Equivariant Poincaré Duality

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1. Introduction

The classical starting point of surgery is the theory of Poincaré duality spaces. In the case of spaces with G-action, we are aware of no adequate analogue. By "adequate" we mean that every compact G-manifold should be a G-Poincaré duality space and that the Spivak normal fibration of every G-Poincaré duality space should be an equivariant spherical fibration. This paper and its sequel [CW2] are an attempt to remedy the situation. Throughout this paper, G will be a finite group.

Of course, the main requirement for an adequate theory of equivariant Poincaré duality spaces is a suitable ordinary equivariant homology theory. The construction of such a theory, as well as the Thom isomorphism and Poincaré duality theorems, is the subject of this first paper.

In [CW1] the authors described an ordinary equivariant cohomology theory indexed on (equivalence classes of) G-vector bundles over a given space. In this paper, we give a more elementary cellular construction of that theory, together with an associated homology theory, and at the same time extend these to theories graded on representations of the fundamental groupoid (this phrase will be explained in §2). We then prove Poincaré duality and Thom isomorphism theorems. In [CW2] we shall use this theory to construct an equivariant analog of the Spivak normal bundle for a finite Poincaré complex, and show that under suitable hypotheses this leads to an equivariant normal map as in [DR] and [LM].

The main properties of the homology and cohomology theories we construct are summarized in the following theorem, from which it follows that they are generalized theories in the sense of Wirthmüller [W4]. (Precise definitions of as yet unexplained terms are given in subsequent sections.)

THEOREM A. *Let (X,A) be a pair of G-spaces, let γ be a representation of the fundamental groupoid of X, and let T be a local coefficient system on X. Then there are abelian group-valued homotopy functors $H_\gamma^G(X,A;T)$ and $H^\gamma_G(X,A;T)$. These satisfy the following properties.*

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(i) *They extend Bredon's ordinary homology and cohomology with twisted coefficients* [B].

(ii) *There are the expected long exact sequences.*

(iii) *There are isomorphisms*

$$\sigma_V: H_\gamma^G(X,A;T) \cong H_{\gamma+V}^G((X,A)\times(D(V),S(V));T)$$

and

$$\sigma_V: H_G^\gamma(X,A;T) \cong H_G^{\gamma+V}((X,A)\times(D(V),S(V));T)$$

for any G-representation V. These satisfy $\sigma_W\sigma_V=\sigma_{V+W}$.

(iv) *If* $K\subset G$ *then there is a restriction homomorphism*

$$\rho: H_\gamma^G(X,A;T) \to H_{\gamma|K}^K(X,A;T\,|\,K)$$

and a similar one in cohomology; we shall usually write $\rho(a)=a\,|\,K$. *The composite*

$$H_\gamma^G(G\times_K X, G\times_K A;T)\to H_{\gamma|K}^K(G\times_K X, G\times_K A;T\,|\,K)\to H_{\gamma|K}^K(X,A;T\,|\,K)$$

is an isomorphism, as is the similar map in cohomology. (These are sometimes referred to as the Wirthmüller isomorphisms.)

(v) *If* $K\subset G$ *then there is a restriction to fixed sets*

$$\phi: H_\gamma^G(X,A;T)\to H_{\gamma^K}^{WK}(X^K,A^K;T^K)$$

where $WK=NK/K$, *and similarly in cohomology; we shall usually write* $\phi(a)=a^K$.

(vi) *There is a cup product*

$$-\cup-: H_G^\gamma(X,A;S)\otimes H_G^\delta(Y,B;T)\to H_G^{\gamma+\delta}((X,A)\times(Y,B);S\,\square\,T).$$

If T is a $\hat\pi X$*-ring then there is a cup product*

$$-\cup-: H_G^\gamma(X,A;T)\otimes H_G^\delta(X,B;T)\to H_G^{\gamma+\delta}(X,A\cup B;T).$$

This product satisfies $(\alpha\cup\beta)\,|\,K=(\alpha\,|\,K)\cup(\beta\,|\,K)$ *and* $(\alpha\cup\beta)^K=\alpha^K\cup\beta^K$.

(vii) *There is a cap product*

$$-\cap-: H_G^\delta(X,B;S)\otimes H_{\gamma+\delta}^G(X,A\cup B;T)\to H_\gamma^G(X,A;S\otimes_{\hat\pi X}\Delta_*T)$$

satisfying $(\alpha\cup\beta)\cap a=\alpha\cap(\beta\cap a)$, $(\alpha\cap a)\,|\,K=(\alpha\,|\,K)\cap(a\,|\,K)$, *and* $(\alpha\cap a)^K=\alpha^K\cap a^K$.

Our Poincaré duality and Thom isomorphism theorems then take the following form (the first of these being stated here only in the case of compact G-manifolds for simplicity). In both theorems, $\mathfrak{A}_G$ will denote the Burnside coefficient system, defined in Section 6.

THEOREM B (Poincaré duality). *If M is a compact G-manifold and* τ *is the representation of the fundamental groupoid of M associated with its tangent bundle, then there is a class* $[M,\partial M]\in H_\tau^G(M,\partial M;\mathfrak{A}_G)$ *such that*

$$-\cap[M,\partial M]: H_G^{\gamma}(M;\mathfrak{A}_G)\to H^G_{\tau-\gamma}(M,\partial M;\mathfrak{A}_G)$$

and

$$-\cap[M,\partial M]: H_G^{\gamma}(M,\partial M;\mathfrak{A}_G)\to H^G_{\tau-\gamma}(M;\mathfrak{A}_G)$$

are isomorphisms. $[M,\partial M]$ *is called a* fundamental class *for M. Moreover,* $[M,\partial M]\,|\,K$ *is a fundamental class for M as a K-manifold, and* $[M,\partial M]^K$ *is a fundamental class for* M^K*.*

A more general form of Theorem B is proved in Section 7.

THEOREM C (Thom isomorphism). *If* ξ *is a G-vector bundle over X and* ρ *is the corresponding representation of the fundamental groupoid of X, then there is a Thom class* $t_\xi\in H_G^{\rho}(D(\xi),S(\xi);\mathfrak{A}_G)$*. For any Thom class,*

$$-\cup t_\xi: H_G^{\gamma}(X;T)\to H_G^{\gamma+\rho}(D(\xi),S(\xi);T)$$

is an isomorphism. Moreover, $t_\xi\,|\,K$ *is a Thom class for* ξ *as a K-bundle, and* t_ξ^K *is a Thom class for* ξ^K*.*

Theorem C is proved in Section 8.

One could take a slightly different viewpoint toward these theories, one that we shall simply mention here. That is, by using the Wirthmüller isomorphisms and stability, we can view the homology and cohomology as Mackey-functor valued. Write $\underline{H}_*$ and $\underline{H}^*$ for these Mackey-functor valued theories. The cup product can then be interpreted as a pairing $\underline{H}^{\gamma}\,\square\,\underline{H}^{\delta}\to \underline{H}^{\gamma+\delta}$ ($\square$ being the usual box product of Mackey functors [L]), one of the necessary identities being given by $(\alpha\cup\beta)\,|\,K=(\alpha\,|\,K)\cup(\beta\,|\,K)$. Similarly, the usual naturality of the cap product implies that it gives a pairing $\underline{H}^{\delta}\otimes_{\hat{\mathbb{G}}}\Delta_*\underline{H}_{\gamma+\delta}\to\underline{H}_{\gamma}$. From this viewpoint the fundamental class of a manifold can be thought of as a map $[M,\partial M]:\mathfrak{A}_G\to\underline{H}_{\tau}(M,\partial M)$, cap product with which gives isomorphisms of Mackey functors $\underline{H}^{\gamma}(M)\cong\underline{H}_{\tau-\gamma}(M,\partial M)$ and so on. There is a similar interpretation of Thom classes and the Thom isomorphism.

2. Groupoids

The following definitions from [CMW] are fundamental to the theory of equivariant orientations. If X is a G-space, the *fundamental groupoid* $\pi(X;G)$ (or just πX if G is understood) *of* X is the category whose objects are the G-maps $x: G/H\to X$, where H ranges over the subgroups of G; equivalently, x is a point in X^H. A morphism $x\to y$, $y: G/K\to X$, is the equivalence class of a pair (σ,ω), where $\sigma: G/H\to G/K$ is a G-map and where $\omega: G/H\times I\to X$ is a G-homotopy from x to $y\circ\sigma$. Two such maps are equivalent if there is a G-homotopy $k:\omega\simeq\omega'$ such that $k(\alpha,0,t)=x(\alpha)$ and $k(\alpha,1,t)=y\circ\sigma(\alpha)$ for $\alpha\in G/H$ and $t\in I$.

For intuition, it is probably best to think of the objects of πX as points in the fixed sets of X. However, in all technical discussions we will use maps from orbits into X. Thus, for example, $y \circ \sigma$ above should be thought of as a translate of the point $y(eK)$ corresponding to y, since $(y \circ \sigma)(eH) = gy(eK)$ if $\sigma(eH) = gK$. We will also call the homotopy ω a *path* from x to $y \circ \sigma$, since it is determined by the path $\omega(eH, t)$ in X^H from $x(eH)$ to $gy(eK)$.

Let $\mathcal{G}$ be the category of G-orbits and G-maps between them. There is a functor $\phi: \pi X \to \mathcal{G}$, given by $\phi(x: G/H \to X) = G/H$ on objects and by $\phi(\sigma, \omega) = \sigma$ on morphisms. Although we will not fully use this level of generality, we note that this turns πX into a groupoid over $\mathcal{G}$ in the sense of [CMW].

DEFINITION 2.1. A *groupoid* over $\mathcal{G}$ consists of a small category $\mathcal{C}$ and a functor $\phi: \mathcal{C} \to \mathcal{G}$ satisfying the following properties.

(a) For each object b of $\mathcal{G}$, the fiber $\phi^{-1}(\mathrm{id}_b)$ is a groupoid in the classical sense (i.e., all morphisms in $\mathcal{C}$ covering an identity map in $\mathcal{G}$ are isomorphisms).

(b) (Source-lifting) For each object y of $\mathcal{C}$ and each morphism $\alpha: a \to \phi(y)$ in $\mathcal{G}$, there exists a morphism $\beta: x \to y$ in $\mathcal{C}$ such that $\phi(\beta) = \alpha$.

(c) (Divisibility) For each pair of morphisms $\alpha: x \to y$ and $\alpha': x' \to y$ in $\mathcal{C}$ and each morphism $\beta: \phi(x) \to \phi(x')$ such that $\phi(\alpha) = \phi(\alpha')\beta$, there exists a morphism $\gamma: x \to x'$ in $\mathcal{C}$ such that $\alpha = \alpha'\gamma$ and $\phi(\gamma) = \beta$.

A groupoid $\mathcal{C}$ has *unique divisibility* if the map γ asserted to exist in (c) is unique. All the groupoids we consider will have this property. (When one considers compact Lie groups, most of the groupoids one considers do not, so we do not include this as part of the general definition.) Notice in particular that πX is a groupoid over $\mathcal{G}$ with unique divisibility, and that the fiber $\phi^{-1}(G/H)$ is the usual fundamental groupoid of X^H.

A *map* of groupoids over $\mathcal{G}$ is a functor $\zeta: \mathcal{C} \to \mathcal{C}'$ such that $\phi' \circ \zeta = \phi$. If $f: X \to Y$ is a G-map, then $f_*: \pi X \to \pi Y$ is a map of groupoids over $\mathcal{G}$.

Let $h\mathcal{O}_n$ be the category of n-dimensional orthogonal G-bundles over G-orbits and G-homotopy classes of linear maps, so there is again a functor $\phi: h\mathcal{O}_n \to \mathcal{G}$ giving the base-space. An *n-dimensional representation of* $\mathcal{C}$ is a functor $\rho: \mathcal{C} \to h\mathcal{O}_n$ such that $\phi\rho = \phi$; that is, it is a functor over $\mathcal{G}$. A *map* of representations of $\mathcal{C}$ is then a natural transformation over the identity. More generally, if ρ is a representation of $\mathcal{C}$ and ρ' is a representation of $\mathcal{C}'$, then a map $\rho \to \rho'$ is given by a pair (ζ, η), where $\zeta: \mathcal{C} \to \mathcal{C}'$ is a map of groupoids and $\eta: \rho \to \rho' \circ \zeta$ is a natural transformation over the identity. If ξ is an n-dimensional G-bundle over the G-space X, then ξ determines a representation $\rho(\xi)$ of πX given by $\rho(\xi)(x: G/H \to X) = x^*(\xi)$ on objects. $\rho(\xi)$ is defined on maps using the lifting property for G-bundles. Similarly, a map of G-bundles gives rise to a map of induced representations.

If V is a representation of G, then there is a representation of any groupoid $\mathcal{C}$ over $\mathcal{G}$ given by letting $\rho(c) = \phi(c) \times V$. We call this representation V again. If M is any smooth G-manifold, then its *tangent representation* τ is defined to be the representation of πM associated with the tangent bundle of M.

We also need the following variations defined in [CMW]. There is a category $v\mathcal{O}_n$ of *virtual* bundles over orbits, for every integer n, positive or negative. It is constructed as follows: Let $\mathcal{U}$ be a *G-universe,* that is, the direct sum of countably many copies of each irreducible representation of G. The category $v\mathcal{O}_n$ has as its objects pairs $(p: E \to G/H, q: F \to G/H) \in h\mathcal{O}_{r+n} \times h\mathcal{O}_r$, where $r \geq 0$ and $r+n \geq 0$. The morphisms, called *virtual maps,* are equivalence classes of pairs of G-bundle maps (f_1, f_2) where

$$f_1: E \oplus W \to E' \oplus W' \quad \text{and} \quad f_2: F \oplus W \to F' \oplus W',$$

where W is an H-invariant subspace of $\mathcal{U}$ and W' is an H'-invariant subspace of $\mathcal{U}$; here $-\oplus W$ means the Whitney sum with $G \times_H W$. The equivalence relation is given by three basic relations. First, (f_1, f_2) is identified with its suspension by any G-representation $Y \subset \mathcal{U}$ orthogonal to both W and W'. Next, (f_1, f_2) is identified with (k_1, k_2),

$$k_1: E \oplus Z \to E' \oplus Z' \quad \text{and} \quad k_2: F \oplus Z \to F' \oplus Z',$$

if there is an H-linear isometry $\mu: W \to Z$ and an H'-linear isometry μ': $W' \to Z'$ such that $k_i \circ (1 \oplus \mu) \simeq (1 \oplus \mu') \circ f_i$ as G-bundle maps, for $i = 1$ and 2. Third, we identity homotopic pairs of maps. This specifies a well-defined category over $\mathcal{G}$. We will often write a pair (E, F) as $E - F$.

A *virtual representation* of $\mathcal{C}$ is then a functor $\mathcal{C} \to v\mathcal{O}_n$ over $\mathcal{G}$; we will call a map $\mathcal{C} \to h\mathcal{O}_n$ an *actual* representation to distinguish it from a virtual one. Maps of virtual representations are defined in the same way as maps of actual representations. The set of isomorphism classes of virtual representations of $\mathcal{C}$ of all dimensions forms a group under direct sum, called $\mathrm{RO}(\mathcal{C})$. If $\mathcal{C}$ has only finitely many isomorphism classes, then $\mathrm{RO}(\mathcal{C})$ is isomorphic to the Grothendieck group of the monoid of isomorphism classes of actual representations of $\mathcal{C}$, under direct sum. In particular, if X is a compact G-space then $\mathrm{RO}(\pi X)$ can be constructed in either way.

We will, however, still need the *category* of virtual representations of fundamental groupoids. Precisely, let $G\mathcal{R}\mathcal{U}$ be the category whose objects are pairs (X, γ), where X is a G-space and γ is a virtual representation of πX. A morphism $(X, \gamma) \to (Y, \delta)$ is given by a G-map $f: X \to Y$ and a map of representations $\gamma \to \delta$ covering $f_*: \pi X \to \pi Y$. For technical reasons, we shall only consider as objects in $G\mathcal{R}\mathcal{U}$ those (X, γ) for which there exists a G-representation V such that, for all objects $x \in \pi X$, if $\phi(x) = G/H$ then $(\gamma \oplus V)(x) \cong G \times_H W - \mathbf{R}^n$ for some representation W of H and some n. This will be needed when we extend our cellular cohomology from cell complexes to arbitrary objects in $G\mathcal{R}\mathcal{U}$. If X is compact this is no restriction at all; in general a more sophisticated approach could probably do without it.

In order to pass to an associated homotopy category, we make the following definition. If $(X, \gamma) \in G\mathcal{R}\mathcal{U}$, then define $(X, \gamma) \times I$ to be the pair $(X \times I, \gamma')$ where $\gamma' = \gamma \circ p_*$, $p: X \times I \to X$ the projection. This gives us the notion of homotopy and the homotopy category $hG\mathcal{R}\mathcal{U}$. The homology and cohomology functors we shall construct will be functorial on this category, rather than on the category of pairs $(X, \alpha \in \mathrm{RO}(\pi X))$ as might be expected.

Finally, we can define *spherical* representations by repeating all of the above using the category $h\mathfrak{F}_n$ of spherical bundles over G-orbits and spherical maps between them. Likewise, there is the category $v\mathfrak{F}_n$ of virtual spherical bundles, which gives us virtual spherical representations. These categories are all related by a commutative diagram

$$\begin{array}{ccc} h\mathcal{O}_n & \longrightarrow & h\mathfrak{F}_n \\ \downarrow & & \downarrow \\ v\mathcal{O}_n & \longrightarrow & v\mathfrak{F}_n \end{array}$$

of categories over $\mathcal{G}$.

3. CW-Complexes

In this section we define a generalized notion of equivariant cell complex which will allow us to define cellular homology and cohomology theories. We use here the category $G\mathcal{R}\mathcal{U}$ of G-spaces and representations of their fundamental groupoids introduced in the previous section.

DEFINITION 3.1. A *cell* in the category $G\mathcal{R}\mathcal{U}$ is a space of the form $(G\times_H D(V), \alpha)$, where $D(V)$ is the unit disc of the H-representation V and α is a representation of $\pi(G\times_H D(V))$ isomorphic to $(G\times_H V)\pm \mathbf{R}^n$ for some n. The *dimension* of such a cell is its nonequivariant dimension.

Let $e=(G\times_H D(V), \alpha)$ and $\partial e=(G\times_H S(V), \alpha\,|\,G\times_H S(V))\subset e$. Suppose that we have a map $\partial e\to(X,\gamma)$. Then we say that (Y,δ) is (X,γ) *with e adjoined* if there is a pushout diagram of the following form in the category $G\mathcal{R}\mathcal{U}$:

$$\begin{array}{ccc} \partial e & \longrightarrow & e \\ \downarrow & & \downarrow \\ (X,\gamma) & \longrightarrow & (Y,\delta). \end{array}$$

Such pushouts always exist and are unique up to isomorphism. Y is the usual pushout of spaces, and by the Van Kampen theorem its fundamental groupoid is the pushout of the fundamental groupoids of X and e. It should be remarked, however, that there is not a canonical choice for δ. The problem is that in the map $\partial e\to(X,\gamma)$, the map $\alpha\to\gamma$ is a virtual isomorphism at each point z, but this does not say that the pair representing $\alpha(z)$ is the same pair of bundles as $\gamma(z)$. We may construct δ by choosing either pair.

DEFINITION 3.2. A *$G\mathcal{R}$-CW complex* is an object (X,γ) in $G\mathcal{R}\mathcal{U}$ that has a filtration $(X^0,\gamma^0)\subset(X',\gamma')\subset\cdots\subset(X,\gamma)$ in $G\mathcal{R}\mathcal{U}$ such that $(X,\gamma)=\operatorname{colim}(X^n,\gamma^n)$, (X^0,γ^0) is a union of 0-cells, and each (X^n,γ^n) for $n\geq 1$ is obtained from (X^{n-1},γ^{n-1}) by attaching n-cells.

Notice that, in order for (X,γ) to have the structure of a $G\mathcal{R}$-CW complex, γ must have a special form: each $\gamma(x)$ must be isomorphic to $(G\times_H V)\pm\mathbf{R}^n$

for some H-representation V and some n, where $\phi(x)=G/H$. Thus, even infinite $G\mathfrak{R}$-CW complexes satisfy the technical condition introduced in the definition of $G\mathfrak{R}\mathfrak{U}$.

Smooth G-manifolds provide the motivation for and important examples of $G\mathfrak{R}$-CW complexes. If M is a smooth G-manifold, let τ be the tangent representation of πM. An explicit $G\mathfrak{R}$-CW structure on (M,τ) may be obtained by starting with a G-triangulation in the sense of Bredon [B] and then passing to the dual cell complex. Thus the top-dimensional cells are of the form GD where D is the closed star, taken in the first barycentric subdivision, of a vertex of the original triangulation.

In order to show that this is a well-behaved theory of cell complexes—that is, that we have cellular approximation and the Whitehead theorem—we now define certain homotopy groups.

DEFINITION 3.3. Let $f:(X,\gamma)\to(Y,\delta)$ be a map in $G\mathfrak{R}\mathfrak{U}$ and let $e=(G\times_H D(V),\alpha)$ be a cell. We define $\pi_e(f)$ to be the set of homotopy classes of commutative diagrams

$$\begin{array}{ccc} e & \longrightarrow & (Y,\delta) \\ \uparrow & & \uparrow \\ \partial e & \longrightarrow & (X,\gamma) \end{array}$$

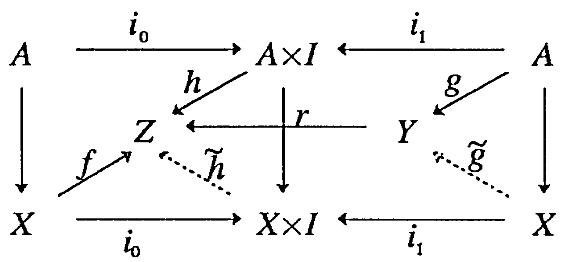
in $G\mathfrak{R}\mathfrak{U}$.

This is functorial in f in the evident way. There is a distinguished subset S of $\pi_e(f)$ consisting of the image of $\pi_e(1_X)$, and we regard $\pi_e(f)$ as *trivial* if $\pi_e(f)=S$.

DEFINITION 3.4. A map $f:(X,\gamma)\to(Y,\delta)$ in $G\mathfrak{R}\mathfrak{U}$ is an *n-equivalence* if $\pi_e(f)$ is trivial for all e of dimension $\le n$; f is a *weak $G\mathfrak{R}$-equivalence* if it is an n-equivalence for all n.

The following is a variant of the "homotopy extension and lifting property" due to May [M].

LEMMA 3.5 (H.E.L.P.). *Let $r:(Y,\delta)\to(Z,\epsilon)$ be an n-equivalence. Let (X,γ) be a $G\mathfrak{R}$-CW complex and let $A\subset X$ be a subcomplex, such that all of the cells of X not in A have dimension $\le n$. If the following diagram commutes in $G\mathfrak{R}\mathfrak{U}$ without the arrows $\tilde{g}$ and $\tilde{h}$, then we can complete it by filling in the arrows $\tilde{g}$ and $\tilde{h}$:*



Moreover, the result remains true when $n=\infty$.

Proof. The proof, as in [M], is by induction over the cells of X not in A. One can first extend $\tilde{h}$ over each new cell using the fact that the inclusion $\partial e \to e$ of the boundary of a cell e is a cofibration in $G\mathfrak{R}\mathfrak{U}$, and then extend $\tilde{g}$ by using triviality of $\pi_e(r)$. □

The main consequences are these.

DEFINITION 3.6. A map $f: X \to Y$ of $G\mathfrak{R}$-CW complexes in $G\mathfrak{R}\mathfrak{U}$ is *cellular* if $f(X^n) \subset Y^n$ for each n.

THEOREM 3.7 (Cellular approximation). *If $f: X \to Y$ is any map of $G\mathfrak{R}$-CW complexes in $G\mathfrak{R}\mathfrak{U}$, then f is homotopic in $G\mathfrak{R}\mathfrak{U}$ to a cellular map.*

Proof. Observe that the inclusion $Y^n \to Y$ is an n-equivalence, using (for example) an elementary G-transversality argument. Wassermann's controlled G-transversality works in this context because maps in $G\mathfrak{R}\mathfrak{U}$ between cell complexes are required to preserve the local representations, modulo addition of trivial summands. □

Now let $[X, Y]_{G\mathfrak{R}}$ denote the set of homotopy classes of maps $X \to Y$ in $G\mathfrak{R}\mathfrak{U}$, suppressing the representations from the notation.

THEOREM 3.8 (Whitehead).

(i) *If $f: Y \to Z$ is an n-equivalence and X is an $(n-1)$-dimensional $G\mathfrak{R}$-CW complex, then $[X, Y]_{G\mathfrak{R}} \to [X, Z]_{G\mathfrak{R}}$ is an isomorphism. It is an epimorphism if X is n-dimensional.*

(ii) *If $f: Y \to Z$ is a weak $G\mathfrak{R}$-equivalence and X is a $G\mathfrak{R}$-CW complex, then $[X, Y]_{G\mathfrak{R}} \to [X, Z]_{G\mathfrak{R}}$ is an isomorphism. In particular, any weak equivalence of $G\mathfrak{R}$-CW complexes is a homotopy equivalence in $G\mathfrak{R}\mathfrak{U}$.*

Proof. (i) Apply H.E.L.P. to $(X, \emptyset)$ to get surjectivity, and to $(X \times I, X \times \partial I)$ to get injectivity. (ii) follows in the same way from the last part of H.E.L.P. □

We now claim that, by mimicking the usual technique of killing homotopy groups, we can "approximate" any object of $G\mathfrak{R}\mathfrak{U}$ by a $G\mathfrak{R}$-CW complex, and that this approximation procedure behaves well. To see this, let (X, γ) be an object in $G\mathfrak{R}\mathfrak{U}$. We start with a union of 0-cells $\Gamma_0 X$ and a map f_0: $\Gamma_0 X \to X$ in $G\mathfrak{R}\mathfrak{U}$, with $\pi_e(f_0)$ trivial for all 0-dimensional cells e. We then inductively attach cells to $\Gamma_{n-1} X$ in order to trivialize $\pi_e(f_{n-1})$ for arbitrary n-dimensional cells e, assuming that an $(n-1)$-connected $f_{n-1}: \Gamma_{n-1} X \to X$ has already been constructed. Summarizing, we have the following theorem.

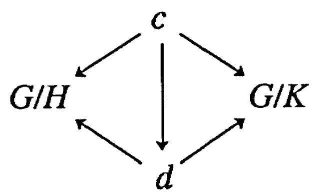
THEOREM 3.9. *Let (X, γ) be any object in $G\mathfrak{R}\mathfrak{U}$. Then there exists a $G\mathfrak{R}$-CW complex ΓX and a weak $G\mathfrak{R}$-equivalence $\Gamma X \to X$.*

By Whitehead's theorem, $\Gamma X \to X$ is unique up to canonical homotopy equivalence. In addition, if $X \to Y$ is in $G\mathfrak{R}\mathfrak{U}$ then there is a map $\Gamma X \to \Gamma Y$, unique up to homotopy, such that the expected diagram commutes.

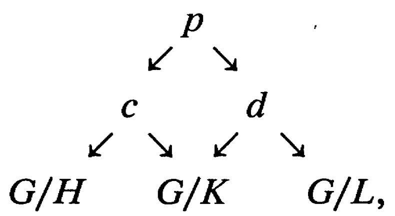
4. Construction of Equivariant Twisted Homology Graded on $\mathrm{RO}(\pi X)$

In order to discuss the appropriate coefficient systems, we introduce a "stable" version of πX analogous to the stable orbit category.

We first recall the construction of the stable orbit category $\hat{\mathcal{G}}$. The objects of $\hat{\mathcal{G}}$ are those of $\mathcal{G}$. The quickest way to describe the morphisms is to define $\hat{\mathcal{G}}(G/H, G/K) = \{G/H^+, G/K^+\}_G$, the group of stable G-homotopy classes of based G-maps $G/H^+ \to G/K^+$, where the superscript "+" denotes addition of a disjoint basepoint. Here one stabilizes by suspending with spheres of arbitrary representations of G. Alternatively, we can define $\hat{\mathcal{G}}(G/H, G/K)$ as follows. Consider equivalence classes of diagrams $G/H \leftarrow c \to G/K$, where c is a finite G-set, and where the diagrams $G/H \leftarrow c \to G/K$ and $G/H \leftarrow d \to G/K$ are equivalent if there is a G-isomorphism $c \to d$ making the diagram



commute. Taking disjoint unions of finite G-sets makes the set of these equivalence classes into a monoid. $\hat{\mathcal{G}}(G/H, G/K)$ is then the Grothendieck group of this monoid. The composite of $G/H \leftarrow c \to G/K$ and $G/K \leftarrow d \to G/L$ is given by the diagram



where the top square is a pullback. The isomorphism relating these two descriptions of $\hat{\mathcal{G}}(G/H, G/K)$ is given by sending the diagram $G/H \leftarrow c \to G/K$ to an associated composite $\Sigma^V G/H^+ \to \Sigma^V c^+ \to \Sigma^V G/K^+$. Here the first arrow is the collapse map associated with an embedding of c in $G/H \times V$ covering the given map $c \to G/H$, while the second is induced by the given map $c \to G/K$.

A *Mackey functor* may then be characterized as an additive contravariant functor $T\colon \hat{\mathcal{G}} \to \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups (see [LMM, L]). Mackey functors are the coefficients needed to define $\mathrm{RO}(G)$-graded ordinary homology and cohomology. Our coefficients incorporate Mackey functor structure together with possible twisting. To define these, first let X be a G-space and let $\hat{\pi} X$ be the following category. For the objects, take those of πX. For the morphisms from x to y, consider the set of equivalence classes of diagrams $x \leftarrow z \to y$ in πX, where the diagrams $x \leftarrow z \to y$ and $x \leftarrow z' \to y$ are equivalent if there is an isomorphism $z \to z'$ in πX making the expected diagram commute as above. $\hat{\pi} X(x, y)$ is then taken to be the free abelian group on the set of equivalence classes. Composition is essentially the same as in $\hat{\mathcal{G}}$; one first takes pullbacks on the level of orbits, then uses the source lifting property and divisibility to obtain the summands of the composite. There is an evident functor $\hat{\phi}: \hat{\pi}X \to \hat{\mathcal{G}}$.

DEFINITION 4.1. A (*stable*) *local coefficient system on* X is an additive contravariant functor $T: \hat{\pi}X \to \mathcal{A}b$. We shall sometimes refer to such a functor instead as a $\hat{\pi}X$-*group*, depending on the context.

The next step is to define an equivariant analogue of the universal covering space of a G-space X; we assume that each fixed set of X is semilocally simply connected. In order to construct the desired G-space, we use the following generalization of Elmendorf's coalescence functor [E].

For $H \subset G$, let $\mathcal{G} \downarrow G/H$ be the category of G-orbits over G/H. Thus the objects of $\mathcal{G} \downarrow G/H$ are the arrows $G/J \to G/H$ in $\mathcal{G}$, and the morphisms are G-maps over G/H. Given a contravariant functor $\Psi: \mathcal{G} \downarrow G/H \to \mathcal{U}$ (where $\mathcal{U}$ is the category of unbased topological spaces), we define an associated G-space $C\Psi$ over G/H to be the geometric realization of the simplicial G-space $B_*(\Psi, \mathcal{G} \downarrow G/H, \mathcal{J})$ over G/H given by

$$B_n(\Psi, \mathcal{G} \downarrow G/H, \mathcal{J}) = \{x[f_n, \ldots, f_1]gJ_0 : x \in \Psi(G/J_n), f_i: G/J_{i-1} \to G/J_i, gJ_0 \in G/J_0\}.$$

We are suppressing the structure over G/H in the notation. (For example, the f_i's are maps over G/H.) The G-action on B_n is given by its action on the last coordinate; the map to G/H is given by projection of the last coordinate.

In order to describe the general properties of this construction, we first introduce some notation. Let Φ be the functor which assigns to each G-space Y over G/H the contravariant functor $\Phi Y: \mathcal{G} \downarrow G/H \to \mathcal{U}$ given by taking $\Phi Y(G/J \to G/H)$ to be the space of G-maps $G/J \to Y$ over G/H. One now has, as in [E], a natural spacewise homotopy equivalence $\Phi C\Psi \to \Psi$. Thus $C\Psi$ is a G-space over G/H with fixed-set data given, up to homotopy, by Ψ.

If x is an object in πX, let $\phi(x) = G/H$. Define a functor $\Psi(x): \mathcal{G} \downarrow G/H \to \mathcal{U}$ by taking $\Psi(x)(\alpha: G/J \to G/H) = \tilde{X}^J_{x\alpha}$, the universal covering space of X^J with basepoint at $x \circ \alpha(eJ)$, that is, the set of homotopy classes of paths in X^J starting at $x \circ \alpha(eJ)$. $\Psi(x)$ is defined on morphisms in the obvious way. Now let $\hat{X}(x) = C\Psi(x)$.

Let $G\mathcal{U} \downarrow \mathcal{G}$ be the category of G-spaces over orbits. Then $\hat{X}$ is in fact a covariant functor $\pi X \to G\mathcal{U} \downarrow \mathcal{G}$ over $\mathcal{G}$. To see this, let (σ, ω) be a morphism in πX, where $\sigma: G/H \to G/K$ and ω is a path from x to $y \circ \sigma$ in X^H. This gives, for each $\alpha: G/J \to G/H$, a map $\tilde{X}^J_{x\alpha} \to \tilde{X}^J_{y\sigma\alpha}$ given by preceding each path with $\omega^{-1} \circ \alpha$. This specifies a natural transformation $\Psi(x) \to \Psi(y) \circ \sigma_*$. Application of C in turn gives a G-map $\hat{X}(x) \to \hat{X}(y)$ over σ, which is what we seek. In addition to this structure, one has a canonical G-map $q: \hat{X}(x) \to X$ for each $x \in \pi X$ such that $q\hat{X}(f) = q$ for any map f in πX. Explicitly, q is given on the simplicial level by $q(\omega[f_n, \ldots, f_1]gJ_0) = \omega(f_n \circ \cdots \circ f_1(gJ_0), 1)$, where $\omega: G/J_n \times I \to X$ is a point in $\tilde{X}^{J_n}_{x\alpha}$.

In summary, the above construction gives us the following. If $\alpha:\hat{X}(x)\to G/H$ is the given structure map, then $\alpha^{-1}(eH)$ is an H-space whose fixed set by any $J\subset H$ is homotopy equivalent to the nonequivariant universal cover $\tilde{X}^J_x$. Moreover, the projection q agrees with the usual universal covering map $\tilde{X}^J_x\to X^J_x$, where X^J_x is the component of X^J containing x. It follows that the restriction of q to $\alpha^{-1}(eH)$ is an H-equivariant quasifibration whose fiber has homotopically discrete fixed sets. Therefore we let $p:\tilde{X}(x)\to X$ be the associated G-fibration (see e.g. [W2]). Since the construction of associated fibrations is functorial in G-spaces over X, $\tilde{X}$ is a functor $\pi X\to G\mathfrak{U}\downarrow\mathcal{G}$. Furthermore, the fixed sets of p are equivalent to the nonequivariant universal covers as in the case of $\hat{X}$.

We are now ready to construct the cellular theory promised above. Suppose then that (X,γ) is a $G\mathfrak{R}$-CW complex. We first construct the cellular chain complex as a complex of $\hat{\pi}X$-groups. If X^n is the n-skeleton of X, let $\tilde{X}^n=p^{-1}(X^n)$, and observe that $\tilde{X}^n$ is a subfunctor of $\tilde{X}$. Now let x be an object in πX with $\phi(x)=G/H$. Write $\gamma(x)=G\times_H V-G\times_H W$, and let

$$C_n(X,\gamma)(x)=\{G\times_H(S^V\wedge S^n),\tilde{X}^n(x)/\tilde{X}^{n-1}(x)\wedge_{G/H}G\times_H(S^W\wedge S^{|\gamma(x)|})\}_{G/H}.$$

Here, $\wedge_{G/H}$ denotes fiberwise smash over G/H, and $\{-,-\}_{G/H}$ denotes stable G-homotopy classes of based maps over G/H. Before describing the behavior on morphisms, we consider the structure of the groups $C_n(X,\gamma)(x)$. By the *center* of the n-cell $e=(G\times_K D(U),\alpha)$ we mean the map $x_0: G/K\to X$ given by the orbit $G\times_K 0\subset e$.

LEMMA 4.2. $C_n(X,\gamma)(x)\cong\Sigma\,\hat{\pi}X(x,x_0)$, *where the sum runs over the centers x_0 of the n-cells of X.*

Proof. Define $F:\Sigma\,\hat{\pi}X(x,x_0)\to C_n(X,\gamma)(x)$ as follows. Take a generator $x\leftarrow y\to x_0$ of $\hat{\pi}X(x,x_0)$ for the center x_0 of some n-cell e. By divisibility, we may assume that the map $y\to x_0$ has the form (σ,c), where $\sigma: G/L\to G/K$ and c is the constant path, so that $y=x_0\sigma$. We now use the representation γ of πX to obtain corresponding maps between disc bundles. First, write $\gamma(x_0)=G\times_K(V_0-W_0)$, so that the cell e has the form $G\times_K D(U)$, where U is K-equivalent to $(V_0-W_0)\oplus\mathbf{R}^{n-|\gamma|}$. This means that we can choose a K-isomorphism $\kappa: V_0\oplus\mathbf{R}^n\cong U\oplus W_0\oplus\mathbf{R}^{|\gamma|}$. Applying γ to the map $y\to x_0$, we get a virtual G-map $\gamma(\sigma,c): G\times_L(V_1-W_1)\to G\times_K(V_0-W_0)$. κ now gives a G-bundle map $G\times_K(V_0\oplus\mathbf{R}^n)\to G\times_K(U\oplus W_0\oplus\mathbf{R}^{|\gamma|})$. Pulling this back over σ and using universality of the pullback, together with $\gamma(\sigma,c)$, gives a stable G-map $\lambda: G\times_L(V_1\oplus\mathbf{R}^n)\to G\times_L(U'\oplus W_1\oplus\mathbf{R}^{|\gamma|})$, where $\sigma^*(G\times_K U)=G\times_L U'$. Now let

$$\gamma(y\to x)=(f_1,f_2): G\times_L(V_1-W_1)\to G\times_H(V-W).$$

Dualizing f_1, we obtain a stable G-map

$$G\times_H S^{V+n}\xrightarrow{\hat{f}_1}G\bar{\times}_L S^{V_1+n}\xrightarrow{\lambda}G\bar{\times}_L S^{U'+W_1+|\gamma|}\xrightarrow{f_2}(G\times_K S^U)\wedge(G\times_H S^{W+|\gamma|}).$$

Here, $\bar{\times}_L$ indicates that base-points in the fibers are identified that map to the same coset in G/H upon composition with the map $G/L \to G/H$. Projecting to the cell e now gives a stable G-map

$$\xi : G \times_H S^{V+n} \to (G \times_K S^U) \wedge (G \times_H S^{W+|\gamma|}) \to X^n/X^{n-1} \wedge (G \times_H S^{W+|\gamma|}).$$

Notice that ξ factors through a map

$$G \bar{\times}_L S^{U'+W_1+|\gamma|} \to X^n/X^{n-1} \wedge (G \times_H S^{W+|\gamma|}).$$

In addition, we are given a map $G/L \to \tilde{X}^n(x)$ over G/H specified by the path $y \to x$. This specifies a lifting of the zero section of $G \bar{\times}_L U'$ to $\tilde{X}^n(x)$, which extends to a lift $G \bar{\times}_L S^{U'} \to \tilde{X}^n(x)/\tilde{X}^{n-1}(x)$, over G/H, of the projection $G \bar{\times}_L S^{U'} \to X^n/X^{n-1}$ onto the cell e. Putting this all together gives a lift of ξ to a stable map $\tilde{\xi} : G \times_H S^{V+n} \to \tilde{X}^n(x)/\tilde{X}^{n-1}(x) \wedge (G \times_H S^{W+|\gamma|})$ over G/H. We take $F(x \leftarrow y \to x_0) = [\tilde{\xi}]$.

We now construct $E = F^{-1} : C_n(X, \gamma)(x) \to \Sigma\, \hat{\pi}X(x, x_0)$ as follows. Consider a stable G-map

$$f : G \times_H (S^V \wedge S^n) \to \tilde{X}^n(x)/\tilde{X}^{n-1}(x) \wedge_{G/H} G \times_H (S^W \wedge S^{|\gamma(x)|}).$$

First, we may compose with p to get

$$pf : G \times_H (S^V \wedge S^n) \to X^n/X^{n-1} \wedge G \times_H (S^W \wedge S^{|\gamma(x)|}).$$

We now claim that we can G-homotope pf to make it transverse to the centers of all the n-cells in X. This assertion rests on the following observation. If z is any point with isotropy subgroup L such that $pf(z)$ is the center x_0 of some n-cell e in X, then one has G-maps $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ where $K = G_{x_0}$. The fact that pf factors through f shows that there are stable G-bundle maps $\alpha^*(G \times_H V \oplus \mathbf{R}^n) \to (G \times_K U) \times G \times_H (W \oplus \mathbf{R}^{|\gamma(x)|})$, where the cell e is $G \times_K D(U)$ (these bundle maps coming from the path defined by $f(z)$ and the representation γ). The existence of these bundle maps means that the various local representations match up where necessary in order to do equivariant transversality à la Wasserman [W3], establishing the claim. Since p is a G-fibration, we can now assume that f is such that pf is transverse to the centers of the cells. It follows that f is G-homotopic to a sum of G-maps f_i, such that each pf_i is the composite of a collapse map associated with a G-orbit Ω in $G \times_H S^{V+n}$ and a map induced by a G-bundle map over orbits. In addition, $f_i(\Omega)$ specifies an object $y \in \pi X$ together with a morphism $y \to x$, and the projection $\Omega \to x_0(G/K)$ induces a morphism $y \to x_0$, defining an element of $\hat{\pi}X(x, x_0)$. By the construction of F above, this element determines a stable G-map $G \times_H S^{V+n} \to X^n/X^{n-1} \wedge G \times_H (S^W \wedge S^{|\gamma(x)|})$ which, due to our use of Wassermann's controlled G-transversality, agrees with pf_i up to a sign $\epsilon = \pm 1$. We now define $E(f_i) = \epsilon[x \leftarrow y \to x_0]$ and $E(f) = \Sigma_i E(f_i)$. That E is the inverse of F should now be clear from the constructions. □

We can now use the isomorphism $C_n(X, \gamma)(x) \cong \Sigma\, \hat{\pi}X(x, x_0)$ to make $C_n(X, \gamma)$ into a contravariant functor on $\hat{\pi}X$. An explicit description of the action of morphisms on $C_n(X,\gamma)$ can be obtained from the proof of the lemma above.

COROLLARY 4.3. *$C_n(X,\gamma)$ is a projective $\hat{\pi}X$-group.*

We now define the boundary homomorphism $\partial_*: C_n(X,\gamma)\to C_{n-1}(X,\gamma)$ to be induced by the usual map $\tilde{X}^n(x)/\tilde{X}^{n-1}(x)\to \Sigma\tilde{X}^{n-1}(x)/\tilde{X}^{n-2}(x)$.

If A is a sub-$G\mathfrak{R}$-CW complex of (X,γ) then $C_*(A,\gamma)$ is a summand of $C_*(X,\gamma)$, and we can define $C_*(X,A,\gamma)=C_*(X,\gamma)/C_*(A,\gamma)$ as usual. More generally, we can consider a *relative* $G\mathfrak{R}$-CW complex (X,A,γ) with filtration $\{X^n\}$, each X^n containing A, and then define $C_*(X,A,\gamma)$ in the same way as we did $C_*(X,\gamma)$.

In order to describe the associated homology and cohomology groups, we make the following standard definitions.

DEFINITIONS 4.4. If A and B are $\hat{\pi}X$-groups, define $\mathrm{Hom}_{\hat{\pi}X}(A,B)$ to be group of natural transformations $A\to B$, and

$$A\otimes_{\hat{\pi}X}B=\frac{\Sigma_{\hat{\pi}X}A(x)\otimes B(x)}{\approx},$$

where the equivalence is given by $f^*(a)\otimes b\approx a\otimes f_*(b)$ for a morphism f in $\hat{\pi}X$. Here $f_*=B(\hat{f})$, where $\hat{f}$ is the dual of f; the dual of a generator $[x\leftarrow y\rightarrow z]$ is $[z\leftarrow y\rightarrow x]$.

Finally, we define cellular homology and cohomology.

DEFINITION 4.5. Let (X,γ) be a $G\mathfrak{R}$-CW complex, and let T be a local coefficient system on X. Define groups $H^G_{\gamma+n}(X;T)$ and $H_G^{\gamma+n}(X;T)$ as the $(|\gamma|+n)$-dimensional homology groups of $C_*(X,\gamma)\otimes_{\hat{\pi}X}T$ and $\mathrm{Hom}_{\hat{\pi}X}(C_*(X,\gamma),T)$ respectively. The relative groups $H^G_{\gamma+n}(X,A;T)$ and $H_G^{\gamma+n}(X,A;T)$ are defined similarly, using $C_*(X,A,\gamma)$.

Before extending these to theories defined on arbitrary G-spaces, we make two observations.

PROPOSITION 4.6.

(i) *If V is any representation of G then there is a natural isomorphism*

$$\sigma: H^G_{\gamma+n}(X,A;T)\to H^G_{\gamma+V+n}((X,A)\times(D(V),S(V));T),$$

and similarly for cohomology. (Here $\gamma+V$ means $\gamma\oplus V$.)

(ii) *$H^G_{\gamma+n}(X;T)$ is functorial on $hG\mathfrak{R}\mathfrak{W}$, the full subcategory of $hG\mathfrak{R}\mathfrak{U}$ whose objects are the $G\mathfrak{R}$-CW complexes, and similarly for pairs.*

Proof. (i) follows from the fact that $((X,A)\times(D(V),S(V)),\gamma+V)$ has a natural $G\mathfrak{R}$-CW structure for which the desired isomorphism can be seen on the chain level. For (ii), one uses the usual arguments involving cellular approximation of maps and homotopies. □

We now extend these functors to the category $hG\mathcal{R}\mathcal{U}$ of all G-spaces with specified representations of their fundamental groupoids. If $(X,\gamma)\in hG\mathcal{R}\mathcal{U}$, define

$$H^G_{\gamma+n}(X,A;T)=\operatorname*{colim}_V H^G_{\gamma+V+n}(\Gamma((X,A)\times(D(V),S(V)));T),$$

where $\Gamma(Y,B)=(\Gamma Y,\Gamma B)$ is the $G\mathcal{R}$-CW approximation of Theorem 3.9, and where the colimit runs over the finite-dimensional G-invariant subspaces of a G-universe. Cohomology is defined similarly, using the inverse limit.

REMARKS 4.7. (i) In view of Proposition 4.6(i), the need for taking (co)-limits over V might seem surprising. However, suspension by V does not in general preserve weak equivalence in $G\mathcal{R}\mathcal{U}$ even though it does in $G\mathcal{R}\mathcal{W}$. The reason for this is that components of fixed sets corresponding to virtual representations not equivalent to actual ones (modulo trivial representations) are not detected by weak homotopy type.

(ii) By the same token, suspension does preserve weak equivalence when γ has the property that

$$\gamma(x)\cong G\times_H V-\mathbf{R}^n \quad \text{for all } x.$$

Thus, by the technical condition introduced in the definition of $G\mathcal{R}\mathcal{U}$, the limit in the definition of cohomology is achieved, and so we will get long exact sequences of pairs. A more sophisticated approach might consider the *spectrum* $\{\Gamma((X,A)\times(D(V),S(V)))\}$, but this is beyond what we need for our purposes here.

(iii) If $\gamma=0$, then $H^G_{\gamma+n}(-;T)$ and $H_G^{\gamma+n}(-;T)$ coincide with Bredon homology and cohomology with twisted coefficients T.

The theories just described are now easily seen to exhibit the first three properties listed in Theorem A. We finish this section with some remarks on the relationship between our definition and the usual one when G is trivial. Thus, let X be a CW-complex, and let γ be a representation of πX, its fundamental groupoid. It is trivial to see that (X,γ) is an $\mathcal{R}$-CW complex in our sense. For simplicity, assume that X is connected, and choose a basepoint $x\in X$. Then γ is determined up to natural isomorphism by the homomorphism $w\colon \pi_1(X,x)\to \mathbf{Z}_2=\{1,-1\}$ given by $w(\lambda)=1$ if $\gamma(\lambda)$ preserves orientation, and by $w(\lambda)=-1$ if $\gamma(\lambda)$ reverses orientation. Now consider $C_n(X,\gamma)(x)$. This is the usual group of cellular n-chains of the universal cover $\tilde{X}$; however, the action of $\pi_1(X,x)$ on this group is slightly different from the usual. If $(c,\lambda)\mapsto c\lambda\colon C_n(X,\gamma)(x)\times\pi_1(X,x)\to C_n(X,\gamma)(x)$ denotes the usual permutation action, then the action we consider is $(c,\lambda)\mapsto w(\lambda)c\lambda$. If now T is a coefficient system on X, then T is determined up to isomorphism by the (right) $\pi_1(X,x)$-module $T(x)$, and the homology groups we construct are the homology of $C_*(X,\gamma)(x)\otimes_{\pi_1(X,x)}T(x)$, where we let $\pi_1(X,x)$ act on the left on $T(x)$ by $\lambda t=t\lambda^{-1}$. Comparing with [W1], we see that this is isomorphic to Wall's twisted homology $H^t_*(X;T)$, where w plays the same role as in [W1]. Similar remarks can be made about cohomology; although Wall does not use this case, one might denote this $H_t^*(X;T)$. Thus our construction is a generalization of Wall's construction. However, we consider his twisting information w to be part of the grading, and this is reflected in the action of πX on the chains.

5. Restriction to Subgroups

In this section we shall construct the two kinds of restriction mentioned in Theorem A: restriction to subgroups and restriction to fixed sets. To make clear what group we are talking about, we will write $C_*^G(X,A,\gamma)$ for the G-equivariant chain complex of (X,A,γ).

Let $K\subset G$ and let $\mathcal{K}$ be its orbit category. There is a functor $i:\mathcal{K}\to\mathcal{G}$ defined by $i(-)=G\times_K -$. If X is a G-space then there is a functor ι: $\pi(X;K)\to\pi(X;G)$ covering i and given by sending $x:K/J\to X$ to $G\times_K x$: $G/J\to X$; the definition on morphisms is similar. ι now extends to a functor $\hat{\iota}$: $\hat{\pi}(X;K)\to\hat{\pi}(X;G)$. If T: $\hat{\pi}(X;G)\to\mathcal{A}b$ is a coefficient system, then the composite $T\,|\,K=T\circ\hat{\iota}$ is a K-equivariant coefficient system. Composing natural transformations with $\hat{\iota}$ makes this construction a functor from $\hat{\pi}(X;G)$-groups to $\hat{\pi}(X;K)$-groups, which we call restriction to K.

In order to describe restriction of representations, let $hG\mathcal{O}_n$ be the category of n-dimensional orthogonal G-bundles over G-orbits and G-homotopy classes of linear maps (what we called $h\mathcal{O}_n$ in §2). Then there is a functor j: $hK\mathcal{O}_n\to hG\mathcal{O}_n$ given again by $G\times_K -$. The diagram

$$\begin{array}{ccc} hK\mathcal{O}_n & \xrightarrow{j} & hG\mathcal{O}_n \\ \downarrow & & \downarrow \\ \mathcal{K} & \xrightarrow{i} & \mathcal{G} \end{array}$$

is then a pullback diagram of categories. Thus if γ: $\pi(X;G)\to hG\mathcal{O}_n$ is a representation, the functors $\gamma\circ\iota$ and ϕ: $\pi(X;K)\to\mathcal{K}$ specify a map $\gamma\,|\,K$: $\pi(X;K)\to hK\mathcal{O}_n$.

We are now ready to define the restriction homomorphism

$$\rho: H_\gamma^G(X,A;T)\to H_{\gamma|K}^K(X,A;T\,|\,K)$$

of Theorem A, and its cohomological analog. First observe that, if (X,A,γ) is a $G\mathcal{R}$-CW pair, then the K-equivariant chains $C_n^K(X,A,\gamma\,|\,K)$ are naturally isomorphic to $C_n^G(X,A,\gamma)\,|\,K$. This gives a chain map

$$\begin{aligned}\mathrm{Hom}_{\hat{\pi}(X;G)}(C_n^G(X,A,\gamma),T) &\xrightarrow{\hat{\iota}^*} \mathrm{Hom}_{\hat{\pi}(X;K)}(C_n^G(X,A,\gamma)\,|\,K,T\,|\,K)\\ &\cong \mathrm{Hom}_{\hat{\pi}(X;K)}(C_n^K(X,A,\gamma\,|\,K),T\,|\,K),\end{aligned}$$

where $\hat{\iota}^*$ is composition with $\hat{\iota}$. This describes the restriction homomorphism for cohomology.

For homology, if M and N are $\hat{\pi}(X;G)$-groups then define

$$r: M\otimes_{\hat{\pi}(X;G)}N\to M\,|\,K\otimes_{\hat{\pi}(X;K)}N\,|\,K$$

as follows. If $x: G/J \to X$ is an object in $\hat{\pi}(X; G)$, decompose G/J as a union of K-orbits: $G/J = \amalg_\alpha K/J_\alpha$. Let $\eta_\alpha: G\times_K (K/J_\alpha) \to G/J$ be the restriction of the obvious projection $G\times_K (G/J) \to G/J$. If $m\otimes n \in M(x)\otimes N(x)$ then define

$$r(m\otimes n) = \sum_\alpha \eta_\alpha^*(m)\otimes \eta_\alpha^*(n),$$

where $\eta_\alpha^*(m) \in M(x\circ\eta_\alpha) = M\,|\,K(x\,|\,(K/J_\alpha))$. The desired homomorphism ρ is now obtained by applying r to $C_n^G(X,A,\gamma)\otimes_{\hat{\pi}(X;G)} T$. Notice that $\hat{\pi}(X;G)(-,x_0)\,|\,K \cong \Sigma_\alpha\, \hat{\pi}(X;K)(-,x_\alpha)$ if $x_0: G/J \to X$ and $x_\alpha = x_0\,|\,K/J_\alpha$, and that the map

$$\begin{aligned} T(x_0) &\cong \hat{\pi}(X;G)(-,x_0)\otimes_{\hat{\pi}(X;g)} T \xrightarrow{r} \hat{\pi}(X;G)(-,x_0)\,|\,K\otimes_{\hat{\pi}(X;K)} T\,|\,K \\ &\cong \sum_\alpha T(G\times_K x_\alpha) \end{aligned}$$

is given by $t \mapsto \Sigma_\alpha\, \eta_\alpha^*(t)$.

Proof of Theorem A(iv). The construction of ρ having been given above, it is easy to see that it is natural, and it remains only to check the Wirthmüller isomorphism. This may be checked first on cells, and then inductively up the skeleta using the usual long exact sequences. □

We now wish to construct the restriction ϕ. First, note that there is a natural functor $e: \hat{\pi}(X^K; WK) \to \hat{\pi}(X; G)$ given by inclusion of X^K in X and extension of NK-maps to G-maps. If T is a $\hat{\pi}(X; G)$-group then the composite $T\circ e$ is a $\hat{\pi}(X^K; WK)$-group. We define the coefficient system $T^K: \hat{\pi}(X^K; WK) \to \mathcal{A}b$ by taking $T^K(x) = T\circ e(x)/I(x)$, where, for $x: NK/L \to X$, $I(x)$ is the subgroup generated by elements of the form $(\sigma, c)_*(t)$ where $t \in T(x\circ\sigma)$, $\sigma: G/J \to G/L$ for a (proper) subgroup J of L not containing K, and c denotes the constant path at $x\circ\sigma$. If we do the same construction with the equivariant chains, we have the following result.

LEMMA 5.1. *$C_n^G(X,A,\gamma)^K \cong C_{n-|\gamma_K|}^{WK}(X^K, A^K, \gamma^K)$ as $\hat{\pi}(X^K; WK)$-groups.*

Here, $|\gamma_K| = |\gamma| - |\gamma^K|$, and is only locally constant.

Proof. Let x be an object of $\hat{\pi}(X^K; WK)$ so that $x: NK/L \to X^K$ with $L \supset K$, and as usual let $\gamma(e(x)) = G\times_L V - G\times_L W$. Consider the map $C_n^G(X,A,\gamma)(ex) \to C_{n-|\gamma_K|}^{WK}(X^K, A^K, \gamma^K)(x)$ given by taking a stable G-map

$$f: G\times_L S^{V+n} \to \tilde{X}^n(ex)/\tilde{X}^{n-1}(ex)\wedge_{G/L} G\times_L S^{W+|\gamma|}$$

to the stable WK-map

$$f^K: NK\times_L S^{V^K+n} \to (\tilde{X}^K)^{n-|\gamma_K|}(x)/(\tilde{X}^K)^{n-|\gamma_K|-1}(x)\wedge_{NK/L} NK\times_L S^{W^K+|\gamma|}$$

given by restricting f to the fiber over the identity coset of L and taking K-fixed points. Here we use the WK-homotopy equivalences $\tilde{X}^n(ex)^K \simeq (\tilde{X}^K)^{n-|\gamma_K|}(x)$. Using the suspension isomorphism

$$
\begin{aligned}
&C^{WK}_{n-|\gamma_K|}(X^K, A^K, \gamma^K)(x)\\
&\quad\cong \{S^{V^K+n-|\gamma_K|}, (\tilde{X}^K)^{n-|\gamma_K|}(x)/(\tilde{X}^K)^{n-|\gamma_K|-1}(x)\wedge S^{W^K+|\gamma^K|}\}_L\\
&\quad\cong \{S^{V^K+n}, (\tilde{X}^K)^{n-|\gamma_K|}(x)/(\tilde{X}^K)^{n-|\gamma_K|-1}(x)\wedge S^{W^K+|\gamma|}\}_L,
\end{aligned}
$$

we have indeed a map $C_n^G(X, A, \gamma)(ex) \to C^{WK}_{n-|\gamma_K|}(X^K, A^K, \gamma^K)(x)$. In fact, this defines a map of $\hat{\pi}(X^K; WK)$-groups.

This map is onto, by the following argument. Let

$$f'\colon S^{V^K+n-|\gamma_K|} \to (\tilde{X}^K)^{n-|\gamma_K|}(x)/(\tilde{X}^K)^{n-|\gamma_K|-1}(x)\wedge S^{W^K+|\gamma^K|}$$

be any stable L-map. The NK-homotopy equivalence

$$(\tilde{X}^K)^{n-|\gamma_K|}(x)/(\tilde{X}^K)^{n-|\gamma_K|-1}(x) \simeq \tilde{X}^n(ex)^K/\tilde{X}^{n-1}(ex)^K$$

gives us a stable L-map $S^{V^K+n-|\gamma_K|} \to \tilde{X}^n(ex)^K/\tilde{X}^{n-1}(ex)^K \wedge S^{W^K+|\gamma^K|}$, which extends to a stable G-map

$$f''\colon G\times_L S^{V^K+n-|\gamma_K|} \to \tilde{X}^n(ex)/\tilde{X}^{n-1}(ex)\wedge_{G/L} G\times_L S^{W+|\gamma^K|},$$

and hence $f'''\colon G\times_L S^{V^K+n} \to \tilde{X}^n(ex)/\tilde{X}^{n-1}(ex)\wedge_{G/L} G\times_L S^{W+|\gamma|}$. The projection of f''' onto $X^n/X^{n-1}\wedge G\times_L S^{W+|\gamma|}$ maps the K-fixed set of S^{V+n} into that of isomorphic spheres, and can therefore be extended normally to all of S^{V+n}. This now gives an element

$$f\colon G\times_L S^{V+n} \to \tilde{X}^n(x)/\tilde{X}^{n-1}(x)\wedge_{G/L} G\times_L S^{W+|\gamma|}$$

of $C_n^G(X, A, \gamma)(ex)$ such that $f^K = f'$ in $C^{WK}_{n-|\gamma_K|}(X^K, A^K, \gamma^K)(x)$.

The kernel of this map is exactly $I(x) \subset C_n^G(X, A, \gamma)(ex)$. For suppose that $f_1^K = f_2^K$; then f_1 and f_2 differ by elements of the Burnside ring of L induced from the Burnside rings of subgroups of L not containing K, and hence $f_1 - f_2 \in I(x)$. Alternatively, one can use the transversality argument of Lemma 4.2 to show that, if $f_1^K = 0$, then f_1 is a sum of maps induced by imbeddings of orbits L/J in $V\oplus \mathbf{R}^n$, for subgroups J of L not containing K.

Therefore, $C_n^G(X, A, \gamma)^K \cong C^{WK}_{n-|\gamma_K|}(X^K, A^K, \gamma^K)$ as claimed. □

We can now construct ϕ in cohomology. It is induced by the chain level map $\phi\colon \mathrm{Hom}_{\hat{\pi}(X;G)}(C_n^G(X, A, \gamma), T) \to \mathrm{Hom}_{\hat{\pi}(X^K;WK)}(C^{WK}_{n-|\gamma_K|}(X^K, A^K, \gamma^K), T^K)$ given by $\phi(F) = F^K$, using Lemma 5.1.

For homology, we need to consider the map

$$z\colon A\otimes_{\hat{\pi}(X;G)} B \to A^K \otimes_{\hat{\pi}(X^K;WK)} B^K$$

given as follows. Let $a\otimes b \in A(y)\otimes B(y)$, where $y\colon G/J \to X$. Decompose $(G/J)^K$ into NK-orbits; $(G/J)^K = \amalg_\alpha NK/J_\alpha$. Let $\eta_\alpha\colon G/J_\alpha \to G/J$ be the evident extension of the inclusion, and let

$$z(a\otimes b) = \sum_\alpha \eta_\alpha^*(a)^K \otimes \eta_\alpha^*(b)^K.$$

Here the superscript K denotes the reduction map $T(ex) \to T^K(x)$. One can check that this is well defined, and this uses the fact that we have reduced mod I. ϕ is the map induced by the chain level map

$$\phi: C_n^G(X,A,\gamma)\otimes_{\hat{\pi}(X;G)} T \to C_{n-|\gamma_K|}^{WK}(X^K,A^K,\gamma^K)\otimes_{\hat{\pi}(X^K;WK)} T^K$$

given by z and Lemma 5.1. Theorem A(v) can now be checked.

6. Multiplicative Structure

In order to describe cup and cap products in the theories we have constructed, it is easiest to allow arbitrary G-sets in places where we have heretofore allowed only G-orbits. Precisely, we make the following definitions.

DEFINITION 6.1. Let $\mathcal{G}^+$ be a small category of G-sets and G-maps that contains $\mathcal{G}$ and is closed under disjoint union and product. If X is a G-space, let π^+X be the groupoid over $\mathcal{G}^+$ formed in the same way as πX, except that its objects are G-maps $A\to X$, where A can be any G-set in $\mathcal{G}^+$.

We can now form the categories $\hat{\mathcal{G}}^+$ and $\hat{\pi}^+X$ in essentially the same way as before, except that we make the identifications $[x\leftarrow y\to z]+[x\leftarrow y'\to z]=[x\leftarrow y\amalg y'\to z]$. Here, if $y:A\to X$ and $y':A'\to X$, then $y\amalg y':A\amalg A'\to X$ is the obvious object of $\hat{\pi}^+X$. We define a *$\hat{\pi}^+X$-group* to be a contravariant additive functor $\hat{\pi}^+X\to\mathcal{Ab}$ that takes disjoint unions to direct sums. It is familiar from the theory of Mackey functors [L] that this last requirement makes the categories of $\hat{\pi}X$-groups and $\hat{\pi}^+X$-groups equivalent.

In order to define cup products, we will need an internal tensor product of $\hat{\pi}^+X$-groups. For this and later constructions we need the following notation: If $x:A\to X$ and $y:B\to Y$ then $x\times y$ will denote the map $x\times y:A\times B\to X\times Y$. Notice that objects of $\pi^+(X\times Y)$ do not necessarily have this form, but are given by pairs (x,y) where $x:A\to X$ and $y:A\to Y$ for the same G-set A.

DEFINITION 6.2. If S is a $\hat{\pi}^+X$-group and T is a $\hat{\pi}^+Y$-group, then the $\hat{\pi}^+(X\times Y)$-group $S\,\square\,T$ is defined by

$$(S\,\square\,T)(z)=\sum_{z\to x\times y} S(x)\otimes T(y)/\approx,$$

where $(s\otimes t)_{z\to x'\times y'\xrightarrow{f\times g}x\times y}\approx(f^*s\otimes g^*t)_{z\to x'\times y'}$.

The importance to us of this "box product" comes from the easily checked observation that

$$\hat{\pi}^+X(-,x_0)\,\square\,\hat{\pi}^+Y(-,y_0)\cong\hat{\pi}^+(X\times Y)(-,x_0\times y_0),$$

from which it follows that $C_*(X\times Y,\gamma+\delta)\cong C_*(X,\gamma)\,\square\,C_*(Y,\delta)$. Moreover, it is easy to write down an explicit isomorphism, given from right to left, by taking smash products of maps. (Note that, if (X,γ) and (Y,δ) are $G\mathcal{R}$-CW complexes then $(X\times Y,\gamma+\delta)$ has an obvious $G\mathcal{R}$-CW structure, and this is the one we use. This is simplified somewhat if we allow cells to have the form of disc bundles over finite G-sets, rather than orbits, so that a cell in X and one in Y give a product cell in $X\times Y$.) The analogous statement holds for products of pairs.

This much allows us to define the external cup product in cohomology: Let S be a $\hat{\pi}^+X$-group and let T be a $\hat{\pi}^+Y$-group. The chain level pairing

$$\begin{aligned}&\mathrm{Hom}_{\hat{\pi}X}(C_*(X,A,\gamma),S)\otimes\mathrm{Hom}_{\hat{\pi}Y}(C_*(Y,B,\delta),T)\\ &\quad\to\mathrm{Hom}_{\hat{\pi}X\times Y}(C_*((X,A)\times(Y,B),\gamma+\delta),S\,\square\,T)\end{aligned}$$

given by $E\otimes F\mapsto E\,\square\,F$ passes to cohomology to give the external cup product

$$-\cup-:H^{\gamma}_G(X,A;S)\otimes H^{\delta}_G(Y,B;T)\to H^{\gamma+\delta}_G((X,A)\times(Y,B);S\,\square\,T).$$

To internalize this product we need a little more algebra. Let $\Delta: X\to X\times X$ denote the diagonal map. This induces the diagonal $\Delta:\hat{\pi}^+X\to\hat{\pi}^+(X\times X)$, and composition with Δ takes any $\hat{\pi}^+(X\times X)$-group U to the $\hat{\pi}^+X$-group Δ^*U. In particular, if T is a $\hat{\pi}^+X$-group, then $\Delta^*(T\,\square\,T)$ is also a $\hat{\pi}^+X$-group. We say that T is a *$\hat{\pi}^+X$-ring* if there is a homomorphism $\mu:\Delta^*(T\,\square\,T)\to T$ satisfying the usual associativity diagram. This is made clearer by noting (from the definitions) that μ is determined by the products $T(x)\otimes T(x)\to T(x)$ given by looking at elements in $\Delta^*(T\,\square\,T)$ indexed by the diagonal map $(x,x)\to x\times x$. This makes each $T(x)$ a ring, and the ring structures are related by certain naturality conditions; this is, in other words, a generalization of the definition of a *Green functor* on $\mathcal{G}^+$ [L].

If now T is a $\hat{\pi}^+X$-ring, then we have the internal cup product

$$-\cup-:H^{\gamma}_G(X,A;T)\otimes H^{\delta}_G(X,B;T)\to H^{\gamma+\delta}_G(X,A\cup B;T),$$

which is obtained from the external product by restricting along the diagonal and using the multiplication of T.

Finally, we would like to have a unit for this multiplication. For this, let $\mathfrak{A}_G=\phi^*\hat{\mathcal{G}}^+(-,G/G)$, so that $\mathfrak{A}_G(x)=\hat{\mathcal{G}}^+(\phi(x),G/G)$; if $\phi(x)=G/H$ then $\mathfrak{A}_G(x)$ is the Burnside ring of H. We call $\mathfrak{A}_G$ the *Burnside ring coefficient system;* it plays much the same role in our theory as $\mathbf{Z}$ does nonequivariantly. If T is any $\hat{\pi}^+X$-group, then there is a homomorphism $\Delta^*(\mathfrak{A}_G\,\square\,T)\to T$ defined as follows. Suppose that $(f\otimes t)_{(x,x)\to a\times b}\in\Delta^*(\mathfrak{A}_G\,\square\,T)(x)$, so that $f\otimes t\in\hat{\mathcal{G}}^+(\phi(a),G/G)\otimes T(b)$. Write $\phi(a)\times b$ for the map $b\circ p_2:\phi(a)\times\phi(b)\to\phi(b)\to X$; this is also the projection of $a\times b$ onto the second factor of X. Then f defines a map $f\times 1:\phi(a)\times b\to b$ in $\hat{\pi}^+X$ and the given $(x,x)\to a\times b$ defines a map $g:x\to\phi(a)\times b$ (project onto the second factor of X). We send $f\otimes t$ to $g^*(f\times 1)^*(t)\in T(x)$. This map has the property that $(1_{\phi(x)}\otimes t)_{\Delta:(x,x)\to x\times x}\mapsto t$. There is a similar map $\Delta^*(T\,\square\,\mathfrak{A}_G)\to T$. We now say that a $\hat{\pi}^+X$-ring is *unital* if there is a homomorphism $\eta:\mathfrak{A}_G\to T$ such that the diagrams

$$\begin{array}{ccc}\Delta^*(\mathfrak{A}_G\,\square\,T) & \longrightarrow & \Delta^*(T\,\square\,T)\\ & \searrow\quad\swarrow & \\ & T & \end{array}\qquad\text{and}\qquad\begin{array}{ccc}\Delta^*(T\,\square\,\mathfrak{A}_G) & \longrightarrow & \Delta^*(T\,\square\,T)\\ & \searrow\quad\swarrow & \\ & T & \end{array}$$

both commute. With the map $\Delta^*(\mathfrak{A}_G\,\square\,\mathfrak{A}_G)\to\mathfrak{A}_G$ constructed above, one can check that $\mathfrak{A}_G$ in particular is a commutative unital $\hat{\pi}^+X$-ring, and that any $\hat{\pi}^+X$-group is naturally a module over $\mathfrak{A}_G$.

The cup products satisfy the following properties:

PROPOSITION 6.3. *The external and internal cup products are natural in all variables. If T is a unital $\hat{\pi}^+X$-ring, then there is a unit $1 \in H_G^0(X;T)$ for the internal product. Further, these products are respected by the restrictions of Theorem A; that is, if $K \subset G$ then $(\alpha \cup \beta)\,|\,K = (\alpha\,|\,K) \cup (\beta\,|\,K)$ and $(\alpha \cup \beta)^K = \alpha^K \cup \beta^K$.*

Proof. Naturality is obvious from the definitions. The unit is the image under η_* of a unit $1 \in H_G^0(X;\mathcal{A}_G)$ which is in turn the image of the unit $1 \in H_G^0(*;\mathcal{A}_G) = A(G)$ (the Burnside ring of G) under the map induced by the projection $X \to *$. That this is a unit for the internal cup product now follows by first checking the external product (of X and $*$) and then using an obvious naturality argument.

That ρ respects the product follows from the definitions and the natural isomorphism $(S \,\square\, T)\,|\,K \cong (S\,|\,K) \,\square\, (T\,|\,K)$. Similarly, that ϕ respects the product follows from the definitions and the fact that $(S \,\square\, T)^K \cong S^K \,\square\, T^K$. □

Now we wish to construct a cap product making cohomology act on homology. We first record that we have an evaluation homomorphism.

PROPOSITION 6.4. *There is a natural homomorphism*

$$\langle -, - \rangle : H_G^\gamma(X,A;S) \otimes H_\gamma^G(X,A;T) \to S \otimes_{\hat{\pi}X} T$$

extending the usual evaluation homomorphism of the nonequivariant theory. It is respected by restriction to subgroups and fixed sets in the sense that

$$r\langle \alpha, a \rangle = \langle \alpha\,|\,K, a\,|\,K \rangle \quad \textit{and} \quad z\langle \alpha, a \rangle = \langle \alpha^K, a^K \rangle,$$

where $r: S \otimes_{\hat{\pi}(X;G)} T \to S\,|\,K \otimes_{\hat{\pi}(X;K)} T\,|\,K$ and $z: S \otimes_{\hat{\pi}X} T \to S^K \otimes_{\pi X^K} T^K$ are the homomorphisms defined in Section 5.

Proof. On the chain level, this is defined by

$$\mathrm{Hom}_{\hat{\pi}X}(C_n(X,A,\gamma), S) \otimes (C_n(X,A,\gamma) \otimes_{\hat{\pi}X} T) \to S \otimes_{\hat{\pi}X} T,$$

given by $F \otimes c \otimes t \mapsto F(c) \otimes t$. The stated properties of the evaluation can now be checked from the definitions. □

The cap product comes from a generalization of this evaluation. Some preliminary algebra is necessary. Suppose that U is a $\hat{\pi}^+(X \times Y)$-group and that T is a $\hat{\pi}^+Y$-group. Then we define the $\hat{\pi}^+X$-group $T \otimes_{\hat{\pi}Y} U$ by

$$(T \otimes_{\hat{\pi}Y} U)(x) = T \otimes_{\hat{\pi}Y} U(x \times -)$$

where, on the right, we regard $U(x \times -)$ as a $\hat{\pi}^+Y$-group and use $\otimes_{\hat{\pi}Y}$ in its previous sense. One of the properties of this tensor is

$$(S \,\square\, T) \otimes_{\hat{\pi}X \times Y} U \cong S \otimes_{\hat{\pi}X} (T \otimes_{\hat{\pi}Y} U);$$

this follows by playing with the definitions. Another piece of algebra we need is this: Suppose that $f: X \to Y$ is a G-map. Given a $\hat{\pi}^+Y$-group T, we can form the $\hat{\pi}^+X$-group f^*T by composing with the induced map $\hat{\pi}^+X \to \hat{\pi}^+Y$. There is a left adjoint to this construction. Given a $\hat{\pi}^+X$-group S, we can form a $\hat{\pi}^+Y$-group f_*S by letting

$$(f_*S)(y) = \sum_{y \to fx} S(x)/\approx,$$

where $s_{y \to fx' \xrightarrow{fh} fx} \approx (h^*s)_{y \to fx'}$. The homomorphisms of $\hat{\pi}^+X$-groups $S \to f^*T$ are in one-to-one correspondence with the homomorphisms of $\hat{\pi}^+Y$-groups $f_*S \to T$. The case that interests us here is Δ_*S, where $\Delta: X \to X \times X$ is the diagonal. Δ_*S is just S along the diagonal, and is extended to other points in $X \times X$ in the minimal way possible. Thus, if $(x, y): G/H \to X \times Y$ can be connected by a path to a point in the diagonal (i.e., if x and y can be connected by a path), then (x, y) gets assigned the same group that S assigns to that diagonal point, which is the same group assigned to x and to y.

We can now define a generalized evaluation

$$\langle -, - \rangle : H_G^\delta(Y, B; T) \otimes H^G_{\gamma+\delta}((X, A) \times (Y, B); U) \to H^G_\gamma(X, A; T \otimes_{\hat{\pi}Y} U).$$

It is given on the chain level by

$$\begin{aligned}
&\mathrm{Hom}_{\hat{\pi}Y}(C_n(Y, B, \delta), T) \otimes (C_{m+n}((X, A) \times (Y, B), \gamma + \delta) \otimes_{\hat{\pi}X \times Y} U) \\
&\quad \to \mathrm{Hom}_{\hat{\pi}Y}(C_n(Y, B, \delta), T) \otimes ((C_m(X, A, \gamma) \,\Box\, C_n(Y, B, \delta)) \otimes_{\hat{\pi}X \times Y} U) \\
&\quad \cong \mathrm{Hom}_{\hat{\pi}Y}(C_n(Y, B, \delta), T) \otimes (C_m(X, A, \gamma) \otimes_{\hat{\pi}X} (C_n(Y, B, \delta) \otimes_{\hat{\pi}Y} U)) \\
&\quad \to C_m(X, A, \gamma) \otimes_{\hat{\pi}X} (T \otimes_{\hat{\pi}Y} U),
\end{aligned}$$

the last step being given by evaluation. We define the cap product

$$- \cap - : H_G^\delta(X, B; S) \otimes H^G_{\gamma+\delta}(X, A \cup B; T) \to H^G_\gamma(X, A; S \otimes_{\hat{\pi}X} \Delta_*T)$$

by $\alpha \cap a = \langle \alpha, \Delta_* a \rangle$, where

$$\Delta_* : H^G_*(X, A \cup B; T) \to H^G_*((X, A) \times (X, B); \Delta_*T)$$

is induced by the diagonal map and the map $T \to \Delta^*\Delta_*T$ (which is actually an isomorphism) adjoint to the identity $\Delta_*T \to \Delta_*T$.

To simplify the coefficients of the result, one can note the following: If $S = \mathfrak{A}_G$ then there is a natural homomorphism $\mathfrak{A}_G \otimes_{\hat{\pi}X} \Delta_*T \to T$ which comes about from the isomorphism

$$(\mathfrak{A}_G \otimes_{\hat{\pi}X} \Delta_*T)(x) \cong \Delta^*(\mathfrak{A}_G \,\Box\, \hat{\pi}^+X(-, x)) \otimes_{\hat{\pi}X} T$$

and the map $\Delta^*(\mathfrak{A}_G \,\Box\, \hat{\pi}^+X(-, x)) \to \hat{\pi}^+X(-, x)$ constructed earlier. In particular, the map $\mathfrak{A}_G \otimes_{\hat{\pi}X} \Delta_*\mathfrak{A}_G \to \mathfrak{A}_G$ can, with a little effort, be shown to be an isomorphism.

PROPOSITION 6.5. *The cap product is natural in the usual sense and satisfies*

$$(\alpha \cup \beta) \cap a = \alpha \cap (\beta \cap a);$$

with $\mathfrak{A}_G$ coefficients $1 \cap a = a$. Further, the cap product is respected by the restrictions, in the sense that

$$(\alpha \cap a) \mid K = (\alpha \mid K) \cap (a \mid K) \quad \textit{and} \quad (\alpha \cap a)^K = \alpha^K \cap a^K.$$

Proof. Naturality and associativity are straightforward from the definition; in showing associativity we use the isomorphism $\Delta^* S \otimes_{\hat{\pi} X} T \cong S \otimes_{\hat{\pi} X \times X} \Delta_* T$ for appropriate S and T. The action of the unit is seen as follows: Let $p: X \to *$ and let $q: X \times X \to X$ be projection onto the first factor. Then

$$1 \cap a = \langle 1, \Delta_* a \rangle = \langle p^* 1, \Delta_* a \rangle = \langle 1, q_* \Delta_* a \rangle = \langle 1, a \rangle = a,$$

since $q \circ \Delta$ is the identity.

That ρ and ϕ respect the cap product is easy to check, as it was for evaluation. □

7. Poincaré Duality

We now show that arbitrary G-manifolds exhibit Poincaré duality in the theory we have constructed, using the line of argument in [MS]. Note that, since twisting information is built in to the grading, we shall make no assumptions about orientability, nor shall we need to use twisted coefficients. Throughout we shall use Burnside ring coefficients, so we write $H_*^G(X)$ for $H_*^G(X; \mathfrak{A}_G)$, and so on. Until the end of this section we will deal strictly with manifolds without boundary. We start with a technical lemma.

LEMMA 7.1. *Let M be a G-manifold, let τ be the representation of πM associated to the tangent bundle, and let $K \subset M$ be a compact G-invariant subset. Then*

(i) $H_{\tau+n}^G(M, M-K) = 0$ *if* $n > 0$, *and*

(ii) *a class* $\alpha \in H_\tau^G(M, M-K)$ *is* 0 *if and only if the restriction* $i^*\alpha \in H_\tau^G(M, M-s)$ *is zero for every finite G-set* $s \subset K$, *where* $i: (M, M-s) \hookrightarrow (M, M-K)$ *is the inclusion.*

Proof. We prove the lemma by considering a succession of cases. For the first, we introduce the following terminology: An invariant subset $C \subset G \times_K V$ of the total space of a vector bundle is *elementary* if it is the finite disjoint union of convex subsets. It follows that C has a finite G-set as a G-deformation retract.

Case 1: $M = G \times_J V$ and $K \subset M$ elementary. As mentioned above, K has some finite G-set s as a deformation retract, so the restriction

$$H_*^G(M, M-K) \to H_*^G(M, M-s)$$

is an isomorphism. (ii) follows immediately. (i) follows from excision and the easy computation $H_{\tau+n}^G(D(\tau(s)), S(\tau(s))) = 0$ for $n > 0$ (here, as in §6, we think of πM as having objects the maps from G-sets into M).

Case 2: M arbitrary and $K = K_1 \cup K_2$, where the lemma is known to be true for K_1, K_2, and $K_1 \cap K_2$. This is an easy consequence of the Mayer–Vietoris exact sequence of the triad $(M; M-K_1, M-K_2)$.

Case 3: K any G-invariant compact subset of $M = G \times_J V$. Let $\alpha \in H^G_{\tau+n}(M, M-K)$, where $n \geq 0$. We first claim that there exists an invariant open neighborhood U of K and an element $\beta \in H^G_{\tau+n}(M, M-U)$ such that β restricts to α. This follows from the fact that $M-K$ is σ-compact, which implies that $H^G_*(M-K) \cong \operatorname{colim} H^G_*(C)$, the colimit running over the compact subsets of $M-K$ (take a sequence $C_1 \subset C_2 \subset \cdots$ whose union is $M-K$, and inductively construct CW approximations); a diagram chase comparing the long exact sequence of $(M, M-K)$ and (M, C) now gets us β. Cover K by a finite collection $\{K_1, \ldots, K_r\}$ of elementary subspaces of M contained in U, so that if $L = K_1 \cup \cdots \cup K_r$ then $K \subset L \subset U$; let $\gamma \in H^G_{\tau+n}(M, M-L)$ be the restriction of β.

If $n > 0$ then $\gamma = 0$ by the first two cases, and so $\alpha = 0$, showing (i). For (ii), we can suppose that every component of every fixed set of K_i meets K, so that the hypothesis of (ii) holds for γ, and the first two cases again imply that $\gamma = 0$, and so $\alpha = 0$.

Case 4: M and K arbitrary. By the compactness of K, we can assume that $K = K_1 \cup \cdots \cup K_r$ where each K_i is contained in a neighborhood of the form $G \times_J V$. The result now follows by excision and cases 2 and 3. □

For the following definition, note that, if $s \subset M$ is a G-orbit G/J, then $H^G_\tau(M, M-s) \cong A(J)$, the Burnside ring of J.

DEFINITION 7.2. Let s be a G-orbit in M. Then a *local homological orientation of M at s* is an element $\mu \in H^G_\tau(M, M-s)$ generating this group as a free module over the Burnside ring of J, $s \cong G/J$. A *global homological orientation of M* is a collection of elements $\mu_K \in H^G_\tau(M, M-K)$, one for each G-invariant compact subset $K \subset M$, that are compatible under the restrictions induced by the inclusions $K_1 \subset K_2$, and have the property that, for each G-orbit s in M, μ_s is a local homological orientation at s. If M itself is compact, we write $[M] = \mu_M \in H^G_\tau(M)$ and call this element a *fundamental class for M*.

THEOREM 7.3. *The global homological orientations of M are in one-to-one correspondence with the virtual spherical self-equivalences of τ covering the identity of πM. In particular, M has a canonical global homological orientation corresponding to the identity map $\tau \to \tau$.*

Proof. Suppose we are given a virtual spherical equivalence $\xi: \tau \to \tau$ covering the identity of πM. To produce a global homological orientation of M it suffices, by Lemma 7.1 and a standard Mayer–Vietoris argument, to produce the classes μ_K for compact K of the form $D(\tau(s))$, where s is a

G-orbit G/J in M. Combining excision, the exponential map and ξ gives an isomorphism

$$H_\tau^G(M, M-K) \cong H_\tau^G(D(\tau(s)), S(\tau(s))) \cong A(J),$$

and we let μ_K be the element corresponding to $1 \in A(J)$. The naturality of ξ implies that this does determine a global homological orientation.

Conversely, suppose we are given the classes μ_K. If s is a G-set G/J in M, then μ_s is a generator of $H_\tau^G(D(\tau(s)), S(\tau(s))) \cong A(J)$ via the exponential map. μ_s therefore gives us a virtual spherical equivalence $\xi(s): \tau(s) \to \tau(s)$. The compatibility of the classes μ_K implies that ξ is a natural transformation, and the one-to-one correspondence between generators of the Burnside ring and virtual spherical equivalences shows that the two constructions above are inverses. □

As in [MS], we will express Poincaré duality in terms of "cohomology with compact support."

DEFINITION 7.4. If X is any G-space, let

$$\mathcal{H}_G^\gamma(X) = \operatorname*{colim}_K H_G^\gamma(X, X-K),$$

where the colimit runs over the G-invariant compact subsets of X.

Note, of course, that if X is compact then $\mathcal{H}_G^\gamma(X) = H_G^\gamma(X)$. Now if $\{\mu_K\}$ is a global homological orientation of M, then we have maps

$$-\cap \mu_K : H_G^\delta(M, M-K) \to H_{\tau-\delta}^G(M)$$

which, by the compatibility of the μ_K, pass to the colimit to give a map

$$-\cap [M] : \mathcal{H}_G^\delta(M) \to H_{\tau-\delta}^G(M).$$

THEOREM 7.5 (Poincaré duality). $-\cap[M]$ *is an isomorphism for any G-manifold M.*

Proof. The proof is another Mayer–Vietoris argument, essentially the same as the one in [MS, §A.9]. The only point at which the proof differs is the local case, so we assume that $M = G \times_J V$. In this case, the collection of closed discs $G \times_J D_r(V)$ of varying radii r is cofinal in the collection of invariant compact subsets, and by excision one sees that

$$\mathcal{H}_G^\delta(M) \cong H_G^\delta(G \times_J D(V), G \times_J S(V)).$$

The result now follows from the commutative diagram

$$\begin{array}{ccc} H_G^{\delta-\tau}(G/J) & \xrightarrow[\cong]{-\cup t} & H_G^\delta(G \times_J D(V), G \times_J S(V)) \\ {\scriptstyle -\cap 1}\downarrow{\scriptstyle \cong} & & {\scriptstyle -\cap[M]}\downarrow \\ H_{\tau-\delta}^G(G/J) & \xrightarrow{\cong} & H_{\tau-\delta}^G(G \times_J D(V)), \end{array}$$

where t is the generator of $H_G^\tau(G \times_J D(V), G \times_J S(V)) \cong A(J)$. □

THEOREM 7.6 (Addendum to Poincaré duality). *If $\{\mu_K\}$ is a global homological orientation of the G-manifold M and $H \subset G$ is a subgroup, then $\{\mu_K \mid H\}$ determines a global homological orientation of M as an H-manifold, and $\{(\mu_K)^H\}$ determines a global homological orientation of M^H.*

Proof. These observations follow from the local case: If μ is a local orientation at the orbit $s \cong G/J$ in M, then $\mu \mid H$ is a local orientation at the H-set s and μ^H is a local orientation at s^H. For the first, suppose that $t \subset s$ is an H-orbit H/L; then restriction to H followed by restriction to t induces the restriction $A(J) \to A(L)$ given by the inclusion of a conjugate of L in J. This restriction is a ring homomorphism, so takes units to units, so takes μ to a generator. For the second, let $t \subset s^H$ be a WH-orbit; $t \cong WH/L$. t determines the inclusion of H in $g^{-1}Jg$, and $L = (NH \cap g^{-1}Jg)/H$. Restriction to H-fixed sets followed by restriction to t induces the map $A(J) \to A(L)$ given by regarding a finite J-set as a $g^{-1}Jg$-set, taking the H-fixed set and regarding that as an L-set. This is another ring homomorphism. □

It is worth noting the following special case.

COROLLARY 7.7. *If M is a closed G-manifold and $[M]$ is a fundamental class for M, then*

$$-\cap [M]: H_G^\delta(M) \to H_{\tau-\delta}^G(M)$$

is an isomorphism. Further, $[M] \mid K$ is a fundamental class for M as a K-manifold, and $[M]^K$ is a fundamental class for M^K. □

In a similar vein, one can prove relative, or Lefschetz, duality. We state the result without proof. To explain the statement, suppose that M is a G-manifold with boundary, and that $\{\mu_K\}$ is a global homological orientation of $M - \partial M$. Let U be a collar neighborhood of ∂M. Then

$$H_\tau^G(M, \partial M \cup (M-K)) \cong H_\tau^G(M, M-(K-U)),$$

and so we can consider the element $\mu_{K-U} \in H_\tau^G(M, \partial M \cup (M-K))$. Cap product with these elements defines a homomorphism

$$-\cap [M, \partial M]: \mathcal{H}_G^\delta(M) \to H_{\tau-\delta}^G(M, \partial M).$$

On the other hand, if we let $\mathcal{H}_G^\delta(M, \partial M) = \operatorname{colim}_K H_G^\delta(M, \partial M \cup (M-K))$, then cap product with the elements described above defines

$$-\cap [M, \partial M]: \mathcal{H}_G^\delta(M, \partial M) \to H_{\tau-\delta}^G(M).$$

THEOREM 7.8 (Lefschetz duality). *If M is a G-manifold with boundary, and if $\{\mu_K\}$ is a global homological orientation of $M - \partial M$, then*

$$-\cap [M, \partial M]: \mathcal{H}_G^\delta(M) \to H_{\tau-\delta}^G(M, \partial M)$$

and

$$-\cap [M, \partial M]: \mathcal{H}_G^\delta(M, \partial M) \to H_{\tau-\delta}^G(M)$$

are isomorphisms.

In the case where M is compact, consider $K = M - U$ where U is a collar neighborhood of ∂M; by excision $H^G_\tau(M, \partial M) \cong H^G_\tau(M, M-K)$ and $-\cap[M, \partial M]$ is given by $-\cap \mu_K$, so we define $[M, \partial M] \in H^G_\tau(M, \partial M)$ to be the element corresponding to μ_K.

It follows from Theorem 7.6 that Lefschetz duality is also respected by restriction to subgroups and fixed sets.

8. The Thom Isomorphism

Here we give a quick proof of the Thom isomorphism, along the lines of our proof of Poincaré duality.

DEFINITION 8.1. Suppose that ξ is a G-vector bundle over X and ρ is the corresponding representation of the fundamental groupoid of X. Then a *Thom class for* ξ is an element $t_\xi \in H^\rho_G(D(\xi), S(\xi); \mathfrak{A}_G)$ such that, for each G-map $x: G/K \to X$, $\bar{x}^*(t_\xi) \in H^\rho_G(D(x^*\xi), S(x^*\xi); \mathfrak{A}_G) \cong A(K)$ is a generator, where $\bar{x}$ is the canonical map $x^*\xi \to \xi$.

The following two results prove Theorem C.

THEOREM 8.2. *For a given G-vector bundle ξ over X, Thom classes $t \in H^\rho_G(D(\xi), S(\xi); \mathfrak{A}_G)$ are in one-to-one correspondence with stable spherical self-maps of ρ covering the identity on πX. Moreover, multiplication by any Thom class t induces an isomorphism*

$$H^\gamma_G(X; T) \to H^{\gamma+\rho}_G(D(\xi), S(\xi); T).$$

Proof. That each Thom class t gives a stable spherical self-map of ρ follows from the definitions. The converse, and the fact that multiplication by Thom classes yields an isomorphism, is clear in the special case where ξ is induced from a G-vector bundle over an orbit. The general result follows by induction on skeleta using standard Mayer–Vietoris patching arguments. □

THEOREM 8.3. *If $t \in H^\rho_G(D(\xi), S(\xi); \mathfrak{A}_G)$ is a Thom class for ξ and $K \subset G$, then $t_\xi | K$ is a Thom class for ξ as a K-bundle, and t^K_ξ is a Thom class for ξ^K.*

Proof. A Thom class is defined in terms of its local properties; as in the proof of Theorem 7.6, generators of the cohomology of spheres are taken to generators of the cohomology of spheres under either kind of restriction. □

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