Equivariant map superalgebras

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Outline

Summary: We classify the irreducible finite dimensional representations of a certain class of Lie superalgebras (equivariant map superalgebras).

Overview

1. Lie superalgebras
2. Equivariant map superalgebras
3. Modules/representations for equivariant map superalgebras
4. Generalized evaluation modules
5. Kac modules
6. Classification Theorem
7. Further directions

Lie superalgebras

$\Bbbk$ - algebraically closed field of characteristic zero

Definition (Lie superalgebra)

A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with a bilinear superbracket

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying: If $u, v, z \in \mathfrak{g}$ are pure in the $\mathbb{Z}_2$-grading, then

- Super skew-symmetry: $[u, v] = -(-1)^{|u||v|}[v, u]$,
- Super Jacobi identity:

$$(-1)^{|w||u|}[u, [v, w]] + (-1)^{|v||w|}[w, [u, v]] + (-1)^{|u||v|}[v, [w, u]] = 0.$$

Lie superalgebras play an important role in the math of supersymmetry.

Lie superalgebras: Examples

Example: Lie algebras

If $\mathfrak{g}_{\bar{1}} = 0$, then $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$ is simply a Lie algebra.

Every Lie algebra is a Lie superalgebra (with zero odd part).

Lie superalgebras arising from associative superalgebras

Recall that any associative algebra A can be given the structure of a Lie algebra with bracket

$$[a, b] = ab - ba, \qquad a, b \in A.$$

If A is an associative superalgebra (i.e. a $\mathbb{Z}_2$-graded associative algebra), then it can be given the structure of a Lie superalgebra with superbracket

$$[a, b] = ab - (-1)^{|a||b|} ba \quad \text{for all} \quad a, b \in A \text{ pure in the grading.}$$

Representations of Lie superalgebras

Endmorphisms of a "super" space

If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space, then

$$\operatorname{End} V = (\operatorname{End}_{\bar{0}} V) \oplus (\operatorname{End}_{\bar{1}} V)$$

is naturally an associative (hence Lie) superalgebra. Here

$$\operatorname{End}_\varepsilon V := \{u \in \operatorname{End} V \mid uV_{\varepsilon'} \subseteq V_{\varepsilon+\varepsilon'}\}, \quad \varepsilon \in \mathbb{Z}_2.$$

Definition (Representation)

A representation of a Lie superalgebra $\mathfrak{g}$ is a homomorphism of Lie superalgebras

$$\mathfrak{g} \to \operatorname{End} V,$$

where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded space.

We say V is a $\mathfrak{g}$-module.

Classical Lie superalgebras

It follows from the definition of a Lie superalgebra that:

- $\mathfrak{g}_{\bar{0}}$ is a Lie algebra,
- $\mathfrak{g}_{\bar{0}}$ acts on $\mathfrak{g}_{\bar{1}}$.

Suppose $\mathfrak{g}$ is finite dimensional and simple.

$\mathfrak{g}$ is classical if the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible.

If $\mathfrak{g}$ is a classical Lie superalgebra, then the rep of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is either

- irreducible ($\mathfrak{g}$ is of type II), or
- the direct sum of two irreducible reps ($\mathfrak{g}$ is of type I).

A classical Lie superalgebra is basic if it has a non-degenerate invariant bilinear form. (Otherwise, it is strange.)

Examples of basic classical Lie superalgebras

Fix a $\mathbb{Z}_2$-graded vector space

$$V = V_{\bar{0}} \oplus V_{\bar{1}}, \quad \dim V_{\bar{0}} = m,\ \dim V_{\bar{1}} = n.$$

Fixing a basis of V compatible with the $\mathbb{Z}_2$-grading, we have

$$\operatorname{End}_{\bar{0}} V = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \in M_{m,m},\ D \in M_{n,n} \right\},$$
$$\operatorname{End}_{\bar{1}} V = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B \in M_{m,n},\ C \in M_{n,m} \right\}.$$

Define the super trace

$$\operatorname{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{tr} A - \operatorname{tr} D.$$

Examples of basic classical Lie superalgebras

We set

$$\mathfrak{sl}(m, n) = \{X \in \mathrm{End}\, V \mid \mathrm{str}\, V = 0\}$$

If $m \neq n$, $m, n \geq 0$, then

$$A(m, n) \stackrel{\text{def}}{=} \mathfrak{sl}(m + 1, n + 1)$$

is simple.

If $m = n > 0$, then

$$A(n, n) \stackrel{\text{def}}{=} \mathfrak{sl}(n + 1, n + 1)/\mathbb{k} I_{2n+2}$$

is simple.

In both cases, $A(m, n)$ is a basic classical Lie superalgebra.

Basic classical Lie superalgebras

The following table contains all basic classical Lie superalgebras that are not Lie algebras.

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	Type
$A(m,n),\ m > n \geq 0$	$A_m \oplus A_n \oplus \mathbb{k}$	I
$A(n,n),\ n \geq 1$	$A_n \oplus A_n$	I
$\mathfrak{sl}(n,n),\ n \geq 1$	$A_{n-1} \oplus A_{n-1} \oplus \mathbb{k}$	N/A
$C(n+1),\ n \geq 1$	$C_n \oplus \mathbb{k}$	I
$B(m,n),\ m \geq 0,\ n \geq 1$	$B_m \oplus C_n$	II
$D(m,n),\ m \geq 2,\ n \geq 1$	$D_m \oplus C_n$	II
$F(4)$	$A_1 \oplus B_3$	II
$G(3)$	$A_1 \oplus G_2$	II
$D(2,1;\alpha),\ \alpha \neq 0, -1$	$A_1 \oplus A_1 \oplus A_1$	II

Note: $\mathfrak{sl}(n,n)$ is not a basic classical Lie superalgebra. It is a one-dimensional central extension of $A(n,n)$.

Map superalgebras

$\mathfrak{g}$ a Lie superalgebra

X a scheme (or algebraic variety) with coordinate ring A

Definition (Map superalgebra)

The Lie superalgebra $\mathfrak{g} \otimes A$ (regular functions on X with values in $\mathfrak{g}$) is a map superalgebra.

- $\mathbb{Z}_2$-grading: $(\mathfrak{g} \otimes A)_\varepsilon = \mathfrak{g}_\varepsilon \otimes A$ for $\varepsilon \in \mathbb{Z}_2$.
- Pointwise multiplication: Extend by linearity the product

$$[u_1 \otimes f_1, u_2 \otimes f_2] = [u_1, u_2] \otimes f_1 f_2, \quad u_1, u_2 \in \mathfrak{g}, \; f_1, f_2 \in A.$$

Map superalgebras – Examples

Example (Loop superalgebras)

Take $A = \mathbb{k}[t, t^{-1}]$, so $X = \operatorname{Spec} A$ is the torus.

Then $\mathfrak{g} \otimes \mathbb{k}[t, t^{-1}]$ is a loop superalgebra.

Play an important role in the theory of affine Lie superalgebras.

Example (Multiloop superalgebras)

Take $A = \mathbb{k}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, so $X = \operatorname{Spec} A$ is a higher dimensional torus.

Then $\mathfrak{g} \otimes \mathbb{k}[t_1^{\pm 1}, \ldots, t_n^{\pm n}]$ is a multiloop superalgebra.

Example (Current superalgebras)

Take $A = \mathbb{k}[t]$, so $X = \operatorname{Spec} A$ is the affine plane.

Then $\mathfrak{g} \otimes \mathbb{k}[t]$ is a current superalgebra.

Equivariant map superalgebras

Suppose Γ is a group acting on X (hence on A) and $\mathfrak{g}$ by automorphisms.

Then Γ acts naturally on $\mathfrak{g} \otimes A$ (diagonally).

Definition (Equivariant map superalgebra)

The Lie superalgebra

$$(\mathfrak{g} \otimes A)^\Gamma = \{\mu \in \mathfrak{g} \otimes A \mid \gamma\mu = \mu \text{ for all } \gamma \in \Gamma\}$$

is an equivariant map superalgebra.

It is the Lie superalgebra of equivariant regular maps from X to $\mathfrak{g}$ (with pointwise multiplication).

Example: Twisted multiloop superalgebras

Fix positive integers $n, m_1, \dots, m_n$ and let

$$\Gamma = \langle \gamma_1, \dots, \gamma_n \mid \gamma_i^{m_i} = 1,\ \gamma_i\gamma_j = \gamma_j\gamma_i \ \forall\ i,j \rangle \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}.$$

Suppose Γ acts on $\mathfrak{g}$ (e.g. by diagram automorphisms).

Let $X = \operatorname{Spec} A$, where $A = \Bbbk[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. So X is the n-dimensional torus.

Define an action of Γ on X by

$$\gamma_i(z_1, \dots, z_n) = (z_1, \dots, z_{i-1}, \xi_i z_i, z_{i+1}, \dots, z_n),$$

where ξ_i is a primitive m_i-th root of unity for $1 \le i \le n$.

Then $(\mathfrak{g} \otimes A)^\Gamma$ is a twisted multiloop superalgebra. If $n = 1$, it is a twisted loop superalgebra. These are related to twisted affine Lie superalgebras.

Previous work

Equivariant map algebras

When $\mathfrak{g}$ is a finite dimensional Lie algebra, the irreducible finite dimensional modules of equivariant map algebras have been classified (Neher-S.-Senesi '09).

In that case, all the irred f.d. modules are tensor products of evaluation modules and one-dimensional modules.

We wish to classify the irred f.d. modules of equivariant map superalgebras.

We start with the case where $\mathfrak{g}$ is a basic classical Lie superalgebra, since these behave the most like the semisimple Lie algebras.

Goal

Today's goal

Classify the irreducible finite dimensional representations of the equivariant map superalgebras $(\mathfrak{g} \otimes A)^\Gamma$ under the assumptions that

- $\mathfrak{g}$ is a basic classical Lie superalgebra,
- A is finitely generated (so $\operatorname{maxSpec} A = X_{\mathrm{rat}}$),
- Γ is finite abelian and acts freely on X_{rat}.

In the case that $A = \mathbb{k}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, Γ is trivial and $\mathfrak{g}$ is basic classical (i.e. for untwisted multiloop superalgebras), the classification has been given by Rao and Zhao.

However, even the classification for the twisted loop superalgebras is (was) not known.

Support of a module

Suppose V is a $(\mathfrak{g} \otimes A)^\Gamma$-module.

Definition ($\operatorname{Ann}_A V$)

$$\operatorname{Ann}_A V \stackrel{\text{def}}{=} \text{largest } \Gamma\text{-invariant ideal } I \text{ of } A \text{ satisfying } (\mathfrak{g} \otimes I)^\Gamma V = 0.$$

Definition (Support of a module)

The support of V is

$$\operatorname{Supp}_A V \stackrel{\text{def}}{=} \operatorname{Supp} \operatorname{Ann}_A V = \{\mathsf{m} \in X_{\text{rat}} \mid \operatorname{Ann}_A V \subseteq \mathsf{m}\}.$$

We say V has reduced support if $\operatorname{Ann}_A V$ is a radical ideal (i.e. if $\operatorname{Spec}(A/\operatorname{Ann}_A V)$ is a reduced scheme).

Weight modules

Assume $\mathfrak{g}$ is a either a reductive Lie algebra, a basic classical Lie superalgebra, or $\mathfrak{sl}(n, n)$.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Note that $\mathfrak{h}$ here is just a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$.

We identify $\mathfrak{g}$ with $\mathfrak{g} \otimes \Bbbk \subseteq \mathfrak{g} \otimes A$.

Suppose V is a $(\mathfrak{g} \otimes A)$-module.

Definition (Weight module)

V is a weight module if its restriction to $\mathfrak{g}$ is a weight module, i.e., if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \quad V_\lambda \stackrel{\text{def}}{=} \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Weight modules

Definition (Quasifinite module)

V is quasifinite if

- it is a weight module,
- all weight spaces are finite dimensional.

Definition (Highest weight module)

V is highest weight if there is a nonzero $v \in V$ such that

- $(\mathfrak{n}^+ \otimes A)v = 0$,
- $(\mathfrak{h} \otimes A)v = \mathbb{k}v$,
- $U(\mathfrak{g} \otimes A)v = V$.

Such a vector v is called a highest weight vector.

Remark

All irreducible finite dimensional modules are highest weight modules.

Irreducible highest weight modules

Fix $\psi \in (\mathfrak{h} \otimes A)^*$.

Define an action of $(\mathfrak{h} \oplus \mathfrak{n}^+) \otimes A$ on $\mathbb{k}$ by letting

- $\mathfrak{h} \otimes A$ act by ψ,
- $\mathfrak{n}^+ \otimes A$ act by zero.

The induced module

$$U(\mathfrak{g} \otimes A) \otimes_{U((\mathfrak{h} \oplus \mathfrak{n}^+) \otimes A)} \mathbb{k}$$

has a unique maximal submodule $N(\psi)$.

We define

$$V(\psi) \overset{\text{def}}{=} \left(U(\mathfrak{g} \otimes A) \otimes_{U((\mathfrak{h} \oplus \mathfrak{n}^+) \otimes A)} \mathbb{k} \right) / N(\psi).$$

Characterization of quasifinite modules

Proposition (S. '12)

The tensor product of two irreducible highest weight $(\mathfrak{g} \otimes A)$-modules with disjoint supports is irreducible.

Theorem (S. '12)

Suppose $\mathfrak{g}$ is a basic classical Lie superalgebra or $\mathfrak{sl}(n, n)$ and V is an irreducible highest weight $\mathfrak{g}$-module.

Then V is quasifinite if and only if it has finite support.

Generalized evaluation map

Suppose

- $\mathsf{m}_1, \ldots, \mathsf{m}_\ell \in X_{\mathrm{rat}}$ are pairwise distinct,
- $n_1, \ldots, n_\ell$ are positive integers.

The associated generalized evaluation map is

$$\mathrm{ev}_{\mathsf{m}_1^{n_1}, \ldots, \mathsf{m}_\ell^{n_\ell}} : \mathfrak{g} \otimes A \twoheadrightarrow (\mathfrak{g} \otimes A) / \left(\mathfrak{g} \otimes \prod_{i=1}^{\ell} \mathsf{m}_i^{n_i} \right) \cong \bigoplus_{i=1}^{\ell} \left(\mathfrak{g} \otimes (A / \mathsf{m}_i^{n_i}) \right).$$

We define

$$\mathrm{ev}^{\Gamma}_{\mathsf{m}_1^{n_1}, \ldots, \mathsf{m}_\ell^{n_\ell}} \overset{\mathrm{def}}{=} \mathrm{ev}_{\mathsf{m}_1^{n_1}, \ldots, \mathsf{m}_\ell^{n_\ell}} \big|_{(\mathfrak{g} \otimes A)^{\Gamma}}.$$

Remark: Evaluation map

If $\mathsf{m} \in X_{\mathrm{rat}}$, then the map $A \twoheadrightarrow A/\mathsf{m}$ is simply evaluation at the point m.

Thus, if $n_1, \ldots, n_\ell = 1$, the above is called an evaluation map.

Generalized evaluation modules

Suppose, for $i = 1, \dots, \ell$, V_i is a f.d. $(\mathfrak{g} \otimes (A/\mathsf{m}_i^{n_i}))$-module with corresponding representation $\rho_i : (\mathfrak{g} \otimes (A/\mathsf{m}_i^{n_i})) \to \operatorname{End} V_i$.

The composition

$$\mathfrak{g} \otimes A \xrightarrow{\operatorname{ev}_{\mathsf{m}_1^{n_1}, \dots, \mathsf{m}_\ell^{n_\ell}}} \bigoplus_{i=1}^{\ell} (\mathfrak{g} \otimes (A/\mathsf{m}_i^{n_i})) \xrightarrow{\bigotimes_{i=1}^{\ell} \rho_i} \operatorname{End} \left(\bigotimes_{i=1}^{\ell} V_i \right)$$

is a generalized evaluation representation and is denoted

$$\operatorname{ev}_{\mathsf{m}_1^{n_1}, \dots, \mathsf{m}_\ell^{n_\ell}}(\rho_1, \dots, \rho_\ell).$$

The corresponding module is a generalized evaluation module and is denoted

$$\operatorname{ev}_{\mathsf{m}_1^{n_1}, \dots, \mathsf{m}_\ell^{n_\ell}}(V_1, \dots, V_\ell).$$

Generalized evaluation modules

We denote the restrictions to $(\mathfrak{g} \otimes A)^\Gamma$ by

$$\mathrm{ev}^\Gamma_{\mathsf{m}_1^{n_1},\dots,\mathsf{m}_\ell^{n_\ell}}(\rho_1,\dots,\rho_\ell) \quad \text{and} \quad \mathrm{ev}^\Gamma_{\mathsf{m}_1^{n_1},\dots,\mathsf{m}_\ell^{n_\ell}}(V_1,\dots,V_\ell).$$

Remark: Evaluation modules

If $n_1,\dots,n_\ell = 1$, then the above are called evaluation representations and evaluation modules.

Remark: Single point generalized evaluation modules

Since

$$\mathrm{ev}_{\mathsf{m}_1^{n_1},\dots,\mathsf{m}_\ell^{n_\ell}}(V_1,\dots,V_\ell) = \bigotimes_{i=1}^{\ell} \mathrm{ev}_{\mathsf{m}_i^{n_i}}(V_i),$$

every generalized evaluation rep is a tensor product of single point generalized evaluation reps.

Evaluation and support

Proposition (S. '12)

Suppose $\mathfrak{g}$ is a reductive Lie algebra, a basic classical Lie superalgebra, or $\mathfrak{sl}(n, n)$.

An irreducible f.d. $(\mathfrak{g} \otimes A)$-module is

1. a generalized evaluation module if and only if it has finite support, and
2. an evaluation module if and only if it has finite reduced support.

Parametrization of evaluation modules

$\mathcal{R}(\mathfrak{g})$ = set of isomorphism classes of irred f.d. reps of $\mathfrak{g}$

The group Γ acts on $\mathcal{R}(\mathfrak{g})$ by

$$\gamma[\rho] = [\rho \circ \gamma^{-1}], \quad \gamma \in \Gamma,$$

where $[\rho]$ denotes the isom class of a rep ρ.

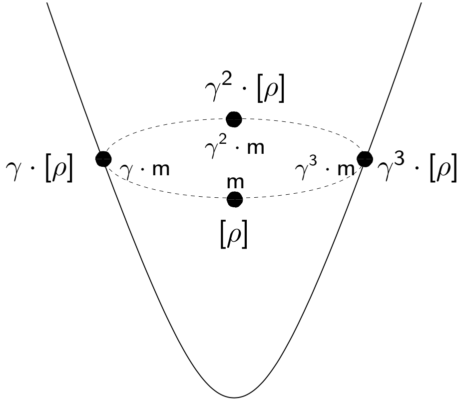
Definition ($\mathcal{E}(X, \mathfrak{g})^\Gamma$)

Let $\mathcal{E}(X, \mathfrak{g})^\Gamma$ be the set of maps $\Psi : X_{\mathrm{rat}} \to \mathcal{R}(\mathfrak{g})$ such that

- Ψ has finite support,
- Ψ is Γ-equivariant.

Parametrization of evaluation modules

We think of $\Psi \in \mathcal{E}(X, \mathfrak{g})^\Gamma$ as assigning a finite number of (isom classes of) reps of $\mathfrak{g}$ to points $\mathsf{m} \in X_{\mathrm{rat}}$ in a Γ-equivariant way.



Parametrization of evaluation modules

Definition (The class ev_Ψ^Γ)

Suppose $\Psi \in \mathcal{E}(X, \mathfrak{g})^\Gamma$.

Choose $\mathsf{M} \subseteq X_{\mathrm{rat}}$ containing one point in each Γ-orbit in the support of Ψ.

Define

$$\mathrm{ev}_\Psi^\Gamma \stackrel{\mathrm{def}}{=} \mathrm{ev}_\mathsf{M}^\Gamma(\Psi(\mathsf{m}))_{\mathsf{m}\in\mathsf{M}}.$$

The class ev_Ψ^Γ is independent of the choice of M.

Proposition (S. '12)

Suppose $\mathfrak{g}$ is a reductive Lie algebra, a basic classical Lie superalgebra, or $\mathfrak{sl}(n, n)$.

The map $\Psi \mapsto \mathrm{ev}_\Psi^\Gamma$ is a bijection from $\mathcal{E}(X, \mathfrak{g})^\Gamma$ to the set of isomorphism classes of irred eval reps of $(\mathfrak{g} \otimes A)^\Gamma$.

Distinguished $\mathbb{Z}$-grading

Assume $\mathfrak{g}$ is a basic classical Lie superalgebra of type I or $\mathfrak{sl}(n, n)$.

Then $\mathfrak{g}$ has a distinguished $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0, \qquad \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1.$$

Example

Suppose $V = V_0 \oplus V_1$ with $\dim V_0 = m$, $\dim V_1 = n$.

Thinking of V as $\mathbb{Z}_2$-graded, $\mathfrak{sl}(m, n)$ is the subsuperalegbra of End V with supertrace zero. So

$$\mathfrak{sl}(m, n)_\varepsilon = \{u \in \mathfrak{sl}(m, n) \mid uV_{\varepsilon'} \subseteq V_{\varepsilon'+\varepsilon}\}.$$

However, thinking of V as $\mathbb{Z}$-graded, induces the distinguished $\mathbb{Z}$-grading on $\mathfrak{sl}(m, n)$.

Distinguished $\mathbb{Z}$-grading

Example (cont.)

If $m \neq n$, we have

$$\mathfrak{sl}(m,n)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \in M_{m,m},\ D \in M_{n,n},\ \operatorname{tr} A = \operatorname{tr} D \right\},$$

$$\mathfrak{sl}(m,n)_{-1} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \middle| B \in M_{m,n} \right\},$$

$$\mathfrak{sl}(m,n)_1 = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \middle| C \in M_{n,m} \right\}.$$

Kac modules

M - irred f.d. $(\mathfrak{g}_0 \otimes A)$-module

Let $\mathfrak{g}_1 \otimes A$ act trivially on M.

The induced module

$$U(\mathfrak{g} \otimes A) \otimes_{U((\mathfrak{g}_0 \oplus \mathfrak{g}_1) \otimes A)} M$$

has a unique maximal submodule $N(M)$.

We define

$$V(M) \stackrel{\text{def}}{=} \left(U(\mathfrak{g} \otimes A) \otimes_{U((\mathfrak{g}_0 \oplus \mathfrak{g}_1) \otimes A)} M\right) / N(M)$$

Remark

When $A = \mathbb{k}$, the $U(\mathfrak{g} \otimes A) \otimes_{U((\mathfrak{g}_0 \oplus \mathfrak{g}_1) \otimes A)} M$ are called Kac modules.

Kac modules

Some properties of the modules $V(M)$:

1. $M_1 \cong M_2$ if and only if $V(M_1) \cong V(M_2)$.
2. $\mathrm{Ann}_A\, M = \mathrm{Ann}_A\, V(M)$.
3. $\mathrm{Supp}_A\, M = \mathrm{Supp}_A\, V(M)$.
4. M is an evaluation (resp. generalized evaluation) module if and only if $V(M)$ is.
5. $V(M)$ is finite dimensional if and only if M is a generalized evaluation module.
6. If M_1, M_2 are generalized evaluation modules with disjoint supports, then
$$V(M_1 \otimes M_2) \cong V(M_1) \otimes V(M_2).$$

Remark: Kac modules in type II

If $A = \Bbbk$, Kac modules are also defined for type II basic classical Lie superalgebras (although the definition is a bit more complicated). We don't need the generalization for type II.

One-dimensional representations

Suppose $\mathfrak{l}^{\mathrm{ab}}$ is an abelian Lie algebra.

Let $\mathcal{L}(X, \mathfrak{l}^{\mathrm{ab}})$ be the space of linear forms on $\mathfrak{l}^{\mathrm{ab}} \otimes A$ with finite support:

$$\mathcal{L}(X, \mathfrak{l}^{\mathrm{ab}}) \stackrel{\text{def}}{=} \{\theta \in (\mathfrak{l}^{\mathrm{ab}} \otimes A)^* \mid \theta(\mathfrak{l}^{\mathrm{ab}} \otimes I) = 0 \text{ for some ideal } I \text{ of } A \text{ with finite support}\}.$$

The group Γ acts naturally on $\mathcal{L}(X, \mathfrak{l}^{\mathrm{ab}})$ via

$$\gamma\theta \stackrel{\text{def}}{=} \theta \circ \gamma^{-1}, \quad \gamma \in \Gamma,\ \theta \in \mathcal{L}(X, \mathfrak{l}^{\mathrm{ab}}).$$

Let

$$\mathcal{L}(X, \mathfrak{l}^{\mathrm{ab}})^{\Gamma} \stackrel{\text{def}}{=} \{\theta \in \mathcal{L}(X, \mathfrak{l}^{\mathrm{ab}}) \mid \gamma\theta = \theta\ \forall\ \gamma \in \Gamma\}.$$

The modules $V^\Gamma(\theta, \Psi)$

Suppose $\mathfrak{g}$ is of type I, $\theta \in \mathcal{L}(X, \mathfrak{g}_{\bar{0}}^{\mathrm{ab}})^\Gamma$, and $\Psi \in \mathcal{E}(X, \mathfrak{g}_{\bar{0}}^{\mathrm{ss}})^\Gamma$.

Let $S' \subseteq X_{\mathrm{rat}}$ contain one point in each Γ-orbit of

$$S = (\mathrm{Supp}_A \theta) \cup (\mathrm{Supp}\, \Psi).$$

The representation $\theta \otimes \mathrm{ev}_\Psi$ factors (for some $n \in \mathbb{N}$) as

$$\mathfrak{g} \otimes A \twoheadrightarrow \bigoplus_{\mathsf{m} \in S} (\mathfrak{g} \otimes (A/\mathsf{m}^n)) \to \mathrm{End}\left(\bigotimes_{\mathsf{m} \in S} V_{\mathsf{m}} \right).$$

Consider the rep obtained by restricting to $S' \subseteq S$ above. Define $M_{S'}$ to be the corresponding module.

Define $V^\Gamma(\theta, \Psi)$ to be $(\mathfrak{g} \otimes A)^\Gamma$-module obtained by restriction from $V(M_{S'})$.

This is independent of the choice of S'.

Classification Theorem

Theorem (S. '12)

Suppose $\mathfrak{g}$ is a basic classical Lie superalgebra and let $\mathcal{R}(X,\mathfrak{g})^\Gamma$ be the set of isomorphism classes of irred f.d. reps of $(\mathfrak{g} \otimes A)^\Gamma$.

1. If $\mathfrak{g}_{\bar{0}}$ is semisimple (i.e. $\mathfrak{g}$ is of type II or is $A(n,n)$), then we have a bijection
$$\mathcal{E}(X,\mathfrak{g})^\Gamma \to \mathcal{R}(X,\mathfrak{g})^\Gamma, \quad \Psi \mapsto \mathrm{ev}_\Psi^\Gamma.$$
2. If $\mathfrak{g}$ is of type I, then we have a bijection
$$\mathcal{L}(X,\mathfrak{g}_{\bar{0}}^{\mathrm{ab}})^\Gamma \times \mathcal{E}(X,\mathfrak{g}_{\bar{0}}^{\mathrm{ss}})^\Gamma \to \mathcal{R}(X,\mathfrak{g})^\Gamma, \quad (\theta,\Psi) \mapsto V^\Gamma(\theta \otimes \mathrm{ev}_\Psi^\Gamma).$$

In particular, in both cases all irred f.d. reps are generalized eval reps.

Remark: $\mathfrak{sl}(n,n)$

If Γ is trivial, the theorem (case 2) also applies to $\mathfrak{sl}(n,n)$.

Application: Twisted multiloop superalgebras

A special case of the Classification Theorem gives a classification of the irred f.d. reps of twisted multiloop superalgebras.

The irred f.d. reps of untwisted multiloop superalgebras (i.e. Γ trivial) were classified by Rao-Zhao (2004) and Rao (2011).

In the twisted case, the classification seems to be new.

Comparison to related classifications

Equivariant map algebras

When $\mathfrak{g}$ is a f.d. Lie algebra (no assumptions on A or Γ), the irred f.d. reps have been classified (Neher-S.-Senesi '09).

They are all tensor products of one-dimensional reps and eval reps.

Generalized eval reps don't play a major role in the classification.

Map Virasoro algebras

The irred quasifinite reps of $\mathrm{Vir} \otimes A$, where Vir is the Virasoro algebra have been classified (S. '11).

Here all reps are generalized eval reps.

Further directions

- Develop a classification where $\mathfrak{g}$ is a more arbitrary type of Lie superalgebra (e.g. classical, but not necessarily basic classical).
- Describe the extensions between irred f.d. reps. This has been done for equivariant map algebras (Neher-S. '11).
- Describe the block decomposition of the category of f.d. reps. This had been done for equivariant map algebras (Neher-S. '11).