# Equivariant pretheories and invariants of torsors

Kirill Zaynullin

Department of Mathematics and Statistics
University of Ottawa

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This is a report on the joint paper with Stefan Gille (http://arxiv.org/abs/1007.3780).

- All schemes/varieties are defined over the base field $k$.
- By a scheme over a field $k$ ($k$-scheme) we mean a reduced separated Noetherian scheme over $k$.
- By a variety over a field $k$ ($k$-variety) we mean a quasi-projective scheme of finite type over $k$.
- By $\mathrm{pt}$ we denote $\mathrm{Spec}\, k$.
- If $l/k$ is a field extension and $X$ is a $k$-scheme, we define $X_l = X \times_{pt} \mathrm{Spec}\, l$ to be the respective base change.
- By an algebraic group we mean an *affine smooth group scheme* over $k$.
- By an action of an algebraic group $G$ on a scheme $X$ we mean a morphism $G \times_{pt} X \to X$ of schemes over $k$ (all group actions are assumed to be on the left), subject to the usual axioms.
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# Part I. Equivariant pretheories.

Examples:

- equivariant Chow- and K-theory
- equivariant algebraic cobordism
- equivariant cycle (co)homology
- spectral sequence for equivariant cycle homology

Let $G$ be an algebraic group over a field $k$.

Consider a contravariant functor from the category of smooth $G$-varieties over $k$ to the category of abelian groups

$$\mathrm{h}_G\colon\ G\text{-}\mathrm{Sm}_k \longrightarrow \mathfrak{Ab}, \quad X \mapsto \mathrm{h}_G(X).$$

Given $X, Y \in G\text{-}\mathrm{Sm}_k$ and a $G$-equivariant map $f\colon X \to Y$ the induced functorial map $\mathrm{h}_G(Y) \to \mathrm{h}_G(X)$ is called a *pull-back* and is denoted by $f_G^*$.

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# Equivariant pretheories. Definition.

The functor $\mathtt{h}_G\colon\ G\text{-}\mathrm{Sm}_k \to \mathfrak{Ab}$ is called a *G-equivariant pretheory* over $k$ if it satisfies the following two axioms:

H. (homotopy invariance) For a $G$-equivariant map $p\colon \mathbb{A}^n_k \to \mathrm{pt}$ (where $G$ acts trivially on $\mathrm{pt}$) the induced pull-back

$$p_G^*\colon \mathtt{h}_G(\mathrm{pt}) \longrightarrow \mathtt{h}_G(\mathbb{A}^n_k)$$

is an isomorphism.

L. (localization) For a smooth $G$-variety $X$ and a $G$-equivariant open embedding $\iota\colon U \hookrightarrow X$ the induced pull-back

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is surjective.

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# Essential pretheories

A $G$-equivariant pretheory $\mathrm{h}_G$ is called *essential* if it can be extended to the category of essentially smooth varieties $G\text{-}\mathrm{Ess}_k$ with $G$-equivariant flat morphisms, i.e.

$$\mathrm{h}_G\colon\; G\text{-}\mathrm{Ess}_k \longrightarrow \mathfrak{Ab},$$

in such a way that

C. For every $f\colon X \to Y$ and $U \subseteq Y$ as above the canonical morphism induced by the pull-backs

$$\bar{\mathrm{h}}_G(X_K) \to \mathrm{h}_{G_K}(X_K)$$

is an isomorphism.

Here $X_K$ denotes the generic fiber of $f$ and $\bar{\mathrm{h}}_G(X_K)$ denotes the colimit $\varinjlim_{U\subseteq Y} \mathrm{h}_G(f^{-1}(U))$, where $U$ ranges over the set of non-empty open $G$-equivariant subsets of $Y$.

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# Examples: Equivariant Chow-theory and $K$-theory

Let $G$ be an algebraic group over $k$ and let $X$ be a smooth $G$-variety. The functor

$$\mathrm{h}_G \colon X \longmapsto \mathrm{CH}^*_G(X) = \bigoplus_{i \in \mathbb{Z}} \mathrm{CH}^i_G(X)$$

where $\mathrm{CH}^*_G(X)$ is the equivariant Chow-theory of Totaro, Edidin and Graham, provides an example of an essential $G$-equivariant pretheory.

Consider the category $\mathcal{P}(G, X)$ of locally free $G$-modules on $X$. It is an exact category and following Thomason one defines the $i$-th $G$-equivariant $K$-group $\mathrm{K}_i(G, X)$ as Quillen's $i$-th $K$-group of $\mathcal{P}(G, X)$. The functor

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# Example: Equivariant algebraic cobordism

Assume that $char(k) = 0$. Consider a family of pairs $(V_i, U_i)_{i\in\mathbb{N}}$ of vector spaces with $U_i \subseteq V_i$ endowed with an action of an algebraic group $G$ such that

(i) $G$ acts freely on $U_i$ and $U_i \to U_i/G$ is a $G$-torsor,

(ii) $V_{i+1} = V_i \oplus W_i$ for some $k$-subspace $W_i$, such that $U_i \oplus W_i \subseteq U_{i+1}$,

(iii) $\sup \dim V_i = \infty$, and

(iv) $\mathrm{codim}_{V_i}(V_i \setminus U_i) < \mathrm{codim}_{V_{i+1}}(V_{i+1} \setminus U_{i+1})$, where we consider $V_i$ as an affine space over $k$.

Observe that assumption (i) ensures that the quotient $X \times^G U := (X \times_k U)/G$ is a quasi-projective variety over $k$. Moreover, it is smooth over $k$ by the descent.

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# Examples: Equivariant algebraic cobordism

Let $G$ be connected. Then the $n$-th *equivariant cobordism group* of a smooth $G$-variety $X$ is defined [Heller and Malagón-López] by

$$\Omega_n^G(X) := \varprojlim_i \Omega_{n-\dim G+\dim U_i}(X \times^G U_i),$$

where $\Omega_*(X)$ denotes the ring of algebraic cobordism of Levine-Morel.

The functor

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# Equivariant cycle cohomology

Following Rost we consider a *cycle module* over the field $k$ that is a (covariant) functor $\mathrm{M}_*$ from the category of field extensions of $k$ to the category of graded abelian groups satisfying several axioms.

The prototype of such a functor is *Milnor K-theory* $\mathrm{K}_*^M$, and by the very definition $\mathrm{M}_*(E) = \bigoplus_{i \in \mathbb{Z}} \mathrm{M}_i(E)$ is a graded $\mathrm{K}_*^M(E)$-module for all field extensions $E \supseteq k$.

Given a quasi-projective $k$-variety $X$ and a cycle module $\mathrm{M}_*$ Rost had defined a complex, the so called *cycle complex* (generalizing a construction of Kato for Milnor $K$-theory):

$$\ldots \to \bigoplus_{x \in X_{(2)}} \mathrm{M}_{n+2}(k(x)) \stackrel{d_2}{\to} \bigoplus_{x \in X_{(1)}} \mathrm{M}_{n+1}(k(x)) \stackrel{d_1}{\to} \bigoplus_{x \in X_{(0)}} \mathrm{M}_n(k(x)),$$

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# Equivariant cycle cohomology

We denote this complex by $\mathrm{C}_\bullet(X, \mathrm{M}_n)$ and consider it as a homological complex with the direct sum $\bigoplus_{x \in X_{(i)}} \mathrm{M}_{n+i}(k(x))$ in degree $i$.

The $i$-th cycle homology group $\mathrm{H}_i(X, \mathrm{M}_n)$ of $\mathrm{M}_n$ over $X$ is then defined as $\mathrm{H}_i(\mathrm{C}_\bullet(X, \mathrm{M}_n))$.

**Example.** Note that there is a natural isomorphism

$$\mathrm{H}_i(X, \mathrm{K}^M_{-i}) \simeq \mathrm{CH}_i(X)$$

for all $i \geq 0$, where we have set $\mathrm{K}^M_{-i} \equiv 0$ for $i < 0$.

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To introduce the equivariant cycle homology we adapt the definition of equivariant Chow groups due to Edidin and Graham (see also works by Guillot and Totaro).

Let $G$ be an algebraic group over $k$ of dimension $s$ and $X$ a $G$-variety. To define an equivariant $i$-th cycle homology group with coefficients in the cycle module $\mathrm{M}_*$ we chose a linear representation $V$ of $G$, such that there is an open subscheme $U \hookrightarrow V$ with $\mathrm{codim}_V(V \setminus U) \geq c = \dim X$ on which $G$ acts freely.

By shrinking $U$ we can assume that $U \longrightarrow U/G$ is a principal bundle. The later assures that $X \times^G U := (X \times_k U)/G$ exists in the category of $k$-varieties.

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We call the pair $(U, V)$ an $(X, G)$-*admissible pair* for the $G$-variety $X$. Note that for a finite number of $G$-varieties there always exist a pair $(U, V)$ which is admissible for all of them.

**Definition.** The $i$-th *$G$-equivariant cycle homology group* with values in the cycle module $\mathrm{M}_*$ over $k$ is defined as

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where $s = \dim G$ and $(U, V)$ is a $(X, G)$-admissible pair with $\dim V = l$ and $\mathrm{codim}_V V \setminus U \geq \dim X$. If $X$ is smooth, we define

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**Theorem (Gille, Z.)** Definitions of the cyclic (co)homology don't depend on the choice of an admissible pair and the functor

$$\mathtt{h}_G \colon X \mapsto \mathrm{H}^*_G(X, \mathrm{M}_*)$$

provides an example of a (graded) essential $G$-equivariant pretheory.

We provide now a version of Merkurjev's equivariant $K$-theory spectral sequence for the equivariant cycle homology.

Let $X$ be a $G$-variety, where $G$ is an algebraic group over $k$, and $T \subseteq G$ a split torus of rank $m$. Let $\chi_1, \ldots, \chi_m$ be a basis of the character group $T^* = \mathrm{Hom}(T, \mathbb{G}_m)$.

Let $T$ act on the affine space $\mathbb{A}^m_k = \mathrm{Spec}\, k[x_1, \ldots, x_m]$ by

$$t \cdot (a_1, \ldots, a_m) \longmapsto (\chi_1(t) \cdot a_1, \ \ldots \ , \chi_m(t) \cdot a_m) \, ,$$

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Let $Z_i \subset \mathbb{A}_k^m$ be the hyperplane defined by $x_i = 0$ for $i = 1, \dots, m$. Then $X \times_k Z_i$ are $T$-subvarieties of $\mathbb{A}_X^m$ and, therefore, we have closed subschemes

$$(X \times_k Z_1) \times^G U\,,\ \dots\ ,(X \times_k Z_m) \times^G U$$

of $\mathbb{A}_X \times^G U$, where $(U, V)$ is a $(\mathbb{A}_X^m, T)$-admissible pair.

Since $U \to U/T$ is a $T$-torsor we have

$$\bigcap_{j \notin I} (X \times_k Z_j) \times^G U \ = (X \times_k Z_I) \times^G U$$

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Following Levine's construction of spectral sequence for Quillen $K$-theory we obtain then a convergent spectral sequence

$$\tilde{E}_1^{p,q} = \bigoplus_{|I|=p} \mathrm{H}^T_{-q-m}(X \times_k Z_I, \mathrm{M}_n) \implies \mathrm{H}^T_{-p-q}(X \times_k T, \mathrm{M}_n).$$

Using $\mathrm{H}^T_{-p-q}(X \times_k T, \mathrm{M}_n) \simeq \mathrm{H}_{-p-q-m}(X, \mathrm{M}_{n+m})$ and the fact that $Z_I = \mathbb{A}_k^{|I|} = \mathrm{Spec}\, k[x_i, i \in I] \hookrightarrow \mathbb{A}_k^m$ is a $T$-equivariant vector bundle over $k$, we obtain that the pull-back

$$\pi_{I\,T}^{*} \colon \mathrm{H}^T_{-q-m-|I|}(X, \mathrm{M}_{n+|I|}) \longrightarrow \mathrm{H}^T_{-q-m}(X \times_k Z_I, \mathrm{M}_n)$$

is an isomorphism, where $\pi_I \colon X \times_k Z_I \to X$ is the projection. Replacing $q$ by $q+m$ the spectral sequence then takes the following form

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**Theorem (Gille, Z.)** There is a convergent spectral sequence

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# Part II. Torsors and equivariant maps

Let $S = \mathrm{GL}(V)$ be the group of automorphisms of a finite dimensional $k$-vector space $V$. Let $H$ be an algebraic subgroup of $S$. Consider $S$ as a (left) $H$-variety.

Let $\mathrm{h}_H$ be a $H$-equivariant pretheory over $k$. We embed $S$ into the affine space $\mathrm{End}_k(V)$ as a $S$-equivariant (and, hence, $H$-equivariant) open subset.

Let $\phi\colon S \to \mathrm{pt}$ denote the structure map. The induced pull-back $\phi_H^*$ factors as the composite of pull-backs

$$\mathrm{h}_H(\mathrm{pt}) \xrightarrow{\simeq} \mathrm{h}_H(\mathrm{End}_k(V)) \twoheadrightarrow \mathrm{h}_H(S),$$

where the first map is an isomorphism by homotopy invariance and the second map is surjective by the localization property.

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Therefore we obtain the following

**Lemma 1.** The induced pull-back $\phi_H^*$ is surjective.

Let $\mu_s \colon S \to S$ denote the right multiplication by $s \in S(k)$. Since $\phi \circ \mu_s = \phi$ as morphisms over $k$ and $\mu_s$ is $H$-equivariant, we have $(\mu_s)_H^* \circ \phi_H^* = \phi_H^*$. Since $\phi_H^*$ is surjective by Lemma 1, this proves that

**Lemma 2.** The induced pull-back $(\mu_s)_H^* \colon \mathtt{h}_H(S) \to \mathtt{h}_H(S)$ is the identity.

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Let $G$ be an algebraic subgroup of $S$ such that $H \subseteq G \subseteq S$ so that $G$ is considered as a (left) $H$-variety.

Let $E$ be a (left) $G$-variety over $k$ and let $\eta_E \colon \operatorname{Spec} K \to E$ denote its generic point, where $K = k(E)$.

Consider the $G$-equivariant (and, hence, $H$-equivariant) map

$$\psi_E \colon G_K = G \times_{\operatorname{Spec} k} \operatorname{Spec} K \xrightarrow{(\mathrm{id}, \eta_E)} G \times_{\operatorname{Spec} k} E \longrightarrow E$$

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Suppose that there is a $G$-equivariant map $\rho\colon E \to S$ over $k$.
Then there is a commutative diagram of $H$-equivariant maps

$$\begin{array}{ccccc} G_K & \xrightarrow{\psi_E} & E & \xrightarrow{\rho} & S \\ {\scriptstyle i}\downarrow & & & & \uparrow{\scriptstyle p} \\ S_K & \longrightarrow & \cdot\rho(\eta_E) & \longrightarrow & S_K \end{array}$$

where the map $i$ is the embedding, $p$ is the projection $S_K = S \times_{\operatorname{Spec} k} \operatorname{Spec} K \to S$ to the first factor and the bottom horizontal map is the multiplication by $\rho(\eta_E)$.

By the diagram $(\psi_E)^*_H \circ \rho^*_H = (\rho \circ \psi_E)^*_H$ coincides with the pull back $(p \circ \mu_{\rho(\eta_E)} \circ i)^*_H$. By Lemma 2 the latter coincides with the pull-back $i^*_H \circ p^*_H$, hence, proving the following

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**Lemma 3.** Let $E$ be a $G$-variety together with a $G$-equivariant map $\rho\colon E \to S$. Then we have

$$(\psi_E)^*_H \circ \rho^*_H = i^*_H \circ p^*_H \colon \mathtt{h}_H(S) \to \bar{\mathtt{h}}_H(G_K).$$

We are now in position to prove the main result of this part

**Theorem (Gille, Z.)** Let $H \subset G$ be algebraic groups and let $\mathtt{h}_H(-)$ be a $H$-equivariant pretheory. Then for any $G$-torsor $E$ with $K = k(E)$ we have

$$\mathrm{Im}(\varphi^*_H) \subseteq \mathrm{Im}((\psi_E)^*_H) \text{ in } \bar{\mathtt{h}}_H(G_K),$$

where $\varphi\colon G_K \to \mathrm{pt}$ is the structure map.

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We are now in position to prove the main result of this part

**Theorem (Gille, Z.)** Let $H \subset G$ be algebraic groups and let $\mathtt{h}_H(-)$ be a $H$-equivariant pretheory. Then for any $G$-torsor $E$ with $K = k(E)$ we have

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Theorem then follows from Lemma 3 and the fact that there exists a finite dimensional $k$-vector space $V$ and a $G$-equivariant map $E \longrightarrow S = \mathrm{GL}(V)$.

**Corollary.** Let $H \subset G$ be algebraic groups over $k$ and let $\mathrm{h}_H(-)$ be an essential $H$-equivariant pretheory. Then there exists a field extension $l/k$ and a $G$-torsor $E$ over $l$ with $L = l(E)$ such that

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Since $S \longrightarrow G\backslash S$ is a $G$-torsor, the map $E \longrightarrow \mathrm{Spec}\, l$ is a $G$-torsor. The $l$-scheme $E$ is a localization of $S$ and, therefore, by (C) and (L) the pull-back $\rho_H^* \colon \mathrm{h}_H(S) \to \bar{\mathrm{h}}_H(E) \to \mathrm{h}_H(E)$ is surjective. This implies that the pull-back $\rho_{H_l}^* \colon \mathrm{h}_{H_l}(S_l) \to \mathrm{h}_{H_l}(E)$ is surjective. It remains to apply the proof of the theorem over $l$ and to observe that $\mathrm{Im}(\varphi_{H_l}^*) = \mathrm{Im}((\psi_E)_{H_l}^* \circ \rho_{H_l}^*) = \mathrm{Im}((\psi_E)_{H_l}^*)$.

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The latter theorem and the corollary can be viewed as a generalization of the following result

**Theorem (Karpenko-Merkurjev).** Consider the characteristic map for the Chow theory

$$c\colon S^*(T^*) \to CH(G/B),$$

where $G/B$ is the variety of Borel subgroups. Let ${}_\xi G/B$ be a twisted form by means of a $G$-torsor $\xi$. Then

$$\mathrm{Im}(res) \supseteq \mathrm{Im}(c),$$

where $res\colon CH({}_\xi G/B) \to CH(G/B)$ is the restriction map. Moreover, there exists a torsor over a field extension of $k$ such that the equality $\mathrm{Im}(res) = \mathrm{Im}(c)$ holds.

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Indeed, we have

$$T = H,\ G = G,\ S^*(T^*) = CH_T(pt),\ CH(G/B) = \overline{CH}_T(G)$$

$$c = \varphi_H^* \text{ and } res = (\psi_E)_H^*.$$

Note that the result by Karpenko-Merkurjev plays a fundamental role in computations of canonical/essential dimensions, discrete motivic invariants of $G$ and in the study of splitting properties of $G$-torsors.

To measure the difference between $\bar{\mathrm{h}}_H(G_K)$ and the image $\mathrm{Im}((\psi_E)_H^*)$ we introduce the following

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$$\widehat{\mathtt{h}}_H(E) := \bar{\mathtt{h}}_H(G_K) \otimes_{\mathtt{h}_H(E)} \bar{\mathtt{h}}_H(H_K),$$

where $\bar{\mathtt{h}}_H(G_K)$ is the $\mathtt{h}_H(E)$-module via $(\psi_E)^*_H$ and $\bar{\mathtt{h}}_H(H_K)$ is the $\mathtt{h}_H(E)$-module via the composite $\mathtt{h}_H(E) \overset{(\psi_E)^*_H}{\to} \bar{\mathtt{h}}_H(G_K) \to \bar{\mathtt{h}}_H(H_K)$ with the last map induced by the embedding $H \subset G$.

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# Part III. Applications to equivariant oriented cohomology.

In this part we investigate the case of a $B$-equivariant oriented cohomology, where $B$ is a Borel subgroup of a split semisimple linear algebraic group.

Let $G$ be a split semisimple linear algebraic group of rank $n$ over a field $k$ and let $T$ be a split maximal torus of $G$. Following the construction of the spectral sequence we consider the action of $T$ on the affine space $\mathbb{A}^n_k$ with weights $\chi_1, \ldots, \chi_n$ together with an action of $T$ on $G$ by left multiplication. Then $T$ embeds into $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ as the complement of the coordinates hyperplanes $Z_i$, $i = 1, \ldots, n$.

Let $V = \mathbb{A}^n_k \times^T G$ be the associated vector bundle over $G/T$. By definition $V = L_{G/T}(\chi_1) \oplus \ldots \oplus L_{G/T}(\chi_n)$, where $L_{G/T}$ is the associated line bundle, and $G = T \times^T G$ embeds into $V$ as the complement of the union of zero-sections

$$V_j = \bigoplus_{j \neq i} L_{G/T}(\chi_i) = Z_j \times^T G\,, \quad i = 1, \ldots, n\,.$$

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In particular, if $X$ is a $k$-variety with an open subvariety $\iota\colon U \hookrightarrow X$ there is an exacts sequence

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Having such a theory $\mathtt{h}(-)$ we get from the localization sequence (by induction) an exact sequence

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By the properties of the first Chern class we have

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# Oriented cohomology of a group

Let $B$ be a Borel subgroup of $G$ containing $T$ and let $G/B$ be the variety of Borel subgroups. The composite of projections $V \to G/T \to G/B$ is a chain of affine bundles. Therefore, by the homotopy invariance there is an isomorphism $\mathrm{h}(G/B) \xrightarrow{\simeq} \mathrm{h}(V)$ compatible with the Chern classes and we obtain the following

**Proposition (Gille, Z.)** There is an isomorphism of rings

$$\mathrm{h}(G) \simeq \mathrm{h}(G/B)/\left(c_1^{\mathrm{h}}(L_{G/B}(\chi_1)), \ldots, c_1^{\mathrm{h}}(L_{G/B}(\chi_n))\right),$$

where $\chi_1, \ldots, \chi_n$ is a basis of the character group $T^*$.

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Let $\mathrm{h}_B(-)$ be an $B$-equivariant pretheory to the category of commutative rings such that

(i) $\mathrm{h}_B(E) = \mathrm{h}(E/B)$ for every $G$-torsor $E$, where $\mathrm{h}(-)$ is an oriented cohomology in the sense of Levine-Morel.

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Then the ring $\hat{\mathrm{h}}_B(E) = \bar{\mathrm{h}}_B(G_K) \otimes_{\mathrm{h}_B(E)} \bar{\mathrm{h}}_B(B_K)$ can be identified with a quotient of $\bar{\mathrm{h}}_B(G_K) \simeq \mathrm{h}(G/B)$ modulo the ideal generated by non-constant elements from the image of the restriction $(\psi_E)^*_B \colon \mathrm{h}(E/B) \to \mathrm{h}(G/B)$.

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Then by the proposition we obtain the following

**Corollary (Gille, Z.).** Assume that the image of $\varphi_B^*$ is generated by the Chern classes $c_1^{\mathrm{h}}(L_{G/B}(\chi_i))$ of line bundles associated to the characters $\chi_i \in T^*$ $(i = 1 \dots n)$.
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# Examples: Equivariant Chow-theory

Consider the equivariant Chow groups $\mathtt{h}_B(-) = \mathrm{CH}^B(-)$.

Let $E$ be a $G$-torsor.

The ring $\mathtt{h}_B(\mathrm{pt})$ can be identified with the symmetric algebra $S(T^*)$ and the map

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where the last isomorphism follows by the corollary.

# Examples: Equivariant Chow-theory

The map $(\psi_E)^*_B$ coincides with the restriction map

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# Examples: Equivariant Chow-theory

Therefore, for an arbitrary $G$-torsor $E$ the ring

$$\widehat{\mathrm{h}}_B(E) = \mathrm{CH}(G/B) \otimes_{\mathrm{Im(res)}} \mathbb{Z}$$

is a quotient ring of $\mathrm{CH}(G/B)$ modulo the ideal generated by non-constant elements from the image of the restriction $\mathrm{CH}(E/B) \to \mathrm{CH}(G/B)$.

# Examples: Equivariant Chow-theory

Observe that the characteristic map $\varphi_B^*$ is not surjective in general. However, its image is a subgroup of finite index in $\mathrm{CH}(G/B)$ measured by the torsion index $\tau$ of $G$. This implies that for a $G$-torsor $E$ we have $\widehat{\mathrm{h}}_B(E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.

If $p \mid \tau$, then there is an isomorphism

$$\widehat{\mathrm{h}}_B(E) \otimes_{\mathbb{Z}} \mathbb{Z}/p \simeq \frac{\mathbb{Z}/p\,[x_1, \ldots, x_r]}{(x_1^{p^{j_1}}, \ldots, x_r^{p^{j_r}})},$$

where $(j_1, \ldots, j_r)$ is the *J-invariant* of $G$ twisted by $E$ [Petrov, Semenov, Z.]. Observe that $j_i \leq k_i$, $i = 1 \ldots r$, where $k_i$ are defined via the *p-exceptional degrees* introduced by Kac, and for a generic torsor $E$ we have equalities $j_i = k_i$ for each $i$.

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# Examples: Equivariant $K_0$

Consider the equivariant $K_0$-groups $\mathtt{h}_B(-) = \mathrm{K}_0(B, -)$.

Let $E$ be a $G$-torsor.

The ring $\mathrm{h}_B(\mathrm{pt})$ can be identified with the integral group ring $\mathbb{Z}[T^*]$ and with the representation ring $\mathrm{Rep}\, T$ of $T$, i.e.

$$\mathrm{h}_B(\mathrm{pt}) = \mathbb{Z}[T^*] = \mathrm{Rep}\, T.$$

The map

$$\varphi_B^* \colon \mathbb{Z}[T^*] = \mathrm{h}_B(\mathrm{pt}) \longrightarrow \bar{\mathrm{h}}_B(G_K) \simeq \mathrm{K}_0(G/B)$$

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As before the map $(\psi_E)^*_B$ coincides with the restriction map

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and applying the main theorem we obtain the following $K_0$-analogue of the Karpenko-Merkurjev result:

**Corollary (Gille, Z.)** Let $E$ be a $G$-torsor over $k$ and let $E/B$ be a twisted form of $G/B$ by $E$.
Then

(i) $\mathfrak{c}(\mathbb{Z}[T^*]) \subseteq \mathrm{res}(\mathrm{K}_0(E/B))$;

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According to a result of Panin the image of the restriction map is given by the sublattice

$$\{i_{w,E} \cdot g_w\}_{w \in W},$$

where $W$ is the Weyl group of $G$, $\{g_w\}_{w \in W}$ is the Steinberg basis of $\mathrm{K}_0(G/B)$ and $\{i_{w,E}\}$ are indexes of the respective Tits algebras.

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Then by the corollary proven in the last part there exists a maximal set of indexes $\{m_w\}_{w\in W}$ such that

(i) $i_{w,E} \leq m_w$ for every $w \in W$ and every torsor $E$;

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The indexes $m_w$ are called the *maximal Tits indexes*. They have been extensively studied by Merkurjev, Panin and Wadsworth. They are closely related to the dimensions of irreducible representations of $G$. Comparing with the case of Chow groups one observes that

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where the last isomorphism follows by the corollary.

Hence, for an arbitrary $G$-torsor $E$

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Consider the equivariant algebraic cobordism $\mathtt{h}_B(-) = \Omega^B(-)$.

Let $E$ be a $G$-torsor.

The completion $\mathtt{h}_B(\mathrm{pt})^\wedge$ of $\mathtt{h}_B(\mathrm{pt})$ at the augmentation ideal, (the kernel of $\mathtt{h}_B(pt) \to \mathtt{h}_B(B)$) can be identified with the formal group ring $\mathbb{L}[[T^*]]_U$ introduced by [Calmes, Petrov, Z.], where $\mathbb{L}$ is the Lazard ring and $U$ denotes the universal formal group law.

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- What is the analogue of $p$-exceptional degrees/maximal Tits indices for algebraic Morava $K$-theories/cobordisms ? The same question for cycle (co)homology theories.
- How to compute $\mathrm{h}(G)$ in general ?
- Using the isomorphism $\hat{\mathrm{h}}_B(E) \simeq \mathrm{h}(G)$ how to construct a direct sum decomposition in the category of $\mathrm{h}$-motives ? Say, for $K_0$ it should give a different proof of Panin's result.

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- What is the analogue of $p$-exceptional degrees/maximal Tits indices for algebraic Morava $K$-theories/cobordisms ? The same question for cycle (co)homology theories.
- How to compute $\mathtt{h}(G)$ in general ?
- Using the isomorphism $\hat{\mathtt{h}}_B(E) \simeq \mathtt{h}(G)$ how to construct a direct sum decomposition in the category of $\mathtt{h}$-motives ? Say, for $K_0$ it should give a different proof of Panin's result.