PACIFIC JOURNAL OF MATHEMATICS
Vol. 56, No 1, 1975

EQUIVARIANT MAPS BETWEEN REPRESENTATION SPHERES

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Let G be a finite group, V and W be finite representations of G, $S(V)$ and $S(W)$ be the unit spheres in V and W. Suppose $\dim V^H \leqq \dim W^H$ for every subgroup H of G. We seek to classify the G-equivariant homotopy classes of G-equivariant maps from $S(V)$ to $S(W)$.

Introduction. We wish to consider the following problem: Let G be a finite group, and V and W be finite dimensional orthogonal representations of G. Let $S(V)$ and $S(W)$ denote the unit spheres of V and W respectively. Then $S(V)$ and $S(W)$ inherit G-actions. Classify the equivariant homotopy classes of G-maps from $S(V)$ to $S(W)$.

The case where $G = Z_p$ and $S(V)$ and $S(W)$ have free Z_p actions was done by Olum **[6]** and was used to give a classification of lens spaces up to homotopy equivalence. In this paper we generalize Olum's result.

Our approach is to consider the behavior of an equivariant map restricted to the various fixed point sets. Explicity, if X is a space with a left action of the group G, and H is a subgroup of G, we denote by X^H the set of points in X left fixed by each element of H. If $f: S(V) \to S(W)$ is a G-equivariant map (i.e., $f(gv) = gf(v)$ for all $g \in G$ and $v \in S(V)$), then f induces maps $f^H: S(V^H \to S(W)^H$ for each subgroup H. Since V and W are linear representations, these fixed point sets $S(V)^H$ and $S(W)^H$ are again spheres, and we may choose an orientation for each $S(V)^H$ and $S(W)^H$. If X is a manifold, denote by $\dim X$ the (real) dimension of X. When $\dim S(V)^H = \dim S(W)^H$, f^H has a well-determined degree, denoted by $\deg f^H$.

Our major theorem asserts that, under suitable hypotheses, the homotopy classes of the maps f^H for all H determine the equivariant homotopy class of f:

DEFINITION. Let G be a finite group. If H is a subgroup of G, denote by $N(H)$ the normalizer of H in G. An orthogonal representation V of G is *completely orientable* if for every subgroup H of G, the induced action of $N(H)$ on $S(V)^H$ is orientation-preserving.

EXAMPLE. Any unitary representation of G is clearly completely orientable.

THEOREM A. *Let G be a finite group, and let V and W be orthogonal representations of G. Assume $V \oplus W$ is completely orientable and that for each subgroup H of G we have* $\dim V^H \leqq \dim W^H$. *Suppose h and k are G-equivariant maps from $S(V)$ to $S(W)$ and* $\deg h^H = \deg k^H$ *whenever* $\dim V^H = \dim W^H$. *Then h and k are G-equivariantly homotopic.*

We note that G. Segal [7] has obtained Theorem A for the case $V = W$. It is clear that if V and W are each completely orientable, then so is $V \oplus W$.

Theorem A may be restated as follows: Denote by $[S(V), S(W)]_G$ the class of G-equivariant homotopy classes of G-equivariant maps from $S(V)$ to $S(W)$. For each subgroup H of G denote by $[S(V)^H, S(W)^H]$ the set of (not necessarily equivariant) homotopy classes of (not necessarily equivariant) maps from $S(V)^H$ to $S(W)^H$. There is a natural map

$$v\colon [S(V), S(W)]_G \to \prod_H [S(V)^H, S(W)^H]$$

where the product runs over all subgroups H of G; the map v is defined by setting the H component of $v(f)$ equal to the class of f^H when $f\colon S(V) \to S(W)$ is a G-equivariant map. Theorem A then has the following formulation:

THEOREM A (*restated*). *Let G be a finite group, and let V and W be orthogonal representations of G so $V \oplus W$ is completely orientable. Suppose* $\dim S(V)^H \leqq \dim S(W)^H$ *for each subgroup H of G. Then v is one-to-one.*

Given this formulation, the classification problem reduces to the computation of the image of v. It is not a difficult exercise, although somewhat cumbersome, to compute this image in the case where G is a finite abelian group.

The hypothesis that $\dim S(V)^H \leqq \dim S(W)^H$ for all H cannot be eliminated. An easy counterexample is provided as follows: Let $G = Z_2$. Let V be the antipodal representation on R^8 (i.e., if τ generates G, let $\tau v = -v$ where $v \in R^8$.) Let W be the constant representation on R^7 (i.e., τ leaves every point of W fixed). Define $f\colon S(V) \to S(W)$ by the composition

$$S(V) \xrightarrow{p} S(V)/G = RP^7 \xrightarrow{k} S^7 \xrightarrow{h} S^6 = S(W)$$

where p is the projection map, k is the map of degree one which collapses all but a single 7-cell to a point, and h is the appropriate suspension of the Hopf map. Clearly f is G-equivariant. Since p has degree 2 and h has order 2, f is (nonequivariantly) homotopically trivial. If f were equivariantly trivial, then the induced map hk on the orbit spaces would be trivial. Using the notation of Steenrod **[9]**, we observe that hk is trivial on the 6-skeleton of RP^7; the difference cocycle $d(hk, t)$ is then defined using the trivial homotopy, where t is the constant map, and we find directly that it corresponds to $h \in H^7(RP^7; \pi_7(S^6)) = Z_2$ where $h \neq 0$. By Steenrod **[9]** it follows that hk is homotopically trivial only if h lies in $Sq^2(H^5(RP^7; Z_2))$. But $Sq^2(H^5(RP^7; Z_2)) = 0$ (see, for example, Steenrod [**8**, page 5].). Hence hk is essential.

Nor can we eliminate the hypothesis of complete orientability. A simple example follows: Let $G = Z_2$ with generator τ. Take $V = \mathbf{R}^3$ with Z_2 action $\tau v = -v$, and take $W = \mathbf{R}^3$ with the trivial action. Let $h: S(V) \to S(W)$ be defined by $S(V) \xrightarrow{P} S(V)/Z_2 = RP^2 \xrightarrow{j} S^2 = S(W)$ where P is the projection map and j is *not* homotopically trivial. Then $j \cdot P$ is trivial since $\deg P = 0$, but h is not equivariantly trivial since its orbit map j is nontrivial.

Details. Let G be a finite group acting on a space X. If $x \in X$, the *isotropy subgroup* of G at x, denoted G_x, is the set of $g \in G$ such that $gx = x$. A *principal isotropy* subgroup H is a subgroup H so $H = G_x$ for some x and whenever $G_y \subset H$, then $G_y = H$. It is well-known that if X is a connected smooth manifold and the action of G on X is smooth, then any two principal isotropy subgroups are conjugate. We shall use $o(G)$ to denote the order of the group G.

We are going to utilize equivariant obstruction theory and such notions as "equivariant triangulation", "obstruction cocycle", and "equivariant coboundary". A good background source for such material is Bredon **[1]**. Note that for us, an equivariant triangulation of a space X with an action of G consists of a way of expressing X as a simplicial complex in such a manner that (1) for each $g \in G$, the map $g: X \to X$ is a simplicial map; and (2) if Δ^r is an r-simplex of X and s lies in the interior of Δ^r (denoted $\operatorname{Int} \Delta^r$), then all points in $\operatorname{Int} \Delta^r$ have isotropy subgroup G_s, and the natural equivariant map from $G/G_s \times \Delta^r$ into X (where Δ^r is here regarded as having no G-action) is a homeomorphism of $G/G_s \times \operatorname{Int} \Delta^r$ onto its image. The r-skeleton of X will be denoted X^r.

The basic tool for our analysis will be the following proposition:

PROPOSITION 1. *Let G be a finite group. Let S_1 and S_2 be spheres with smooth actions of G such that each element of G preserves orientation on S_1 if and only if it preserves orientation on S_2.*[1] *Suppose the principal isotropy group of G on S_1 is e. Let $L = \bigcup_{H \neq e} S_1^H$. Let h and k be G-equivariant maps from S_1 to S_2 satisfying that $h|L = k|L$. Then*

(1) *if* $\dim S_1 < \dim S_2$, *h and k are G-homotopic* rel L;

(2) *if* $\dim S_1 = \dim S_2$, *then* $\deg h \equiv \deg k \mod o(G)$ *and* $\deg h = \deg k$ *if and only if h and k are G-homotopic* rel L.

Proof. By Illman **[4]** or Matumoto **[5]** we may obtain an equivariant triangulation of S_1. We observe that L is a subcomplex. We try to construct an equivariant homotopy $F: S_1 \times I \to S_2$ by equivariant obstruction theory. We set $F|S_1 \times 0 = h$, $F|S_1 \times 1 = k$, $F|L \times I = h \cdot$projection. Assume that F has been extended equivariantly over $(S_1 \times I)^r$, the r-skeleton of $S_1 \times I$. If $G/e \times \Delta^r \times I$ is an $(r+1)$ cell of $S_1 \times I - (S_1 \times \partial I \cup L \times I)$ then F induces a map from S^r to S_2 by $S^r = \partial(\Delta^r \times I) \to e \times \Delta^r \times I \xrightarrow{F} S_2$. If $r < \dim S_2$, then clearly the map extends over $\Delta^r \times I$, and we can easily extend equivariantly over $G/e \times \Delta^r \times I$. This proves (1) by induction; and in case (2), it shows there is an equivariant extension of F over $(S_1 \times I)^n$ where $n = \dim S_1 = \dim S_2$. Continue to call the extension F.

In this latter case, we try to extend over $(S_1 \times I)^{n+1}$. Define a function c (the obstruction cochain) which assigns to each cell $g \times \Delta^n \times I \subset G/e \times \Delta^n \times I$ the integer which is the degree of the map $F|\partial(g \times \Delta^n \times I) \to S_2^n$. (Here, each cell $g \times \Delta^n \times I$ has been assigned an orientation consistent with the orientation of $S_1 \times I$.) It follows that $c(g_1 \times \Delta^n \times I) = c(g_2 g_1 \times \Delta^n \times I)$ for any g_1 and g_2 since F is equivariant and g_2 reverses orientation on S_2 if and only if it reverses orientation as a map from $\partial(g_1 \times \Delta^n \times I)$ to $\partial(g_2 g_1 \times \Delta^n \times I)$. Hence it is clear that F extends equivariantly over $(S_1 \times I)^{n+1}$ if and only if each such integer is 0.

From the proof that the degree map is a homomorphism from $\pi_n(S^n)$ to Z, we see

$$\sum \deg F\,|\,\partial(g \times \Delta^n \times I) = \deg k - \deg h$$

where the summation runs over all cells of form $g \times \Delta^n \times I$. Since $\deg F\,|\,\partial(g \times \Delta^n \times I) = \deg F\,|\,\partial(e \times \Delta^n \times I)$, we obtain $o(G)\Sigma_{\Delta^n} \deg F\,|\,\partial(e \times \Delta^n \times I) = \deg k - \deg h$. This proves $\deg k \equiv \deg h \mod o(G)$.

[1] In fact S_1 need not be a sphere, but only a smooth closed orientable manifold; the same proof then applies.

If $\deg k = \deg h$, we shall see that c is an equivariant coboundary δl, where l is a cochain containing no terms involving cells of $L \times I$. Hence we may modify F equivariantly on $(S_1 \times I)^n - ((S_1 \times I)^{n-1} \cup S_1 \times \partial I \cup L \times I)$ so that the new map extends equivariantly over $(S_1 \times I)^{n+1}$. Note that the new map agrees with F on $L \times I$. For further details, see Bredon [1].

To see that c is an equivariant coboundary, we choose a fixed cell $\sigma_0 = G/e \times \Delta_0^n \times I$. For each cell $\sigma_1 = G/e \times \Delta_1^n \times I$, obtain a path from one of the simplices of σ^1 to $e \times \Delta_0^n \times I$, which passes only through the interiors of cells of form $G/e \times \Delta^n \times I$ and $G/e \times \Delta^{n-1} \times I$. Such a path exists since the image in the orbit space S_1/G of the set of points with isotropy group e is connected. (See Bredon [2, p. 179].) For any cell $\tau^n = G/e \times \Delta^{n-1} \times I$, we may let d_{τ^n} be the cochain which assigns 1 to each cell $g \times \Delta^{n-1} \times I \subset \tau^n$ and 0 to all other cells. Then ∂d_{τ^n} assigns 1 to each $g \times \Delta^n \times I$ on one side of τ^n and -1 to each on the other side. Following the path from each σ_1 to σ_0 and adding appropriate multiples of δd_{τ^n} for the various $G/e \times \Delta^{n-1} \times I$ which are crossed by those paths, we obtain a cochain which assigns $(\deg k - \deg h)/o(G)$ to each cell $g \times \Delta_0^n \times I \subset \sigma_0$ and 0 to all others. Hence, if $\deg h = \deg k$, we may express $c = \Sigma\, a_i \delta d_i$ where $a_i \in Z$, δd_i is the cochain obtained from some $G/e \times \Delta^{n-1} \times I$.

COROLLARY 2. *Let G be a finite group. Let V and W be orthogonal representations of G, where $V \oplus W$ is completely orientable. Suppose the principal isotropy group of G on $S(V)$ is e. Suppose $\dim S(V)^H \leqq \dim S(W)^H$ for all $H \subset G$. Let $h, k\colon S(V) \to S(W)$ be G-maps. Suppose that whenever $\dim S(V)^H = \dim S(W)^H$ ($H \neq e$) then $\deg h^H = \deg k^H$. Then*

(1) *if $\dim S(V) < \dim S(W)$, h and k are G-homotopic;*

(2) *if $\dim S(V) = \dim S(W)$, then $\deg h \equiv \deg k \mod o(G)$ and $\deg h = \deg k$ iff h and k are G-homotopic.*

Proof. We merely apply the Equivariant Homotopy Extension Property (see Illman [4] or Willson [10]) and Proposition 1. Explicitly, let H_1 be a maximal isotropy group. By Proposition 1, $h \mid S(V)^{H_1}$ and $k \mid S(V)^{H_1}$ are $N(H_1)/H_1$ homotopic. Extend the homotopy over $GS(V)^{H_1}$ to a G-homotopy by equivariance; there is no difficulty since it is defined on cells of form $N(H_1)/H_1 \times \Delta^i \times I \subset G/H_1 \times \Delta^i \times I$. Now by the Homotopy Extension Property, we may equivariantly homotope h to h_1 so that $h_1 | GS(V)^{H_1} = k \mid GS(V)^{H_1}$.

Let H_2 be an isotropy group maximal among those remaining. By Proposition 1, $h_1 | S(V)^{H_2}$ and $k \mid S(V)^{H_2}$ are $N(H_2)/H_2$

homotopic. Continue as above to G-homotope h_1 rel $S(V)^{H_1}$ to a G-map h_2 so $h_2|GS(V)^{H_2} = k\,|GS(V)^{H_2}$.

Continue inductively.

REMARK. Observe that in this corollary we use the assumption that the actions are orthogonal to ensure that $S(V)^{H_i}$ and $S(W)^{H_i}$ are spheres; and we use the assumption that $V \oplus W$ is completely orientable to ensure that any element of $N(H_i)$ either preserves orientation on both $S(V)^{H_i}$ and $S(W)^{H_i}$ or preserves orientation on neither.

Proof of Theorem A. Let K denote the set of $g \in G$ such that $gx = x$ for all $x \in S(V)$. Then K is a normal subgroup of G. Replacing G by G/K if necessary, we may assume the action of G on $S(V)$ is effective. Now Newman's Theorem [3, p. 204] implies that the principal isotropy group on $S(V)$ is e. Then Corollary 2 yields the Theorem.

COROLLARY 3. *Let G be a finite group. Let V and W be unitary representations of G. Suppose* $\dim V^H < \dim W^H$ *for all H. Then any two G-equivariant maps h and k from $S(V)$ to $S(W)$ are G-equivariantly homotopic.*

I wish to acknowledge the assistance of A. G. Wasserman, C. N. Lee, A. Edmonds, and R. Schultz; and to thank the referee for very helpful comments. Part of this research was done while the author held an NSF Graduate Fellowship.

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Received December 11, 1973.

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