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EQUIVARIANT K-THEORY

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§ 1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is to set down the basic facts about equivariant K-theory. The theory was invented by Professor Atiyah, and most of the results are due to him. Various applications can be found in [2], [6], [11]. I shall assume the reader has some acquaintance with ordinary K-theory ([5], [3], [4], [2]), and shall only sketch the development of the equivariant theory when it is parallel to the ordinary case.

The theory is defined on spaces with group action: Let us choose a fixed topological group G ; then a G -space is a topological space X together with a continuous action $G \times X \rightarrow X$, written $(g, x) \mapsto g \cdot x$, satisfying the usual conditions $g \cdot (g' \cdot x) = (gg') \cdot x$ and $1 \cdot x = x$.

We are going to construct a cohomology theory by considering the equivariant vector bundles on G -spaces: I shall have to begin with a collection of definitions and simple facts concerning equivariant vector bundles, modelled precisely on the discussion in [3].

If X is a G -space, a G -vector bundle on X is a G -space E together with a G -map $p : E \rightarrow X$ (i.e. $p(g \cdot \xi) = g \cdot p(\xi)$) such that

(i) $p : E \rightarrow X$ is a complex vector bundle on X , i.e. the fibres $E_x = p^{-1}(x)$ for $x \in X$ are finite-dimensional complex vector spaces, and the situation is locally trivial in a familiar sense [3], and

(ii) for any $g \in G$ and $x \in X$ the group action $g : E_x \rightarrow E_{gx}$ is a homomorphism of vector spaces.

If, for example, X is a trivial G -space (i.e. $g \cdot x = x$ for all $g \in G$ and $x \in X$) a G -vector bundle is a family of representations E_x of G parametrized by the points x of X and varying continuously with x in a certain sense.

G -vector bundles are fairly common in nature. I shall mention three kinds of example:

a) If X is a differentiable manifold and G is a Lie group which acts smoothly on X then the complexified tangent bundle $T_X \otimes \mathbf{C}$ of X is a G -vector bundle, and so are all the associated tensor bundles.

b) If E is any vector bundle on a space X then the k -fold tensor product $E \otimes \dots \otimes E$

is in a natural way an S_k -vector bundle on X , where S_k is the symmetric group which permutes the factors of the product, and X is regarded as a trivial S_k -space.

c) Homogeneous vector bundles. Let us determine the G -vector bundles on the space of cosets G/H , when H is a closed subgroup of G . If $\pi : E \rightarrow G/H$ is such a G -vector-bundle then the fibre E_0 over the neutral coset is an H -module which, as we shall see, determines E completely. The action of G on E induces a map $G \times E_0 \rightarrow E$, which can be regarded as a map $\alpha : G \times_H E_0 \rightarrow E$, where $G \times_H E_0$ means the space of orbits of $G \times E_0$ under H , when H acts on it by $(h, g, \xi) \mapsto (gh^{-1}, h\xi)$. If G acts on $G \times_H E_0$ by $(g, g', \xi) \mapsto (gg', \xi)$ then α is a G -map, and is a homeomorphism: for one can construct its inverse as follows. Consider the homeomorphism $\beta : G \times E \rightarrow G \times E$ defined by $\beta(g, \xi) = (g, g^{-1}\xi)$. The inverse image under β of $G \times E_0$ is $G \times_{G/H} E = \{(g, \xi) \in G \times E : gH = \pi\xi\}$. The natural map $G \times_{G/H} E \rightarrow G \times_H E_0$ factorizes through the projection $(g, \xi) \mapsto \xi$ of $G \times_{G/H} E$ on to E , which is an open map. The resulting map $\tilde{\beta} : E \rightarrow G \times_H E_0$ is the inverse of α . Thus any G -vector bundle on G/H it is of the form $G \times_H E_0$ for some H -module E_0 .

Conversely, if H is locally compact, and E_0 is any H -module, $G \times_H E_0$ is a G -vector-bundle on G/H . The only thing in question is local triviality. Now if $G \rightarrow G/H$ is locally trivial then $G \times_H E_0$ looks locally like $(U \times H) \times_H E_0$, i.e. $U \times E_0$, where U is an open set of G/H , and so it is also locally trivial. This deals with the case when H is a Lie group ([13], p. 315). But in general one can write $G \times_H E_0 = (G/N) \times_{(H/N)} E_0$, where N is the kernel of the action of H on E_0 . H/N is a Lie group, so we are reduced to the earlier case.

Of course if M is any G -module (finite-dimensional complex representation space of G) and X is any G -space one can form the G -vector bundle $X \times M$ on X , which I shall call *trivial*, and denote by \mathbf{M} when there is no risk of confusion.

The sections of a G -vector bundle $E \xrightarrow{p} X$ are the maps $s : X \rightarrow E$ such that $ps = \text{id}$. They form a vector space ΓE . If a section is a G -map it is called *equivariant*: the equivariant sections form a vector subspace $\Gamma^G E$ of ΓE which is the space of fixed points of the natural action of G on ΓE .

If E and F are two G -vector bundles on X one can form their sum $E \oplus F$, a G -vector bundle on X with $(E \oplus F)_x = E_x \oplus F_x$; and similarly the tensor product $E \otimes F$, and a bundle $\text{Hom}(E; F)$ with $(\text{Hom}(E; F))_x = \text{Hom}(E_x; F_x)$.

A homomorphism $f : E \rightarrow F$ of G -vector bundles on X is a continuous G -map which induces a homomorphism of vector spaces $f_x : E_x \rightarrow F_x$ for each $x \in X$. The homomorphisms form a vector space isomorphic to $\Gamma^G \text{Hom}(E; F)$.

If $\varphi : Y \rightarrow X$ is a G -map of G -spaces, and E is a vector bundle on X , then one can form a G -vector bundle $\varphi^* E$ on Y with $(\varphi^* E)_y = E_{\varphi(y)}$, just as in the ordinary case. More generally, if Y is an H -space, X a G -space, $\alpha : H \rightarrow G$ a homomorphism, and $\varphi : Y \rightarrow X$ such that $\varphi(h \cdot y) = \alpha(h) \cdot \varphi(y)$, then $\varphi^* E$ is an H -vector bundle on Y . If $i : Y \rightarrow X$ is the inclusion of a subspace, $i^* E$ is often written $E|Y$.

For the rest of this paper I shall assume that G is a *compact* group, and I shall continually perform integrations over G with respect to the Haar measure. (One can integrate any continuous function $G \rightarrow \Gamma$ with values in a hausdorff, locally convex, and complete topological vector space ([8], Chap. 3, § 4).) Also, for the most part I shall confine myself to compact G -spaces X .

Let E be a G -vector bundle on a compact G -space X . If the vector space ΓE is given the compact-open topology then G acts continuously on it in the sense that $G \times \Gamma E \rightarrow \Gamma E$ is continuous. For the continuity of the G -action $G \times \text{Map}(X; E) \rightarrow \text{Map}(X; E)$ follows from that of the map $G \times X \times \text{Map}(X; E) \rightarrow E$ defined by $(g, x, s) \mapsto g.s(g^{-1}x)$. It is obvious that ΓE is hausdorff, locally convex, and complete. (It becomes a Banach space if one chooses a hermitian metric in E .) So one can “average” a section of E over the group to obtain an equivariant section.

We need a string of lemmas generalizing those of [3].

Proposition (1.1). — *If E is a G -vector bundle on a compact G -space X , and A is a closed G -subspace of X , then an equivariant section of $E|A$ can be extended to an equivariant section of E .*

One simply extends the section arbitrarily, as in [3], and then averages it over G .

By applying Proposition (1.1) to the G -vector bundle $\text{Hom}(E; F)$ we obtain, just as in [3]:

Proposition (1.2). — *In the situation of (1.1), if F is another G -vector bundle on X and $f: E|A \rightarrow F|A$ is an isomorphism then there is a G -neighbourhood U of A in X and an isomorphism $f: E|U \rightarrow F|U$ extending f .*

And Proposition (1.2) implies in turn [3]:

Proposition (1.3). — *If $\varphi_0, \varphi_1: Y \rightarrow X$ are G -homotopic G -maps, and Y is compact, and E is a G -vector bundle on X , then $\varphi_0^*E \cong \varphi_1^*E$.*

Example. — (1.3) implies that the representations of a compact group are “discrete”. For if X is a path-connected trivial G -space then E is just a continuous family of G -modules $\{E_x\}_{x \in X}$, and (1.3) implies that $E_x \cong E_y$ for any $x, y \in X$.

We need to know also that G -vector-bundles can be constructed by clutching: if X is the union of compact G -subspaces X_1, X_2 with intersection A , and E_1, E_2 are G -vector bundles on X_1, X_2 , and $\alpha: E_1|A \rightarrow E_2|A$ is an isomorphism, then there is a unique G -vector bundle E on X with isomorphisms $E|X_1 \cong E_1, E|X_2 \cong E_2$ compatible with α . The group G is irrelevant in the proof of this proposition, so I shall not repeat it.

Finally, if $f: E \rightarrow F$ is a morphism of G -vector bundles on X such that $f_x: E_x \rightarrow F_x$ is an isomorphism for each $x \in X$, then f is an isomorphism, i.e. it has an inverse. Again G is irrelevant.

§ 2. EQUIVARIANT K-THEORY

Let X be a compact G -space.

The set of isomorphism classes of G -vector bundles on X forms an abelian semi-group under \oplus . The associated abelian group is called $K_G(X)$: its elements are formal differences $E_0 - E_1$ of G -vector bundles on X , modulo the equivalence relation $E_0 - E_1 = E'_0 - E'_1 \Leftrightarrow E_0 \oplus E'_1 \oplus F \cong E'_0 \oplus E_1 \oplus F$ for some G -vector bundle F on X .

The tensor product of G -vector bundles induces a structure of commutative ring in $K_G(X)$.

If $\varphi : Y \rightarrow X$ is a G -map of compact G -spaces the functor $E \mapsto \varphi^* E$ induces a morphism of rings $\varphi^* : K_G(X) \rightarrow K_G(Y)$, so that K_G is a contravariant functor from compact G -spaces to commutative rings. A homomorphism $\alpha : H \rightarrow G$ induces a morphism of “restriction” $K_G(X) \rightarrow K_H(X)$; and, more generally, if $\varphi : Y \rightarrow X$ is a map from an H -space to a G -space compatible with α , one has $\varphi^* : K_G(X) \rightarrow K_H(Y)$.

If $G = 1$ one writes, of course, $K(X)$ for $K_G(X)$.

Examples. — (i) If X is a point then $K_G(X) \cong R(G)$, the *representation ring*, or *character ring*, of G (cf. [5], [16]) — for a G -vector bundle is then just a G -module. As a group $R(G)$ is the free abelian group generated by the set \hat{G} of simple G -modules. In general $K_G(X)$ is an algebra over $R(G)$, because any G -space X has a natural map on to a point. (The morphism $R(G) \rightarrow K_G(X)$ is just $M \mapsto \mathbf{M}$.)

(ii) $K_G(G/H) \cong R(H)$ when H is a closed subgroup of G . For we have seen that the category of G -vector bundles on G/H is equivalent to the category of H -modules.

(iii) More generally, if X is a compact H -space one can form a compact G -space $(G \times X)/H = G \times_H X$. There is an embedding $\varphi : X \rightarrow G \times_H X$ which identifies X with the H -subspace $H \times_H X$ of $G \times_H X$. The restriction φ^* is an equivalence between G -vector bundles on $G \times_H H$ and H -vector bundles on X , inverse to the extension $E \mapsto G \times_H E$: the argument of § 1, ex. c) applies without change.

For any compact G -space X the projection of X onto its orbit space X/G induces a morphism $\text{pr}^* : K(X/G) \rightarrow K_G(X)$. Now if G acts *freely* on X (i.e. $g.x = x \Leftrightarrow g = 1$) and E is a G -vector bundle on X , then E/G is a vector bundle on X/G . The only non-trivial point is to show that E/G is locally trivial, which is always the case if G is a compact *Lie* group (see [7], Chap. 7). But we shall see presently that a G -vector bundle on X is always pulled back from a G/N -vector bundle on X/N , where N is some normal subgroup of G such that G/N is a Lie group, so E/G is locally trivial in any case. The functor $E \mapsto E/G$ is inverse to the functor pr^* , in fact the natural G -maps $E \rightarrow X$, $E \rightarrow E/G$ induce an isomorphism $E \rightarrow X \times_{X/G} E/G = \text{pr}^*(E/G)$; while if F is a vector bundle on X/G , the projection on to the second factor induces an isomorphism $(X \times_{X/G} F)/G = (\text{pr}^* F)/G \rightarrow F$.

Thus we have proved

Proposition (2.1). — If G acts freely on X then $\text{pr}^* : K(X/G) \xrightarrow{\cong} K_G(X)$. More generally, if N is a normal subgroup of G which acts freely on X then $\text{pr}^* : K_{G/N}(X/N) \xrightarrow{\cong} K_G(X)$.

(Observing that $X/N \cong (G/N) \times_G X$, one can combine everything said so far into the statement that a homomorphism $\alpha : G \rightarrow G'$ induces an isomorphism $K_{G'}(G' \times_G X) \rightarrow K_G(X)$ if $\ker(\alpha)$ acts freely on X .)

Now let us consider the other extreme case, when G acts trivially on X . Then we have a homomorphism $K(X) \rightarrow K_G(X)$ which gives a vector bundle the trivial G -action. Combining this with the natural map $R(G) \rightarrow K_G(X)$ we have a morphism of rings $R(G) \otimes K(X) \rightarrow K_G(X)$. In fact

Proposition (2.2). — If X is a trivial G -space the natural map

$$\mu : R(G) \otimes K(X) \rightarrow K_G(X)$$

is an isomorphism of rings.

Proof. — I shall prove this by constructing an inverse to μ . The point is to show that a G -vector bundle can be decomposed into isotypical pieces which are locally trivial vector bundles. Because G acts trivially on X , it acts in each fibre of a G -vector bundle E on X , and there is an operation of averaging over G in each fibre, varying continuously. That is to say, there is a projection operator (cf. [3]) in E whose image is the subset E^G of E pointwise invariant under G . So ([3], Lemma (1.4)) E^G is a vector bundle on X , and the functor $E \mapsto E^G$ induces a homomorphism of abelian groups $\varepsilon : K_G(X) \rightarrow K(X)$. And similarly for any G -module M the functor $E \mapsto \text{Hom}^G(M; E) = (\text{Hom}(M; E))^G$ induces a homomorphism $\varepsilon_M : K_G(X) \rightarrow K(X)$. I assert that the map $\nu : K_G(X) \rightarrow R(G) \otimes K(X)$ defined by $\nu(\xi) = \sum_{[M] \in \hat{G}} [M] \otimes \varepsilon_M(\xi)$ is the inverse of μ . If E is a G -vector bundle on X we have a canonical isomorphism $\bigoplus_{[M] \in \hat{G}} (M \otimes \text{Hom}^G(M; E)) \rightarrow E$ (it is an isomorphism because $\bigoplus (M \otimes \text{Hom}(M; E_x)) \xrightarrow{\cong} E_x$ for each fibre E_x), so $\mu \circ \nu = \text{id}$. On the other hand $\text{Hom}^G(M_1; M_2 \otimes E) \cong \text{Hom}^G(M_1; M_2) \otimes E$ if G acts trivially on E , and the last bundle is E or 0 according as the simple G -modules M_1, M_2 are isomorphic or not. So $\nu \circ \mu = \text{id}$, also.

Example. — If E is a vector bundle on a space X , I have mentioned that $E^{\otimes k} = E \otimes \dots \otimes E$ is an S_k -vector bundle, where S_k is the symmetric group. The functor $E \mapsto E^{\otimes k}$ induces a natural transformation $K(X) \rightarrow K_{S_k}(X)$. We know now that $K_{S_k}(X) \cong R(S_k) \otimes K(X)$, so for each element of $R'(S_k) = \text{Hom}(R(S_k); \mathbf{Z})$ we obtain a natural transformation $K(X) \rightarrow K(X)$. It turns out that the operations of this type generate in a certain sense all the operations in K -theory [1].

Remark. — Proposition (2.2) is one of the few statements in this paper which does not generalize directly to *real* equivariant K -theory. If E is a real G -vector bundle on

a trivial G -space X then it must be decomposed $\coprod_M (\mathbf{M} \otimes_{D_M} \text{Hom}^G(\mathbf{M}; E)) \xrightarrow{\cong} E$, where M runs through the simple real G -modules, D_M is the field of endomorphisms of M (i.e. $D_M = \mathbf{R}, \mathbf{C}$, or \mathbf{H}), and $\text{Hom}^G(\mathbf{M}; E)$ is a G -vector bundle over the field D_M . Thus $K\mathbf{R}_G(X) \cong (K\mathbf{R}(X) \otimes R(G; \mathbf{R})) \oplus (K(X) \otimes R(G; \mathbf{C})) \oplus (K\mathbf{H}(X) \otimes R(G; \mathbf{H}))$, where $R(G; D)$ is the free abelian group generated by the simple real G -modules M with $D_M = D$.

I should record also the following consequence of (1.3).

Proposition (2.3). — *If $\varphi_0, \varphi_1 : Y \rightarrow X$ are G -homotopic G -maps then*

$$\varphi_0^* = \varphi_1^* : K_G(X) \rightarrow K_G(Y).$$

Despite the simple results we have obtained we still know very little about the elements of $K_G(X)$. The following proposition is fundamental for the development of the theory.

Proposition (2.4). — *If E is a G -vector bundle on X there is a G -module \mathbf{M} and a G -vector bundle E^\perp such that $E \oplus E^\perp \cong \mathbf{M}$.*

Proof. — Observe that it suffices to embed E in some \mathbf{M} . For one can choose a G -invariant hermitian metric in M and can define E^\perp as the orthogonal complement of E in \mathbf{M} . Similarly it suffices to find a surjection $\mathbf{M} \rightarrow E$: one defines E^\perp as its kernel.

The proof depends on the following formulation of the Peter-Weyl theorem.

Theorem (2.5) ([12], p. 31). — *Let Γ be a topological vector space which is locally convex, hausdorff, and complete. If G acts continuously on Γ (i.e. $G \times \Gamma \rightarrow \Gamma$ is continuous), and Γ_a is the union of the finite-dimensional invariant subspaces of Γ , then Γ_a is dense in Γ . (Γ_a is the image of the canonical injection $\bigoplus_{[\mathbf{M}] \in \hat{G}} (\mathbf{M} \otimes \text{Hom}^G(\mathbf{M}; \Gamma)) \rightarrow \Gamma$.)*

I apply the theorem when $\Gamma = \Gamma E$ is the Banach space of sections of E . For any $x \in X$ one can choose a finite set σ_x of sections of E such that $\{s(x)\}_{s \in \sigma_x}$ spans E_x . Because Γ_a is dense in Γ , and the evaluation map $\Gamma \rightarrow E_x$ is continuous, one can suppose $\sigma_x \subset \Gamma_a$. The set $\{s(y)\}_{s \in \sigma_x}$ spans E_y for all y in a neighbourhood U_x of x . Suppose U_{x_1}, \dots, U_{x_n} cover X . Let $\sigma = \bigcup_i \sigma_{x_i}$, and let M be the finite-dimensional G -subspace of Γ generated by σ . Then the evaluation map $X \times M \rightarrow E$ is the required surjection.

Of numerous consequences of Proposition (2.4) I shall mention two.

(i) Two G -vector bundles E, E' on X are called *stably equivalent* if there exist G -modules M, M' such that $E \oplus \mathbf{M} \cong E' \oplus \mathbf{M}'$. Proposition (2.4) implies that the stable equivalence classes of G -vector bundles on X form an abelian group under \oplus . This group is called $\tilde{K}_G(X)$; it can be identified naturally with a quotient group of $K_G(X)$.

(ii) Let M be a G -module, and let $\text{Gr}(n, M)$ be the G -space of n -dimensional subspaces of M , with the usual topology, and let $\text{Gr}(M)$ be the topological sum of

all $\text{Gr}(n, M)$. There is a canonical G -vector bundle $E_M = \{(\xi, A) : \xi \in A\} \subset M \times \text{Gr}(M)$ on $\text{Gr}(M)$, and Proposition (2.4) can be interpreted as the statement that any E on X is of the form $\varphi^* E_M$ for some M and some G -map $\varphi : X \rightarrow \text{Gr}(M)$. (This justifies the reduction principle which I used in proving (2.1) above, because any G -module M is really a (G/N) -module, where G/N is the image of G in $\text{Aut}(M)$, which is a Lie group.)

Now we can begin the topological study of the functor K_G . As the discussion to follow is unaffected by the presence of the group G , I shall be fairly brief.

To begin with I shall work in the category of compact G -spaces with base point. (I shall call all base points o ; of course $g \cdot o = o$ for all $g \in G$.) If X is such a space I write CX for the *reduced cone* on X , i.e. CX is obtained from $X \times [0, 1]$ by shrinking to a point the subspace $(X \times 0) \cup (o \times [0, 1])$. ($[0, 1]$ is the unit interval in \mathbf{R} .) If $i_1 : X \rightarrow Y_1$, $i_2 : X \rightarrow Y_2$ are two inclusions of compact G -spaces with base point then $Y_1 \amalg_X Y_2$ means the space obtained from the topological sum $Y_1 \amalg Y_2$ by identifying $i_1(x)$ with $i_2(x)$ for each $x \in X$. There is an obvious embedding of X in CX , and $CX \amalg_X CX$ is called the *reduced suspension* of X , and written SX .

Proposition (2.6). — *If X is a compact G -space with base point, and A is a closed G -subspace (with the same base point), then the sequence*

$$\tilde{K}_G(X \amalg_A CA) \rightarrow \tilde{K}_G(X) \rightarrow \tilde{K}_G(A)$$

is exact.

Proof. — The composition is zero because $A \rightarrow X \amalg_A CA$ is null-homotopic. On the other hand if the bundle E on X represents an element of $\tilde{K}_G(X)$ which vanishes in $\tilde{K}_G(A)$ then $E|_A \oplus \mathbf{M} \cong \mathbf{N}$ for some M, N . Form a bundle E on $X \amalg_A CA$ by clutching $E \oplus \mathbf{M}$ to \mathbf{N} on CA by this isomorphism. Then E represents the desired element of $\tilde{K}_G(X \amalg_A CA)$.

Let us iterate this proposition: first attach a cone to $X \amalg_A CA$ on the subspace X to obtain $CX \amalg_A CA$; then attach a cone to $CX \amalg_A CA$ on the subspace $X \amalg_A CA$ to obtain $CX \amalg_X C(X \amalg_A CA) \cong CX \amalg_X CX \amalg_{CA} C(CA)$. There is a natural map $SX = CX \amalg_X CX \rightarrow CX \amalg_X (CX \amalg_{CA} CCA)$, and the diagram

$$\begin{array}{ccc} SA & \longrightarrow & CX \amalg_A CA \\ \downarrow & & \downarrow \\ SX & \longrightarrow & CX \amalg_X (CX \amalg_{CA} CCA) \end{array}$$

commutes up to G -homotopy. In fact on the left-hand CA in SA it commutes trivially; on the right-hand CA we have the two different natural maps $CA \rightarrow C(CA)$, which are homotopic relative to A . Moreover $SA \rightarrow CX \amalg_A CA$ is a G -homotopy-equivalence; and so is $SX \rightarrow CX \amalg_X (CX \amalg_{CA} CCA)$, because $CX \rightarrow CX \amalg_{CA} C(CA)$ is a homotopy-

equivalence relative to X , because $C(CA)$ collapses to CA leaving CA fixed. So we have an exact sequence

$$(*) \quad \tilde{K}_G(SX) \rightarrow \tilde{K}_G(SA) \rightarrow \tilde{K}_G(X \amalg_A CA) \rightarrow \tilde{K}_G(X) \rightarrow \tilde{K}_G(A),$$

where the first map is induced by $A \rightarrow X$.

Definition (2.7). — If X is a compact G -space with base point, and A is a closed G -subspace, define (for any $q \in \mathbf{N}$)

$$\tilde{K}_G^{-q}(X) = \tilde{K}_G(S^q X), \quad \text{where } S^q X = S(\dots S(SX) \dots),$$

$$\text{and} \quad \tilde{K}_G^{-q}(X, A) = \tilde{K}_G(S^q(X \amalg_A CA)).$$

$$\text{Thus} \quad \tilde{K}_G^{-q}(X, o) = \tilde{K}_G^{-q}(X).$$

Because $S^q(X \amalg_A CA) = S^q X \amalg_{S^q A} CS^q A$ one has at once by iterating the sequence $(*)$ an exact sequence, infinite to the left

$$\dots \rightarrow \tilde{K}_G^{-q}(X, A) \rightarrow \tilde{K}_G^{-q}(X) \rightarrow \tilde{K}_G^{-q}(A) \rightarrow \tilde{K}_G^{-q+1}(X, A) \rightarrow \dots \rightarrow \tilde{K}_G(X, A) \rightarrow \tilde{K}_G(X) \rightarrow \tilde{K}_G(A).$$

By the device of [10], Chap. 10 one can obtain from a cohomology theory defined on compact spaces with base point a theory defined on locally compact spaces without base point. If X is a locally compact G -space which is not compact, let X^+ denote its one-point compactification, a compact G -space with base point. If X is already compact, define $X^+ = X \amalg o$, the sum of X and a base point.

Definition (2.8). — If X is a locally compact G -space, and A is a closed subspace, define $K_G^{-q}(X) = \tilde{K}_G^{-q}(X^+)$ and $K_G^{-q}(X, A) = \tilde{K}_G^{-q}(X^+, A^+)$. Thus $K_G^{-q}(X, o) = K_G^{-q}(X)$.

Example. — $K_G^{-q}(X) = K_G(X \times \mathbf{R}^q)$ and $K_G^{-q}(X, A) = K_G(X \times \mathbf{R}^q, A \times \mathbf{R}^q)$ for any locally compact G -space X and closed G -subspace A , for $(X \times \mathbf{R}^q)^+ \xrightarrow{\cong} S^q(X^+)$.

The groups so defined should be thought of as “ K_G with compact supports”. (They form an “LC-theory” in the sense of [10].) They are functorial only for *proper* G -maps. However if X is compact the new $K_G^0(X)$ coincides with the original $K_G(X)$: there is a homomorphism $K_G(X) \rightarrow \tilde{K}_G(X \amalg o)$ defined by extending G -vector bundles by giving them the fibre zero at the point o ; and its inverse is defined by assigning to a G -vector bundle E on $X \amalg o$ the element $(E|X) - (E_o \times X)$ of $K_G(X)$, where E_o is the fibre of E at o . A G -vector bundle E on X does not define an element of $K_G(X)$ unless X is compact, but, as we shall see, it does define a multiplication $\xi \mapsto \xi \cdot [E]$ in $K_G(X)$.

$(X, A) \mapsto K_G(X, A)$ is a contravariant functor for proper maps. It is also a *covariant* functor for *open embeddings*, for if U is an open G -subspace of a locally compact G -space X there is a natural G -map $X^+ \rightarrow U^+$. We have the following excision theorem.

Proposition (2.9). — *If A is a closed G -subspace of a locally compact G -space X then the natural map*

$$K_G^{-q}(X-A) \rightarrow K_G^{-q}(X, A)$$

is an isomorphism.

Proof. — $(X-A)^+ \cong (X^+-A^+)^+ \cong X^+/A^+$, so it suffices to show that

$$S^q(X^+ \amalg_A CA^+) = S^q X^+ \amalg_{S^q A^+} CS^q A^+ \rightarrow S^q(X^+/A^+) \cong S^q X^+/S^q A^+ \cong (S^q X^+ \amalg_{S^q A^+} CS^q A^+)/CS^q A^+$$

induces an isomorphism in \tilde{K}_G . That follows from

Proposition (2.10). — *If A is a closed G -contractible subspace of a compact G -space X then $K_G(X/A) \xrightarrow{\cong} K_G(X)$.*

Proof. — Given a G -vector bundle E on X we construct a bundle \check{E} on X/A as follows. Because A is contractible, $E|_A \cong \mathbf{M}$ for some G -module M . Extend this isomorphism to an open G -neighbourhood U of A in X . Now $X-A \cong X/A-A/A$. Construct \check{E} by clutching $E|_{X-A}$ and $M \times (U/A)$ by the isomorphism between them on $(X/A-A/A) \cap (U/A) \cong U-A$. One must check that the isomorphism class of \check{E} depends only on E ; $E \mapsto \check{E}$ is then obviously additive, and defines a map $K_G(X) \rightarrow K_G(X/A)$ inverse to the natural map.

The following continuity property of K_G is often useful.

Proposition (2.11). — *If \mathcal{J} is a filtering family of pairs of closed G -subspaces of a locally compact G -space X then*

$$\lim_{(Y,B) \in \mathcal{J}} K_G^*(Y, B) \xrightarrow{\cong} K_G^*\left(\bigcap_{(Y,B) \in \mathcal{J}} Y, \bigcap_{(Y,B) \in \mathcal{J}} B\right).$$

In particular, if A is a closed G -subspace of X then $\lim_{\rightarrow} K_G^(U) \xrightarrow{\cong} K_G^*(A)$ when U runs through the closed G -neighbourhoods of A .*

(Filtering means that if $(Y, B), (Y', B') \in \mathcal{J}$ then there is $(Y'', B'') \in \mathcal{J}$ such that $(Y'', B'') \subset (Y \cap Y', B \cap B')$.)

Proof. — Because $\bigcap S^q(Y^+) = S^q(\bigcap Y^+)$, and

$$\bigcap (Y^+ \amalg_{B^+} CB^+) = (\bigcap Y^+) \amalg_{(\bigcap B^+)} C(\bigcap B^+)$$

it suffices to show that $\lim_{\rightarrow} \tilde{K}_G(Y) \xrightarrow{\cong} \tilde{K}_G(\bigcap Y)$ when all the Y are compact and have a common base point.

If any bundle on $A = \bigcap Y$ can be extended to a neighbourhood of A in X then the last map is surjective, because any neighbourhood contains some Y . On the other hand, if E and E' in $\tilde{K}_G(Y)$ define the same element of $\tilde{K}_G(A)$, then $(E|_A) \oplus \mathbf{N} \cong (E'|_A) \oplus \mathbf{N}'$ for some G -modules N and N' , and this isomorphism can be extended to some Y' , so that E and E' have the same image in $\tilde{K}_G(Y')$.

To prove that a bundle E on a closed subspace A can be extended to a neighbourhood of A one can, for example, proceed as follows. Express E as the image of a projection operator in a trivial bundle \mathbf{M} . The operator can be identified with a continuous G -map $\alpha : A \rightarrow \text{Proj}(\mathbf{M})$, where $\text{Proj}(\mathbf{M})$ is the set of projection operators in \mathbf{M} , regarded as a closed G -subspace of the vector space $\text{End}(\mathbf{M})$ of endomorphisms of \mathbf{M} . It suffices to show that α can be extended to a neighbourhood of A . First extend α to a G -map $\beta : X \rightarrow \text{End}(\mathbf{M})$. Let V be the open subset of the vector-space $\text{End}(\mathbf{M})$ consisting of endomorphisms with no eigenvalues on the circle $\gamma = \left\{ z \in \mathbf{C} : |z-1| = \frac{1}{2} \right\}$. Then $T \mapsto \rho(T) = \frac{1}{2\pi i} \int_{\gamma} (z-T)^{-1} dz$ is an equivariant retraction of V onto $\text{Proj}(\mathbf{M})$, and $\rho\beta : \beta^{-1}(V) \rightarrow \text{Proj}(\mathbf{M})$ is the required extension of α .

Remark. — With a little more effort one can show that $\varinjlim K_G^*(X_\alpha) \xrightarrow{\cong} K_G^*(\varprojlim X_\alpha)$ for any directed inverse system of compact G -spaces.

Corollary (2.12). — If X is a locally compact G -space, then $\varinjlim K_G^*(U) \xrightarrow{\cong} K_G^*(X)$, where U runs through the relatively compact open G -subspaces of X , or, more generally, through any antilimiting open covering of X .

(A covering \mathcal{U} is antilimiting if for all $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $U \cap V \subset W$.)

§ 3. COMPLEXES, THE THOM HOMOMORPHISM, PERIODICITY

For many purposes it is convenient to know that K_G can be defined by complexes of G -vector bundles. Once again the group G is not relevant, so I shall simply state the result here. There is a proof in the compact case in [2]; and a proof for CW-complexes in [4]. The general case is no more difficult, but nevertheless I shall give in an appendix to this paper a proof in a slightly different spirit from that of [2] and [4].

A complex on a G -space is a sequence

$$E^* : \dots \xrightarrow{d} E^{i-1} \xrightarrow{d} E^i \xrightarrow{d} E^{i+1} \xrightarrow{d} \dots \quad (i \in \mathbf{Z})$$

of G -vector bundles on X such that $E^i = 0$ when $|i|$ is large, and of homomorphisms d such that $d^2 = 0$. A morphism of complexes $f : E^* \rightarrow F^*$ is a sequence of morphisms $f^i : E^i \rightarrow F^i$ such that $f^i d = d f^{i+1}$. The complex E^* is called *acyclic* if the sequence E_x^* of vector spaces is exact for all x in X .

The *support* of a complex E^* is the closed G -subset of X consisting of the points x for which E_x^* is not exact. I shall write it $\text{supp}(E^*)$. It is closed because if f_x is a homomorphism of vector spaces depending continuously on x then $\text{rank}(f_x)$ is a lower-semi-continuous function of x ; and $\dim(\ker f_x)$ is upper-semi-continuous.

If A is a closed G -subspace of a locally compact G -space X , let $L_G(X, A)$ be the set of isomorphism classes of complexes E^* on X such that $\text{supp}(E^*)$ is a compact subset of $X - A$. The set $L_G(X, A)$ is a semi-group under direct sum. Two elements E_0^*, E_1^* of

$L_G(X, A)$ are called *homotopic*, $E_0^* \simeq E_1^*$, if there is an object E^* of $L_G(X \times [0, 1], A \times [0, 1])$ such that $E_0^* = E^*|_{(X \times 0)}$ and $E_1^* = E^*|_{(X \times 1)}$. Introduce the equivalence relation \sim in $L_G(X, A)$ defined by

$$E_0^* \sim E_1^* \Leftrightarrow E_0^* \oplus F_0^* \simeq E_1^* \oplus F_1^*$$

for some acyclic complexes F_0^* and F_1^* on X .

Proposition (3.1). — $L_G(X, A)/\sim$ is an abelian group naturally isomorphic to $K_G(X, A)$.

This is easy when X is compact and $A = \emptyset$. The map $L_G(X, \emptyset) \rightarrow K_G(X)$ is simply $E^* \mapsto \sum_k (-1)^k E^k$; it is trivially surjective, and is injective because a complex is homotopic to the complex obtained by replacing its differential by zero.

If E^* and F^* are complexes on X one can form their tensor product $E^* \otimes F^*$, with $(E^* \otimes F^*)^k = \bigoplus_{p+q=k} E^p \otimes F^q$. One has $\text{supp}(E^* \otimes F^*) = \text{supp}(E^*) \cap \text{supp}(F^*)$. In view of Proposition (3.1) the tensor product of complexes induces a homomorphism

$$K_G(X, A) \otimes K_G(X, B) \rightarrow K_G(X, A \cup B),$$

which, when $A=B=\emptyset$, reduces to the product in the ring $K_G(X)$. This pairing is associative, and in particular it makes $K_G(X, A)$ into a commutative ring, which has a unit element if and only if $X-A$ is compact. (The product for relative groups can also be expressed as a product $K_G(U) \otimes K_G(V) \rightarrow K_G(U \cap V)$ for open G -subsets U, V of X .)

The product in $K_G(X)$ extends to make $K_G^*(X)$ into a graded ring. If $\xi_i \in K_G^{-p_i}(X)$ (for $i=1, 2$) is represented by a complex E_i^* on $X \times \mathbf{R}^{p_i}$ with compact support, then the product $\xi_1 \cdot \xi_2$ in $K_G^{-p_1-p_2}(X)$, is represented by the complex $\text{pr}_1^* E_1^* \otimes \text{pr}_2^* E_2^*$ on $X \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2}$, which also has compact support. ($\text{pr}_i : X \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \rightarrow X \times \mathbf{R}^{p_i}$ is the projection.) To relativize this definition is automatic, and in any case unnecessary. The graded ring $K^*(X)$ is anticommutative: to see that one has to look at the effect of permuting the factors \mathbf{R} in $K_G(X \times \mathbf{R}^p)$, and is immediately reduced to showing that if $\theta : X \times \mathbf{R} \rightarrow X \times \mathbf{R}$ is defined by $\theta(x, t) = (x, -t)$ then $\theta^* E^* = -E^*$ in $K_G(X \times \mathbf{R})$, when E^* is a complex on $X \times \mathbf{R}$ with compact support. But it is easy to see that $E^* \oplus \theta^* E^*$ is homotopic to an acyclic complex.

The most important application of Proposition (3.1) is the definition of the Thom homomorphism for K_G . First observe that if E is a G -vector bundle on X and s is an equivariant section of E one can form the *Koszul complex*

$$\dots \rightarrow 0 \rightarrow \mathbf{C} \xrightarrow{d} \Lambda^1 E \xrightarrow{d} \Lambda^2 E \xrightarrow{d} \dots$$

where d is defined by $d(\xi) = \xi \wedge s(x)$ if $\xi \in \Lambda^i E_x$. This complex is acyclic at all points x at which $s(x) \neq 0$, so its support is the set of zeros of s .

Now, if $p : E \rightarrow X$ is the projection, the bundle $p^* E$ on E has a natural section which is the diagonal map $\delta : E \rightarrow E \times_X E = p^* E$. This section δ vanishes precisely on

the zero-section of E . I shall denote by Λ_E^\bullet the Koszul complex on E formed from p^*E and δ .

If F^\bullet is a complex with compact support on X then p^*F^\bullet is a complex on E with support $p^{-1}(\text{supp}(F^\bullet))$, and $\Lambda_E^\bullet \otimes p^*F^\bullet$ is a complex with compact support on E . The assignment $F^\bullet \mapsto \Lambda_E^\bullet \otimes p^*F^\bullet$ induces an additive homomorphism $\varphi_* : K_G(X) \rightarrow K_G(E)$ which is called the *Thom homomorphism*. (It is a homomorphism of $K_G(X)$ -modules in the obvious sense.)

If $\varphi : X \rightarrow E$ is the zero-section, then $\varphi^* \varphi_*(F^\bullet)$ is just the alternating sum of the complexes $\Lambda^i E \otimes F^\bullet$, i.e. $\varphi^* \varphi_*(\xi) = \xi \cdot \lambda_{-1} E$ for any $\xi \in K_G(X)$.

If X is compact, Λ_E^\bullet has compact support and defines the *Thom class* $\varphi_*(1) = \lambda_E$ in $K_G(E)$.

By replacing X and E by $X \times \mathbf{R}^q$ and $E \times \mathbf{R}^q$ one obtains a Thom homomorphism $\varphi_* : K_G^{-q}(X) \rightarrow K_G^{-q}(E)$ for each $q \in \mathbf{N}$.

The most important theorem in equivariant K-theory is

Proposition (3.2). — *The Thom homomorphism $\varphi_* : K_G^*(X) \rightarrow K_G^*(E)$ is an isomorphism for any G-vector bundle E on a locally compact G-space X .*

I shall not prove this theorem here, as the proof for a general group G uses families of elliptic differential operators. But I shall perform some reductions, and in particular prove it when G is abelian.

First observe that it suffices to prove

Proposition (3.3). — *The Thom homomorphism $\varphi_* : K_G(X) \rightarrow K_G(E)$ is an isomorphism for any G-vector bundle on a compact G-space X .*

Proof that (3.3) \Rightarrow (3.2). — By the continuity (2.12) of K_G^* it suffices to show $\varphi_* : K_G^*(U) \xrightarrow{\cong} K_G^*(E|U)$ when U is a relatively compact open G-subspace of X . Then by the exact sequence for the pair $(\bar{U}, \bar{U} - U)$ one is reduced to the case of a compact base space. Finally because of the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_G^{-q}(E) & \longrightarrow & K_G(E \times S^q) & \longrightarrow & K_G(E) \longrightarrow 0 \\
 & & \uparrow \varphi_* & & \uparrow \varphi_* & & \uparrow \varphi_* \\
 0 & \longrightarrow & K_G^{-q}(X) & \longrightarrow & K_G(X \times S^q) & \longrightarrow & K_G(X) \longrightarrow 0,
 \end{array}$$

where the rows are split exact sequences, one is reduced to the case of (3.3).

When E is a G -line bundle, Proposition (3.3) is a generalization of the Bott periodicity theorem, and is proved in [3]. (The equivariant case is not considered

in [3], but the argument applies without change.) To be precise, in [3] it is proved that $K_G(P(E \oplus \mathbf{C}))$ is a free $K_G(X)$ -module generated by the unit element $\mathbf{1}$ and the class of the Hopf bundle H . (If E is a G -vector bundle on X , $P(E)$ denotes the associated bundle of projective spaces, whose points are lines in E . There is a canonical G -line bundle $H^* = \{(\xi, x) \in P(E) \times E : x \in \xi\}$ on $P(E)$. The Hopf bundle H is the dual of H^* .) Now E can be identified with $P(E \oplus \mathbf{C}) - P(E)$, so $K_G(E)$ is the kernel of the restriction $K_G(P(E \oplus \mathbf{C})) \rightarrow K_G(P(E)) = K_G(X)$, and is generated by $H^* - \pi^*E$ or $\mathbf{C} - \pi^*E \otimes H$, where $\pi : P(E \oplus \mathbf{C}) \rightarrow X$ is the projection. Because H^* is a sub-bundle of $\pi^*(E \oplus \mathbf{C})$, there is a canonical morphism $H^* \rightarrow \pi^*E$ or $\mathbf{C} \rightarrow \pi^*E \otimes H$. Restricted to $E \subset P(E \oplus \mathbf{C})$ the last thing becomes the complex Λ_E^\bullet : observe that $H|_E$ is canonically trivial. So $K_G(E)$ is the free $K_G(X)$ -module generated by λ_E , as desired.

If (3.3) is known for line bundles then so is (3.2). And hence (3.2) is true whenever E is a sum of line bundles, because the Thom homomorphism is transitive:

Proposition (3.4). — *If E and F are bundles on X , and $p : E \oplus F \rightarrow E$, $q : E \oplus F \rightarrow F$ are the projections, then $\Lambda_{E \oplus F}^\bullet \cong p^* \Lambda_E^\bullet \otimes q^* \Lambda_F^\bullet$, and the diagram*

$$\begin{array}{ccc} K_G(X) & \xrightarrow{\varphi_*} & K_G(E \oplus F) \\ & \searrow \varphi_* \quad \nearrow \varphi_* & \\ & K_G(E) & \end{array}$$

commutes.

Proof. — The first statement is trivial; the second follows from $\Lambda_{E \oplus F}^\bullet \cong p^* \Lambda_E^\bullet \otimes \Lambda_{\pi^*F}^\bullet$ (where $\pi : E \rightarrow X$), which is true because $\Lambda_{\pi^*F}^\bullet \cong q^* \Lambda_F^\bullet$.

Applying (3.3) to the trivial bundle \mathbf{C} one finds

Proposition (3.5). — $K_G^{-q}(X)$ is naturally isomorphic to $K_G^{-q-2}(X)$, the map being multiplication by a certain element of $K_G^{-2}(\text{point})$.

Proposition (3.5) suggests that one should define $K_G^q(X)$ for positive q as $K_G^{q-2n}(X)$, where $n \geq q/2$. Then one has cohomological exact sequences extending infinitely in both directions, which are very much more powerful tools than the semi-infinite ones which exist not only for K_G but for any “half-exact functor” ⁽¹⁾ For example they permit one to prove the following:

Proposition (3.6). — (3.2) is true when E is locally a sum of G -line bundles. (Locally means “in a neighbourhood of each orbit”.)

⁽¹⁾ But nevertheless it is usually convenient to regard $K_G^*(X)$ as graded modulo two. In the sequel $K_G^*(X)$ will mean $K_G^0(X) \oplus K_G^{-1}(X)$.

Proof. — One reduces oneself very simply to showing that if Y is a closed G -subspace of X such that $E|(X-Y)$ is a sum of line bundles, and $\varphi_* : K_G^*(Y) \rightarrow K_G^*(E|Y)$ is an isomorphism, then $\varphi_* : K_G^*(X) \rightarrow K_G^*(E)$ is an isomorphism. Because

$$\varphi_* : K_G^*(X-Y) \xrightarrow{\cong} K_G^*(E|(X-Y)),$$

that follows on applying the 5-lemma to the exact sequences for the pairs (X, Y) and $(E, E|Y)$.

In particular (3.2) is true when G is abelian, because

Proposition (3.7). — *If G is abelian then any G -vector bundle E on X is locally a sum of G -line bundles.*

Given $x \in X$, let G_x be its stabilizer or isotropy group. The fibre E_x is a G_x -module, but one can extend the G_x -action to make it a G -module, because an inclusion $G_x \rightarrow G$ of abelian groups induces a surjection $\hat{G} \rightarrow \hat{G}_x$. Then $E|Gx \cong G \times_{G_x} E_x \cong (G/G_x) \times E_x$. So E and $X \times E_x$ are isomorphic on the orbit Gx , and hence in a neighbourhood of it. But $X \times E_x$ is a sum of G -line bundles, because E_x is a sum of one-dimensional modules.

As to the proof of (3.2) or (3.3) in the general case, it depends on the following proposition, whose proof involves families of elliptic differential operators.

Proposition (3.8). — *If G is a compact connected Lie group, and $i : T \rightarrow G$ is the inclusion of a maximal torus, then for each locally compact G -space X there is a natural homomorphism of $K_G^*(X)$ -modules $i_* : K_T^*(X) \rightarrow K_G^*(X)$ such that $i_*(1) = 1$, and hence $i_* i^* = \text{identity}$.*

Observe that by considering $U \times_G X$ instead of X , and using Example (iii) of § 2, one need prove (3.8) only when G is a unitary group U .

If one allows (3.8) the proof of (3.3) is very simple. For (3.3) is stable under the operation of extending the group, and any compact Lie group can be embedded in a unitary group, so if G is a Lie group one reduces oneself first to a unitary group, and then to a torus, and then applies (3.7). In fact one can even avoid (3.7), for the G -bundle E on X can be lifted to a trivial $(G \times U(n))$ -bundle \mathbf{M} on its principal bundle P , so one can reduce oneself to the case $K_G^*(X) \rightarrow K_G^*(X \times M)$ when G is abelian; and the M is a sum of one-dimensional modules. The case when G is not a Lie group follows by a simple continuity argument.

To conclude this section I should point out that by standard arguments [2] using (3.2) and (3.8) one can calculate $K_G^*(P)$ in terms of $K_G^*(X)$ when P is a bundle of projective spaces, Grassmannians, Stiefel manifolds, flag manifolds, or lens spaces associated to a G -vector bundle on X . For example

Proposition (3.9). — *If E is a G -vector bundle on X then $K_G^*(P(E))$ is generated as $K_G^*(X)$ -algebra by the Hopf bundle H , modulo the relation $\sum_k (-1)^k \Lambda^k E \cdot H^k = 0$.*

First proof. — I shall confine myself to the case when E is a sum of line bundles, $E = L_1 \oplus \dots \oplus L_n$. Then one can proceed by induction on n . Let

$E_0 = L_1 \oplus \dots \oplus L_{n-1}$. Then $E_0 \cong P(E) - P(E_0)$. The Hopf bundle on $P(E)$ satisfies the relation $\sum_k (-1)^k \Lambda^k E \cdot H^k = 0$ over $K_G^*(X)$, because the bundle $\pi^* E \otimes H$ on $P(E)$ has a natural non-vanishing section, and so its Koszul complex is acyclic. Also, H restricts to the Hopf bundle on $P(E_0)$, and is trivial on E_0 . Write $A = K_G^*(X)$, and consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & A(H)/(f_n) & \xrightarrow{\beta} & A(H)/(f_{n-1}) \longrightarrow 0 \\ & & \downarrow \varphi_* & & \downarrow \theta_n & & \downarrow \theta_{n-1} \\ \dots & \longrightarrow & K_G^*(E_0) & \longrightarrow & K_G^*(P(E)) & \xrightarrow{\rho} & K_G^*(P(E_0)) \longrightarrow \dots \end{array}$$

Here $f_n = \sum_k (-1)^k H^k \cdot \Lambda^k E$, $f_{n-1} = \sum_k (-1)^k H^k \cdot \Lambda^k E_0$, so that $f_n = (1 - L_n \cdot H) \cdot f_{n-1}$; α is defined by $\alpha(1) = f_{n-1}$, making the top line exact; φ_* is the Thom isomorphism. The diagram commutes. We suppose θ_{n-1} is an isomorphism, and want to prove θ_n is one, too. Because β and θ_{n-1} are surjective, ρ is surjective, so one can add 0 at each end of the lower line. Then θ_n is an isomorphism by the 5-lemma.

Second proof. — $P(E)$ is the quotient by the group \mathbf{T} of complex numbers of modulus 1 of $S(E)$, the sphere bundle of E . Because \mathbf{T} acts freely on $S(E)$ one has $K_G(P(E)) = K_{G \times \mathbf{T}}(S(E))$. Let $D(E)$ be the disc bundle associated to E , so that $S(E)$ is the “boundary” of $D(E)$, and $D(E) - S(E) = E$. Then $D(E)$ is contractible to X , so $K_{G \times \mathbf{T}}^*(D(E)) \cong K_{G \times \mathbf{T}}^*(X) \cong K_G^*(X) \otimes R(\mathbf{T}) \cong A[H, H^{-1}]$, where $A = K_G^*(X)$. The group $K_{G \times \mathbf{T}}^*(E)$ is the same, by (3.2), and the map $K_{G \times \mathbf{T}}^*(X) \rightarrow K_{G \times \mathbf{T}}^*(X)$ corresponding to the inclusion $E \rightarrow D(E)$ is the multiplication by $\lambda_{-1} E$, where E is regarded as a $(G \times \mathbf{T})$ -bundle. That means $\lambda_{-1} E = \sum_k (-1)^k \Lambda^k E \cdot H^k = f_n \in A[H, H^{-1}]$.

One has an exact sequence

$$\dots \rightarrow K_{G \times \mathbf{T}}^*(E) \rightarrow K_{G \times \mathbf{T}}^*(D(E)) \rightarrow K_{G \times \mathbf{T}}^*(S(E)) \rightarrow \dots,$$

and can identify it with

$$\dots \rightarrow A[H, H^{-1}] \xrightarrow{\alpha/f_n} A[H, H^{-1}] \rightarrow K_G^*(P(E)) \rightarrow \dots$$

The multiplication by f_n is injective, so one can add 0 at each end of the sequence, so $K_G^*(P(E))$ can be identified with $A[H, H^{-1}]/(f_n) = A[H]/(f_n)$, as desired. I leave it to the reader to check that the H in this proof is the same as the earlier one.

§ 4. LOCALIZATION

I have pointed out that $K_G^*(X)$ is an algebra over $R(G)$. It turns out that localizing it at a prime ideal of $R(G)$ corresponds to restricting one's attention to the set of fixed points of a conjugacy class of elements of G associated to the ideal.

An obvious prime of $R(G)$ consists of all the elements whose characters vanish at a certain element g of G . This prime depends only on the conjugacy class of the closed subgroup of G generated by g . In [16] I described all the prime ideals of $R(G)$, and showed that to each prime \mathfrak{p} is associated a cyclic ⁽¹⁾ subgroup S of G determined up to conjugation, which I called the *support* of \mathfrak{p} . The subgroup S is characterized as minimal among the subgroups of G such that \mathfrak{p} is the inverse image of a prime of $R(S)$. If \mathfrak{p} is the ideal of characters vanishing at g , then S is the cyclic subgroup generated by g . I proved in [16] that, if H is any subgroup of G , then $R(H)_{\mathfrak{p}}$, i.e. $R(H)$ regarded as an $R(G)$ -module and localized at \mathfrak{p} , is non-zero if and only if S is conjugate to a subgroup of H .

I shall write G_x for the stabilizer or isotropy group of a point x of a G -space X , and, if S is a cyclic subgroup of G , I shall write $X^{(S)}$ for the closed G -subset of X consisting of points x for which S is conjugate to a subgroup of G_x . That means $X^{(S)} = G \cdot X^S$, where X^S is the set of points left fixed by S .

The localization theorem is

Proposition (4.1). — *If X is a locally compact G -space, and \mathfrak{p} is a prime of $R(G)$ with support S , then the restriction*

$$K_G^*(X)_{\mathfrak{p}} \rightarrow K_G^*(X^{(S)})_{\mathfrak{p}}$$

is an isomorphism.

Remark. — In view of the preceding remarks, $X^{(S)}$ is precisely the union of the orbits T of X such that $K_G^*(T)_{\mathfrak{p}} \neq 0$, for $K_G^*(Gx) \cong R(G_x)$.

Proof of (4.1). — Because of the cohomological exact sequence it suffices to show that $K_G^*(X - X^{(S)})_{\mathfrak{p}} = 0$, i.e. one can suppose $X^{(S)} = \emptyset$. So one has to show that $K_G^*(X)_{\mathfrak{p}} = 0$ if $K_G^*(T)_{\mathfrak{p}} = 0$ for each orbit T in X . By the continuity of K_G^* (2.11) it suffices to show that $K_G^*(U)_{\mathfrak{p}} = 0$ for all relatively compact open G -subspaces of X . Then by the exact sequence for the pair $(\bar{U}, \bar{U} - U)$ one reduces oneself to the case of a compact G -space.

If X is a compact G -space then there exists a *slice* at each point x of X , i.e. a G_x -subspace S of X containing x such that the natural map $G \times_{G_x} S \rightarrow X$ is an open embedding ([7], Chap. 7). The projection $G \times_{G_x} S \rightarrow G/G_x$ is an equivariant retraction of a neighbourhood of the orbit Gx on to Gx . In view of this one can choose a finite number of points x_1, \dots, x_n of X with compact G -neighbourhoods X_1, \dots, X_n which cover X and are such that X_i admits a G -retraction onto the orbit T_i of x_i . Now, assuming that each $K_G^*(T_i)_{\mathfrak{p}}$ is zero, I want to prove $K_G^*(X)_{\mathfrak{p}} = 0$. It suffices to show that if Y is a G -subspace of X such that $K_G^*(Y)_{\mathfrak{p}} = 0$ then $K_G^*(Y \cup X_i)_{\mathfrak{p}} = 0$. So it

⁽¹⁾ S is *cyclic* if it contains an element g whose powers are dense in S , i.e. if it is the product of a torus and a finite cyclic group.

suffices to show $K_G^*(Y \cup X_i, Y)_p = 0$. But this is $K_G^*(X_i, X_i \cap Y)_p$, which is a unitary module over the ring $K_G^*(X_i)_p$. The projection $X_i \rightarrow T_i$ induces a homomorphism of rings $K_G^*(T_i)_p \rightarrow K_G^*(X_i)_p$; as the first ring is zero so is the second, and so therefore is the module $K_G^*(X_i, X_i \cap Y)_p$.

Some interesting applications of the localization theorem can be found in [6] and [11].

§ 5. THE FILTRATION AND THE SPECTRAL SEQUENCE

If X is a CW-complex, which is filtered by its skeletons $\{X^p\}$, it is customary to define [5] a filtration of $K^*(X)$ by setting $K_p^*(X) = \text{kernel}(K^*(X) \rightarrow K^*(X^{p-1}))$. Then

$$K^*(X) = K_0^*(X) \supset K_1^*(X) \supset K_2^*(X) \supset \dots,$$

and $K^*(X)$ is a filtered ring in the sense that $K_p^*(X) \cdot K_q^*(X) \subset K_{p+q}^*(X)$.

In the equivariant theory there are several quite different filtrations of $K_G^*(X)$. The one I am going to discuss corresponds to filtering X by the G -subspaces $\pi^{-1}(Y^p)$ when the orbit space $Y = X/G$ is a CW-complex. ($\pi : X \rightarrow Y$ is the projection.) But to avoid making assumptions about the orbit space I shall define the filtration by a Čech method. For a fuller and more sophisticated discussion of the construction I refer the reader to [15].

To each finite covering $U = \{U_\alpha\}_{\alpha \in A}$ of a compact G -space X by G -stable closed sets I am going to associate a compact G -space W_U with a G -map $w : W_U \rightarrow X$ and a filtration by G -subspaces $W_U^0 \subset W_U^1 \subset \dots \subset W_U$, so that the following conditions are satisfied:

(i) $w^* : K_G^*(X) \rightarrow K_G^*(W_U)$ is an isomorphism, and

(ii) when V is a refinement of U there is a G -map $W_V \rightarrow W_U$, defined up to G -homotopy, respecting the filtrations and the projections on to X .

Then I shall say that an element of $K_G^*(X)$ is in $K_{G,p}^*(X)$ if, for some finite covering U , it is in the kernel of $w^* : K_G^*(X) \rightarrow K_G^*(W_U^{p-1})$. Thus $K_{G,p}^*(X)$ is an ideal in $K_G^*(X)$. To see that $K_{G,p}^*(X) \cdot K_{G,q}^*(X) \subset K_{G,p+q}^*(X)$ one needs a further property of W_U :

(iii) the diagonal map $W_U \rightarrow W_U \times W_U$ is G -homotopic to a filtration preserving map, when the filtration of $W_U \times W_U$ is defined by $(W_U \times W_U)^n = \bigcup_{p+q=n} W_U^p \times W_U^q$.

The definition of W_U is as follows. Let N_U be the *nerve* of U , a finite simplicial complex whose simplexes are the finite subsets σ of A such that $U_\sigma = \bigcap_{\alpha \in \sigma} U_\alpha$ is non-empty. Let $|N_U|$ be the geometrical realization of N_U , a compact space. Then W_U is the closed subspace $\bigcup_{\sigma} (U_\sigma \times |\sigma|)$ of the product $X \times |N_U|$, and $w : W_U \rightarrow X$ is the projection onto the first factor. Define $W_U^p = \bigcup_{\dim(\sigma) \leq p} (U_\sigma \times |\sigma|)$, i.e. it is the inverse image of the p -skeleton of $|N_U|$.

To prove (i) above, define X_k as the subset of points of X which are contained in at least $k+1$ of the sets U_α . Thus $X = X_0 \supset X_1 \supset X_2 \supset \dots$. Define also $W_k = w^{-1}(X_k) \subset W_U$. Consider the diagram

$$\begin{array}{ccc} \prod_{\sigma} ((U_{\sigma} - U'_{\sigma}) \times |\sigma|) & \longrightarrow & W_k - W_{k+1} \\ \downarrow & & \downarrow w \\ \prod_{\sigma} (U_{\sigma} - U'_{\sigma}) & \longrightarrow & X_k - X_{k+1}, \end{array}$$

where σ runs through the p -simplexes of N_U , and $U'_{\sigma} = U_{\sigma} \cap X_{k+1}$. The horizontal arrows are homeomorphisms, and the vertical arrow on the left is a proper homotopy-equivalence, so $w^* : K_G^*(X_k - X_{k+1}) \rightarrow K_G^*(W_k - W_{k+1})$ is an isomorphism. As this is true for all k , it follows from the cohomology exact sequence that $w^* : K^*(X) \xrightarrow{\cong} K_G^*(W_U)$.

I shall not give here the proofs of the statements (ii) and (iii). They are obtained in the same way as the analogous ones for $|N_U|$, and the details can be found in [15].

I should record the following simple facts about the filtration of $K_G^*(X)$.

Proposition (5.1). — *If X is a compact G -space, then*

- (i) *an element of $K_G^*(X)$ is in $K_{G,1}^*(X)$ if and only if its restriction to each orbit is zero, i.e. $K_{G,1}^*(X) = \text{kernel } (K_G^*(X) \rightarrow \prod_{x \in X} R(G_x))$;*
- (ii) *the elements of $K_{G,1}^*(X)$ are nilpotent.*

Proof. — (i) Because $W_U^0 = \prod_{\alpha \in A} U_{\alpha}$, an element ξ belongs to $K_{G,1}^*(X)$ if there is a finite G -stable covering U such that ξ restricts to zero in each $K_G^*(U_{\alpha})$. This is equivalent to (i) because K_G^* is continuous.

(ii) If $\xi \in K_{G,1}^*(X)$, choose U so that ξ vanishes in $K_G^*(W_U^0)$. The covering U has some finite dimension n , so that $W_U^n = W_U$. Then ξ^{n+1} vanishes in $K_G^*(W_U) = K_G^*(X)$, so ξ is nilpotent.

This proposition implies the localization theorem of § 4, at least in the form that if $R(G_x)_p = 0$ for all $x \in X$ then $K_G^*(X)_p = 0$. For by localizing the exact sequence $0 \rightarrow K_{G,1}^*(X) \rightarrow K_G^*(X) \rightarrow \prod R(G_x)$ at p one finds that every element of $K_G^*(X)_p$, including the unit element, is nilpotent.

To the filtration of the space W_U there corresponds a spectral sequence, defined by the method of [9], p. 333, terminating in $K_G^*(W_U) \cong K_G^*(X)$, and with $E_1^{pq} = K_G^{p+q}(W^p - W^{p-1})$. There is a homeomorphism $\prod_{\sigma} (U_{\sigma} \times \overset{\circ}{\sigma}) \rightarrow W^p - W^{p-1}$, where σ runs through the p -simplexes of N_U , and $\overset{\circ}{\sigma}$ is the interior of $|\sigma|$. So $E_1^{pq} \cong \prod_{\sigma} K_G^{p+q}(U_{\sigma} \times \overset{\circ}{\sigma}) \cong \prod_{\sigma} K_G^q(U_{\sigma})$. One can verify that the differential $d : E_1^{pq} \rightarrow E_1^{p+1,q}$ corresponds to the differential of the complex of cochains of N_U with coefficients in the system $\sigma \mapsto K_G^q(U_{\sigma})$. That is :

Proposition (5.2). — *If U is a finite closed G -stable covering of a compact G -space X , there is a spectral sequence $H^p(N_U; K_G^q(U)) \Rightarrow K_G^*(X)$, where $K_G^q(U)$ means the coefficient system $\sigma \mapsto K_G^q(U_\sigma)$.*

If one lets U run through the directed family of closures of the finite open G -stable coverings of X , and takes the direct limit of the family of spectral sequences corresponding, then, because G -stable coverings of X can be identified with coverings of X/G , one obtains

Proposition (5.3). — *If X is a compact G -space there is a spectral sequence $H^p(X/G; \mathcal{K}_G^q) \Rightarrow K_G^*(X)$, where \mathcal{K}_G^q is the sheaf on X/G associated to the presheaf $V \mapsto K_G^q(\pi^{-1}V)$. ($\pi: X \rightarrow X/G$ is the projection.) The stalk of \mathcal{K}_G^q at an orbit $Gx = G/G_x$ is $R(G_x)$ if q is even, and $\mathcal{K}_G^q = 0$ if q is odd.*

For the details I refer again to [15]. The assertion about the stalk follows from the continuity of K_G^* . (Prop (2.11)).

Remark. — More generally, if $f: X \rightarrow Y$ is a map of compact G -spaces and G acts trivially on Y , the argument shows there is a spectral sequence $H^p(Y; \mathcal{K}_G^q f) \Rightarrow K_G^*(X)$, where $\mathcal{K}_G^q f$ is a sheaf on Y whose stalk at y is $K_G^q(f^{-1}y)$.

One application of the spectral sequence (5.3) is to prove the following useful finiteness theorem.

Let us call a G -space X *locally G -contractible* if each point $x \in X$ has arbitrarily small G_x -stable neighbourhoods which are G_x -contractible in themselves to x , or, what is the same thing, if each orbit has arbitrarily small G -neighbourhoods of which it is a G -deformation retract. For example, a differentiable manifold X on which a compact Lie group G acts smoothly is locally G -contractible. Then one has

Proposition (5.4). — *If X is a locally G -contractible compact G -space such that X/G has finite covering dimension, then $K_G^*(X)$ is a finite $R(G)$ -module.*

To prove this one observes first that because X/G has finite dimension the spectral sequence $H^*(X/G; \mathcal{K}_G^*) \Rightarrow K_G^*(X)$ is convergent, and so it suffices to show that $H^*(X/G; \mathcal{K}_G^*)$ is finite over $R(G)$. ($R(G)$ is noetherian [16].) Because X is locally G -contractible one can show that each point Gx of X/G has arbitrarily small neighbourhoods U such that $\mathcal{K}_G^*(U) \cong R(G_x)$, which is finite over $R(G)$. This implies, after a little manipulation, that $H^*(X/G; \mathcal{K}_G^*)$ is finite over $R(G)$; but I shall not give the details here.

The hypothesis that X/G has finite dimension is satisfied in the case of a smooth G -manifold, because X/G is then a finite union of open manifolds (cf. [13], (1.7.31)).

APPENDIX

Proof of Proposition (3.1)

I shall begin with some definitions.

A complex E^\bullet is *elementary* if $E^i = 0$ except for two values $i = n, n+1$, and $d : E^n \rightarrow E^{n+1}$ is an isomorphism; it is *trivial* if it is elementary with trivial bundles. Because exact sequences of G -vector bundles split, any acyclic complex is a sum of elementary complexes.

Two morphisms $f_0, f_1 : E^\bullet \rightarrow F^\bullet$ are *equivalent* if there is a sequence of homomorphisms $h^i : E^i \rightarrow F^{i-1}$ such that $f_1 - f_0 = dh + hd$. Complexes on X and equivalence classes of morphisms form a category denoted by $C_G(X)$. To avoid confusion I shall use the word *equivalence* for an isomorphism in this category. An elementary complex is equivalent to 0, and hence so is any acyclic complex.

If A is a subspace of X , I shall write $C_G(X, A)$ for the full subcategory of $C_G(X)$ whose objects are the complexes E^\bullet such that $E^\bullet|_A$ is equivalent to zero, or, what is the same, acyclic.

Two objects E_0^\bullet, E_1^\bullet of $C_G(X, A)$ are *homotopic* if there is an object E^\bullet of $C_G(X \times [0, 1], A \times [0, 1])$ with equivalences $E_i^\bullet \rightarrow E^\bullet|_{(X \times i)}$ for $i = 0, 1$.

Proposition (3.1) can be reformulated in the following way, which seems to me more appealing.

Proposition (A.1). — *If X is a compact G -space and A a closed G -subspace, then the set of homotopy classes of objects of $C_G(X, A)$ forms an abelian group $Q_G(X, A)$ under \oplus . This group is naturally isomorphic to $K_G(X, A)$.*

I shall prove the equivalence of (3.1) and (A.1) after proving (A.1); and before proving (A.1) I need a few more definitions, and a lemma.

If E^\bullet is a complex, TE^\bullet denotes the complex with the grading translated: $(TE)^\bullet = E^{\bullet-1}$.

A morphism $f : E^\bullet \rightarrow F^\bullet$ of complexes has a *mapping cone* C_f^\bullet , which is the complex obtained by regarding the double complex $\dots \rightarrow 0 \rightarrow E^\bullet \rightarrow F^\bullet \rightarrow 0 \rightarrow \dots$ as a single complex; C_f^\bullet is acyclic if and only if f is an equivalence, for a family of splitting maps for C_f^\bullet is the same thing as an inverse equivalence to f .

If $f, g : E^\bullet \rightarrow F^\bullet$ are two morphisms then the complexes C_f^\bullet, C_g^\bullet are obviously homotopic. In particular, taking $E^\bullet = F^\bullet$, the complexes C_{id}^\bullet and $C_0^\bullet = E^\bullet \oplus TE^\bullet$ are homotopic. C_{id}^\bullet is equivalent to zero, so $E^\bullet = -TE^\bullet$ in $C_G(X, A)$. This proves that $Q_G(X, A)$ is an abelian group.

Lemma (A.2). — *If a G -space is the union of two compact G -subspaces X and Y with intersection A , and if E^\bullet is a complex of $C_G(X, A)$, then there is a complex \tilde{E}^\bullet of*

$C_G(X \amalg_A Y, Y)$ such that $\tilde{E}^*|X$ is equivalent to E^* . The complex \tilde{E}^* is unique up to equivalence. That is, the categories $C_G(X, A)$ and $C_G(X \amalg_A Y, Y)$ are equivalent.

Proof. — To see \tilde{E}^* exists it suffices to show that one can add an acyclic complex to E^* so that $E^*|A$ becomes isomorphic to a sum of trivial complexes, for the latter can be extended over Y . First add elementary complexes to E^* so as to make all the bundles trivial except the first, say E^a . Then $\sum_k (-1)^k (E^k|A) = 0$ in $\tilde{K}_G(A)$ because $E^*|A$ is acyclic. $E^k|A$ represents zero in $K_G(A)$ for $k \neq a$, and hence also for $k = a$, i.e. $E^a|A$ is stably trivial. One sees inductively that when $E^*|A$ is expressed as a sum of elementary complexes all the bundles occurring are stably trivial, and so by adding trivial complexes one can make $E^*|A$ into a sum of trivial complexes.

As to the uniqueness of \tilde{E}^* , if \tilde{E}_1^* is another candidate, then one has an equivalence $f: \tilde{E}^*|X \rightarrow \tilde{E}_1^*|X$ and would like to extend it over Y . But $f|A$ is equivalent to the zero-morphism (because $\tilde{E}_1^*|A$ is equivalent to 0), so one can write $f|A = dh + hd$ for suitable h defined on A . Then any extension of h over Y defines an extension of f by the formula $dh + hd$.

Now I can prove (A.1). By excision $K_G(X, A) = K_G(X \amalg_A X, X)$, where the subspace is the second summand, and by (A.2) $Q_G(X, A) = Q_G(X \amalg_A X, X)$, so it suffices to show that $Q_G(X \amalg_A X, X)$ is naturally isomorphic to $K_G(X \amalg_A X, X)$. This is more convenient because $K_G(X \amalg_A X, X)$ can be identified with the kernel of the split restriction $i_2^*: K_G(X \amalg_A X) \rightarrow K_G(X)$ onto the second summand. In fact one is reduced to proving the following:

Proposition (A.3). — *If A is a closed G -subspace of a compact G -space X , and is a retract of X by a map $p: X \rightarrow A$, then the sequence*

$$0 \rightarrow Q_G(X, A) \xrightarrow{\alpha} K_G(X) \rightarrow K_G(A) \rightarrow 0,$$

where α is $E^* \mapsto \sum_k (-1)^k E^k$, is split exact.

Proof. — If E is a bundle on X then $E|A$ and $(p^* i^* E)|A$, where $i: A \rightarrow X$ is the inclusion, are isomorphic. Form a complex $\dots \rightarrow 0 \rightarrow E \rightarrow p^* i^* E \rightarrow 0 \rightarrow \dots$ on X by extending arbitrarily this isomorphism. Because different extensions lead to homotopic complexes this construction defines a homomorphism $\beta: K_G(X) \rightarrow Q_G(X, A)$. It remains to see that $p^* i^* + \alpha\beta = 1$ and that $\alpha\beta = 1$. The first is trivial. As to the second: if E^* is a complex of $C_G(X, A)$, choose a morphism $f: E^* \rightarrow p^* i^* E^*$ which extends the canonical isomorphism on A . (That is possible because $p^* i^* E^*$ is acyclic and hence a sum of elementary complexes.) The mapping cone C_f^* is equivalent to E^* , and on the other hand is homotopic in $C_G(X, A)$ to the mapping cone of $f: E_0^* \rightarrow p^* i^* E_0^*$, where E_0^* is obtained from E^* by replacing the differential by zero. This last mapping cone, however, represents $\beta\alpha(E^*)$.

It remains to show that $Q_G(X, A)$ is the same as the semi-group $L_G(X, A)/\sim$ introduced in § 3. There is an obvious surjection $L_G(X, A)/\sim \rightarrow Q_G(X, A)$, and one has only to show that, if $f: E_0^* \rightarrow E_1^*$ is an equivalence, then $E_0^* \sim E_1^*$, i.e. $E_0^* \oplus F_0^* \simeq E_1^* \oplus F_1^*$ for some acyclic complexes F_0^*, F_1^* . But in fact $TE_0^* \oplus C_I^* \simeq TE_1^* \oplus C_I^*$, where I means the identity morphism of E_0^* , for $TE_0^* \oplus C_I^*$ is the mapping cone of $0 \oplus f: E_0^* \rightarrow E_0^* \oplus E_1^*$, and $TE_1^* \oplus C_I^*$ is the mapping cone of $I \oplus 0: E_0^* \rightarrow E_0^* \oplus E_1^*$.

So far in this appendix I have confined myself to compact G -spaces. The generalization to locally compact spaces amounts to the proof of the following lemma.

Proposition (A.4). — $Q_G(X, A) \xrightarrow{\cong} Q_G(X-A)$ when X and A are compact G -spaces. ($Q_G(X-A)$ is formed, of course, from the category of complexes with compact support on $X-A$.)

Proof. — In fact the categories $C_G(X, A)$ and $C_G(X-A)$ are equivalent. I shall define a functor $C_G(X-A) \rightarrow C_G(X, A)$. Let E^* be a complex of $C_G(X-A)$, and let K be a compact G -neighbourhood of its support. Apply (A.2) to $E^*|K$, which is acyclic on $K-\bar{K}$, and extend it to \tilde{E}^* defined on X . Then $\tilde{E}^*|(X-A)$ is canonically equivalent to E^* , for if $\theta: E^*|K \rightarrow \tilde{E}^*|K$ is the canonical equivalence and $\varphi: X-A \rightarrow \mathbf{R}_+$ is a function vanishing outside K and equal to 1 on $\text{supp}(E^*)$ then $\varphi\theta$ is an equivalence between E^* and $\tilde{E}^*|(X-A)$ and does not depend on the choice of φ . If one chooses such an \tilde{E}^* for each E^* , one can clearly define a functor $\varepsilon: C_G(X-A) \rightarrow C_G(X, A)$ with $\varepsilon(E^*) = \tilde{E}^*$ which is inverse to the restriction, and is the required equivalence of categories. By (A.2) the composition $C_G(X, A) \rightarrow C_G(X-A) \rightarrow C_G(X, A)$ is the “identity”. As to the composition in the other order, let $\theta: E^*|K \rightarrow \tilde{E}^*|K$ be the canonical equivalence, and let $\varphi: X-A \rightarrow \mathbf{R}_+$ be a function equal to 1 on $\text{supp}(E^*)$, and with $\text{supp}(\varphi) \subset K$. Then $\varphi\theta$ is a canonical equivalence between E^* and $\tilde{E}^*|(X-A)$.

To conclude this appendix I would like to point out that the lemma (A.2) is just a special case of the following clutching property of complexes, which illustrates the naturalness of the categories $C_G(X)$.

Proposition (A.5). — If a G -space X is the union of compact G -subspaces X_1, X_2 with intersection A , and if E_1^*, E_2^* are complexes on X_1, X_2 , and $\alpha: E_1^*|A \rightarrow E_2^*|A$ is an equivalence, then there is a complex E^* on X with equivalences $\beta_i: E^*|X_i \rightarrow E_i^*$ such that $(\beta_2|A) = \alpha \cdot (\beta_1|A)$. The complex E^* is unique up to canonical equivalence.

Proof. — First replace the situation by an equivalent one in which each $\alpha^k: E_1^k|A \rightarrow E_2^k|A$ is surjective — for example by adding trivial complexes to E_1^* . Then, as in (A.2), add an acyclic complex to E_1^* so that $K^* = \ker(\alpha)$ becomes a sum of trivial complexes. Now whenever one has a short exact sequence of complexes $0 \rightarrow F_1^* \rightarrow F_2^* \rightarrow F_3^* \rightarrow 0$ one can identify F_2^q with $F_1^q \oplus F_3^q$ for each q , and this identifies F_2^*

with the mapping-cone of a map $f: F_3^\bullet \rightarrow F_1^\bullet$. Extend K^\bullet over X_2 and extend the corresponding f to $f: E_2^\bullet \rightarrow K^\bullet$. Then C_i^\bullet is a complex on X_2 canonically equivalent to E_2^\bullet , and such that $C_i^\bullet|_A$ can be identified with $E_1^\bullet|_A$. So form E^\bullet by joining together E_1^\bullet and C_i^\bullet . This proves the existence of E^\bullet .

One can prove the uniqueness of E^\bullet just as in (A.2).

REFERENCES

- [1] M. F. ATIYAH, Power operations in K-theory, *Quart. J. of Math.* (Oxford), **17** (1966), 165-193.
- [2] M. F. ATIYAH, *Lectures on K-theory*, mimeographed, Harvard, 1964.
- [3] M. F. ATIYAH and R. BOTT, On the periodicity theorem for complex vector bundles, *Acta mathematica*, **112** (1964), 229-247.
- [4] M. F. ATIYAH, R. BOTT and A. SHAPIRO, Clifford modules, *Topology*, **3** (Suppl. 1) (1964), 3-38.
- [5] M. F. ATIYAH and F. HIRZEBRUCH, Vector bundles and homogeneous spaces, *Differential geometry, Proc. of Symp. in Pure Math.*, **3** (1961), Amer. Math. Soc., 7-38.
- [6] M. F. ATIYAH, I. M. SINGER, etc., The index of elliptic operators I, II (To appear).
- [7] A. BOREL et al., Seminar on transformation groups, *Ann. of Math. Studies*, n° 46, Princeton, 1960.
- [8] N. BOURBAKI, *Intégration*, chap. 1-4, Paris, Hermann, 1952, A.S.I., 1175.
- [9] H. CARTAN and S. EILENBERG, *Homological algebra*, Princeton University Press, 1956.
- [10] S. EILENBERG and N. E. STEENROD, *Foundations of algebraic topology*, Princeton University Press, 1952.
- [11] L. ILLUSIE, Nombres de Chern et groupes finis (To appear).
- [12] G. D. MOSTOW, Cohomology of topological groups and solvmanifolds, *Ann. of Math.*, **73** (1961), 20-48.
- [13] R. S. PALAIS, The classification of G-spaces, *Mem. Amer. Math. Soc.*, n° 36, 1960.
- [14] R. S. PALAIS, On the existence of slices for actions of non-compact Lie groups, *Ann. of Math.*, **73** (1961), 295-323.
- [15] G. B. SEGAL, Classifying-spaces and spectral sequences, *Publ. Math. Inst. des Hautes Études Scient.* (Paris), **34** (1968).
- [16] G. B. SEGAL, The representation-ring of a compact Lie group, *Publ. Math. Inst. des Hautes Études Scient.* (Paris), **34** (1968).

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