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Equivariant s-cobordism theorems

Dedicated to Professor Itiro Tamura on his 60th birthday

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§ 1. Introduction.

The classical h-cobordism theorem and the s-cobordism theorem have played an important role in studying differential topology [15], [16].
In the present paper, we discuss equivariant versions of these theorems.
Let G be a compact Lie group and X a finite G-CW-complex. In 1974, S. Illman [6] defined the equivariant Whitehead group WhG(X) of X and the equi-variant Whitehead torsion TG(f) for a G-homotopy equivalence f : X-+Y between finite G-CW-complexes X, Y as an element of WhG(X). When DG(f)=O, f is called a simple G-homotopy equivalence. In this paper, we deal with only smooth G-manifolds.

Let (W; X, Y) be a smooth G-h-cobordism. Namely W is a compact G-manifold with boundary 6W=X1LY (disjoint union) and the inclusions
iX: X -~W and iY: Y---~W

are G-homotopy equivalences.
When G is a finite group, W admits a unique smooth G-triangulation [7]. Accordingly the equivariant Whitehead torsion TG(iZ) is well-defined. On the other hand, the recent investigation of Matumoto and Shiota [13] enables us to define the equivariant Whitehead torsion rG(i X) even when G is a compact Lie group. Notice that DG(iI) is often written as DG(W, X).
As in the non-equivariant case, a G-h-cobordism (W; X, Y) is called a G-s-cobordism when DG(iX) vanishes.
We say that a G-h-cobordism (resp. G-s-cobordism) theorem holds if a G-h-cobordism (resp. G-s-cobordism) (W; X, Y) implies a G-dill eomorphism

W=XxI rel X

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where 1 is the interval [0, 1] with trivial G-action.
Unfortunately the G-h-cobordism theorem and the G-s-cobordism theorem do not hold in general [12]. Accordingly we need to add some assumptions for a theorem of this sort.

Let H, K be isotropy groups appearing in W and

WH=LW, WK=11W~ A it

be the decompositions to connected components. We now consider two condi-tions.

(*1) If W~Wf, , then dimWR-dimWH>_dimG+3 for any pair of com-ponents W j5 and W'.
(*2) If H is a maximal isotropy group, then
dimWf >_ dimG+6

for any component W~ .
Then our first theorem is the following

THEOREM 1. Let G be a compact Lie group and (W; X, Y) a G-s-cobordism.
If W satisfies the conditions (*1) and (*2) above, then we have a G-diffeomorphism

W NXXI rel X.

In particular, X is G-di ffeomorphic to Y.

If we stabilize a G-h-cobordism (W; X, Y) with respect to disks of suitable representations, then the conditions (*1) and (*2) are automatically satisfied. However the restriction homomorphism (to a closed subgroup H of G) Wh0(X)
- *WhH(X) is defined only for the case of the index I G/HI being finite, and we need to use such restriction homomorphisms to diagonal actions in stable ver-sions. Thus we assume hereafter that the group G is finite and have the fol-lowing

THEOREM 2 (stable equivariant s-cobordism theorem). Let G be a finite group and (W; X, Y) a G-s-cobordism. Then there exist an orthogonal G-representation space V and a G-diffeomorphism

W X V (1) N XX V(1) X I rel X X V (1) .

In particular, we have G-diffeomorphisms

XXV(1) ~ YXV(1) and XXSV(1) N YXSV(1).

Here V(1) (resp. SV(1)) denotes the closed unit disk (resp. the unit sphere) of V.

Let M,, M2 be closed G-manifolds. A G-homotopy equivalence f : M1-~M2 will be called a tangential G-homotopy equivalence if there exist a G-representa-

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tion space V and a G-vector bundle isomorphism :

T(M1) ®V ~ f *T (M2)® V

where T (M) are tangent G-vector bundles of M1 (i=1, 2), V is the trivial G-vector bundle M1 X V-~M1 and f *T (M2) is the induced G-vector bundle of T(M2) via the map f .
A tangential G-homotopy equivalence f : M1-+M2 is called a tangential simple G-homotopy equivalence if f is a simple G-homotopy equivalence.
Then we have the following equivariant version of [5], [14].

THEOREM 3. Let G be a finite group. Let M1 and M2 be closed G-manifolds and f : M1->M2 a G-map. Then f is tangential simple G-homotopy equivalence i f and only if there exist an orthogonal G-representation space V and a G-diffeomor-phism
J: M1 X V(1) -> M2 X V(1)
such that the following diagram
f M
1 X V(1) ---> M2 X V(1)

n

f

M1 -> M2

is G-homotopy commutative, where n are the projection maps.

REMARK. Browder and Quinn had an isovariant s-cobordism theorem in [20].

REMARK. An equivariant s-cobordism theorem is stated in [17]. Unfortu-nately the assumption of the theorem is not stated in terms of the equivariant torsion r0(W, X) in the sense of Illman [6]. One of our tasks for the proofs of Theorems 1 and 2 is to show that a filtration inherits the property of G-deforma-tion retractions (see § 4). Accordingly we can define equivariant torsions suc-cessively. The other task is to show that it follows from the assumption DG(W, X)=0 that these successive equivariant torsions also vanish.

REMARK. An equivariant s-cobordism theorem for finite G in the category
of PL and Top is studied in [18].

§ 2. Naturality of equivariant Whitehead torsions.

We first review some of the basic facts about equivariant simple homotopy theory for the benefit of the reader. For further details we refer to [2].
In [6], Illman described the basic properties of the equivariant Whitehead group WhG(X) for a finite G-CW-complex X, got a decomposition of WhG(X) and

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described it algebraically for abelian G.
Each element of WhG(X) is represented by a finite G-CW-pair (V, X) such that X is a strong G-deformation retract of V. The element represented by such a pair (V, X) is denoted by TG(V, X) and is called the Whitehead G-torsion of (V, X).
By a family EF of closed subgroups of G, we understand a . collection of closed subgroups H of G such that HEE EF implies (H)c F, where (H) denotes the conjugacy class of H.
For a family F of closed subgroups of G, Illman introduced the notion of restricted Whitehead group WhG(X,) consisting of those elements TG(V, X) such that all the isotropy groups of V -X belong to F. Then WhG(X,) is a sub-group of WhG(X).
In 1978, H. Hauschild [4] gave the natural direct sum decomposition

WhG(X) . Ji WhG(X, (H))
(H)

where (H) runs over all conjugacy classes of closed subgroups of G. He de-scribed WhG(X) algebraically based on this decomposition in a way.
Let H be a closed subgroup of G and X a G-space. We denote by XH the
H -fixed point set of X and by WH the quotient group NH/H where NH is the normalizer of H in G.
Then the G-action on X induces a WH action on XH and there holds the

following natural isomorphism

WhG(X, (H)) = WhwH(~'H, {e})

which is also due to Hauschild [4].
The WH action on XH induces the WH action on the set of connected com-

ponents of XH. Taking WH orbits of the induced action, we get a decomposi-tion
XH = Jt WHXa

a as a topological sum of WH subspaces, where the Xa 's are connected components of XH. Denote by AH the index set {a} of the above decomposition. We call each summand WH. X« a WH component of XH and Xa a representative component of the WH component WH• Xa .
Then there holds a direct sum decomposition [2]

WhWH(XH, {e}) ~ Jt WhWH(WH'Xa, {e}).

aEAN

We now put
WaH= {w~WHJ w.XHCXa }

which is a closed subgroup of WH. Xa is a WaH space and we can express

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WH' Xa = WH x XH .
WaH

Then there holds a kind of Shapiro isomorphism [2]

WhWH(WH•Xa, {e}) WhWaH(Xa, {e}).

We are now in a position to pass to universal covering spaces.

N

Denote by Xa the universal covering space of XH. Choose a point xo of
N XH. Then x1=7r1(XH, xo) operates on X« as the covering transformation group.
By [1], [8], we have a Lie group Pa satisfying the short exact sequence

1--~ 1 -~ T a-* W H -~ 1

N and X« is a Pa-space such that the Pa-action contains the 7r1-action and the
N covering projection p : X«-*X« is q-equivariant.
Then there holds an isomorphism [2]

N

WhWaH(X«, {e}) Whra(X«, {e}).

We now consider the final step of reductions of WhG(X).
Denote by l a, o the component of Pa including the unit element. As is well-known, Ta, o is a closed normal subgroup of Pa. Then we have the fol-lowing isomorphism [2]

N

Whra(X«, {e}) ^' Wh(ra/l a,o)

where the right hand side is the Whitehead group defined algebraically (see [3]). Putting all this together, we have the following theorem.

THEOREM 4. Let X be a finite G-CW-complex. Then we have a direct sum decomposition
WhG(X) N JL 11 W h(ral ra, o)
(H) aEAH

Since one verifies the naturalities of all the processes of the reductions above, one has the following theorem on which our theorems are based.

THEOREM 5 ([2]). Let f : X--~Y be a G-map between finite G-CW-complexes and H a closed subgroup of G. Suppose that the restriction f H : XH_-~YH gives a bijection of the connected components and induces isomorphisms of fundamental groups for any base points. Then there holds the isomorphism
IV

f,~ : WhG(X, (H)) --~ WhG(Y, (H)).

For the detailed proof of Theorems 4 and 5, see [2]. Theorem 4 is proved also by Illman [8] in a different approach.

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§ 3. Decomposition of G-manifolds.
In this section, we recall the decomposition theorem of smooth G-manifolds
of [11] for the benefit of the reader.
Let G be a compact Lie group. There is a partial order among the set of conjugacy classes of closed subgroups of G, i.e., (Hl)s(H2) if and only if there exists gEG such that gHlg-1CH2.
Let W be a compact G-manifold. We shall denote the isotropy group at
x W by G x, namely
Gx = {gE G I gx=x} .

For a closed subgroup H of G, we shall put
W(H) = {xEW I (Gx)=(H)} .
Since W is compact, there are only finitely many isotropy types, say

{(Gx) I xEW } = {(H1), (H2), ... , (Hk)} .
It is possible to arrange {(H1)} in such order that (Hti)>_(H;) implies i< j.
We get a filtration
W = Wl J W2 D ,.. Wk

consisting of compact G-manifolds Wi with corners such that

{(Gx) I x~W1} = {(Hz), (II +), ... , (Hk)}
as follows.
For this, we introduce some notations. Let 'r : E-~M be a differentiable G-vector bundle over a compact G-manifold M. As is well known, there is a G-invariant Riemannian metric < ,> on E. Concerning the metric < , >, we set

II v I = ./(v, v> for vEE E .
Then we put for r> 0,
E(r) = {vEE I Ilvli<<-_r},
SE(r) = {v~E I Iivll=r},
E(r) = E(r)-SE(r) = {vEE I Ilvll<r}

Obviously E(r) and SE(r) are compact G-manifolds.
Since (Hl) is a maximal conjugacy class, W(H1) is a compact G-invariant submanifold of W. We identify the normal bundle vl of W (H1) in W with an open tubular neighborhood of W(H1) in W and impose a G-invariant Riemannian metric on vl.

Concerning the metric on vl, we set

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W2 = W-(1).
Then W2 is a compact G-manifold with corner and satisfies

.

{(Gx)I x~W2} = {(H2), (Hs), ... , (11)}
Suppose that we get a filtration
W=W1~W2~...iWz

consisting of compact G-manifolds W; with corners such that
{(Gx)I x~W;} = {(H,), (M;+1), ... , (Hk)}
for every 3<_i. Since (Hi) is a maximal conjugacy class among the set

W1(H1) is a compact G-invariant submanif old of W1. We identify the normal bundle vi of W1(H1) in W1 with an open tubular neighborhood of W1(H1) in W1 and impose a G-invariant Riemannian metric on vi. Concerning this metric, we
set

W1+1= W1--v1(1)
Then Wi+1 is a compact G-manifold with corner and satisfies
{(Gx)I xEWi+1} = {(11+1), (Hti+2), ... , (Hk)} .
This completes the inductive construction.
Thus we have shown the following decomposition theorem.

THEOREM 6 ([11]). Let W be a compact G-manifold and (H1), ..., (Hi) the isotropy types appearing in W. Arrange {(H1)} in such order that (H1)>_(H~) im-plies i< j. Then there exist compact G-manifolds M1 with corners and G-vector bundles ,.-*M1 ~for 1 <_ i<_ k such that
M1(H1) = Mi = W (H1)
G and that we have a decomposition
W ` y1(1)Uy2(1)u ...
Moreover i f we set

Wi = y1(1)Uyi+1(1)U ... Uyk(1),
then we have
{(Gx)I x~W1} = {(H~), (I-I+\ ..., (Hk)} and a G-di ffeomorphism
Mi = W1(H1)

:

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4. Excision theorem of G-deformation retractions.

Let W be a compact G-manifold with boundary dW=XJIY (disjoint union). Let
W =W1 WW2 ~... ~Wk

be the filtration of Theorem 6.
We now set
Xi = XnWi, Yti = YnW1.

Then we have the following theorem which is crucial for the inductive proof of our equivariant s-cobordism theorem.

THEOREM 7 (Excision theorem of G-deformation retractions). Suppose that (W ; X, Y) is a G-h-cobordism. Namely both X and Y are G-deformation retracts of W. If W satisfies the condition (*1) in § 1, then both Xi and Yi are G-deformation retracts of Wi for each i, 1 <i _<k.

PROOF. Although the assumption of Lemma 3.1 of [11] is slightly different from the condition (*1), we proved it actually under the condition (*1). There-fore Theorem 7 was already shown in the proof of Lemma 3.1 of [11].

REMARK. In [12], we have shown that the excision theorem of G-deforma-tion retractions does not hold in general if the condition (*1) is not satisfied. Accordingly the equivariant torsion itself is not defined in general.
The counter example of the equivariant s-cobordism theorem is provided by making use of the failure of the excision theorem of G-deformation retractions.

REMARK. The excision theorem of G-deformation retractions does not follow from [10].

The rest of the section will be devoted to showing how to employ Matumoto-Shiota's well-definedness of the equivariant Whitehead torsion in our case. Let (W ; X, Y) and (W i ; Xi, Yi) be the G-h-cobordisms in Theorem 7. Then their method is briefly as follows. Take a G-diffeomorphism f : (W; X, Y)-+(W' ; X', Y') such that (W' ; X', Y') is an analytic G-h-cobordism embedded in a representa-tion space V of G analytically and equivariantly. Then (W' ; X', Y') is endowed with an equivariant analytic stratification by isotropy types. Now the quotient W'/G is embedded in Rn subanalytically for some n > 0 and has subanalytic stratification by isotropy types. Moreover the quotient map nv' : W'-+W'/G is subanalytic. Next take a subanalytic triangulation of the triple (W'/G ; X'/G, Y'/G) compatibly with the stratification. After taking a barycentric subdivision of this triangulation they lift each simplex to a subanalytic simplex embedded in W' satisfying certain conditions (cf. [13], Lemma 4.4) and take its G-orbit. The

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collection of them forms a G-CW-subdivision of (W' ; X', Y'). Finally take the pull-back of such a G-CW-complex by f, then we get a G-CW-subdivision of (W ; X, Y). This type of G-CW-subdivisions of (W; X, Y) is unique up to sub-divisions and G-isomorphisms.
In our case we use the above mentioned filtration

W = Wl J W2 J ••• D Wk .

Construct the same filtration

W'=W1~W2~...~yVk

making use of real analytic induced invariant metric from V, which refines subanalytic stratifications of W' and of W'/G respectively. Take Matumoto-Shiota's construction of a G-CW-subdivision of W' so that it is compatible with these refined stratifications, and pull back to W the filtration and G-CW-sub-division of W' by f. Then we get well-defined equivariant Whitehead torsion at each stage of our inductive argument.

§ 5. Equivariant s-cobordism theorem.

Let A be a G-manifold and B a G-invariant submanifold of A. Denote by
I the unit interval [0, 1] with trivial G-action. Then a G-cliff eomorphism f : A
- ~BxI which is an extension of the canonical G-diffeomorphism B(CA)-- Bx {0} is called a G-di ffeomorphism relative B and is denoted by
ANBXI rel B.

For a compact G-manifold M, we denote its boundary by dM. Let W, X, Y, Z,
be compact G-manifolds with corners satisfying

dW = (X1LY)uZ,

aX = XnZ, d Y = YnZ and 6X. Y = 3Z.

We prove Theorem 1 in the following form.

THEOREM 8. Let W, X, Y, Z be as above. Suppose that both X and Y are G-deformation retracts of W and

(i) Z. 9X x 1 rel d X
(ii) r0(W, X) = 0
(iii) the conditions (*1), (*2) are satisfied for W.
Then there exists a G-di ffeomorphism
W =' X x I rel X

which is an extension of the G-diffeomorphism of (i).

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PROOF. We prove Theorem 8 by induction on the number of isotropy types
of W.

Suppose that W has only one isotropy type, say (H). In this case, we have the isomorphism
WhWH(XH, {e}) ^' WhG(X(H), (H)) WhG(X, (H)). Note that both XH and YH are WH deformation retracts of W". It follows from the above isomorphism that
TWH(WH, XH) = r0(W(H), X(H)) = TG(W, X) =0.

Since WH acts freely on W", W"/ WH, XH/ WH and YH/ WH are compact~mani-folds with corners and satisfy :

6WH/WH= (XH/WHILYH/WH)UZH/WH, aXH/WH= XH/WHnZH/WH,
6YH/WH= YH/WHnZH/WH,
aXH/WHJtdYH/WH= dZH/WH.

Obviously we have the induced diffeomorphism
ZH/WH (aXH/WH)xI rel aXH/WH. Moreover one verifies that both XH/WH and YH/WH are deformation retracts of WH/WH and that
T(WH/WH, XH/WH) = 0
by [6]. It follows from the classical s-cobordism theorem that we get a dif-f eomorphism
WH/WH N (XH/WH)xI rel XH/WH

extending the above diffeomorphism since dim(W H/ WH)> 6. For the relative s-cobordism theorem, see for example [19].
The projection it : W H--*W H/ WH is a principal WH bundle. Hence by the homotopy property of principal bundles, we have a WH diff eomorphism WH=XHXI rel XH

extending the induced WH diffeomorphism
ZH ^' 6XHXI rel aXH.

Since W has only one isotropy type (H), there are the canonical G-diff eomor-phisms :
W G/Hx WH, X N G/Hx XH

WH WH

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Y N G/H x YH, Z~ G/H x ZH. WH WH
Thus we get a G-diff eomorphism
W N XXI rel X

extending the given G-diffeomorphism (i).
This completes the first step of the inductive proof.
Next we assume that Theorem 8 holds for the case where the

the isotropy types is less than k.

is k. Let { (Hi) I i=1, •••, k } be the isotropy types indexed as in § 3. W

Z(H1) compact Y( )
have
aW(H1) _ (X(H1)JLY(H1))UZ(H1)
dX(H1) = X(H1)nZ(H1),
aY(H1) = Y(H1)nZ(H1),
6X(H1)j.dY(H1) = aZ(H1)
and we have a G-diff eomorphism
Z(H1) ~ aX(H1) x I rel aX(H1)

which is the restriction of the G-diffeomorphism (i).
exist the canonical G-diffeomorphisms
W(H1) N G/Hl x W', X(H1) G/Hl X XH1,
WH1 WH1

Y(H1) N G/Hl X YH1, Z(H1) G/Hl X ZH1.

WH1 WH1

number

Let: W, X, Y Z be as before such that the number of isotropy types of

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of

w

Since (Hl) is maximal among the set of isotropy types of W, (H1), X(Hl),
l and G-invariant submanifolds of W. Obviously we H are

As is well-known, there

Since both XH1 and YH1 are WH1-deformation retracts of WH1, we may assert
that both X(H1) and Y(H1) are G-deformation retracts of W(H1).
It follows from [6] that

TG(W, X)=0 implies TG(W (Hl), X(H1))=0.

We are now in a position to employ the arguments in the case where W has only one isotropy type and we get a G-diff eomorphism
W (H1) ~ X(H1) x I rel X(H1)

extending the above G-difeomorphism
Z(H1) N 6X(H1)xI rel aX(H1).

Next we consider the normal bundle vl of W(H1) in W. By the G-homotopy

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property of G-vector bundles, we have an isomorphism of G-vector bundles v1 ~ (v1 I X(H1)) x 1

which is an extension of the canonical bundle isomorphism

v1 I Z(H1) "' (v1 I ax(H1)) x I

induced from the product structure above.
In particular, we have G-diffeomorphisms
v1(1) (v1(1) I X(H1)) x I rel v1(1) I X(H1),

Sv1(1) ^' (Sv1(1) I X (H1)) x I rel Sv1(1) I X(H1) .
Therefore we have
r0(v1(1), v1(1) I X(H1)) = 0
and
TG(SL)l(1), Sv1(1) I X(H1)) = 0.
We now set
W2 = W-v1(1), X2 = XnW2i Y2 = YnW2 and
Z2 = (Z-v1(1) I Z(H1))USv1(1) .

Then W2i X2, Y2f Z2 are compact G-manifolds with corners and Z2 has the induced product structure
Z2 ax2x1 rel 6X2.

Moreover it is easy to see that
aW2 = (X2.LY2)UZ2,
axe = X2nZ2 i 6Y2 = Y2nZ2, dX2116Y2 = aZ2.

It follows from Theorem 7 that both X2 and Y2 are G-deformation retracts
of W2.
Next we will show that

rG(W 2, X2) = 0.

For this, we make use of the following geometric sum theorem due to Illman [6].

THEOREM 9 (Illman). Let (A, B) be a finite G-CW-pair and A1, A2 G-sub-complexes of A such that A=A1UA2. Set

Ao = A1f IA2 and Bk = BnAk (k=0, 1, 2).

Denote by ik : Bk-*B the inclusion maps (k=0, 1, 2). Suppose that the inclusion maps 3k : Bk--*Ak are all C-homotopy eguivalences.

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Then the inclusion B-~A is also a G-homotopy equivalence and we have the equality
TG(A, B) = i1*r0(A1, B1)+i2*TG(A2, B2)-io*TG(Ao, Bo).

Apply Theorem 9 to the following case :

A = W , B = X, A1= v1(1), A2 = W2.
Then we have

Ao = A1nA2 = Sv1(1), Bo = XfSv1(1) = Sv1(1) I X(H1),

B1, Xnv1(1) = (1)l X(H1) and B2 = XnW2 = X2.

The maps corresponding to the maps in Theorem 9 are the following inclusion maps :
io : Sv1(1)IX(H1) -~ X,
it : v1(1)IX(Hi) --> X,
i2 : X2 -~ X,

Jo : Sv1(1) I X(H1) -~ Sv1(1),

J1: vi(1) I X(H1) -~ vl(1),
12: X2 -- * W2.

Note that j k are all G-homotopy equivalences (k =0, 1, 2). It follows from Theorem 9 that

TG(W, X) = it*TG(vl(1), ,(1) I X(H1))

+i2*TG(W 2, X2)-i0 rG(Sy1(1), Sv1(1) I X (H1)) Thus we have
i2*TG(W2, X2) = 0.

Consider the Hauschild decomposition :

WhG(X2) ~ 1. WhG(X2, (H))
(H)

Since the set of the isotropy types of W2 is {(H2), (H2), , (Hk)}, the element TG(W2, X2) can be written as

k

(W2, X2) = -L TG(W 2, X2)(HZ)

$=2

corresponding to the Hauschild decomposition. By the assumption (*1), the in-clusion map
i' : XHz --o XHti i>_2

gives a bijection of the connected components and induces isomorphisms of fundamental groups for any base points. It follows from Theorem 5 that the

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homomorphism
i2: Wh0(X2, (Hi)) -~ Wh0(X, (Hi)), i?2,

is an isomorphism. Since i2*TG(W2, X2)(Hi)=0 for any i>_2, we have

v0(W2, X2)(Hi) = 0 for 2i k.
It turns out that
VG(W 2, X2) = 0.

Clearly W2 satisfies the conditions (*1), (*2).
Thus we have shown that W2, X2i Y2, Z2 instead of W, X, Y, Z in Theorem
8 satisfy all the conditions of Theorem 8. Since the number of the isotropy types of W 2 is equal to k-1, we get a G-diff eomorphism
W 2 _ X2 X 1 rel X2

which is an extension of the product structure on Z2, by the inductive hypothesis. Note that

Z c (y~(1)IZ(Hl))uZ2

and that the product structure on the right hand side agrees with that of Z. Thus we obtain a product structure on W which is an extension of the product structure on Z.
This makes the proof of Theorem 8 complete.

§ 6. Equivariant stable s-cobordism theorem.
In this section, we assume that G is a finite group.
First we state the following lemma which follows directly from the defini-tion of an elementary G-collapse and an elementary G-expansion.

LEMMA 1O. Let (W, X) be a finite G-CW -pair such that X is a G-de f orma-tion retract of W. I f VG(W, X)=0, then

zG(W X Y, X x Y) = 0

for any finite G-CW-complex Y.
In view of [ii], any compact G-manifold has a finite G-CW-structure. Hence we have the following corollary.

COROLLARY ii. Let W be a compact G-manifold and X a compact G-sub-mani f old of W such that X is a G-deformation retract of W. I f VG(W, X)=0, then we have
zG(W x Y, X x Y) = 0

for any compact G-manifold Y.

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PROOF OF THEOREM 2. In [11], it is shown that there exists an orthogonal G-representation space V such that W X V (1) and W X SV(1) satisfy the condi-tions (*1), (*2) in § 1.
First we will apply Theorem 8 to the triad

(WXSV(1); XXSV(1), YXSV(1)).

It follows from Corollary 11 that DG(W, X)=0 implies

r0(W XSV(1), XXSV(1)) = 0.

Hence by Theorem 8 we get a G-diffeomorphism

WXSV(1) ~ XXSV(1)XI rel XXSV(1).

Next we will apply Theorem 8 to the triad

(WXV(1);XXV(1), YXV(1)).
As above, we get
r0(WXV(1), XXV(1)) = 0.

Appealing to Theorem 8 again, we have a G-diff eomorphism

WXV(1) N XXV(1)XI rel XXV(1)

which is an extension of the above product structure on W X S V(1).
This makes the proof of Theorem 2 complete.

§ 7. Stable equivalence of G-manifolds.
In this section, we assume that G is a finite group.
Let M1, M2 be closed G-manifolds and f : M1-- M2 a tangential simple G-homotopy equivalence. It is well-known that there exist an orthogonal G-representation space V1 and a G-embedding e : M1-~ V1. We assume that V1 includes R with trivial G-action as a direct summand. For any positive integer m, we denote by V the direct sum of m-copies of V1 and by j : V1--* Vm the inclusion to the first factor. Set V =V r.
Then the composition

f Xe idX j M
1 --- >M2XV1- >M2XV

is a G-embedding. One verifies that the normal bundle of the G-embedding is isomorphic to the product bundle
M1 X V -* M1,

if V1 is sufficiently large. Thus we get a G-embedding

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i : M1xV(1)--~M2XV.

By this embedding, we identify M1 X V(1) with the image i(M1 X V(1)). If we choose a sufficiently large number r, there holds the following inclusion

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M1xV(1) c M2XV(r).
We now set

W= M2 X V (r)-Ml X V (1)
and get a triad
(W; M1XSV(1), M2XSV(r)).
The proof that the triad above is a G-h-cobordism for m>_ 3 is shown in the proof of Lemma 3.2 in [11].
Next we will show that the G-h-cobordism is in fact a G-s-cobordism.

For this, we first show that the G-embedding
i : M1XV(1) --~ M2XV(r)

is a simple G-homotopy equivalence. Consider the following G-homotopy com-mutative diagram
f M
1 ---> M2

~ j
i
M1 X V (1) -----> M2 X V (r)

where r is the projection map and j is the natural inclusion map. In view of [3], [6], we have
TG(z) = DG(j ' f ' 7r) = ZGC7 ' f)+(J ' f)*VG(lr)
= ZG(j)+i *T G(f)+(j' f)*VG(lt)

Since n, f and j are all simple G-homotopy equivalences, we may conclude that i is also a G-simple homotopy equivalence.
Next we will show that

DG(W, M1 x S V(1)) = 0.

For this, we apply Theorem 9 to the following case :
A = M2XV(r), B = M1XV(1),
A1=M1xV(1)=B, A2=W.
Then we have
Ao = A1nA2 = M1xSV(1),
Bo=BfAo=M1xSV(1)=A0,

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B1 = BnA1 = A1 = B = M1xV(1),
B2 = BnA2 = M1xSV(1) = A0.
The maps corresponding to those maps in Theorem 9 are the following inclu-sion maps :
io = i2: M1xSV(1) --~ M1xV(1),
it =id: M1 X V(1) --~ M1 X V(1) and

j o = id : M1 X S V(1) -~ M1 x S V(1) ,
j 1 = id : M1 X V(1) -* M1 X V(1),
12: M1 x SV(1) -~ W .
Note that j k are all G-homotopy equivalences (k =0, 1, 2). It follows from Theorem 9 that there holds

z0(M2 X V (r), M1 x V(1)) = i 1*v0(M1 X V (1), M1 x V (1))
+i2*VG(W, M1 X SV(1))-io*v0(M1 X SV(1), Ml X SV (1)) .
By definition, we have
VG(Ml x V(1), M1 X V(1)) = 0,
DG(M1 X S V(1), M1 X S V(1)) = 0.
Thus we have
i2*r0(W, M1 X S V(1)) = r0(M2 X V (r), M1 X V (1)) = Z'G(i) = 0.

Finally we will show that 1 * is an isomorphism. If the m above is greater than two, we have
dimVG = m dimV >_ m>_ 3.

Hence for any subgroup H of G, SV(1)H is connected and simply connected. It turns out that the inclusion map
a2 : (Ml X SV(1))H = MH X SV(1)H --~ (Ml x V (1))H = MH x V (1)H

gives a bijection of the connected components and induces isomorphisms of fundamental groups for any base points. Accordingly there holds an isomorphism i2*: WhG(MO X S V(1), (H)) N WhG(M1 X V (1), (H)) by Theorem 5. Since H is an arbitrary subgroup of G, it follows from the Hauschild decomposition that

Z2*: Wh0(M1 X S V(1)) = Wh0(M1 X V (1))
is an isomorphism.

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Since i2*TG(W, M1 X SV(1))=0, we conclude that

TG(W, M1XSV(1)) = 0.

Namely the triad (W; M1 X SV(1), M2>< SV(r)) is a G-s-cobordism.
If we take m as m>_6, then the conditions (*1), (*2) of Theorem 1 are satisfied and we have a G-diffeomorphism

W M1XSV(1)XI rel M1XSV(1).

Therefore we obtain the following G-diff eomorphisms

M2 X V (r) = M1 X V (1)UW ~ M1 X V(1)U(M1 X S V(1) X I)
M1 X V(1) .

Obviously M2 X V (r) and M2 X V(1) are G-diff eomorphic and we have the required G-cliff eomorphism
f : M1 X V(1) ---~ M2 X V(1) .
The G-homotopy commutativity of the following diagram :
f M
1 X V (1) ------> M2 X V (1)

I M
1 -- > M2
is obvious.

Conversely suppose that there exists a G-diff eomorphism

f : M1 X V(1) -- > M2 X V(1)

so that the diagram above is G-homotopy commutative. Since two projection maps 2r: M1 X V (1)--~M1, 7r: M2 X V (1)-~M2i and f are all simple G-homotopy equivalences, one can show that f is also a simple G-homotopy equivalence as before.

This makes the proof of Theorem 3 complete.

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