### Equivariant pretheories and invariants of torsors

#### Kirill Zaynullin

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Kirill Zaynullin Equivariant pretheories and invariants of torsors

- All schemes/varieties are defined over the base field k.
- By a scheme over a field *k* (*k*-scheme) we mean a reduced separated Noetherian scheme over *k*.
- By a variety over a field *k* (*k*-variety) we mean a quasi-projective scheme of finite type over *k*.
- By pt we denote Spec k.
- If I/k is a field extension and X is a k-scheme, we define  $X_I = X \times_{pt} \text{Spec } I$  to be the respective base change.
- By an algebraic group we mean an *affine smooth group scheme* over *k*.
- By an action of an algebraic group G on a scheme X we mean a morphism G ×<sub>pt</sub> X → X of schemes over k (all group actions are assumed to be on the left), subject to the usual axioms).
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## Part I. Equivariant pretheories.

Examples:

- equivariant Chow- and K-theory
- equivariant algebraic cobordism
- equivariant cycle (co)homology
- spectral sequence for equivariant cycle homology

#### Let G be an algebraic group over a field k.

Consider a contravariant functor from the category of smooth G-varieties over k to the category of abelian groups

$$h_G: G-Sm_k \longrightarrow \mathfrak{Ab}, X \mapsto h_G(X).$$

Given X,  $Y \in G$ -Sm<sub>k</sub> and a G-equivariant map  $f: X \to Y$  the induced functorial map  $h_G(Y) \to h_G(X)$  is called a *pull-back* and is denoted by  $f_G^*$ .

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## Equivariant pretheories. Definition.

The functor  $h_G: G-Sm_k \to \mathfrak{Ab}$  is called a *G*-equivariant pretheory over k if it satisfies the following two axioms:

H. (homotopy invariance) For a *G*-equivariant map  $p \colon \mathbb{A}_k^n \to \text{pt}$  (where *G* acts trivially on pt) the induced pull-back

$$p_G^*: h_G(\mathrm{pt}) \longrightarrow h_G(\mathbb{A}_k^n)$$

is an isomorphism.

L. (localization) For a smooth G-variety X and a G-equivariant open embedding  $\iota \colon U \hookrightarrow X$  the induced pull-back

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A *G*-equivariant pretheory  $h_G$  is called *essential* if it can be extended to the category of essentially smooth varieties *G*-Ess<sub>k</sub> with *G*-equivariant flat morphisms, i.e.

 $h_G: G-Ess_k \longrightarrow \mathfrak{Ab},$ 

#### in such a way that

C. For every  $f: X \to Y$  and  $U \subseteq Y$  as above the canonical morphism induced by the pull-backs

$$ar{\mathtt{h}}_G(X_K) o \mathtt{h}_{G_K}(X_K)$$

is an isomorphism.

Here  $X_K$  denotes the generic fiber of f and  $\overline{h}_G(X_K)$  denotes the colimit  $\varinjlim_{U\subseteq Y} h_G(f^{-1}(U))$ , where U ranges over the set of non-empty open G-equivariant subsets of Y.

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Let G be an algebraic group over k and let X be a smooth G-variety. The functor

$$h_G: X \longmapsto CH^*_G(X) = \bigoplus_{i \in \mathbb{Z}} CH^i_G(X)$$

where  $CH^*_G(X)$  is the equivariant Chow-theory of Totaro, Edidin and Graham, provides an example of an essential *G*-equivariant pretheory.

Consider the category  $\mathcal{P}(G, X)$  of locally free *G*-modules on *X*. It is an exact category and following Thomason one defines the *i*-th *G*-equivariant *K*-group  $K_i(G, X)$  as Quillen's *i*-th *K*-group of  $\mathcal{P}(G, X)$ . The functor

$$h_G(X) := K(X, G)$$

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Assume that char(k) = 0. Consider a family of pairs  $(V_i, U_i)_{i \in \mathbb{N}}$  of vector spaces with  $U_i \subseteq V_i$  endowed with an action of an algebraic group G such that

- (i) G acts freely on  $U_i$  and  $U_i \rightarrow U_i/G$  is a G-torsor,
- (ii)  $V_{i+1} = V_i \oplus W_i$  for some k-subspace  $W_i$ , such that  $U_i \oplus W_i \subseteq U_{i+1}$ ,
- (iii) supdim  $V_i = \infty$ , and
- (iv)  $\operatorname{codim}_{V_i}(V_i \setminus U_i) < \operatorname{codim}_{V_{i+1}}(V_{i+1} \setminus U_{i+1})$ , where we consider  $V_i$  as an affine space over k.

Observe that assumption (i) ensures that the quotient  $X \times^G U := (X \times_k U)/G$  is a quasi-projective variety over k. Moreover, it is smooth over k by the descent.

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## Examples: Equivariant algebraic cobordism

Let G be connected. Then the *n*-th equivariant cobordism group of a smooth G-variety X is defined [Heller and Malagón-López] by

$$\Omega_n^G(X) := \varprojlim_i \Omega_{n-\dim G+\dim U_i}(X \times^G U_i),$$

where  $\Omega_*(X)$  denotes the ring of algebraic cobordism of Levine-Morel.

The functor

$$h_G: X \longmapsto \bigoplus_{n \in \mathbb{Z}} \Omega_n^G(X)$$

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It seems that this construction can be extended to any oriented cohomology theory in the sense of Levine-Morel.

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## Equivariant cycle cohomology

Following Rost we consider a *cycle module* over the field k that is a (covariant) functor  $M_*$  from the category of field extensions of k to the category of graded abelian groups satisfying several axioms.

The prototype of such a functor is *Milnor K-theory*  $K_*^M$ , and by the very definition  $M_*(E) = \bigoplus_{i \in \mathbb{Z}} M_i(E)$  is a graded  $K_*^M(E)$ -module for all field extensions  $E \supseteq k$ .

$$\ldots \rightarrow \bigoplus_{x \in X_{(2)}} \mathrm{M}_{n+2}(k(x)) \stackrel{d_2}{\rightarrow} \bigoplus_{x \in X_{(1)}} \mathrm{M}_{n+1}(k(x)) \stackrel{d_1}{\rightarrow} \bigoplus_{x \in X_{(0)}} \mathrm{M}_n(k(x)),$$

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We denote this complex by  $C_{\bullet}(X, M_n)$  and consider it as a homological complex with the direct sum  $\bigoplus_{x \in X_{(i)}} M_{n+i}(k(x))$  in degree *i*.

The *i*-th cycle homology group  $H_i(X, M_n)$  of  $M_n$  over X is then defined as  $H_i(C_{\bullet}(X, M_n))$ .

**Example.** Note that there is a natural isomorphism

 $\mathrm{H}_i(X,\mathrm{K}^M_{-i})\simeq \mathrm{CH}_i(X)$ 

for all  $i \ge 0$ , where we have set  $\mathbf{K}_{-i}^M \equiv 0$  for i < 0.

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Let G be an algebraic group over k of dimension s and X a G-variety. To define an equivariant *i*-th cycle homology group with coefficients in the cycle module  $M_*$  we chose a linear representation V of G, such that there is an open subscheme  $U \hookrightarrow V$  with  $\operatorname{codim}_V(V \setminus U) \ge c = \dim X$  on which G acts freely.

By shrinking U we can assume that  $U \longrightarrow U/G$  is a principal bundle. The later assures that  $X \times^G U := (X \times_k U)/G$  exists in the category of k-varieties.

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We call the pair (U, V) an (X, G)-admissible pair for the G-variety X. Note that for a finite number of G-varieties there always exist a pair (U, V) which is admissible for all of them.

**Definition.** The *i*-th *G*-equivariant cycle homology group with values in the cycle module  $M_*$  over k is defined as

$$\mathrm{H}_{i}^{G}(X, \mathrm{M}_{*}) := \mathrm{H}_{i+l-s}(X \times^{G} U, \mathrm{M}_{*-(l-s)}),$$

where  $s = \dim G$  and (U, V) is a (X, G)-admissible pair with dim V = I and  $\operatorname{codim}_V V \setminus U \ge \dim X$ . If X is smooth, we define

$$\mathrm{H}^{i}_{G}(X, \mathrm{M}_{n}) := \mathrm{H}^{G}_{\dim X - i}(X, \mathrm{M}_{n - \dim X}) = \mathrm{H}^{i}(X \times^{G} U, \mathrm{M}_{n})$$

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**Theorem (Gille, Z.)** Definitions of the cyclic (co)homology don't depend on the choice of an admissible pair and the functor

 $\mathtt{h}_{G}\colon X\mapsto \mathrm{H}^{*}_{G}(X, \mathrm{M}_{*})$ 

provides an example of a (graded) essential *G*-equivariant pretheory.

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## We provide now a version of Merkurjev's equivariant K-theory spectral sequence for the equivariant cycle homology.

Let X be a G-variety, where G is an algebraic group over k, and  $T \subseteq G$  a split torus of rank m. Let  $\chi_1, \ldots, \chi_m$  be a basis of the character group  $T^* = \text{Hom}(T, \mathbb{G}_m)$ .

Let T act on the affine space  $\mathbb{A}_k^m = \operatorname{Spec} k[x_1, \dots, x_m]$  by

$$t \cdot (a_1, \ldots, a_m) \longmapsto (\chi_1(t) \cdot a_1, \ldots, \chi_m(t) \cdot a_m),$$

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Let  $Z_i \subset \mathbb{A}_k^m$  be the hyperplane defined by  $x_i = 0$  for i = 1, ..., m. Then  $X \times_k Z_i$  are *T*-subvarieties of  $\mathbb{A}_X^m$  and, therefore, we have closed subschemes

$$(X \times_k Z_1) \times^G U, \ldots, (X \times_k Z_m) \times^G U$$

of  $\mathbb{A}_X \times^G U$ , where (U, V) is a  $(\mathbb{A}_X^m, T)$ -admissible pair.

Since  $U \rightarrow U/T$  is a *T*-torsor we have

$$\bigcap_{j \notin I} (X \times_k Z_j) \times^G U = (X \times_k Z_I) \times^G U$$

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$$\widetilde{E}_1^{p,q} = \bigoplus_{|I|=p} \mathrm{H}_{-q-m}^T(X \times_k Z_I, \mathrm{M}_n) \implies \mathrm{H}_{-p-q}^T(X \times_k T, \mathrm{M}_n).$$

Using  $\operatorname{H}_{-p-q}^{T}(X \times_{k} T, \operatorname{M}_{n}) \simeq \operatorname{H}_{-p-q-m}(X, \operatorname{M}_{n+m})$  and the fact that  $Z_{I} = \mathbb{A}_{k}^{|I|} = \operatorname{Spec} k[x_{i}, i \in I] \hookrightarrow \mathbb{A}_{k}^{m}$  is a *T*-equivariant vector bundle over k, we obtain that the pull-back

$$\pi_{IT}^* \colon \operatorname{H}_{-q-m-|I|}^{T}(X, \operatorname{M}_{n+|I|}) \longrightarrow \operatorname{H}_{-q-m}^{T}(X \times_k Z_I, \operatorname{M}_n)$$

is an isomorphism, where  $\pi_I : X \times_k Z_I \to X$  is the projection. Replacing q by q + m the spectral sequence then takes the following form

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Theorem (Gille, Z.) There is a convergent spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} \mathrm{H}_{-q-p}^{T}(X, \mathrm{M}_{n+p}) \implies \mathrm{H}_{-p-q}(X, \mathrm{M}_{n+m}).$$

### Part II. Torsors and equivariant maps

Let  $h_H$  be a *H*-equivariant pretheory over *k*. We embed *S* into the affine space  $\operatorname{End}_k(V)$  as a *S*-equivariant (and, hence, *H*-equivariant) open subset.

Let  $\phi \colon S \to \text{pt}$  denote the structure map. The induced pull-back  $\phi_H^*$  factors as the composite of pull-backs

$$h_H(\mathrm{pt}) \xrightarrow{\simeq} h_H(\mathrm{End}_k(V)) \twoheadrightarrow h_H(S),$$

where the first map is an isomorphism by homotopy invariance and the second map is surjective by the localization property.

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#### Therefore we obtain the following

#### **Lemma 1.** The induced pull-back $\phi_H^*$ is surjective.

Let  $\mu_s \colon S \to S$  denote the right multiplication by  $s \in S(k)$ . Since  $\phi \circ \mu_s = \phi$  as morphisms over k and  $\mu_s$  is H-equivariant, we have  $(\mu_s)^*_H \circ \phi^*_H = \phi^*_H$ . Since  $\phi^*_H$  is surjective by Lemma 1, this proves that

**Lemma 2.** The induced pull-back  $(\mu_s)^*_H \colon h_H(S) \to h_H(S)$  is the identity.

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## Let G be an algebraic subgroup of S such that $H \subseteq G \subseteq S$ so that G is considered as a (left) H-variety.

Let *E* be a (left) *G*-variety over *k* and let  $\eta_E$ : Spec  $K \to E$  denote its generic point, where K = k(E).

Consider the G-equivariant (and, hence, H-equivariant) map

$$\psi_E \colon G_K = G \times_{\operatorname{Spec} k} \operatorname{Spec} K \stackrel{(\operatorname{id}, \eta_E)}{\longrightarrow} G \times_{\operatorname{Spec} k} E \longrightarrow E$$

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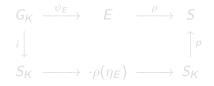
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Suppose that there is a *G*-equivariant map  $\rho: E \to S$  over *k*. Then there is a commutative diagram of *H*-equivariant maps

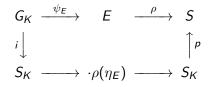


where the map *i* is the embedding, *p* is the projection  $S_K = S \times_{\text{Spec } k} \text{Spec } K \to S$  to the first factor and the bottom horizontal map is the multiplication by  $\rho(\eta_E)$ .

By the diagram  $(\psi_E)_H^* \circ \rho_H^* = (\rho \circ \psi_E)_H^*$  coincides with the pull back  $(\rho \circ \mu_{\rho(\eta_E)} \circ i)_H^*$ . By Lemma 2 the latter coincides with the pull-back  $i_H^* \circ p_H^*$ , hence, proving the following

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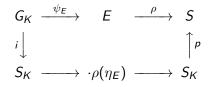


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$$(\psi_E)^*_H \circ \rho^*_H = i^*_H \circ p^*_H \colon h_H(S) \to \bar{h}_H(G_K).$$

We are now in position to prove the main result of this part

**Theorem (Gille, Z.)** Let  $H \subset G$  be algebraic groups and let  $h_H(-)$  be a *H*-equivariant pretheory. Then for any *G*-torsor *E* with K = k(E) we have

$$\operatorname{Im}(\varphi_H^*) \subseteq \operatorname{Im}((\psi_E)_H^*) \text{ in } \bar{\mathbf{h}}_H(G_K),$$

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Theorem then follows from Lemma 3 and the fact that there exists a finite dimensional k-vector space V and a G-equivariant map  $E \longrightarrow S = \operatorname{GL}(V)$ .

**Corollary.** Let  $H \subset G$  be algebraic groups over k and let  $h_H(-)$  be an essential H-equivariant pretheory. Then there exists a field extension 1/k and a G-torsor E over I with L = I(E) such that

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$$\operatorname{Im}(\varphi_{H}^{*}) = \operatorname{Im}(i_{H}^{*} \circ p_{H}^{*} \circ \phi_{H}^{*}) = \operatorname{Im}(i_{H}^{*} \circ p_{H}^{*}).$$

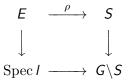
Theorem then follows from Lemma 3 and the fact that there exists a finite dimensional k-vector space V and a G-equivariant map  $E \longrightarrow S = GL(V)$ .

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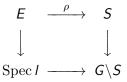


Since  $S \longrightarrow G \setminus S$  is a *G*-torsor, the map  $E \longrightarrow \operatorname{Spec} I$  is a *G*-torsor. The *I*-scheme *E* is a localization of *S* and, therefore, by (C) and (L) the pull-back  $\rho_H^* \colon h_H(S) \to \overline{h}_H(E) \to h_H(E)$  is surjective. This implies that the pull-back  $\rho_{H_I}^* \colon h_{H_I}(S_I) \to h_{H_I}(E)$  is surjective. It remains to apply the proof of the theorem over *I* and to observe that  $\operatorname{Im}(\varphi_{H_I}^*) = \operatorname{Im}((\psi_E)_{H_I}^* \circ \rho_{H_I}^*) = \operatorname{Im}((\psi_E)_{H_I}^*)$ .

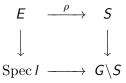


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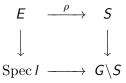
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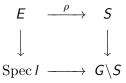
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$$c: S^*(T^*) \to CH(G/B),$$

where G/B is the variety of Borel subgroups. Let  $_{\xi}G/B$  be a twisted form by means of a *G*-torsor  $\xi$ . Then

#### $\operatorname{Im}(\operatorname{res}) \supseteq \operatorname{Im}(c),$

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It will be shown that for most of the examples of equivariant pretheories  $\widehat{h}_{H}(E)$  is a quotient of the cohomology ring h(G) of G.

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# Part III. Applications to equivariant oriented cohomology.

In this part we investigate the case of a B-equivariant oriented cohomology, where B is a Borel subgroup of a split semisimple linear algebraic group.

Let  $V = \mathbb{A}_k^n \times^T G$  be the associated vector bundle over G/T. By definition  $V = L_{G/T}(\chi_1) \oplus \ldots \oplus L_{G/T}(\chi_n)$ , where  $L_{G/T}$  is the associated line bundle, and  $G = T \times^T G$  embeds into V as the complement of the union of zero-sections

$$V_j = \bigoplus_{j \neq i} L_{G/T}(\chi_i) = Z_j \times^T G, \quad i = 1, \dots, n.$$

Let *G* be a split semisimple linear algebraic group of rank *n* over a field *k* and let *T* be a split maximal torus of *G*. Following the construction of the spectral sequence we consider the action of *T* on the affine space  $\mathbb{A}_k^n$  with weights  $\chi_1, \ldots, \chi_n$  together with an action of *T* on *G* by left multiplication. Then *T* embeds into

 $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$  as the complement of the coordinates hyperplanes  $Z_i$ ,  $i = 1, \ldots, n$ .

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Let now h(-) be an oriented cohomology theory in the sense of Levine-Morel, i.e. a contravariant functor from the category of smooth varieties over k to the category of graded commutative rings satisfying certain axioms.

In particular, if X is a k-variety with an open subvariety  $\iota \colon U \hookrightarrow X$  there is an exacts sequence

$$\mathtt{h}(Z) \stackrel{j_*}{\to} \mathtt{h}(X) \stackrel{\iota^*}{\to} \mathtt{h}(U) o 0$$

where  $j: Z = X \setminus U \hookrightarrow X$  is the closed complement of U, and there is also a first Chern class which we denote by  $c_1^{h}$ .

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Having such a theory h(-) we get from the localization sequence (by induction) an exact sequence

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By the properties of the first Chern class we have

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Proposition (Gille, Z.) There is an isomorphism of rings

 $\mathbf{h}(G) \simeq \mathbf{h}(G/B) / \left( c_1^{\mathbf{h}}(L_{G/B}(\chi_1)), \dots, c_1^{\mathbf{h}}(L_{G/B}(\chi_n)) \right),$ 

where  $\chi_1, \ldots, \chi_n$  is a basis of the character group  $T^*$ .

Note that the case of h = CH Chow groups is due to Grothendieck and the case of  $K_0$  – to Merkurjev.

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(i) h<sub>B</sub>(E) = h(E/B) for every G-torsor E, where h(−) is an oriented cohomology in the sense of Levine-Morel.
(ii) h<sub>B</sub>(B<sub>K</sub>) = h(pt) and h<sub>B</sub>(G<sub>K</sub>) ~ h(G/B)

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Then the ring  $\hat{\mathbf{h}}_B(E) = \bar{\mathbf{h}}_B(G_K) \otimes_{\mathbf{h}_B(E)} \bar{\mathbf{h}}_B(B_K)$  can be identified with a quotient of  $\bar{\mathbf{h}}_B(G_K) \simeq \mathbf{h}(G/B)$  modulo the ideal generated by non-constant elements from the image of the restriction  $(\psi_E)_B^* \colon \mathbf{h}(E/B) \to \mathbf{h}(G/B).$ 

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### Consider the equivariant Chow groups $h_B(-) = CH^B(-)$ .

Let E be a G-torsor.

The ring  $h_B(pt)$  can be identified with the symmetric algebra  $S(T^*)$  and the map

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## The map $(\psi_E)^*_B$ coincides with the restriction map res: $\operatorname{CH}(E/B) \longrightarrow \operatorname{CH}(G/B)$ ,

where E/B is the twisted form of G/B by means of E and the map

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Therefore, for an arbitrary G-torsor E the ring

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is a quotient ring of CH(G/B) modulo the ideal generated by non-constant elements from the image of the restriction  $CH(E/B) \rightarrow CH(G/B)$ .

Observe that the characteristic map  $\varphi_B^*$  is not surjective in general. However, its image is a subgroup of finite index in  $\operatorname{CH}(G/B)$ measured by the torsion index  $\tau$  of G. This implies that for a G-torsor E we have  $\widehat{\mathbf{h}}_B(E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$ . If  $p \mid \tau$ , then there is an isomorphism

$$\widehat{\mathrm{h}}_B(E)\otimes_{\mathbb{Z}} \mathbb{Z}/p\simeq rac{\mathbb{Z}/p\left[x_1,\ldots,x_r
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where  $(j_1, \ldots, j_r)$  is the *J-invariant* of *G* twisted by *E* [Petrov, Semenov, Z.]. Observe that  $j_i \leq k_i$ ,  $i = 1 \ldots r$ , where  $k_i$  are defined via the *p-exceptional degrees* introduced by Kac, and for a generic torsor *E* we have equalities  $j_i = k_i$  for each *i*.

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### Consider the equivariant $K_0$ -groups $h_B(-) = K_0(B, -)$ .

Let *E* be a *G*-torsor. The ring  $h_B(pt)$  can be identified with the integral group ring  $\mathbb{Z}[T^*]$  and with the representation ring Rep *T* of *T*, i.e.

$$h_B(pt) = \mathbb{Z}[T^*] = \operatorname{Rep} T.$$

The map

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and applying the main theorem we obtain the following  $K_0$ -analogue of the Karpenko-Merkurjev result:

**Corollary (Gille, Z.)** Let E be a G-torsor over k and let E/B be a twisted form of G/B by E.

Then

(i)  $\mathfrak{c}(\mathbb{Z}[T^*]) \subseteq \operatorname{res}(\operatorname{K}_0(E/B));$ 

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# According to a result of Panin the image of the restriction map is given by the sublattice

 $\{i_{w,E}\cdot g_w\}_{w\in W},$ 

where W is the Weyl group of G,  $\{g_w\}_{w \in W}$  is the Steinberg basis of  $K_0(G/B)$  and  $\{i_{w,E}\}$  are indexes of the respective Tits algebras.

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# Then by the corollary proven in the last part there exists a maximal set of indexes $\{m_w\}_{w\in W}$ such that

- (i)  $i_{w,E} \leq m_w$  for every  $w \in W$  and every torsor E;
- (ii) there exists *E* such that  $i_{w,E} = m_w$  for every  $w \in W$ ;
- (iii) the image of the characteristic map  $\varphi_B^*(\mathbb{Z}[\mathcal{T}^*])$  coincides with the sublattice  $\{m_w \cdot g_w\}_{w \in W}$ , hence, providing a way to compute  $m_w$ .

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The indexes  $m_w$  are called the maximal Tits indexes. They have been extensively studied by Merkurjev, Panin and Wadsworth.

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$$\widehat{\mathrm{h}}_B(E) = \mathrm{K}_0(G/B) \otimes_{\mathbb{Z}[T^*]} \mathbb{Z} \simeq \mathrm{K}_0(G),$$

where the last isomorphism follows by the corollary.

Hence, for an arbitrary G-torsor E

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# Consider the equivariant algebraic cobordism $h_B(-) = \Omega^B(-)$ . Let *E* be a *G*-torsor.

The completion  $h_B(pt)^{\wedge}$  of  $h_B(pt)$  at the augmentation ideal, (the kernel of  $h_B(pt) \rightarrow h_B(B)$ ) can be identified with the formal group ring  $\mathbb{L}[[T^*]]_U$  introduced by [Calmes, Petrov, Z.], where  $\mathbb{L}$  is the Lazard ring and U denotes the universal formal group law.

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## Examples: Equivariant algebraic cobordism

The map

$$\varphi_B^* \colon \mathbb{L}[[T^*]]_U = h_B(\mathrm{pt})^{\wedge} \longrightarrow \bar{h}_B(G_K) = \Omega(G/B)$$

coincides with the characteristic map from [Calmes, Petrov, Z.] and its image is generated by the first Chern classes.

The map  $(\psi_E)_B^*$  coincides with the restriction map

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### Examples: Equivariant algebraic cobordism

By the corollary for an arbitrary G-torsor E we have

$$\widehat{\mathrm{h}}_B(E) = \Omega(G/B) \otimes_{\mathrm{Im}(\mathrm{res})} \mathbb{L}.$$

is a quotient of the ring  $\Omega(G/B)$  modulo the image of the restriction  $\Omega(E/B) \rightarrow \Omega(G/B)$  from the kernel of the augmentation. And for a generic *G*-torsor *E* we obtain an isomorphism

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- What is the analogue of *p*-exceptional degrees/maximal Tits indices for algebraic Morava *K*-theories/cobordisms ? The same question for cycle (co)homology theories.
- How to compute h(G) in general ?
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