^{-a(s)}}\prod_{s\in\Sigma_2(\square_{j,\lambda_j+1})}\frac{1-t_1^{-l(s)}t_2^{a(s)+1}}{1-t_1^{-l(s)}t_2^{a(s)}},$$

where $\chi=t_1^{\lambda_j}t_2^{j-1}$. Applying Proposition 7.1 for computing a_λ and $a_{\lambda+j}$, we see that (23) is equivalent to

(24)
$$1=\sum_{j\leq k+1}\chi^{-1}\prod_{s\in\Sigma_1(\square_{j,\lambda_j+1})}\frac{1-t_1^{-l(s)+1}t_2^{a(s)+1}}{1-t_1^{-l(s)}t_2^{a(s)+1}}\prod_{s\in\Sigma_2(\square_{j,\lambda_j+1})}\frac{1-t_1^{l(s)+1}t_2^{-a(s)+1}}{1-t_1^{l(s)+1}t_2^{-a(s)}}.$$

Denote $x_j=t_1^{\lambda_j}t_2^j$. Let us rewrite the right-hand side of (24):

$$\prod_{s\in\Sigma_2(\square_{j,\lambda_j+1})}\frac{1-t_1^{l(s)+1}t_2^{-a(s)+1}}{1-t_1^{l(s)+1}t_2^{-a(s)}}=\prod_{i<j}\frac{1-t_1^{\lambda_i-\lambda_j}t_2^{1+i-j}}{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j}}=\prod_{i<j}\frac{x_j-t_2x_i}{x_j-x_i},$$

$$\prod_{s\in\Sigma_1(\square_{j,\lambda_j+1})}\frac{1-t_1^{-l(s)+1}t_2^{a(s)+1}}{1-t_1^{-l(s)}t_2^{a(s)+1}}=\frac{1-t_2}{1-t_1^{-\lambda_j}t_2^{k-j+1}}\prod_{k\geq i>j}\frac{1-t_1^{\lambda_i-\lambda_j}t_2^{1+i-j}}{1-t_1^{\lambda_i-\lambda_j}t_2^{i-j}}$$
$$=\prod_{k\geq i>j}\frac{x_j-t_2x_i}{x_j-x_i}\cdot\frac{(1-t_2)x_j}{x_j-t_2^{k+1}}.$$

Hence equality (24) is equivalent to the following identity:

$$1=\sum_{j\leq k+1}\chi^{-1}\prod_{k\geq i\neq j}\frac{x_j-t_2x_i}{x_j-x_i}\cdot\frac{(1-t_2)x_j}{x_j-t_2^{k+1}}. \tag{25}$$

Using $x_{k+1}=t_2^{k+1}$, we have $((1-t_2)x_j)/(x_j-t_2^{k+1})=(x_j-t_2x_{k+1})/(x_j-x_{k+1})\cdot((1-t_2)x_j)/(x_j-t_2^{k+2})$. From this observation and $\chi^{-1}=t_2/x_j$ we are reduced to proving

$$1=\sum_{j\leq k+1}\prod_{k+1\geq i\neq j}\frac{x_j-t_2x_i}{x_j-x_i}\cdot\frac{t_2(1-t_2)}{x_j-t_2^{k+2}}. \tag{26}$$

Let us denote the right-hand side of (26) by $F(x_1,\dots,x_{k+1})$, where $x_1=t_1^{\lambda_1}t_2,\dots,x_k=t_1^{\lambda_k}t_2^k,x_{k+1}=t_2^{k+1}$. If we set $\lambda_1=\dots=\lambda_k=0$, then it is easy to see that $F(x_1,\dots,x_{k+1})=1$. Since F is a rational function in $x_1,\dots,x_k,x_{k+1}=t_2^{k+1}$ the degree of whose numerator is not bigger then the degree of the denominator, to prove identity (26) it is enough to show that F does not have poles. The only possible poles can occur at $x_j=t_2^{k+2}$ and diagonals $x_i=x_j$. In the first case, $x_j-t_2x_{k+1}=0$, and hence there is no pole in fact. It is also obvious that on the diagonals $x_i=x_j$ there are no poles as well. So (26) is proved. This finishes the proof of Step 1.

Step 2: Case $n\geq 2$.

Let $C_n=(-t_1)^{n(n-1)/2}/((1-t_1)^n(1-t_2)\cdots(1-t_2^n))$. We want to prove $K_{-n}(v)=C_n\cdot v$. This implies the statement of the theorem since $\widetilde{K_{-n}}:=1/(d_1\cdots d_n)K_{-n}$.

To prove $K_{-n}(v)=C_n\cdot v$ it is enough to check for any Young diagram λ the following identity: $C_n=\sum K_{-n[\lambda+i_1+\dots+i_n,\lambda]}\cdot a_{\lambda+i_1+\dots+i_n}/a_\lambda$, where the sum is over all sets of indices $i_1\leq\dots\leq i_n$, such that $\lambda+i_1+\dots+i_n$ is a Young diagram. The matrix elements $K_{-n[\lambda+i_1+\dots+i_n,\lambda]}$ are computed similarly to Corollary 5.1. Namely,

$$K_{-n[\lambda+i_1+\dots+i_n,\lambda]}$$
$$=\prod_{1\leq a<b\leq n}\frac{(\chi_a-\chi_b)(\chi_b-t_1\chi_a)}{(\chi_a-t_2\chi_b)(\chi_a-t_1^{-1}t_2^{-1}\chi_b)}\prod_{1\leq r\leq n}f_{0[\lambda+i_1+\dots+i_r,\lambda+i_1+\dots+i_{r-1}]},$$

where $\chi_a=t_1^{\lambda_{i_a}}t_2^{i_a-1}$.

Hence

$$K_{-n[\lambda+i_1+\dots+i_n,\lambda]}\cdot\frac{a_{\lambda+i_1+\dots+i_n}}{a_\lambda}=\prod_{1\leq a<b\leq n}\frac{(\chi_a-\chi_b)(\chi_b-t_1\chi_a)}{(\chi_a-t_2\chi_b)(\chi_a-t_1^{-1}t_2^{-1}\chi_b)} \tag{27}$$

$$\times \prod_{1\le r\le n} f_{0[\lambda+i_1+\cdots+i_r,\lambda+i_1+\cdots+i_{r-1}]}\frac{a_{\lambda+i_1+\cdots+i_r}}{a_{\lambda+i_1+\cdots+i_{r-1}}}.$$

Using the computations from the Case 1, we see

$$f_{0[\lambda+i_1+\cdots+i_r,\lambda+i_1+\cdots+i_{r-1}]}\frac{a_{\lambda+i_1+\cdots+i_r}}{a_{\lambda+i_1+\cdots+i_{r-1}}} = \prod_{k_r\ge i\ne i_r}\frac{x_{i_r}-t_2x_i}{x_{i_r}-x_i}\cdot\frac{t_2(1-t_2)}{x_{i_r}-t_2^{k_r+1}}$$

$$\times(1-t_1)^{-1}(1-t_2)^{-1} = \prod_{k+n\ge i\ne i_r}\frac{x_{i_r}-t_2x_i}{x_{i_r}-x_i}\cdot\frac{t_2}{(1-t_1)(x_{i_r}-t_2^{k+n+1})},$$

where k_r denotes the height of a diagram $\lambda+i_1+\cdots+i_{r-1}$.

So the right-hand side of (27) is equal to

$$\prod_{1\le a<b\le n}\frac{(\chi_a-\chi_b)(\chi_b-t_1\chi_a)}{(\chi_a-t_2\chi_b)(\chi_a-t_1^{-1}t_2^{-1}\chi_b)}$$

$$\times\prod_{1\le r\le n}\left(\frac{t_2}{1-t_1}\frac{1}{x_{i_r}-t_2^{k+n+1}}\prod_{k+n\ge i\ne i_r}\frac{x_{i_r}-t_2x_i}{x_{i_r}-x_i}\right)$$

$$=\prod_{1\le r\le n}\left((-t_1t_2)^{r-1}\frac{t_2}{1-t_1}\frac{1}{(x_{i_r}-t_2^{k+n+1})}\prod_{k+n\ge j\ne i_1,\ldots,i_n}\frac{x_{i_r}-t_2x_j}{x_{i_r}-x_j}\right).$$

Define

$$G(x_1,\ldots,x_{k+n})$$
$$:=\sum_{i_1<\cdots<i_n}\prod_{1\le r\le n}\left((-t_1t_2)^{r-1}\frac{t_2}{1-t_1}\frac{1}{(x_{i_r}-t_2^{k+n+1})}\prod_{k+n\ge j\ne i_1,\ldots,i_n}\frac{x_{i_r}-t_2x_j}{x_{i_r}-x_j}\right).$$

It is a rational function in variables $x_1,\ldots,x_k,x_{k+1}=t_2^{k+1},\ldots,x_{k+n}=t_2^{k+n}$ with the degree of the numerator not bigger than the degree of the denominator. Moreover, we claim that it does not have poles. Indeed, the poles can occur only at $x_{i_s}=t^{k+n+1}$ or on diagonals x_i-x_j. But arguments similar to those from Case 1 show that there are no poles in fact. (It is crucial that $x_{k+1}=t_2^{k+1},\ldots,x_{k+n}=t_2^{k+n}$.)

Thus $G(x_1,\ldots,x_{k+n})$ is constant. Let us calculate its value at $x_m=t_2^m$ $1\le m\le k+n$. In the expression for G survives only a summand corresponding to $i_1=1,\ldots,i_n=n$. (All others are zero.) Hence

$$G=(-t_1t_2)^{n(n-1)/2}\frac{t_2^n}{(1-t_1)^n}\frac{1}{(t_2-t_2^{k+n+1})\cdots(t_2^n-t_2^{k+n+1})}$$

$$\times\prod_{n+1\le j\le n+k}\frac{(t_2-t_2^{j+1})\cdots(t_2^n-t_2^{j+1})}{(t_2-t_2^j)\cdots(t_2^n-t_2^j)}$$

$$=(-t_1t_2)^{n(n-1)/2}\frac{t_2^n}{(1-t_1)^n}t_2^{-n(n+1)/2}\frac{1}{(1-t_2)\cdots(1-t_2^n)}=C_n.$$

This finishes the proof of the theorem. □

Since the action of the negative half of the Heisenberg algebra was constructed using

$$1+\sum_{i>0}\widetilde{K_{-i}}z^i=\exp\Big(\sum_{i>0}\frac{(-1)^{i-1}}{i}\mathfrak{h}_{-i}z^i\Big), \tag{28}$$

we get the following corollary.

COROLLARY 7.1
We have $\mathfrak{h}_{-i}(v)=\alpha_i\cdot v$, *where* $\alpha_i=(-1)^{i-1}((1-t_2)^i/(1-t_2^i))$.

Proof
The fact that the vector v is an eigenvector for all $\mathfrak{h}_{-i}$ immediately follows from Theorem 7.1 and (28). Now we compute the eigenvalues α_i explicitly. Using the well-known formula $\sum_{i\geq 0}\frac{z^i}{(1-t)(1-t^2)\cdots(1-t^i)}=\prod_{i\geq 0}(1/(1-t^iz))$ and Theorem 7.1, we get

$$\sum_{i>0}\frac{(-1)^{i-1}}{i}\alpha_i z^i=\ln\Big(\sum_{i\geq 0}\frac{(1-t_2)^i}{(1-t_2)(1-t_2^2)\cdots(1-t_2^i)}z^i\Big)$$
$$=-\ln\Big(\prod_{i\geq 0}\big(1-t_2^i(1-t_2)z\big)\Big)=\sum_{i\geq 0}\sum_{j\geq 1}\frac{t_2^{ij}(1-t_2)^jz^j}{j}=\sum_{j\geq 1}\frac{(1-t_2)^j}{1-t_2^j}\cdot\frac{z^j}{j}.$$

Thus we get $\alpha_i=(-1)^{i-1}((1-t_2)^i/(1-t_2^i))$. □

Let $[\mathfrak{h}_{-i},\mathfrak{h}_i]=\gamma_i$, and let $v_0=[O_{X^{[0]}}]\in M$. Then we get the following exponential expression for the Whittaker vector v.

PROPOSITION 7.2
We have $v=\exp(\sum_{i>0}(\alpha_i/\gamma_i)\mathfrak{h}_i)v_0$.

Proof
Since our representation is isomorphic to a representation of the Heisenberg algebra in a Fock space, it boils down to a standard fact. □

REMARK 7.1
Completely analogously we can consider the Whittaker vector u in $R:=\bigoplus_n H^{2n}_{\mathbb{T}}(X^{[n]})\otimes_{H_{\mathbb{T}}(\mathrm{pt})}\mathrm{Frac}(H_{\mathbb{T}}(\mathrm{pt}))$. It is even easier to see $u=\exp((1/\hbar\hbar')\mathfrak{h}_1)u_0$, and hence u is also an eigenvector with respect to the negative half of the Heisenberg algebra.

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References

[1] A. Braverman and M. Finkelberg, *Finite difference quantum Toda lattice via equivariant K-theory*, Transform. Groups **10** (2005), 363–386.

[2] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Quantum continuous gl_∞: Semiinfinite construction of representations*, Kyoto J. Math. **51** (2011), 365–392.

[3] B. Feigin, M. Finkelberg, A. Negut, and L. Rybnikov, *Yangians and cohomology rings of Laumon spaces*, Select Math. (N.S.) **17** (2011), 573–607.

[4] B. Feigin, K. Hashizume, J. Shiraishi, and S. Yanagida, *A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials*, J. Math. Phys. **50** (2009), no. 095215.

[5] B. Feigin and A. Odesskii, "Vector bundles on elliptic curve and Sklyanin algebras" in *Topics in Quantum Groups and Finite-Type Invariants*, Amer. Math. Soc. Transl. Ser. 2 **185**, Amer. Math. Soc., Providence, 1998, 65–84.

[6] W. Li, Z. Qin, and W. Wang, "The cohomology rings of Hilbert schemes via Jack polynomials" in *Algebraic Structures and Moduli Spaces*, CRM Proc. Lecture Notes **38**, Amer. Math. Soc., Providence, 2004, 249–258.

[7] I. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford Math. Monogr., Oxford Univ. Press, New York, 1995.

[8] H. Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces*, Univ. Lecture Ser. **18**, Amer. Math. Soc., Providence, 1999.

[9] O. Schiffmann, *Drinfeld realization of the elliptic Hall algebra*, J. Algebr. Comb. (2011) DOI 10.1007/s10801-011-0302-8.

[10] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of $\mathbb{A}^2$*, arXiv:0905.2555v2 [math.QA]

[11] A. Tsymbaliuk, *Quantum affine Gelfand-Tsetlin bases and quantum toroidal algebra via K-theory of affine Laumon spaces*, Selecta Math. (N.S.) **16** (2010), 173–200.

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