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EQUIVARIANT POINT THEOREMS

(Dedicated to Professor A. Komatu on his 70th birthday)

By

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1. Introduction.

This paper is a continuation of my previous paper [13], and is concerned with generalizations of the following two classical theorems on a continuous map f of an n-sphere S^n to itself.

THEOREM 1.1. *If the degree of* f *is even then there exists* $x\in S^n$ *such that* $f(-x)=f(x)$.

THEOREM 1.2. *If the degree of* f *is odd then there exists* $x\in S^n$ *such that* $f(-x)=-f(x)$.

Throughout this paper, a prime p is fixed, and $G=\{1, T, \cdots, T^{p-1}\}$ will denote a cyclic group of order p.

Generalizing the situation in the above theorems, we shall consider the following problems.

PROBLEM 1. *Let* $f: N\rightarrow M$ *be a continuous map between G-spaces. Under what conditions does* f *have an equivariant point, i.e., a point* $x\in N$ *such that*

$$(1.1)\qquad f(T^i x)=T^i f(x)$$

for $i=1,2,\cdots, p-1$?

PROBLEM 2. *Let* $f: L\rightarrow M$ *and* $g: L\rightarrow N$ *be continuous maps of a space* L *to G-spaces* M *and* N. *Under what conditions do there exist* p *points* $x_1, \cdots, x_p\in L$ *such that*

$$(1.2)\qquad f(x_{i+1})=T^i f(x_1),\ g(x_{i+1})=T^i g(x_1)$$

for $i=1,2,\cdots, p-1$?

We shall denote by $A(f)$ the set of points $x\in N$ satisfying (1.1), and by $A(f, g)$ the set of points $(x_1, \cdots, x_p)\in L^p$ satisfying (1.2).

If $L=N$ in Problem 2, then $A(f, \mathrm{id})$ may be identified with $A(f)$. Therefore

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Problem 2 is more general than Problem 1; still Problem 2 can be reduced to Problem 1. In fact, if we define $h: L^p \to M\times N$ by

$$(1.3)\qquad h(x_1, \cdots, x_p)=(f(x_1), g(x_1)) \quad (x_i\in L),$$

and regard L^p and $M\times N$ as G-spaces by cyclic permutations and the diagonal action respectively, then we have $A(h)=A(f, g)$.

Throughout this paper, a manifold will always mean a compact connected topological manifold which is assumed to be oriented if p is odd. The dimension of manifolds $M, N, \cdots$ will be denoted by $m, n, \cdots$. By a G-manifold is meant a manifold on which G acts topologically.

In this paper we shall consider Problems 1 and 2 in case M and N are G-manifolds. Some answers have been obtained by Conner-Floyd [3], Munkholm [10], Fenn [5], Lusk [8] and others with respect to generalizations of Theorem 1.1, and by Milnor [9] and the author [13] with respect to generalizations of Theorem 1.2. By pushing the line of [13] we shall prove in this paper more general results.

Throughout this paper the cohomology stands for the Čech cohomology and it takes coefficients from $\boldsymbol{Z}_p$, the group of integers mod p.

2. Theorems

In this section we shall state our main theorems answering to Problem 2 and then corollaries answering to Problem 1. The main theorems will be proved in §5 and §6.

Let $\omega_k\in H^k(BG)$ $(k=0, 1, \cdots)$ denote the usual generators, where BG is the classifying space for G. If X is a paracompact space on which G acts freely, $H^*(X/G)$ can be regarded as an $H^*(BG)$-module via the homomorphism induced by a classifying map of X; in particular we have $\omega_k=\omega_k\cdot 1\in H^k(X/G)$.

The first main theorem is stated as follows, and it generalizes Theorem 1.1 (see Remark 1 below).

THEOREM A. *Let $f: L\to M$ and $g: L\to N$ be continuous maps of a compact space L to G-manifolds M and N. Suppose that*

i) *the action on M is trivial;*

ii) *the action on N is free and $\omega_n\in H^n(N/G)$ is not zero;*

iii) $n\geqq(p-1)m$;

iv) $f^*: H^q(M)\to H^q(L)$ $(q>0)$ *is trivial;*

v) $g^*: H^n(N)\to H^n(L)$ *is not trivial.*

Then we have $A(f, g)\neq\phi$; if L is moreover a manifold, we have

$$\dim A(f, g)\geqq pl-(p-1)(m+n)\geqq 0,$$

where $\dim A$ *denotes the covering dimension of* A.

Putting $L=N$ and $g=\mathrm{id}$, we get

COROLLARY. *Let* $f: N\to M$ *be a continuous map of a G-manifold* N *to a manifold* M. *Suppose that*

i) *the action on* N *is free and* $\omega_n\in H^n(N/G)$ *is not zero*;

ii) $f^*: H^q(M)\to H^q(N)$ $(q>0)$ *is trivial.*

Then we have

$$\dim A(f)\geqq n-(p-1)m,$$

where M *is regarded as a G-manifold by the trivial action.*

REMARK 1. Taking

$$N=\text{a } \bmod p \text{ homology } n\text{-sphere}$$

in the above corollary, we have the results due to Conner-Floyd [3], Munkholm [10] and the author [12], which are direct generalizations of Theorem 1.1.

REMARK 2. Taking

$$L=N=\text{a } \bmod p \text{ homology } n\text{-sphere},$$

$$M=S^m,\ \deg f=0,\ \deg g\not\equiv 0 \bmod p$$

in Theorem A, we have the results due the to Fenn [5] and Lusk [8].

To state the second main theorem and its corollaries, we shall make some preparations.

For any indexing set I, consider the complement $I_0^p=I^p-dI$ of the diagonal in I^p, and define $(i_1, \cdots, i_p)$, $(i'_1, \cdots, i'_p)\in I_0^p$ to be equivalent if $(i'_1, \cdots, i'_p)$ is a cyclic permutation of $(i_1, \cdots, i_p)$. We denote by $R(I_0^p)$ a set of representatives of the equivalent classes.

Let $f: L\to M$ and $g: L\to N$ be continuous maps of a manifold L to G-manifolds M and N. Given homogeneous bases $\{\alpha_i\}_{i\in I}$, $\{\beta_j\}_{j\in J}$ of $H^*(M)$, $H^*(N)$ and sets $R(I_0^p)$, $R(J_0^p)$, we define $\lambda(f, g)$, $\lambda'(f, g)\in \boldsymbol{Z}_p$ as follows.

Define $\Delta: M\to M^p$ by

$$\Delta(x)=(x, Tx, \cdots, T^{p-1}x)\qquad (x\in M), \tag{2.1}$$

and put

$$\Delta_!(1)=\sum_{(i_1,\cdots,i_p)\in I^p} c_{i_1\cdots i_p}\alpha_{i_1}\times\cdots\times\alpha_{i_p}\qquad (c_{i_1\cdots i_p}\in\boldsymbol{Z}_p) \tag{2.2}$$

for the Gysin homomorphism $\Delta_!: H^*(M)\to H^*(M^p)$.

Similarly, put

$$\Delta_!(1)=\sum_{(j_1,\cdots,j_p)\in J^p} d_{j_1\cdots j_p}\beta_{j_1}\times\cdots\times\beta_{j_p} \qquad (d_{j_1\cdots j_p}\in \boldsymbol{Z}_p)$$

for the homomorphism $\Delta_! : H^*(N)\to H^*(N^p)$.

We define

$$\lambda(f, g)=\langle (f^{*p}\sum_{(i_1,\cdots,i_p)\in R(I_0^p)} c_{i_1\cdots i_p}\alpha_{i_1}\times\cdots\times\alpha_{i_p})(g^{*p}\Delta_!(1)), [L]^p\rangle,$$

$$\lambda'(f, g)=\langle (f^{*p}\Delta_!(1))(g^{*p}\sum_{(j_1,\cdots,j_p)\in R(J_0^p)} d_{j_1\cdots j_p}\beta_{j_1}\times\cdots\times\beta_{j_p}), [L]^p\rangle.$$

Obviously we have $\lambda(f, g)=\lambda'(g, f)$.

If $L=N$ and $g^*=\mathrm{id}$, we write $\lambda(f)=\lambda(f, g)$. It follows that

$$\lambda(f)=\sum_{(i_1,\cdots,i_p)\in R(I_0^p)} c_{i_1\cdots i_p}\langle (f^*\alpha_{i_1})(T^*f^*\alpha_{i_2})\cdots(T^{*p-1}f^*\alpha_{i_p}), [N]\rangle.$$

REMARK 3. By the definition of $\Delta_!$ we have

$$\langle \Delta^*(\alpha_{k_1}\times\cdots\times\alpha_{k_p}), [M]\rangle=\langle (\alpha_{k_1}\times\cdots\times\alpha_{k_p})\Delta_!(1), [M]^p\rangle.$$

From this we get

$$y_{k_1\cdots k_p}=\sum_{(i_1,\cdots,i_p)\in I^p}(-1)^{\varepsilon(i_1,\cdots,i_p,k_1,\cdots,k_p)}c_{i_1\cdots i_p}z_{k_1i_1}\cdots z_{k_pi_p}, \tag{2.3}$$

where

$$y_{k_1\cdots k_p}=\langle \alpha_{k_1}(T^*\alpha_{k_2})\cdots(T^{*p-1}\alpha_{k_p}), [M]\rangle,$$

$$z_{ki}=\langle \alpha_k\alpha_i, [M]\rangle,$$

$$\varepsilon(i_1, \cdots, i_p, k_1, \cdots, k_p)=\sum_{s=1}^{p-1}|\alpha_{i_s}|(|\alpha_{k_{s+1}}|+\cdots+|\alpha_{k_p}|),$$

being $|\alpha|=\deg\alpha$. The relations (2.3) for $(k_1, \cdots, k_p)\in I^p$ characterize the coefficients $c_{i_1\cdots i_p}$ ([6]). In particular, if $p=2$ we see that the matrix (c_{ij}) is the inverse of the matrix (y_{ij}).

Now the second main theorem is stated as follows, and it generalizes Theorem 1.2 (see Remark 5 below).

THEOREM B. *Let $f: L\to M$ and $g: L\to N$ be continuous maps of a manifold L to G-manifolds M and N. Suppose that*

i) *$i^*: H^q(M)\to H^q(M^G)$ is trivial for $q\geqq m/p$, where M^G is the fixed point set of M, and i is the inclusion;*

ii) *the action on N is free;*

iii) *$pl=(p-1)(m+n)$.*

Then $\lambda(f, g)$ and $\lambda'(f, g)$ are independent of the choices of $\{\alpha_i\}_{i\in I}$, $\{\beta_j\}_{j\in J}$, $R(I_0^p)$,

$R(J_0^p)$, and we have $\lambda(f, g)=\lambda'(f, g)$; If $\lambda(f, g)\neq 0$ we have $A(f, g)\neq\phi$.

Putting $L=N$ and $g=\mathrm{id}$ in Theorem B we have

COROLLARY 1. *Let $f: N\to M$ be a continuous map between G-manifolds, and suppose that*

i) *$i^*: H^q(M)\to H^q(M^G)$ is trivial for $q\geqq m/p$;*

ii) *the action on N is free;*

iii) *$n=(p-1)m$.*

Then $\lambda(f)$ is independent of the choices of $\{\alpha_i\}_{i\in I}$ and $R(I_0^p)$, and if $\lambda(f)\neq 0$ we have $A(f)\neq\phi$.

Put $L=M$ and $f=\mathrm{id}$ in Theorem B, and replace the notations M, N, g by N, M, f respectively. Then we get

COROLLARY 2. *Let $f: N\to M$ be a continuous map between G-manifolds, and suppose that*

i) *$i^*: H^q(N)\to H^q(N^G)$ is trivial for $q\geqq n/p$;*

ii) *the action on M is free;*

iii) *$n=(p-1)m$.*

Then the same conclusions as in Corollary 1 hold.

REMARK 4. The above two corollaries for $p=2$ have been obtained in [13].

The following proposition will be proved in §4 (see p. 407 of [2] for $p=2$).

PROPOSITION 2.1. *If M is a G-manifold such that $i^*: H^{m/p}(M)\to H^{m/p}(M^G)$ is trivial, then*

$$\langle\alpha(T^*\alpha)\cdots(T^{*p-1}\alpha), [M]\rangle=0 \quad (\alpha\in H^*(M)).$$

Let M be the one in Proposition 2.1 for $p=2$. Then, the proposition and the Poincaré duality show that $H^*(M)$ has a homogeneous basis $\{\mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_r'\}$ such that

$$\langle\mu_i(T^*\mu_k), [M]\rangle=0,\ \langle\mu_i'(T^*\mu_k'), [M]\rangle=0,\ \langle\mu_i(T^*\mu_k'), [M]\rangle=\delta_{ik}.$$

In terms of this basis we see that

$$\lambda(f)=\sum_{i=1}^{r}\langle(f^*\mu_i)(T^*f^*\mu_i'), [N]\rangle$$

if $p=2$. In particular, if $M=N$ and $f^*=\mathrm{id}$ then $\lambda(f)$ equals the *semi-characteristic*

$$\chi_{1/2}(M)=\dim H^*(M)/2 \bmod 2.$$

Thus, for $p=2$ we have the following

COROLLARY 3. *Let M be a manifold with a free involution T, and assume $\chi_{1/2}(M)\neq 0$. Let $f, g: M\rightarrow M$ be continuous maps such that $f^*=g^*=\mathrm{id}: H^*(M)\rightarrow H^*(M)$. Then there exist $x, x'\in M$ such that $f(x')=Tf(x)$ and $g(x')=Tg(x)$. In particular, there exists a point $x\in M$ such that $fT(x)=Tf(x)$.*

REMARK 5. Taking

$$M=\text{a mod 2 homology } m\text{-sphere}$$

in Corollary 3, we have the result due to Milnor [9], which is a direct generalization of Theorem 1.2.

3. Method.

In this section we shall explain how to prove Theorems A and B.

Let M be a G-manifold. If we regard M^p as a G-manifold by cyclic permutations, the map $\Delta: M\rightarrow M^p$ in (2.1) is an equivariant embedding. Regard S^{2k+1} as a G-manifold by the standard free action. Then we have a pair $(S^{2k+1}\times_G M^p, S^{2k+1}\times_G \Delta M)$ of manifolds, and hence the Thom isomorphism

$$\theta_k: H^q(S^{2k+1}\times_G \Delta M)\cong H^{q+(p-1)m}(S^{2k+1}\times_G (M^p, M^p-\Delta M))$$

which is the composite of the duality isomorphisms for $S^{2k+1}\times_G \Delta M$ and for $(S^{2k+1}\times_G M^p, S^{2k+1}\times_G \Delta M)$ (see p. 353 of [14]). We denote the Thom class $\theta_k(1)$ by $\hat{U}_M^{(k)}$.

The isomorphisms θ_k for sufficiently large k define the Thom isomorphism

$$\theta: H_G^q(\Delta M)\cong H_G^{q+(p-1)m}(M^p, M^p-\Delta M)$$

of the equivariant cohomology. The element $\theta(1)$ is denoted by $\hat{U}_M$, and is called the *equivariant fundamental cohomology class of M.*

The image of $\hat{U}_M$ in $H_G{}^{m(p-1)}(M^p)$ is denoted by $\hat{U}_M'$, and is called the *equivariant diagonal cohomology class* of M.

If the action of G on M is free, the diagonal set dM is in $M^p-\Delta M$. In this case the image of $\hat{U}_M$ in $H_G{}^{m(p-1)}(M^p, dM)$ is denoted by $\hat{U}_M''$, and is called the *modified equivariant diagonal cohomology class of M.*

LEMMA 3.1. *Let M and N be G-manifolds, and regard $M\times N$ as a G-manifold by the diagonal action. If the action on N is free, we have*

$$\hat{U}_{M\times N}''=\pm(q_1^{p*}\hat{U}_M')(q_2^{p*}\hat{U}_N''),$$

where $q_1^{p*}: H_G^*(M^p)\rightarrow H_G^*((M\times N)^p)$ and $q_2^{p*}: H_G^*(N^p, dN)\rightarrow H_G^*((M\times N)^p, d(M\times N))$ are induced by the projections $q_1: M\times N\rightarrow M$, $q_2: M\times N\rightarrow N$.

PROOF. There are the following natural inclusions of manifolds:

$$\begin{array}{ccc} (S^{2k+1}\times_G \Delta M)\times S^{2k+1}\times_G \Delta N) & \subset & (S^{2k+1}\times_G M^p)\times(S^{2k+1}\times_G N^p) \\ \cup & & \cup \\ S^{2k+1}\times_G \Delta(M\times N) & \subset & S^{2k+1}\times_G (M\times N)^p \end{array}$$

From properties of the Thom class (see 325 of [4]), it follows that the Thom class for the pair in the upper line equals $\pm\hat{U}_M^{(k)}\times\hat{U}_N^{(k)}$, and that it is sent to $\pm\hat{U}_{M\times N}^{(k)}$ by the homomorphism i^* induced by the natural inclusion of the lower line to the upper. Therefore we have

$$\hat{U}_{M\times N}^{(k)}=\pm i^*(\hat{U}_M^{(k)}\times\hat{U}_N^{(k)})=\pm i^*(p_1^*\hat{U}_M^{(k)}\cdot p_2^*\hat{U}_N^{(k)})=\pm(q_1^{p*}\hat{U}_M^{(k)})(q_2^{p*}\hat{U}_N^{(k)}),$$

where p_1, p_2 are the projections of $(S^{2k+1}\times_G M^p)\times(S^{2k+1}\times_G N^p)$ to $S^{2k+1}\times_G M^p$, $S^{2k+1}\times_G N^p$. This fact proves immediately the desired result.

LEMMA 3.2. *Let $f: N\rightarrow M$ be a continuous map of a G-space N to a G-manifold M, and define an equivariant map $\hat{f}: N\rightarrow M^p$ by*

$$\hat{f}(x)=(f(x), fT(x), \cdots, fT^{p-1}(x)) \quad (x\in N).$$

If the action on M is free, and if $\hat{f}^*(\hat{U}_M'')\neq 0$ for the homomorphism $\hat{f}^*: H_G^*(M^p, dM)\rightarrow H_G^*(N, N^G)$, then we have $A(f)\neq\phi$. If N is moreover a G-manifold, we have

$$\dim A(f)\geqq n-(p-1)m\geqq 0.$$

PROOF. In virtue of a commutative diagram

$$\begin{array}{ccc} H_G^*(M^p, M^p-\Delta M) & \xrightarrow{i^*} & H_G^*(M^p, dM) \\ \downarrow \hat{f}^* & & \downarrow \hat{f}^* \\ H_G^*(N, N-A(f)) & \xrightarrow{i^*} & H_G^*(N, N^G), \end{array}$$

$\hat{f}^*(\hat{U}_M'')\neq 0$ implies $H_G^{m(p-1)}(N, N-A(f))\neq 0$. Therefore $A(f)\neq\phi$. If N is a G-manifold, we have isomorphisms

$$\begin{aligned} H^{n-m(p-1)}(A(f)/G) &\cong H_{m(p-1)}(N'/G, (N'-A(f))/G) \\ &\cong H^{m(p-1)}(N'/G, (N'-A(f))/G)\cong H_G^{m(p-1)}(N', N'-A(f)) \\ &\cong H_G^{m(p-1)}(N, N-A(f)), \end{aligned}$$

where $N'=N-N^G$. Therefore $H^{n-m(p-1)}(A(f)/G)\neq 0$, and so $\dim A(f)\geqq n-m(p-1)\geqq 0$. This completes the proof.

PROPOSITION 3.3. *Let $f: L\rightarrow M$ and $g: L\rightarrow N$ be continuous maps of a space*

L to G-manifolds M and N. Suppose that the action on N is free. Then if

$$(f^{p*}\hat{U}'_M)(g^{p*}\hat{U}''_N)\in H_G^{(m+n)(p-1)}(L^p, dL)$$

is not zero, we have $A(f, g)\neq\phi$. *If L is moreover a manifold, we have*

$$\dim A(f, g)\geqq pl-(p-1)(m+n)\geqq 0.$$

PROOF. Consider $h: L^p\to M\times N$ defined by (1.3). Then, for the map $\hat{h}: L^p\to (M\times N)^p$ we have $q_1^p\circ\hat{h}=f^p$, $q_2^p\circ\hat{h}=g^p$. Therefore by Lemma 3.1 we have

$$\hat{h}^*(\hat{U}''_{M\times N})=\pm\hat{h}^*((q_1^{p*}\hat{U}'_M)(q_2{}^{p*}\hat{U}''_N))$$

$$=\pm(f^{p*}\hat{U}_M)(g^{p*}\hat{U}''_N).$$

This proves the desired result by Lemma 3.2.

We shall prove Theorems A and B by making use of Proposition 3.3. For this purpose we are asked to examine the following:

(i) structure of the equivariant cohomologies $H_G^*(X^p)$ and $H_G^*(X^p, dX)$ for a compact space X.

(ii) the equivariant diagonal cohomology class $\hat{U}'_M$ and the modified equivariant diagonal cohomology class $\hat{U}''_M$ for a G-manifold M.

As for (i) we have the results due to Steenrod and Thom, which are stated in § 4. Thus Theorems A and B will be proved by examining (ii), as seen in § 5 and § 6.

4. Preparations

In this section we shall recall some facts needed later.

Let X be a paracompact G-space. Then we have

$$H^*(X)=\varinjlim H^*(K),$$

$$H_G^*(X, X^G)=\varinjlim H^*(K/G, K^G/G),$$

where K ranges over the nerves of G-coverings of X (see Chap III, § 6 and Chap VII, § 1 of [2]). For each K a cochain map

$$\varphi_K: C^*(K)\longrightarrow C^*(K/G, K^G/G)$$

is defined by

$$\langle\varphi_K(u), \pi(s)\rangle=\sum_{i=0}^{p-1}u(T^is),$$

where $u\in C^*(K)$, s is a simplex of K, and $\pi: K\longrightarrow K/G$ is the projection. Thus we have a homomorphism

(4.1) $$\pi_!' : H^*(X) \longrightarrow H_G^*(X, X^G)$$

defined by the cochain maps φ_K.

We define

(4.2) $$\pi_! : H^*(X) \longrightarrow H_G^*(X)$$

to be the composite $j^* \circ \pi_!'$, where $j^* : H_G^*(X, X^G) \longrightarrow H_G^*(X)$ is induced by the inclusion. It follows that $\pi_!$ is the composite of the usual transfer $H^*(X) \longrightarrow H^*(X/G)$ and the canonical homomorphism $H^*(X/G) \longrightarrow H_G^*(X)$.

We call $\pi_!$ in (4.2) the *transfer*, and $\pi_!'$ in (4.1) the *modified transfer*.

Put

$$\sigma^* = \sum_{i=0}^{p-1} T^{i*} : H^*(X) \longrightarrow H^*(X).$$

Then it is easily seen that

(4.3) $$\pi^* \circ \pi_! = \sigma^*$$

for the canonical homomorphism $\pi^* : H_G^*(X) \longrightarrow H^*(X)$, and that

(4.4) $$\pi_!(\alpha_1) \cdot \pi_!'(\alpha_2) = \pi_!'(\alpha_1 \cdot \sigma^* \alpha_2) = \pi_!'(\sigma^* \alpha_1 \cdot \alpha_2)$$

$(\alpha_1, \alpha_2 \in H^*(X))$. We have also

(4.5) $$\pi_!(\alpha) \cdot \delta^*(\beta) = 0 \qquad (\alpha \in H^*(X),\ \beta \in H_G^*(X^G))$$

for the coboundary homomorphism $\delta^* : H_G^*(X^G) \longrightarrow H_G^*(X, X^G)$.

In fact

$$\begin{aligned}(-1)^{|\alpha|} \pi_!(\alpha) \cdot \delta^*(\beta) &= \delta^*(i^* \pi_!(\alpha) \cdot \beta) \\ &= \delta^*(i^* j^* \pi_!'(\alpha) \cdot \beta) = 0,\end{aligned}$$

where $i^* : H_G^*(X) \longrightarrow H_G^*(X^G)$.

If X is a paracompact G-space, the Smith special cohomology groups $H_\rho^*(X)$ are defined for $\rho = \sigma = \sum_{i=0}^{p-1} T^i$ and $\rho = \tau = 1 - T$, and we have the exact sequences

$$\cdots \xrightarrow{\rho^*} H^q(X) \xrightarrow{i_\rho^*} H_\rho^q(X) \oplus H^q(X^G) \xrightarrow{\delta_\rho^*} H_{\bar\rho}^{q+1}(X) \xrightarrow{j_\rho^*} H^{q+1}(X) \longrightarrow \cdots$$

for $(\rho, \bar\rho) = (\sigma, \tau)$ and (τ, σ). We have also an isomorphism

$$H_\sigma^*(X) \cong H_G^*(X, X^G).$$

(See p. 143 of [2].)

It follows that

$$(4.6) \qquad i_T^* = (\pi_!', i^*) : H^*(X) \longrightarrow H_G^*(X, X^G) \oplus H^*(X^G).$$

LEMMA 4.1. *If M is a G-manifold such that the action is not trivial, then it holds*

$$\pi_!' : H^m(M) \cong H_G^m(M, M^G).$$

PROOF. In the exact sequence

$$H^m(M) \xrightarrow{i_T^*} H_\sigma^m(M) \oplus H^m(M^G) \xrightarrow{\delta_T^*} H_T^{m+1}(M),$$

we have $H_T^{m+1}(M)=0$, $H^m(M) \cong \boldsymbol{Z}_p$, $H^m(M^G) \cong H_0(M, M-M^G)=0$, and moreover $H_\sigma^m(M) \neq 0$ is proved as follows. Therefore we get the desired result by (4.6).

Suppose $H_\sigma^m(M)=0$. Then, by the Smith cohomology exact sequence, we see that $i_\sigma^* : H^m(M) \cong H_T^m(M)$ and $\tau^* : H_T^m(M) \longrightarrow H^m(M)$ is onto. This implies that $\tau^* : H^m(M) \longrightarrow H^m(M)$ is onto and so $H^m(M)=0$, which is a contradiction.

For a paracompact space X, consider the equivariant cohomology $H_G^*(X^p)$, where G acts on X^p by cyclic permutations. Then we have the external Steenrod p-th power operation

$$P : H^q(X) \longrightarrow H_G^{pq}(X^p),$$

which is related to the Steenrod square Sq^i if $p=2$, and to the reduced p-th power $\mathcal{P}^i$ and the Bockstein operation β^* if $p \neq 2$ as follows ([15]):

$$(4.7) \qquad d^*P(\alpha) = \begin{cases} \sum_{i=0}^{|\alpha|} \omega_{|\alpha|-i} \times Sq^i\alpha & \text{if } p=2, \\ h_q \sum_{i=0}^{[|\alpha|/2]} (-1)^i (\omega_{(|\alpha|-2i)(p-1)} \times \mathcal{P}^i\alpha - \omega_{(|\alpha|-2i)(p-1)-1} \times \beta^*\mathcal{P}^i\alpha) & \text{if } p \neq 2, \end{cases}$$

where $d^* : H_G^*(X^p) \longrightarrow H_G^*(X) = H^*(BG \times X)$ is induced by the diagonal map, and

$$(4.8) \qquad h_q = \begin{cases} (-1)^{q/2} & \text{if } q \text{ is even}, \\ (-1)^{(q-1)/2}((p-1)/2)! & \text{if } q \text{ is odd}. \end{cases}$$

P is natural, and it satisfies also

$$(4.9) \qquad \pi^*P(\alpha) = \alpha^p$$

for the canonical homomorphism $\pi^* : H_G^*(X^p) \longrightarrow H^*(X^p)$.

LEMMA 4.2. *Let M be a G-manifold, and let $\alpha \in H^*(M)$ satisfy $i^*(\alpha)=0$ for $i^*: H^*(M) \longrightarrow H^*(M^G)$. Then $\Delta^*P(\alpha)$ is in the image of $j^*: H_G^*(M, M^G) \longrightarrow H_G^*(M)$ induced by the inclusion.*

PROOF. Consider a diagram

$$\begin{array}{ccccc} & & H_G^*(M^p) & \xrightarrow{d^*} & H^*(BG \times M) \\ & & \downarrow \Delta^* & & \downarrow (id \times i)^* \\ H_G^*(M, M^G) & \xrightarrow{j^*} & H_G^*(M) & \xrightarrow{i^*} & H^*(BG \times M^G), \end{array}$$

in which the rectangle is commutative and the lower sequence is exact. Then it follows from (4.7) that $i^*\Delta^*P(\alpha)=(id \times i)^*d^*P(\alpha)=0$. Therefore $\Delta^*P(\alpha) \in \mathrm{Im} j^*$.

PROOF OF PROPOSITION 2.1. We may assume that the action is non-trivial and $|\alpha|=m/p$. Consider a commutative diagram

$$\begin{array}{ccc} H_G^m(M, M^G) & \xrightarrow{j^*} & H_G^m(M) \\ \uparrow \pi_!' & \nearrow \pi_! & \downarrow \pi^* \\ H^m(M) & \xrightarrow{\sigma^*} & H^m(M) \end{array}$$

By Lemmas 4.1 and 4.2, we see

$$\pi^*\Delta^*P(\alpha) \in \mathrm{Im}\, \sigma^*.$$

Since $\sigma^*H^m(M)=0$ and

$$\pi^*\Delta^*P(\alpha)=\Delta^*(\alpha^p)=\alpha(T^*\alpha)\cdots(T^{p-1*}\alpha)$$

by (4.9), the proof completes.

The following theorem is due to Steenrod [15] (see also [12]).

THEOREM 4.3. *Let X be a compact space, and $\{\alpha_i\}_{i \in I}$ be a homogeneous basis of $H^*(X)$. Then the totality of elements*

$$\omega_j P(\alpha_i) \qquad (i \in I,\ j \geqq 0),$$

$$\pi_!(\alpha_{i_1} \times \cdots \times \alpha_{i_p}) \qquad ((i_1, \cdots, i_p) \in R(I_0^p))$$

is a homogeneous basis of $H_G^*(X^p)$.

The following is due to Thom [16] (see also [1], [11], [17]).

THEOREM 4.4. *Let X be a compact space, and $\{\alpha_i\}_{i\in I}$ be a homogeneous basis of $H^*(X)$. Then the totality of elements*

$$\delta^*(\omega_j\times\alpha_i) \qquad (i\in I,\ 0\leqq j<(p-1)|\alpha_i|),$$

$$\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p}) \qquad ((i_1,\cdots,i_p)\in R(I_0^p))$$

is a homogeneous basis of $H_G^*(X^p, dX)$, where $\delta^*: H^*(BG\times X)=H_G^*(dX)\longrightarrow H_G^*(X^p, dX)$ is the coboundary homomorphism. Furthermore we have

$$\pi_!(\alpha\times\alpha)=\sum_{i=0}^{|\alpha|-1}\delta^*(\omega_{|\alpha|-i-1}\times Sq^i\alpha)$$

if $p=2$, and

$$\pi_!(\alpha\times\cdots\times\alpha)=\sum_{i=0}^{[|\alpha|/2]}\varepsilon_i\delta^*(\omega_{(p-1)(|\alpha|-2i)-1}\times\mathcal{P}^i\alpha)$$

with some $\varepsilon_i\not\equiv 0 \bmod p$ if $p\neq 2$.

REMARK. Theorems 4.3 and 4.4 are proved in the literatures for a compact polyhedron. However we can extend them to compact spaces by the device seen in [2].

5. Proof of Theorem A.

The equivariant diagonal cohomology class $\hat{U}'_M$ in case the action on M is trivial has been studied by Haefliger. By Theorem 3.2 in his paper [6] and

$$\pi^*(\hat{U}')=\Delta_!(1), \tag{5.1}$$

we have the following (see the proof of Theorem 9.1 in [13]).

PROPOSITION 5.1. *If the action on M is trivial, then*

$$\hat{U}'_M=\sum_{k=0}^{[m/2]}\omega_{m-2k}P(V_k)+\sum_{i<j}(c_{ij}-c_{ii}c_{jj})\pi_!(\alpha_i\times\alpha_j)$$

if $p=2$, and

$$\hat{U}'_M=h_m\sum_{k=0}^{[m/2p]}(-1)^k\omega_{(p-1)(m-2kp)}P(V_k)$$
$$+\sum_{(i_1,\cdots,i_p)\in R(I_0^p)}(c_{i_1\cdots i_p}-c_{i_1\cdots i_1}\cdots c_{i_p\cdots i_p})\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p})$$

if $p\neq 2$, where $\{\alpha_i\}_{i\in I}$ is a homogeneous basis of $H^*(M)$, $c_{i_1,\cdots,i_p}$, h_m are those in (2.2), (4.8), *and $V_k\in H^*(M)$ are the Wu classes given by*

$$\langle V_k\cdot\alpha, [M]\rangle=\begin{cases}\langle Sq^k\alpha, [M]\rangle & \text{if } p=2,\\ \langle\mathcal{P}^k\alpha, [M]\rangle & \text{if } p\neq 2.\end{cases}$$

We shall next prove

PROPOSITION 5.2. *If the action on M is free and* $\omega_m \in H^*(M/G)$ *is not zero, it holds*

$$\omega_m \hat{U}''_M = \delta^*(\omega_{(p-1)m-1} \times \mu),$$

where μ *is a generator of* $H^m(M)$.

PROOF. Let V be an equivariant open neighbourhood of dM in M^p, and put

$$W = M^p - \varDelta M - dM, \qquad C = M^p - \varDelta M - V.$$

Then C/G is a closed connected and non-compact subset of W/G, and hence we have $H^{mp}(W/G, W/G - C/G) = 0$ (see p. 260 of [4]). Therefore it follows that

$$H_G^{mp}(M^p - \varDelta M, V) \cong H_G^{mp}(W, W-C) \cong H^{mp}(W/G, W/G - C/G) = 0.$$

This shows that $i^* : H_G^{mp}(M^p, M^p - \varDelta M) \longrightarrow H_G^{mp}(M^p, dM)$ is onto, and so is

$$i^* \circ \theta : H_G^m(\varDelta M) \longrightarrow H_G^{mp}(M^p, dM).$$

It follows from Lemma 4.1 and the assumptions that $H_G^m(\varDelta M) \cong \boldsymbol{Z}_p$ is generated by ω_m. By Theorem 4.4, $H_G^{mp}(M^p, dM) \cong \boldsymbol{Z}_p$ is generated by $\delta^*(\omega_{(p-1)m} \times \mu)$. Since $i^* \circ \theta$ is a homomorphism of $H^*(BG)$-modules and it sends 1 to $\hat{U}''_M$, we have the desired result.

We shall now give

PROOF OF THEOREM A. By the assumption ii) and Proposition 5.2, it holds

$$\omega_n \hat{U}''_N = \delta^*(\omega_{(p-1)n-1} \times \nu),$$

where ν is a generator of $H^n(N)$. Therefore we have

$$\omega_n (g^{p*} \hat{U}''_N) = \delta^*(\omega_{(p-1)n-1} \times g^* \nu),$$

and this is not zero by the assumption v) and Theorem 4.4. Since $n \geqq (p-1)m$ by the assumption iii), it holds

$$\omega_{(p-1)m}(g^{p*} \hat{U}''_N) \neq 0.$$

On the other hand, it follows from the assumptions i), iv) and Proposition 5.1 that

$$f^{p*}(\hat{U}'_M) = h_m \omega_{(p-1)m}$$

with $h_m \not\equiv 0 \bmod p$. Consequently we have

$$f^{p*}(\hat{U}'_M) \cdot g^{p*}(\hat{U}''_N) = h_m \omega_{(p-1)m} g^{p*}(\hat{U}''_N) \neq 0,$$

which completes the proof by Proposition 3.3.

6. Proof of Theorem B and an example.

The following proposition has been proved in [13] if $p=2$. By the similar method we shall prove it for any p.

PROPOSITION 6.1. *If $i^*: H^q(M)\longrightarrow H^q(M^G)$ is trivial for $q\geqq m/p$, then we have*

$$\hat{U}'_M=\sum_{(i_1,\cdots,i_p)\in R(I_0^p)} c_{i_1\cdots i_p}\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p}),$$

$$c_{i\cdots i}=0 \quad (i\in I),$$

where $\{\alpha_i\}_{i\in I}$ is a homogeneous basis of $H^*(M)$, and $c_{i_1\cdots i_p}$ are those in (2.2).

Before we proceed to proof we make some preparations.

The equivariant homology group $H^G_*(X^p)=H_*(EG\underset{G}{\times}X^p)$ is canonically identified with $H_*(G; H_*(X)^p)$, the homology group of the group G with coefficients in $H_*(X)^p=H_*(X)\otimes\cdots\otimes H_*(X)$ on which G acts by cyclic permutations. Taking the standard G-free acyclic complex W, we have an element of $H_*(G; H_*(X)^p)$ represented by $w_k\otimes a\otimes\cdots\otimes a$, where $w_k\in W$ is the basis of degree k and $a\in H_*(X)$. The corresponding element in $H^G_*(X^p)$ will be denoted by $P_k(a)$.

LEMMA 6.2. *Suppose that $i^*: H^q(M)\longrightarrow H^q(M^G)$ is trivial for $q\geqq m/p$. Then, for any $k\geqq 0$ and for any $\alpha\in H^*(M)$, we have*

$$\langle\omega_1\hat{U}'_M, P_{k+1}(\alpha\frown[M])\rangle=0$$

if $p=2$, and

$$\langle\hat{U}'_M, P_{2k+1}(\alpha\frown[M])\rangle=0,$$

$$\langle\omega_1\hat{U}'_M, P_{2k+1}(\alpha\frown[M])\rangle=0$$

if $p\neq 2$.

PROOF. Similarly to Lemma 4.4 in [13], the result for $p\neq 2$ is proved as follows.

It follows that $P_{2k+1}([M])$ is in the image of

$$i_{k*}: H_{2k+1+pm}(S^{2k+1}\underset{G}{\times}M^p)\longrightarrow H^G_{2k+1+pm}(M^p)$$

induced by the inclusion, and that $i_k^*(\hat{U}'_M)$ is the image of 1 under the homomorphism

$$(\mathrm{id}\underset{G}{\times}\Delta)_!: H^*(S^{2k+1}\underset{G}{\times}M)\longrightarrow H^*(S^{2k+1}\underset{G}{\times}M^p).$$

From these facts we see that $\hat{U}'_M\frown P_{2k+1}([M])$ is in the image of

$$H_{2k+1+m}(S^{2k+1}\underset{G}{\times}M)\xrightarrow{i_{k*}}H^G_{2k+1+m}(M)\xrightarrow{\Delta_*}H^G_{2k+1+m}(M^p).$$

Therefore it follows that

$$\langle\hat{U}'_M,\ P_{2k+1}(\alpha\frown[M])\rangle=\langle\hat{U}'_M,\ P(\alpha)\frown P_{2k+1}([M])\rangle$$
$$=\langle P(\alpha),\ \hat{U}'_M\frown P_{2k+1}([M])\rangle=\varepsilon_k\langle P(\alpha),\ \Delta_* i_{k*}[S^{2k+1}\underset{G}{\times}M]\rangle$$
$$=\varepsilon_k\langle\Delta^*P(\alpha),\ i_{k*}[S^{2k+1}\underset{G}{\times}M]\rangle\qquad(\varepsilon_k\in\boldsymbol{Z}_p),$$

and similarly

$$\langle\omega_1\hat{U}'_M,\ P_{2k+1}(\alpha\frown[M])\rangle=\varepsilon_k\langle\omega_1\Delta^*P(\alpha),\ i_{k*}[S^{2k+1}\underset{G}{\times}M]\rangle.$$

To prove the desired two equalities, we may suppose $p|\alpha|\geqq m+1$ in the first, and $p|\alpha|\geqq m$ in the second. Consequently it suffices to prove that

$$\Delta^*P(\alpha)=0\quad\text{if}\quad p|\alpha|\geqq m+1,$$
$$\omega_1\Delta^*P(\alpha)=0\quad\text{if}\quad p|\alpha|\geqq m.$$

By Lemma 4.2 $\Delta^*P(\alpha)$ and $\omega_1\Delta^*P(\alpha)$ are in the image of $j^*: H^*_G(M, M^G)\longrightarrow H^*_G(M)$, and the Smith cohomology exact sequence implies $H^q_G(M, M^G)=0$ $(q>m)$. Therefore we have the desired results, and the proof completes.

PROOF OF PROPOSITION 6.1. In virtue of Theorem 4.3 it can be written uniquely that

$$\hat{U}'_M=\sum_{i,j}\xi_{ij}\omega_jP(\alpha_i)+\sum_{(i_1,\cdots,i_p)\in R(I^p_0)}\eta_{i_1\cdots i_p}\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p})$$

with some ξ_{ij}, $\eta_{i_1\cdots i_p}\in\boldsymbol{Z}_p$. Since it is easily seen that

$$\operatorname{Im}\pi_!\frown P_k(a)=0,$$
$$\langle\omega_jP(\alpha),\ P_k(a)\rangle=\delta_{jk}\langle\alpha,\ a\rangle$$

$(\alpha\in H^*(M),\ a\in H_*(M))$, it follows from Lemma 6.2 that $\xi_{ij}=0$. We see from (5.1) that $\eta_{i_1\cdots i_p}=c_{i_1\cdots i_p}$ if $(i_1,\cdots,i_p)\in R(I^p_0)$ and $c_{i\cdots i}=0$ for any $i\in I$. This completes the proof.

REMARK 1. Working in the smooth category, Hattori [7] has given formulae for $\hat{U}'_M$ with no assumption on M^G.

The following is immediate from Proposition 6.1 and Theorem 4.5.

PROPOSITION 6.3. *If the action on M is free, then it can be written uniquely that*

$$\hat{U}''_M=\sum_{(i_1,\cdots,i_p)\in R(I^p_0)} c_{i_1\cdots i_p}\pi'_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p})$$

$$+\sum_{|\alpha_i|\geqq m-m/p}\varepsilon_i\delta^*(\omega_{(p-1)m-|\alpha_i|-1}\times\alpha_i)$$

with some $\varepsilon_i\in \boldsymbol{Z}_p$.

REMARK 2. The author does not know how to determine ε_i in the above. If M is a mod p homology sphere, it follows from Propositions 5.2 and 6.3 that

$$\hat{U}''_M=\begin{cases}\pi'_!(1\times\mu) & \text{if } p=2,\\ \pi'_!(1\times\mu\times\cdots\times\mu)+\varepsilon\delta^*(\omega_{(p-2)m-1}\times\mu) & \text{if } p\neq 2,\end{cases}$$

where $\varepsilon\not\equiv 0 \bmod p$, and $\mu\in H^m(M)$ is a generator such that $\langle\mu, [M]\rangle=1$.

We shall now give

PROOF OF THEOREM B. By the assumption i) and Proposition 6.1 we have

$$f^{p*}\hat{U}'_M=\pi_! f^{*p}\Big(\sum_{(i_1,\cdots,i_p)\in R(I^p_0)} c_{i_1\cdots i_p}\alpha_{i_1}\times\cdots\times\alpha_{i_p}\Big),$$

and by the assumption ii) and Proposition 6.3 we have

$$g^{p*}\hat{U}''_N=\pi'_! g^{*p}\Big(\sum_{(j_1,\cdots,j_p)\in R(J^p_0)} d_{j_1\cdots j_p}\beta_{j_1}\times\cdots\times\beta_{j_p}\Big)$$

$$+\sum_{|\beta_j|\geqq n-n/p}\varepsilon_j\delta^*(\omega_{(p-)n-|\beta_j|-1}\times g^*\beta_j).$$

It follows from (5.1) and Proposition 6.1 that

$$\sigma^*\sum_{(i_1,\cdots,i_p)\in R(I^p_0)} c_{i_1\cdots i_p}\alpha_{i_1}\times\cdots\times\alpha_{i_p}=\varDelta_!(1),$$

$$\sigma^*\sum_{(j_1,\cdots,j_p)\in R(J^p_0)} d_{j_1,\cdots,j_p}\beta_{j_1}\times\cdots\times\beta_{j_p}=\varDelta_!(1).$$

Thus, by (4.4), (4.5) and the assumption ii), we have

$$(f^{p}\hat{U}'_M)\cdot(g^{p*}\hat{U}''_N)$$

$$=\pi'_!\Big(f^{*p}\sum_{(i_1,\cdots,i_p)\in R(I^p_0)} c_{i_1\cdots i_p}\alpha_{i_1}\times\cdots\times\alpha_{i_p}\Big)(g^{*p}\varDelta_!(1))$$

$$=\pi'_!(f^{*p}\varDelta_!(1))\Big(g^{*p}\sum_{(j_1,\cdots,j_p)\in R(J^p_0)} d_{j_1,\cdots,j_p}\beta_{j_1}\times\cdots\times\beta_{j_p}\Big)$$

in $H^{pl}_G(L^p, dL)$.

It follows from Theorem 4.4 that $H^{pl}_G(L^p, dL)\cong \boldsymbol{Z}_p$ is generated by $\delta^*(\omega_{(p-1)l-1}\times\rho)$ or $\pi'_!(\rho\times\cdots\times\rho)$, where $\rho\in H^l(L)$ is a generator such that $\langle\rho, [L]\rangle=1$.

Consequently we have

$$(f^{p*}\hat{U}'_M)(g^{p*}\hat{U}''_N)=\lambda(f, g)\pi'_!(\rho\times\cdots\times\rho)$$

$$=\lambda'(f, g)\pi'_!(\rho\times\cdots\times\rho),$$

which completes the proof by Proposition 3.3.

Theorem B for $p=2$, particularly corollary 3 in § 2, has interesting applications as is seen in [13]. The author does not know so interesting applications of Theorems B for $p\neq 2$. However there is the following example for which Theorem B for $p=3$ is applicable.

Let $n=1, 3$ or 7, and take in Theorem B

$$L=S^n\times S^n,\ M=S^n\times S^n,\ N=S^n,$$

where the action on N is any free G-action, and action on M is given as follows:

$$T(x, y)=(y, y^{-1}x^{-1}),$$

x, y being complex numbers, quaternions or Cayley numbers according as $n=1, 3$ or 7. It follows that the fixed point set of M is homeomorphic to $S^{n-1}+$point. Thus the assumptions i), ii), iii) in Theorem B are satisfied.

Let $\nu\in H^n(S^n)$ denote a generator, and put $\nu_1=\nu\times 1$, $\nu_2=1\times\nu\in H^n(S^n\times S^n)$. Then, by Remark 3 in § 2, it can be seen that

$$\Delta_!(1)=\sigma^*(1\times\nu_1\nu_2\times\nu_1\nu_2-\nu_1\times\nu_1\times\nu_1\nu_2-\nu_2\times\nu_2\times\nu_1\nu_2-\nu_2\times\nu_1\times\nu_1\nu_2)$$

for the homomorphism $\Delta_! : H^*(M)\longrightarrow H^*(M^3)$, and

$$\Delta_!(1)=\sigma^*(1\times\nu\times\nu)$$

for the homomorphism $\Delta_! : H^*(N)\longrightarrow H^*(N^3)$. Therefore, if continuous maps $f : L\longrightarrow M$, $g : L\longrightarrow N$ satisfy

$$f^*(\nu_i)=a_{i1}\nu_1+a_{i2}\nu_2,\ g^*(\nu)=b_1\nu_1+b_2\nu_2$$

(a_{ij}, $b_i\in \boldsymbol{Z}_3$), simple calculation shows

$$\lambda(f, g)=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\left(\begin{vmatrix} a_{11} & a_{12}\\ b_1 & b_2\end{vmatrix}-\begin{vmatrix} a_{21} & a_{22}\\ b_1 & b_2\end{vmatrix}\right).$$

This yields by Theorem B the following

Theorem 6.4. *Let $n=1, 3$ or 7, and let f_1, f_2, $g : S^n\times S^n\longrightarrow S^n$ be continuous maps of type (a_{11}, a_{12}), (a_{21}, a_{22}), (b_1, b_2) respectively. Let $T : S^n\longrightarrow S^n$ be a homomorphism of period 3 without fixed points. Then, if*

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\left(\begin{vmatrix} a_{11} & a_{12}\\ b_1 & b_2\end{vmatrix}-\begin{vmatrix} a_{21} & a_{22}\\ b_1 & b_2\end{vmatrix}\right)\not\equiv 0 \bmod 3,$$

there exist x, y, $z\in S^n\times S^n$ such that

$$(f_2(x), f_2(y), f_2(z))=(f_1(y), f_1(z), f_1(x)),$$

$$(Tg(x), Tg(y), Tg(z)) = (g(y), g(z), g(x)),$$
$$f_1(x)f_1(y)f_1(z) = 1.$$

In particular, taking f_i=projection to the i-th factor, we have

COROLLARY. *If* $b_1+b_2 \neq 0$ *then there exist* $x, y, z \in S^n$ *such that*

$$Tg(x, y) = g(y, z), \quad Tg(y, z) = g(z, x), \quad xyz = 1,$$

where n, g *and* T *are those in Theorem* 6.4.

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