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# EQUIVARIANT MAPS UP TO HOMOTOPY AND BOREL SPACES

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Equivariant maps between G-spaces induce fiber preserving maps between the associated Borel spaces. We will show that not all fiber preserving maps between Borel spaces are induced that way, not even all fiber homotopy classes of such maps. However there is a one-to-one correspondence between homotopy classes of $G_\infty$-maps (i.e. maps equivariant up to homotopy in a way, see section 1 for definitions) between G-spaces and fiber homotopy classes of maps between Borel spaces. This one-to-one correspondence is obtained by a functor equivalence between the respective categories (Theorem 1 and 2 in section 4). As a result equivariant homotopy theory (in a modified sense) is equivalent to the theory of homotopy fibrations.

To prove these theorems we have to include H-spaces into our discussion: In fact, the functor equivalence mentioned above is an extension of the equivalence between the categories of H-spaces and classifying spaces presented in [2]. Therefore we need the notion of a Borel space for H-spaces.

The Borel space we use, is associated with the modified Dold-Lashof construction in [3].

In section seven we present a number of examples of G-spaces with differing fix point sets, such that these differences cannot be detected by studying the cohomology of their Borel spaces, nor by studying the Borel space itself. The groups in most examples are $\mathbb{Z}_p$ or $S^1$, but the G-spaces are not all of finite dimension. Thus we illustrate the limits of theorems like the localization theorem by Hsiang ([5], p. 47). All the examples arise from the fact that if $h = \{h_n\}$ $n = 0,1,\ldots$ is a $G_\infty$-map between the G-spaces $X_1$ and $X_2$ and $h$ is an ordinary homotopy equivalence, then the fiber map induced between the Borel spaces is a fiber homotopy equivalence.

## 1. Definitions

1.1. The H-spaces $H$ we are using are supposed to be strictly associative and to have a strict unit element $e$. Furthermore we assume $H$ has a homotopy inverse $\nu$ (such that $H \xrightarrow{\Delta} H \times H \xrightarrow{1\times\nu} H \times H \xrightarrow{\mu} H$ is homotopic to $id_H$).

1.2. We say that a topological space $X$ is a G-space, if an H-space $H$ acts on $X$ from the left continuously and in a strictly associative manner. We assume that $ex = x$ for all $x \in X$.

1.3. As usual, an $H_\infty$-map $h$ from $H_1$ to $H_2$ (of length $r$) is a sequence of continuous maps $h_n : (H_1 \times I_r)^n \times H_1 \to H_2$ $(n = 0,1,2,\ldots)$ such that

$$h_n(g_0,t_1,\ldots,t_n,g_n)$$

$$= \begin{cases} h_{n-1}(g_0,t_1,\ldots,g_{i-1}g_i,\ldots,t_n,g_n) & t_i = r \\ \\ h_{i-1}(g_0,t_1,\ldots,g_{i-1})h_{n-i}(g_i,\ldots,g_n) & t_i = 0 \end{cases}$$

for $n > 0$, $g_0,\ldots,g_n \in H_1$, and $t_1,\ldots,t_n \in I_r = [0,r] \subseteq \mathbb{R}$. If $r = 0$, the map $h_0$ is a homomorphism in the usual sense.

1.4. If $H_1$ acts on $X_1$ and $H_2$ acts on $X_2$ from the left, and if $h$ is an $H_\infty$-map from $H_1$ to $H_2$ of length $r$, then we define a $G_\infty$-map $f$ from $X_1$ to $X_2$ of length $r$ associated with $h$ to be a sequence of maps

$$f_n : (H_1 \times I_r)^n \times X_1 \to X_2 \qquad (n = 0,1,2,\ldots)$$

such that for $n > 0$

$$f_n(g_0,t_1,\ldots,g_{n-1},t_n,x)$$

$$= \begin{cases} f_{n-1}(g_0,t_1,\ldots,g_{i-1}g_i,\ldots,g_{n-1},t_n,x) & t_i = r \\ \\ h_{i-1}(g_0,\ldots,g_{i-1})f_{n-i}(g_i,\ldots,g_{n-1},t_n,x) & t_i = 0 \end{cases}$$

Composition of $H_\infty$-maps and $G_\infty$-maps is defined as in [3].

1.5. If $f$ is a $G_\infty$-map from $X_1$ to $X_2$ associated to the $H_\infty$-map $h$ from $H_1$ to $H_2$, then $f$ is called a $G_\infty$-homotopy equivalence if there exists an $H_\infty$-map $k$ from $H_2$ to $H_1$ and a $G_\infty$-map $g$ from $X_2$ to $X_1$

associated with $k$ such that $g \circ f$ and $f \circ g$ are $G_\infty$-homotopic to $\mathrm{id}_{X_1}$ and $\mathrm{id}_{X_2}$ respectively (associated to the $H_\infty$-homotopies between $k \circ h$ respectively $h \circ k$ and $\mathrm{id}_{H_1}$ respectively $\mathrm{id}_{H_2}$).

We are going to use the theorem from [4]:

<u>Theorem</u>. If $H_1$ acts on $X_1$ and $H_2$ acts on $X_2$ and if $h : H_1 \to H_2$ is an $H_\infty$-map such that $h_0$ is an ordinary homotopy equivalence, and if $f : X_1 \to X_2$ is a $G_\infty$-map associated with $h$ such that $f_0$ is an ordinary homotopy equivalence, then $h$ is an $H_\infty$-homotopy equivalence and $f$ is a $G_\infty$-homotopy equivalence associated to $h$.

<u>1.6</u>. H-spaces and $H_\infty$-maps form the category $\mathcal{H}$ and G-spaces and $G_\infty$-maps form the category $\mathcal{G}$. The associated homotopy categories are denoted by $\underline{\mathcal{H}}$ and $\underline{\mathcal{G}}$.

## 2. <u>Construction of the Borel Space</u>

In this section we rely heavily on [3], where many additional details can be found.

<u>2.1</u>. Let $(p,r)$ be an H-principal fibration

$$\begin{tikzcd} E \times H \arrow[r, "r"] \arrow[d, "pr_1"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

as described in [3] and let $X$ be a G-space with respect to $H$ with action $s : H \times X \to X$. Assume that $p_X : EX \to B$ is a fibration with fiber $X$ associated to $p : E \to B$ in the following sense: 1) The two fibrations are fiber homotopy trivial with respect to the same numerable covering $\mathfrak{U}$ of $B$ and every $U \in \mathfrak{U}$ is contractible in $B$. 2) There is a map $r_X : E \times X \to EX$ such that for each $U \in \mathfrak{U}$ the diagram

(1)
$$\begin{tikzcd} U \times H \times X \arrow[r, "1\times s"] \arrow[d, shift left, "\beta"] & U \times X \arrow[d, shift left, "\beta_X"] \\ p^{-1}(U) \times X \arrow[r, "r_X"'] \arrow[u, shift left, "\alpha"] & p_X^{-1}(U) \arrow[u, shift left, "\alpha_X"] \end{tikzcd}$$

is commutative ($(\alpha, \beta, \alpha_X, \beta_X)$ are the obvious coordinate maps). In addition we want

(2)
$$\begin{tikzcd} E \times H \times X \arrow[r, "1\times s"] \arrow[d, "r\times 1"'] & E \times X \arrow[d, "r_X"] \\ E \times X \arrow[r, "r_X"'] & E_X \end{tikzcd}$$

to be commutative.

2.2. For the general step of the Borel space construction we look at the H-principal fibration $(\tilde{p}, \tilde{r})$ as described in [3], p. 329-331.

The base space $\tilde{B}$ of the new fibration is the mapping cone of $p : E \to B$ with the coordinate topology. We consider the covering of $\tilde{B}$ consisting of

$B_1 = \{y \perp t \mid t > \frac{1}{3}\}$ and $B_2 = \{y \perp t \mid t < \frac{2}{3}\}$.

Let $p_1 : E_1 \to B_1$ respectively $p_{1X} : E_1X \to B_1$ be the fibrations induced by $f(y \perp t) = p(y)$, the map collapsing $B_1$ to the range space $B$ of the mapping cone $\tilde{B}$. $p_{1X}$ is associated to $p_1$ if we define $r_{1X} : E_1 \times X \to E_1X$ by

$$r_1(y \perp t, y_1, x) = (y \perp t, r_X(y_1, x)) .$$

Furthermore let $E_2 = B_2 \times H$ and $E_2X = B_2 \times X$. Define $r_{2X}(y \perp t, h, x) = (y \perp t, hx)$. Obviously these fibrations are associated.

We recall from [3], p. 330, that the map $F : p_2^{-1}(B_1 \cap B_2) \to p_1^{-1}(B_1 \cap B_2)$ defined by

$$F(y \perp t, h) = (y \perp t, yh)$$

is a strictly equivariant fiber homotopy equivalence. We define the associated map $F_X : p_{2X}^{-1}(B_1 \cap B_2) \to p_{1X}^{-1}(B_1 \cap B_2)$ by

$$F_X(y \perp t, x) = (y \perp t, r_X(y, x)) .$$

$F_X$ is a map over $B_1 \cap B_2$ and a homotopy equivalence on each fiber (this follows from diagram (1) and the fact that $H$ has a homotopy inverse) and hence is a fiber homotopy equivalence according to Theorem 6.3 in [1].

2.3. As in [3], p. 330 we now form the mapping cylinder of $F$ and of $F_X$ and construct the H-principal fibration $\tilde{p} : \tilde{E} \to \tilde{B}$ and similarly the associated fibration

$\tilde{p}_X : \tilde{E}X \to \tilde{B}X$. With the help of $r_{X1}$ and $r_{X2}$ we construct $\tilde{r}_X : \tilde{E} \times X \to \tilde{E}X$ in the obvious manner. No problem arises since the diagram

$$\begin{array}{ccc} (y \perp t, h, x) & \xrightarrow{r_{2X}} & (y \perp t, hx) \\ {\scriptstyle F\times id}\downarrow & & \downarrow{\scriptstyle F_X} \\ (y \perp t, yh, x) & \xrightarrow[r_{1X}]{} & (y \perp t, r_X(yh,x)) \end{array} \qquad \frac{1}{3} < t < \frac{2}{3}$$

commutes as a consequence of diagram (2). So it is easy to see that $\tilde{E}$ and $\tilde{E}X$ are associated.

2.4. To construct the Borel space of $X$ we start out with $p_0 : E_0 \to B_0$, where $E_0 = H$ and $B_0 = \{*\}$ = point, and with $p_{0X} : E_0X \to B_0$, where $E_0X = X$. From $p_n$ and $p_{nX}$ we construct $p_{n+1}$ and $p_{n+1,X}$ by letting $E_{n+1} = \tilde{E}_n$, $B_{n+1} = \tilde{B}_n$ and $E_{n+1}X = \tilde{E}_nX$. Obviously $p_{n+1} = \tilde{p}_n$ and $p_{n+1,X} = \tilde{p}_{nX}$ are associated. As on p. 333 in [3] we use telescopes to finally get the universal H-principal fibration $p_H : EH \to BH$ and the associated fibration $p_X : EX \to BH$. We call $EX$ the Borel space of $X$ and $p_X$ the Borel fibration of $X$. Notice that $p_X$ is a numerable, locally fiber homotopy trivial fibration with fiber $X$ associated with $p_H$ through the map $r_X : EH \times X \to EX$. $r_X$ is essentially the direct limit of the maps $r_{n,X}$, and it is continuous because we used the telescope construction. (Compare the continuity of $r_H$ in [3], p. 333).

## 3. Induced Maps Between Borel-Spaces

3.1. Before we can discuss G-spaces, we have to know more about H-spaces. So let $h : H_1 \to H_2$ be an $H_\infty$-map between the H-spaces $H_1$ and $H_2$. We define a $G_\infty$-map $E_0 h : E_0 H_1 \to E_0 H_2$ as $E_0 h = h$. (Note that all the spaces $E_n H$ have a right action, so the notion of $G_\infty$-map has to be modified accordingly). Also we let $B_0 h : B_0 H_1 \to B_0 H_2$ be the trivial map.

Assume that $E_0 h$ has been extended to a $G_\infty$-map $E_n h : E_n H_1 \to E_n H_2$ associated with $h$ and $B_0 h$ has been extended to $B_n h$ such that

$$p_{n2} \circ E_n h_k(y, t_1, g_1, \ldots, t_k, g_k) = B_n h \circ p_{n1}(y) .$$

(We will call a $G_\infty$-map with this property fiber preserving). First we extend $B_n h$ from $B_n H_1$ to $\bar{B}_n H_1$ by defining

$$\bar{B}_n h(y \perp t) = (E_n h_0(y) \perp t)$$

On $E_{n1} H_1$ we define

$$E_{n1} h_0(y \perp t, y_0) = (E_n h_0(y) \perp t,\ E_n h_0(y_0))$$

and

$$E_{n1} h_k(y \perp t, y_0, t_1, g_1, \ldots, t_k, g_k)$$

$$= (E_n h_0(y) \perp t,\ E_n h_k(y_0, t_1, g_1, \ldots, t_k, g_k)$$

for $k = 1, 2, \ldots$ .

Recall (from [3], p. 330) that $E_{n2}H_1' = (B_{n2}H_1 \times H_1) \cup (B_{n1} \cap B_{n2} \times I \times H_1)$ and define

$$E_{n2}'h_k(y \perp t, \tau, g_0, t_1, \ldots, t_k, g_k)$$

$$= \begin{cases} (E_n h_0(y) \perp t,\ h_k(g_0, t_1, \ldots, t_k, g_k)) \\ \qquad \tau = 0, \quad 0 \le t \le \frac{1}{3} \\ (E_n h_0(y) \perp t,\ 2\tau, h_k(g_0, t_1, \ldots, t_k, g_k) \\ \qquad \text{when} \quad 0 \le \tau \le \frac{1}{2} \quad \text{and} \quad \frac{1}{3} < t < \frac{2}{3} \\ (E_n h_0(y) \perp t,\ E_n h_{k+1}(y, 2\tau - 1, g_0, t_1, \ldots, t_k, g_k)) \\ \qquad \text{when} \quad \frac{1}{2} \le \tau \le 1 \quad \text{and} \quad \frac{1}{3} < \tau < \frac{2}{3}\ . \end{cases}$$

(When $\tau = 1$ we use that $E_n h_{k+1}(y, 1, g_0, t_1, \ldots) = E_n h_k(yg_0, t_1, \ldots)$. Hence $E_{n2}'h_k$ and $E_{n1}h$ together induce a $G_\infty$-map $\tilde{E}_n h$ from $\tilde{E}_n H_1$ to $\tilde{E}_n H_2$ which satisfies all the conditions mentioned before and hence we get $E_{n+1}h : E_{n+1}H_1 \to E_{n+1}H_2$ together with $B_{n+1}h$. In the obvious manner we obtiain the $G_\infty$-map $Eh : EH_1 \to EH_2$ associated with $h$.

Because of our definition of $E_{n2}'h_k$ on the mapping cylinder part of $\tilde{E}_n H$, we only get $E(h \circ h')$ is $G_\infty$-homotopic to $Eh \circ Eh'$ and similarly $B(h \circ h') \simeq Bh \circ Bh'$. In fact the $G_\infty$-homotopy mentioned is fiber preserving. We get the

_Theorem_. The construction of universal fibrations described in [3] induces a functor $(\underline{E},\underline{B})$ from the category $\underline{\mathfrak{X}}$ as described in 1.6 to the category $\underline{\mathfrak{U}}$ of universal fibrations and fiber homotopy classes of $G_\infty$-maps (with distinguished fiber).

_3.2_. Now let $X$ be a topological space on which the H-space $H$ acts from the left. The map $r_X : EH \times X \to EX$ discussed in section 2 is part of the structure of $EX$. A map between two Borel spaces has to preserve this structure at least up to homotopy. This leads to the following.

_Definition_. Let $Y_1$ and $Y_2$ be topological spaces on which $H_1$ and $H_2$ respectively act from the right, let $X_1$ and $X_2$ be topological spaces on which $H_1$ and $H_2$ respectively act from the left, and let $r_1 : Y_1 \times X_1 \to Z_1$ and $r_2 : Y_2 \times X_2 \to Z_2$ be maps ($Z_1$ and $Z_2$ are topological spaces) such that

$$\begin{tikzcd} Y_i \times H_1 \times X_i \arrow[r, "1\times\mu_{ri}"] \arrow[d, "\mu_{\ell i}\times 1"'] & Y_i \times X_i \arrow[d, "r_i"] \\ Y_i \times X_i \arrow[r, "r_i"'] & Z_i \end{tikzcd}$$

are commutative $(i = 1,2)$. Assume $h : H_1 \to H_2$ is a $G_\infty$-map and $k : Y_1 \to Y_2$ and $f : X_1 \to X_2$ are $G_\infty$-maps associated with $h$, then a $G_\infty$-map associated with

$h, k,$ and $f$ is a sequence of maps $F_0, F_1, \ldots$ such that

$$F_0 : Z_1 \to Z_2$$

and

$$F_k : Y_1 \times I \times (H_1 \times I)^{k-1} \times X_1 \to Z_2 \qquad k = 1,2,\ldots$$

with

$$F_k(y, t_1, g_1, \ldots, g_{k-1}, t_k, x)$$

$$= \begin{cases} r_2(k_{i-1}(y, t_1, \ldots, g_{i-1}), f_{k-i}(g_i, \ldots, t_k, x) & t_i = 0 \\ F_{k-1}(y, t_1, \ldots, g_{i-1}g_i, \ldots, t_k, x) & t_i = 1 \end{cases}$$

and appropriate modifications in special cases (like $k = 1$ or $i = 0$ and $i = k$).

3.3. Now we are ready to disucss Borel fibrations. Let $X_1$ and $X_2$ be topological spaces on which $H_1$ and $H_2$ respectively act from the left. Assume $f : X_1 \to X_2$ is a $G_\infty$-map associated with the $H_\infty$-map $h : H_1 \to H_2$. Again we define the $G_\infty$-map $E_0f : E_0X_1 \to E_0X_2$ by $E_0f = f$.

Assume we defined a $G_\infty$-map $E_nf : E_nH_1 \times X_1 \to E_nX_2$ in the sense of 3.2, associated with $E_nh$, $f$, and $h$. Furthermore we assume that all maps in $E_nf$ are "fiber-maps" over $B_nh$ in the obvious manner. Let us extend $E_nf$ to $\tilde{E}_nf : \tilde{E}_nH_1 \times X_1 \to \tilde{E}_nX_2$. We define $\tilde{E}_nf_0 : \tilde{E}_nX_1 \to \tilde{E}_nX_2$ first on

$$E_{n1}X_1 = \{(y \perp t, x_n) \mid (y \perp t \in B_nH_1,\ x_n \in E_nX,\ p(y) = p_X(y_n)\}$$

as

$$E_{n1}f_0(y \perp t, x_n) = (E_nh_0(y) \perp t,\ E_nf_0(x_n))$$

Then we define for $k = 1,2,\ldots$

$$E_{n1}f_k(y \perp t, y_0, t_1, \ldots, g_{k-1}, t_k, x)$$

$$= (E_nh_0(y) \perp t,\ E_nf_k(y_0, t_1, \ldots, g_{k-1}, t_k, x))$$

where $(y \perp t, y_0) \in E_{n1}H_1$, $x \in X_1$, $g_i \in H_1$ and $t_i \in I$.

On $E_{n2}X_1'$ we define for $k = 0$

$$E_{n2}'f_0(y \perp t, \tau, x)$$

$$= \begin{cases} (E_nh_0(y) \perp t, f_0(x)) & 0 \leq t \leq \frac{1}{3},\quad \tau = 0 \\ (E_nh_0(y) \perp t, 2\tau, f_0(x)) & \frac{1}{3} < t < \frac{2}{3},\quad 0 \leq \tau \leq \frac{1}{2} \\ (E_nh_0(y) \perp t, E_nf_1(y, 2\tau - 1, x) & \frac{1}{3} < t < \frac{2}{3},\quad \frac{1}{2} \leq \tau \leq 1 \end{cases}$$

and for $k = 1,2,\ldots$ we define $E_{n2}'f_k$ just like $E_{n2}'h_k$ with the following changes: replace $h_k$ and $h_{k+1}$ by $f_k$ and $f_{k+1}$ respectively and $g_k$ by $x$. $E_{n2}'f_k$ and $E_{n1}f_k$ can be pieced together to obtain $\tilde{E}_nf_k$ for $k = 0,1,2,\ldots$ . Ultimately we get the $G_\infty$-map $\{Ef\} : EH_1 \times X_1 \to EX_2$ "over" $Bh : BH_1 \to BH_2$ associated with $Eh$, $f$ and $h$.

<u>3.3</u>. We point out that if $h,k : H_1 \to H_2$ are $H_\infty$-maps which are $H_\infty$-homotopic, then $Bh$ is homotopic to $Bk$

leaving the base point fixed, and $Eh$ is $G_\infty$-fiber homotopic to $Ek$ over the homotopy between $Bh$ and $Bk$.

Furthermore if $f, g : X_1 \to X_2$ are $G_\infty$-maps associated to $h$ and $k$, and if $f, g$ are $G_\infty$-homotopic associated to the $H_\infty$-homotopy between $h$ and $k$, then $Ef$ and $Eg$ are fiber homotopic associated with the $G_\infty$-fiber homotopy between $Eh$ and $Ek$ etc. and over the homotopy between $Bh$ and $Bk$.

Definition. Let $\mathcal{F}$ be the category whose objects are fibrations $p : E \to B$ which are locally fiber homotopy trivial with respect to a numerable covering of sets contractible in $B$, and whose morphisms are fiber homotopy classes of fiber preserving maps. Let $\mathcal{F}_*$ be the associated category of fibrations with a distinguished fiber over a basepoint $*$, and let $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}_*$ be the associated homotopy categories.

Theorem. The constructions $EH$, $BH$, and $EX$ define a functor $B : \underline{\mathcal{B}} \to \underline{\mathcal{F}}_*$, the Borel functor.

## 4. The Inverse Functor of $B$

For every topological space $X$ and subsets $A, B \subseteq X$ we recall that

$$L(X;A,B) = \{(\omega,r) \mid \omega : \mathbb{R}^+ \to X, \ \omega(0) \in A, \\ \omega(t) = \omega(r) \in B \ \text{ for } \ t \geq r\}$$

Often we omit $r$ in our notation for the sake of simplicity.

Definition. For every fibration $p : E \to B$ with distinguished fiber $F_* = p^{-1}(*)$ we define

$$\overline{E} = \{(\omega, y) \mid y \in E,\ \omega \in L(B;B,B),\ \omega(r) = p(y)\}$$

and $\overline{p} : \overline{E} \to \overline{B}$ as $\overline{p}(\omega, y) = \omega(0)$.

If the fibration $p : E \to B$ is an object in $\underline{\mathcal{F}}_*$ then the fiber map $\tau : E \to \overline{E}$ defined by $\tau(y) = (\omega_y, y)$ is a fiber homotopy equivalence, see [1], Theorem 6.3 ($\omega_y : \mathbb{R}^+ \to E$ is defined as $\omega_y(t) = y$ for all $t \in \mathbb{R}^+$, $r = 0$).

Let $WE = \overline{p}^{-1}(*)$ be the distinguished fiber of $\overline{p}$, then $\tau|F_*$ is a homotopy equivalence between $F_*$ and $WE$. We observe that the loopspace of $B$, $\Omega(B,*)$, acts on $WE$ from the left ($\Omega(B,*) = L(B;*,*)$ is an H-space). Furthermore if $p$, $p'$ are two fibrations in $\underline{\mathcal{F}}_*$ and if $(F,f)$ is a based fiber map from $p$ to $p'$, then $Wf : WE \to WE'$ defined by $Wf(\omega, y) = (Lf(\omega), F(y))$ is an equivariant map associated with the induced homomorphism $\Omega f : \Omega(B,*) \to \Omega(B',*)$. We summarize this observation in the

Definition. $W$ induces a functor

$$\underline{W} : \underline{\mathcal{F}}_* \to \underline{\mathcal{B}}\ ,$$

the inverse functor to $\underline{B}$, as we shall see in the following

Theorem 1. $\underline{WB}$ is equivalent to $1_{\underline{\mathcal{B}}}$

and

*Theorem 2.* $\underline{\underline{BW}}$ is equivalent to $1_{\underline{\mathcal{F}}}$.

## 5. Proof of Theorem 1

To prove Theorem 1 we have to review the natural transformation $S : H \to \Omega BH$.

*5.1.* We need from [3], p. 333 the

*Theorem.* EH is contractible.

Let $k : EH \times I \to EH$ be a contraction with $k(y,0) = y$ and $k(y,1) = * = k(*,t)$. (For this it is necessary that $* \in H$ is a nondegenerate base point. If necessary one can switch to $H \vee I$, see [2], p. 215).

Associated with the contraction $k$ is the map $K : EH \to L(EH;EH,*)$ defined by $K(y) = (k(y,t),1)$.

*5.2.* Define $S_0 : H \to \Omega(BH,*)$ as

$$S_0(y) = Lp_H \circ K|E_0H$$

with $Lp_H : L(EH;EH,*) \to L(BH;BH,*)$ induced by $p_H$.

*Lemma 1.* $S_0$ is a homotopy equivalence.

*Proof:* $L(BH;BH,*)$ is the total space of a numerable fibration over BH, and so is EH. Both total spaces are contractible. $S_0$ is the restriction of $Lp_H \circ K$, which is a fiber map over $id_{BH}$ and which is also a homotopy equivalence. Theorem 6.1 in [1] implies that $Lp_H \circ K$ is a fiber homotopy equivalence and hence $S_O$ is a homotopy equivalence.

Lemma 2. $S_O$ can be extended to an $H_\infty$-map.

Proof: Let $K \mid E_O H = K \mid H = K_O$. Then we have to find maps $S_1, S_2, \ldots$ which make $S_O = Lp_H \circ K_O : H \to \Omega BH$ into an $H_\infty$-map. Assume we already constructed $S_i = Lp_H \circ K_i$ $(i = 0,1,\ldots,n)$. Then $S_{n+1}$ and hence $K_{n+1}$ is defined on $\partial H(n+1)$ through the maps $S_i$ and $K_i$ respectively $(i = 0,\ldots,n)$.

Associated with $K_i$ are the maps

$$k_i : H(i) \times \mathbb{R}^+ \to EH$$

and

$$r_i : H(i) \to \mathbb{R}^+$$

with $k_i\,(g_O, t_1, \ldots, t_i, g_i, 0) = *$ and $k_i\,(g_O, t_1, \ldots, t_i, g_i, \tau) = g_O \ldots g_i$ for $\tau \geq r_i(g_O, t_1, \ldots, t_i, g_i)$. These maps define $k_{n+1}$ and $r_{n+1}$ respectively on $\partial H(n+1)$. Since $\mathbb{R}^+$ is contractible we can extend $r_{n+1}$ to all of $H(n+1)$. Then we can extend $k_{n+1}$ to all of $H(n+1)$ such that $k_{n+1}(g_O, t_1, \ldots, t_{n+1}, g_{n+1}, 0) = *$ and $k_{n+1}(g_O, t_1, \ldots, t_{n+1}, g_{n+1}, r_{n+1}(\ldots)) = g_O \ldots g_{n+1}$, since $EH$ is contractible.

Define

$$K_{n+1} = (k_{n+1}, r_{n+1}) \quad \text{and} \quad S_{n+1} = Lp_H \circ K_{n+1} .$$

For further details compare [2], p. 214–215. (Note the addition of paths on p. 213 should be reversed.)

5.2. Proposition. $S$ is a natural transformation between

$$1_{\mathfrak{A}} \quad \text{and} \quad \Omega B \ .$$

Proof: In the diagram

$$\begin{tikzcd}
H \arrow[r, "h"] \arrow[d, "K"'] \arrow[dd, bend right=60, "S"'] & H' \arrow[d, "K"] \\
L(EH;EH,*) \arrow[r, "LEh"] \arrow[d, "Lp_H"'] & L(EH';EH',*) \arrow[d, "Lp_{H'}"] \\
\Omega(BH,*) \arrow[r, "\Omega Bh"] & \Omega(BH',*)
\end{tikzcd}$$

the lower portion commutes for all the maps of $LEh$. To see that the upper portion commutes up to an $H_\infty$-homotopy, one has to look again at the associated maps into $EH'$. Since $EH'$ is contractible, all extensions necessary to construct the $H_\infty$-homotopy between $LEh \circ K$ and $K \circ h$ can be carried out. Further details in [2]. (In [2] the $G_\infty$-map $Eh$ was not discussed. Instead the notion of a "regular" H-homomorphism had to be used. Now $EH$ provides the homotopy between formula 2 and 2a on p. 217 in 2, translated from right to left actions.)

5.3. With $S$ out of the way we define for any G-space X:

$$T_0 : X \to WE \quad \text{as} \quad T_0 = \tau \mid X .$$

We already know that $T_0(y) = (*,y)$ is a homotopy equivalence. We define $T_n : (H \times I)^n \times X \to WE$ as

$$T_n(g_0,t_1,\ldots,t_n,x)$$

$$= (pK_{n-1}(g_0,\ldots,t_{n-1},g_{n-1})(t_n+\sigma),$$

$$r_X(K_{n-1}(g_0,\ldots,t_{n-1},g_{n-1})(t_n),x))$$

with $0 \leq t_n \leq r_{n-1}(g_0,\ldots,t_{n-1},g_{n-1})$ and $0 \leq \sigma \leq r_{n-1} - t_n$. Recall $r_X : EH \times X \to EX$. We have

$$T_n(g_0,t_1,\ldots,g_{n-1},t_n,x)$$

$$= \begin{cases} (S_{n-1}(g_0,t_1,\ldots,g_{n-1}),x) & t_n = 0 \\ (*,g_0g_1 \ldots g_{n-1},x) & t_n = r . \end{cases}$$

The "$G_\infty$-homotopy" between $LEh \circ K$ and $K \circ h$ implies that $T$ is a natural transformation between $1_{\underline{\mathcal{B}}}$ and $\underline{WB}$.

## 6. Proof of Theorem 2

6.1. Let $\mathcal{T}_*$ be the category of based topological spaces $X$, which have a numerable covering $\mathfrak{U}$ such that every $U \in \mathfrak{U}$ is contractible in $X$, and based continuous maps. Let $\underline{\mathcal{T}}_*$ be the associated homotopy category.

Remark. It is easy to see that for every $H$ in $\mathcal{H}$ the classifying space $BH$ is in $\mathcal{T}_*$.

In preparation for the proof of Theorem 2 we list three universal fibrations with fiber $\Omega(X,*)$ for $X \in \underline{\mathcal{T}}_*$.

a) Application of the modified Dold-Lashof construction to the trivial fibration $\Omega(X,*) \to *$ leads to

$$p_{\Omega X} : E\Omega X \to B\Omega X$$

b) It is well-known that

$$p_L : L(X;X,*) \to X$$

also classifies numerable $\Omega(X,*)$-fibrations.

c) If we apply the modified Dold-Lashof construction to $p_L$ of b), we get again a universal fibration

$$p_{EL} : ELX \to BLX \ .$$

All three constructions induce functors from $\underline{\mathcal{T}}_*$ to $\underline{\mathcal{F}}_*$.

6.2. The inclusion of $\Omega(X,*)$ as distinguished fiber of $p_L$ $L(X;X,*) \to X$ can be interpreted as a principal map of principal fibrations and hence it induces the fiber map $(f,\overline{f})$:

$$\begin{array}{ccc} E(\Omega X) & \xrightarrow{f} & E(LX) \\ {\scriptstyle p_{\Omega X}}\downarrow & & \downarrow{\scriptstyle p_{LX}} \\ B(\Omega X) & \xrightarrow{\overline{f}} & B(LX) \end{array}$$

which is a principal fiber homotopy equivalence; $(f,\overline{f})$ is an inclusion, hence $p_{\Omega X}$ is principal fiber homotopy equivalent to the pullpack of $p_{LX}$. For universal fibrations this implies $\overline{f}$ is a homotopy equivalence. Let $\overline{g}$ be a homotopy inverse of $\overline{f}$.

As a result, $(f,\overline{f})$ represents a functor equivalence between the functors from $\underline{\mathcal{I}}_*$ to $\underline{\mathcal{F}}_*$ induced by a) and c).

<u>6.3</u>. The inclusion

$$\begin{array}{ccc} L(X;X,*) & \xrightarrow{k} & ELX \\ {\scriptstyle p_L}\downarrow & & \downarrow{\scriptstyle p_{Lx}} \\ X & \xrightarrow{\overline{k}} & BLX \end{array}$$

is a fiber homotopy equivalence by the same reasoning as described in 6.2. So $(k,\overline{k})$ represents a functor equivalence between the functors arising from b) and c).

<u>6.4</u>. Now consider a fibration $p : E \to X$ from the category $\mathcal{F}_*$. The associated Hurewicz-fibration $\overline{p} : \overline{E} \to X$ admits a map

$$r_0 : L(X;X,*) \times WE \to \overline{E}$$

defined through the addition of paths, which makes $\overline{E}$ a look alike of a Borel space associated to $WE$.

Assigning to $p$ the Hurewicz fibration $\overline{p}$ induces a functor $H_r$ on $\mathcal{T}_*$ which is obviously equivalent to $\mathrm{id}_{\mathcal{T}_*}$. We are now going to show $\underline{BW} \simeq \underline{Hr}$. Consider the diagram of Borel spaces:

$$\begin{tikzcd}
L(X;X,*) \times WE \arrow[r, "k\times 1"] \arrow[d] & ELX \times WE \arrow[r, "g\times 1"] \arrow[d] & E\Omega X \times WE \arrow[d] \\
\overline{E} \arrow[r, "K"] \arrow[d, "\overline{p}"'] & EL(WE) \arrow[r, "G"] \arrow[d] & E(WE) \arrow[d] \\
X \arrow[r, "\overline{k}"] & BLX \arrow[r, "\overline{g}"] & B\Omega X
\end{tikzcd}$$

$K$ is induced by applying the Borel space construction to $\overline{p}$ (an obvious modification) and $G$ is induced by $g$, the homotopy inverse of $f$ from 6.2.

$(K,\overline{k})$ and $(G,\overline{g})$ represent functor equivalences associated to the equivalences $(k,\overline{k})$ and $(g,\overline{g})$ discussed in 6.2 and 6.3. Since the right side of the diagram represents $BW$ and the left side represents $H_r$, the proof is complete.

## 7. Two Applications

7.1. Let $G = \mathbb{R}^1$ and $X = \mathbb{R}^2$. Consider the two $\mathbb{R}^1$-spaces $X_1$ and $X_2$ defined by the two actions

$$\mu_1 : \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \mu_1(t, re^{i\varphi}) = re^{i(\varphi+t)},$$

$$\mu_2 : \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \mu_2(t, re^{i\varphi}) = re^{i(\varphi+t(1-r))}$$

The fix point set of $\mu_1$ is just the origin of $\mathbb{R}^2$ and the fix point set of $\mu_2$ is the origin and the unit circle. Obviously we could define actions with more complicated fix point sets.

The constant map from one of these spaces to the origin of the other is an equivariant map which is also an ordinary homotopy equivalence. It induces (according to section four) a homotopy equivalence between the Borel spaces of the two spaces.

7.2. a) Let $P$ be an acyclic finite polyhedron with nontrivial fundamental group. Then the suspension $\Sigma P$ is a contractible $\mathbb{Z}_2$-space with fix point set $P$, and the join $P * S^1$ is a constractible $S^1$ or $\mathbb{Z}_p$-space $(p \neq 2)$ with fix point set $P$ in the obvious manner (notice $P * S^1 \cong \Sigma^2 P$).

b) Let $P$ be any finite polyhedron. The obvious $\mathbb{Z}_2$-action on $\Sigma P$ can be extended to $\Sigma^2 P$ etc. so that $\lim\limits_{n\to\infty} \Sigma^n P$ is a contractible $\mathbb{Z}_2$-space with fix point set $P$.

For $G = \mathbb{Z}_p$ $(p \neq 2)$ and $G = S^1$ we can do the same by reiterating the join with $S^1$.

7.3. Let $G$ be either $\mathbb{Z}_p$ or $S^1$ and let $X$ be a $G$-space with fix points. Let $Y$ be a contractible $G$-space with nonempty fix point set $F$, e.g. let $Y$ be one of the spaces mentioned above. The one point union $W$ of $X$ and $Y$ formed by identifying two fix points is a new G-space in the obvious manner and the inclusion of $X$ into $W$ is an equivariant map and also an ordinary homotopy equivalence.

By the theorem in [4] the inclusion represents an isomorphism in $\mathcal{J}$ and induces a fiber homotopy equivalence between $BX$ and $BW$ by section 4. Hence the cohomology of these Borel spaces carries no information about $F$.

7.4. Assume $G$ is either $\mathbb{Z}_p^k$ or $(S^1)^k$ and $X_1, X_2$ are G-spaces which satisfy the assumptions for Borel's theorem as described in Proposition 1 of Chapter IV in [5], i.e., let $X_1, X_2$ be paracompact G-spaces with finite cohomology dimension. Let $f: X_1 \to X_2$ be an equivariant map which is also an ordinary homotopy equivalence. Again $Ef: EX_1 \to EX_2$ is a fiber homotopy equivalence between Borel spaces. $Ef$ induces isomorphisms between $H_G^*(X_2)$ and $H_G^*(X_1)$ as $H^*(BG)$ modules. Hence Proposition 1 on p.45 in [5] tells us, that $f|F_1 : F_1 \to F_2$ induces an isomorphism of the cohomology rings $H^*(F_2) \otimes_k R_0$ and $H^*(F_1) \otimes_k R_0$ of the fix point sets $F_1$ and $F_2$.

T. Petrie in [7] and elsewhere, Ch. N. Lee and A. Wasserman in [6] have constructed examples of such maps which do not have equivariant homotopy inverses. Hence the fiber homotopy inverse of $Ef$ is not induced by an equivariant map from $X_2$ to $X_1$. This answers the opening statement of the introduction of this paper.

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