EQUIVARIANT STABLE HOMOTOPY THEORY

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Introduction

The study of symmetries on spaces has always been a major part of algebraic and geometric topology, but the systematic homotopical study of group actions is relatively recent. The last decade has seen a great deal of activity in this area. After giving a brief sketch of the basic concepts of space level equivariant homotopy theory, we shall give an introduction to the basic ideas and constructions of spectrum level equivariant homotopy theory. We then illustrate ideas by explaining the fundamental localization and completion theorems that relate equivariant to nonequivariant homology and cohomology.

The first such result was the Atiyah-Segal completion theorem which, in its simplest terms, states that the completion of the complex representation ring $R(G)$ at its augmentation ideal I is isomorphic to the K-theory of the classifying space BG: $R(G)^{\wedge}_{I} \cong K(BG)$. A more recent homological analogue of this result describes the K-homology of BG. As we shall see, this can best be viewed as a localization theorem. These are both consequences of equivariant Bott periodicity, although full understanding depends on the localization away from I and the completion at I of the spectrum K_G that represents equivariant K-theory. We shall explain a still more recent result which states that a similar analysis works to give the same kind of localization and completion theorems for the spectrum MU_G that represents a stabilized version of equivariant complex cobordism and for all module spectra over MU_G. We shall also say a little about equivariant cohomotopy, a theory for which the cohomological completion theorem is true, by Carlsson's proof of the Segal conjecture, but the homological localization theorem is false.

1. Equivariant homotopy

We shall not give a systematic exposition of equivariant homotopy theory. There are several good books on the subject, such as [12] and [17], and a much more thorough expository account will be given in [53]. Some other expository articles are [49, 1]. We aim merely to introduce ideas, fix notations, and establish enough background in space level equivariant homotopy theory to make sense of the spectrum level counterpart that we will focus on later.

The group.

We shall restrict our attention to compact Lie groups G, although the basic unstable homotopy theory works equally well for general topological groups. To retain the homeomorphism between orbits and homogenous spaces we shall always restrict attention to *closed* subgroups.

The class of compact Lie groups has two big advantages: the subgroup structure is reasonably simple ('nearby subgroups are conjugate'), and there are enough representations (any sufficiently nice G-space embeds in one). We shall sometimes restrict to finite groups to avoid technicalities, but most of what we say applies in technically modified form to general compact Lie groups. The reader unused to equivariant topology may find it helpful to concentrate on the case when G is a group of order 2. Even this simple case well illustrates most of the basic ideas.

G-spaces and G-maps

All of our spaces are to be compactly generated and weak Hausdorff.

A G-space is a topological space X with a continuous left action by G; a based G-space is a G-space together with a basepoint fixed by G. These will be our basic objects. We frequently want to convert unbased G-spaces Y into based ones, and we do so by taking the topological sum of Y and a G-fixed basepoint; we denote the result by Y_+.

We give the product $X \times Y$ of G-spaces the diagonal action, and similarly for the smash product $X \wedge Y$ of based G-spaces. We use the notation $\mathrm{map}(X, Y)$ for the G-space of continuous maps from X to Y; G acts via $(\gamma f)(x) = \gamma f(\gamma^{-1}x)$; we let $F(X, Y)$ denote the subspace of based maps. The usual adjunctions apply.

A map of based G-spaces is a continuous basepoint preserving function which commutes with the action of G. A homotopy of based G-maps $f_0 \simeq f_1$ is a G-map $X \wedge I_+ \longrightarrow Y$ whose composites with the inclusions of $X \wedge \{0\}_+$ and $X \wedge \{1\}_+$ are f_0 and f_1. We use the notation $[X, Y]_G$ to denote the set of homotopy classes of based G-maps $X \longrightarrow Y$.

Cells, spheres, and G-CW complexes

We shall be much concerned with cells and spheres. There are two important sorts of these, arising from homogeneous spaces and from representations, and the interplay between the two is fundamental to the subject.

Given any closed subgroup H of G we may form the homogeneous space G/H and its based counterpart, G/H_+. These are treated as 0-dimensional cells, and they play a role in equivariant theory analogous to the role of a point in nonequivariant theory. We form the n-dimensional cells from these homogeneous spaces. In the unbased context, the cell-sphere pair is

$$(G/H \times D^n, G/H \times S^{n-1}),$$

and in the based context

$$(G/H_+ \wedge D^n, G/H_+ \wedge S^{n-1}).$$

We shall always use different notation for different actions, so that when we write D^n and S^n we understand that G acts trivially.

Starting from these cell-sphere pairs, we form G-CW complexes exactly as non-equivariant CW-complexes are formed from the cell-sphere pairs (D^n, S^{n-1}). The usual theorems transcribe directly to the equivariant setting, and we shall say more about them below. Smooth compact G-manifolds are triangulable as finite G-CW complexes, but topological G-manifolds need not be.

We also have balls and spheres formed from orthogonal representations V of G. We shall be concerned especially with the one-point compactification S^V of V, with ∞ as the basepoint; note in particular that the usual convention that n denotes the trivial n-dimensional real representation gives S^n the usual meaning. We may also form the unit disc

$$D(V) = \{v \in V \mid \|v\| \leq 1\},$$

and the unit sphere

$$S(V) = \{v \in V \mid \|v\| = 1\};$$

we think of them as unbased G-spaces. There is a homeomorphism $S^V \cong D(V)/S(V)$. The resulting cofibre sequence

$$S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$$

can be very useful in inductive arguments since there is an equivariant homotopy equivalence $D(V)_+ \simeq S^0$.

Fixed points and quotients.

There are a number of ways to increase or decrease the size of the ambient group. If $f : G_1 \longrightarrow G_2$ is a group homomorphism we may regard a G_2-space Y as a G_1-space f^*Y by pullback along f, and we usually omit f^* when the context makes it clear. The most common cases of this are when G_1 is a subgroup of G_2 and when G_2 is a quotient of G_1; in particular every space may be regarded as a G-fixed G-space.

The most important construction on G-spaces is passage to fixed points:

$$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}.$$

For example, $F(X,Y)^G$ is the space of based G-maps $X \longrightarrow Y$. It is easy to check that the fixed point spaces for the conjugates of H are all homeomorphic; indeed, multiplication by g induces a homeomorphism $g : X^{g^{-1}Hg} \longrightarrow X^H$. In particular X^H is invariant under the action of the normalizer $N_G(H)$, and hence it has a natural action of the Weyl group $W_G(H) = N_G(H)/H$. Passage to H-fixed point spaces is a functor from G-spaces to $W_G(H)$-spaces.

Dually, we have the quotient space X/H of X by H. This is actually a standard abuse of notation, since $H\backslash X$ would be more consistent logically; for example, we are using G/H to denote the quotient of G by its *right* action by H. Again, multiplication by g gives a homeomorphism $X/g^{-1}Hg \longrightarrow X/H$. Thus X/H also has a natural action of the Weyl group, and passage to the quotient by H gives a functor from G-spaces to $W_G(H)$-spaces.

If N is a normal subgroup of G, then it is easy to verify that passage to N-fixed points is right adjoint to pullback along $G \longrightarrow G/N$ and that passage to the quotient by N is left adjoint to this pullback.

Lemma 1.1. *For G-spaces X and G/N-spaces Y, there are natural homeomorphisms*

$$G\text{-}map(Y,X) \cong G/N\text{-}map(Y,X^N) \quad \text{and} \quad G/N\text{-}map(X/N,Y) \cong G\text{-}map(X,Y),$$

and similarly in the based context.

The particular case

$$G\text{-map}(G/H,X) \cong X^H$$

helps explains the importance of the fixed point functor.

Isotropy groups and universal spaces.

An unbased G-space is said to be *G-free* if $X^H = \emptyset$ whenever $H \neq 1$. A based G-space is *G-free* if $X^H = *$ whenever $H \neq 1$. More generally, for $x \in X$ the *isotropy group* at x is the stabilizer G_x; given any collection $\mathcal{F}$ of subgroups of G, we say that X is an $\mathcal{F}$-space if $G_x \in \mathcal{F}$ for every non-basepoint $x \in X$. Thus a G-space is free if and only if it is a $\{1\}$-space. It is usual to think of a G-space as built up from the G-fixed subspace X^G by adding points with successively smaller and smaller isotropy groups. This gives a stratification in which the pure strata consist of points with isotropy group in a single conjugacy class.

A collection $\mathcal{F}$ of subgroups of G closed under passage to conjugates and subgroups is called a *family* of subgroups. For each family, there is an unbased $\mathcal{F}$-space $E\mathcal{F}$, required to be of the homotopy type of a G-CW complex, which is universal in the sense that there is a unique homotopy class of G-maps $X \longrightarrow E\mathcal{F}$ for any $\mathcal{F}$-space X of the homotopy type of a G-CW complex. It is characterized by the fact that the fixed point set $(E\mathcal{F})^{\mathcal{H}}$ is contractible for $H \in \mathcal{F}$ and empty for $H \notin \mathcal{F}$. For example, if $\mathcal{F}$ consists of only the trivial group, then $E\{1\}$ is the universal free G-space EG, and if $\mathcal{F}$ is the family of all subgroups, then $EAll = *$. Another case of particular interest is the family $\mathcal{P}$ of all proper subgroups. If G is finite, then $E\mathcal{P} = \bigcup_{\|\geq\prime} \mathcal{S}(\|\mathcal{V})$, where V is the reduced regular representation of G, and in general $E\mathcal{P} = \operatorname{colim}_{\mathcal{V}} \mathcal{S}(\mathcal{V})$ where V runs over all finite dimensional representations V of G such that $V^G = \{0\}$; to be precise, we restrict V to lie in some complete G-universe (as defined in the next section). Such universal spaces exist for any family and may be constructed either by killing homotopy groups or by using a suitable bar construction [20]. In the based case we consider $E\mathcal{F}_+$, and a very basic tool is the isotropy separation cofibering

$$\boxed{E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}}\,,$$

where the first map is obtained from $E\mathcal{F} \longrightarrow *$ by adding a disjoint basepoint. Note that the mapping cone $\tilde{E}\mathcal{F}$ may alternatively be described as the join $S^0 * E\mathcal{F}$; it is $\mathcal{F}$-contractible in the sense that it is H-contractible for every $H \in \mathcal{F}$. We think of this cofibering as separating a space X into the $\mathcal{F}$-space $E\mathcal{F}_+ \wedge X$ and the $\mathcal{F}$-contractible space $\tilde{E}\mathcal{F} \wedge X$.

Induced and coinduced spaces.

We can use the fact that G is both a left and a right G-space to define induced and coinduced G-space functors. If H is a subgroup of G and Y is an H-space, we define the induced G-space $G \times_H Y$ to be the quotient of $G \times Y$ by the equivalence relation $(gh, y) \sim (g, hy)$ for $g \in G$, $y \in Y$, and $h \in H$; the G-action is defined by $\gamma[g, y] = [\gamma g, y]$.

Similarly the coinduced G-space $\mathrm{map}_H(G,Y)$ is the subspace of $\mathrm{map}(G,Y)$ consisting of those maps $f: G \longrightarrow Y$ such that $f(gh^{-1}) = hf(g)$ for $h \in H$ and $g \in G$; the G-action is defined by $(\gamma f)(g) = f(\gamma^{-1}g)$. When these constructions are applied to a G-space, the actions may be untwisted, and it is well worth writing down the particular homeomorphisms.

Lemma 1.2. *If X is a G-space then there are homeomorphisms*

$$G \times_H X \cong G/H \times X \quad \text{and} \quad map_H(G,X) \cong map(G/H,X),$$

natural for G-maps of X.

Proof. In the first case, the maps are $[g,x] \longmapsto (gH, gx)$ and $[g, g^{-1}x] \longleftarrow (gH, x)$. In the second case, $f \longmapsto a(f)$, where $a(f)(gH) = gf(g)$, and $b(f') \longleftarrow f'$, where $b(f')(g) = g^{-1}f'(gH)$. We encourage the reader to make the necessary verifications. □

The induced space functor is left adjoint to the forgetful functor and the coinduced space functor is right adjoint to it.

Proposition 1.3. *For G-spaces X and H-spaces Y, there are natural homeomorphisms*

$$G\text{-}map(G\times_H Y, X) = H\text{-}map(Y,X) \quad \text{and} \quad H\text{-}map(X,Y) = G\text{-}map(X, map_H(G,Y)).$$

Proof. The unit and counit for the first adjunction are the H-map $\eta: Y \longrightarrow G\times_H Y$ given by $y \longmapsto [e,y]$ and the G-map $\varepsilon: G\times_H X \longrightarrow X$ given by $[g,x] \longmapsto gx$. For the second, they are the G-map $\eta: X \longrightarrow \mathrm{map}_H(G,X)$ that sends x to the constant function at x and the H-map $\varepsilon: \mathrm{map}_H(G,Y) \longrightarrow Y$ given by $f \longmapsto f(e)$. We encourage the reader to make the necessary verifications. □

Analogous constructions and homeomorphisms apply in the based case. If Y is a based H-space, it is usual to write $G_+ \wedge_H Y$ or $G \ltimes_H Y$ for the induced based G-space, and $F_H(G_+, Y)$ or $F_H[G,Y)$ for the coinduced based G-space.

Homotopy groups, weak equivalences, and the G-Whitehead theorem

One combination of the above adjunctions is particularly important. To define H-equivariant homotopy groups, we might wish to define them G-equivariantly as $[G/H_+ \wedge S^n, \cdot]_G$, or we might wish to define them H-equivariantly as $[S^n, \cdot]_H$; fortunately these agree, and we define

$$\pi_n^H(X) = [G/H_+ \wedge S^n, X]_G \cong [S^n, X]_H \cong [S^n, X^H].$$

Using the second isomorphism, we may apply finiteness results from non-equivariant homotopy theory. For example, if X and Y are finite G-CW complexes and double suspensions, then $[X,Y]_G$ is a finitely generated abelian group.

A G-map $f : X \longrightarrow Y$ is a weak G-equivalence if $f^H : X^H \longrightarrow Y^H$ is a weak equivalence for all closed subgroups H. As in the non-equivariant case one proves that any G-CW pair has the homotopy extension and lifting property and deduces that a weak equivalence induces a bijection of $[T, \cdot]_G$ for every G-CW complex T. The G-Whitehead theorem follows: a weak G-equivalence of G-CW complexes is a G-homotopy equivalence. Similarly, the cellular approximation theorem holds: any map between G-CW complexes is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic. Also, by the usual construction, any G-space is weakly equivalent to a G-CW complex.

The generalization to families $\mathcal{F}$ is often useful. We say that a G-map f is a weak $\mathcal{F}$ -equivalence if f^H is a weak equivalence for $H \in \mathcal{F}$; the principal example of an $\mathcal{F}$-equivalence is the map $E\mathcal{F}_+ \wedge X \longrightarrow X$. A based $\mathcal{F}$-CW complex is a G-CW complex whose cells are all of the form $G/H_+ \wedge S^n$ for $H \in \mathcal{F}$; note that an $\mathcal{F}$-CW complex is an $\mathcal{F}$-space. The usual proofs show that a weak $\mathcal{F}$-equivalence induces a bijection of $[T, \cdot]_G$ for every $\mathcal{F}$-CW complex T and that any G-space is $\mathcal{F}$-equivalent to an $\mathcal{F}$-CW complex.

To state a quantitative version of the G-Whitehead theorem, we consider functions n on the set of subgroups of G with values in the set $\{-1, 0, 1, 2, 3, \ldots, \infty\}$ that are constant on conjugacy classes. For example if X is a G-space, we can view dimension and connectivity as giving such functions by defining $dim(X)(H) = dim(X^H)$ and $conn(X)(H)$ to be the connectivity of X^H. The value -1 allows the possibility of empty or of non-connected fixed point spaces. Now the standard proof gives the following result.

Theorem 1.4. *If T is a G-CW complex and $f : X \longrightarrow Y$ is n-connected, then the induced map*

$$f_* : [T, X]_G \longrightarrow [T, Y]_G$$

is surjective if $dim(T^H) \leq n(H)$ for all $H \subseteq G$, and bijective if $dim(T^H) \leq n(H) - 1$.

The G-Freudenthal suspension theorem

In the stable world, we shall want to desuspend by spheres of representations. Accordingly, for any orthogonal representation V, we define the Vth suspension functor by $\Sigma^V X = X \wedge S^V$. This gives a map

$$\Sigma^V : [X, Y]_G \longrightarrow [\Sigma^V X, \Sigma^V Y]_G.$$

We shall be content to give the version of the Freudenthal Theorem, due to Hauschild [36], that gives conditions under which this map is an isomorphism. However, we note in passing that the presence of S^V gives the codomain a richer algebraic structure than the domain, and it is natural to seek a theorem stating that Σ^V may be identified with an algebraic enrichment of the domain even when it is not an isomorphism. L.G.Lewis [38] has proved versions of the Freudenthal Theorem along these lines when X is a sphere.

Just as nonequivariantly, we approach the Freudenthal Theorem by studying the adjoint map $\eta : Y \longrightarrow \Omega^V \Sigma^V Y$.

Theorem 1.5. *The map $\eta : Y \longrightarrow \Omega^V \Sigma^V Y$ is an n-equivalence if n satisfies the following two conditions:*

1. *$n(H) \leq 2conn(Y^H) + 1$ for all subgroups H with $V^H \neq 0$, and*
2. *$n(H) \leq conn(Y^K)$ for all pairs of subgroups $K \subseteq H$ with $V^K \neq V^H$.*

Therefore the suspension map

$$\Sigma^V : [X, Y]_G \longrightarrow [\Sigma^V X, \Sigma^V Y]_G$$

is surjective if $dim(X^H) \leq n(H)$ for all H, and bijective if $dim(X^H) \leq n(H) - 1$.

This is proven by reduction to the non-equivariant case and obstruction theory. When G is finite and X is finite dimensional, it follows that if we suspend by a sufficiently large representation, then all subsequent suspensions will be isomorphisms.

Corollary 1.6. *If G is finite and X is finite dimensional, there is a representation $V_0 = V_0(X)$ such that, for any representation V,*

$$\Sigma^V : [\Sigma^{V_0} X, \Sigma^{V_0} Y]_G \xrightarrow{\cong} [\Sigma^{V_0 \oplus V} X, \Sigma^{V_0 \oplus V} Y]_G$$

is an isomorphism.

If X and Y are finite G-CW complexes, this stable value $[\Sigma^{V_0} X, \Sigma^{V_0} Y]_G$ is a finitely generated abelian group. If G is a compact Lie group and X has infinite isotropy groups, there is usually no representation V_0 for which all suspensions Σ^V are isomorphisms, and the colimit of the $[\Sigma^V X, \Sigma^V Y]_G$ is usually not finitely generated.

The direct limit $\mathrm{colim}_V [S^V, S^V]_G$ is a ring under composition, and it turns out to be isomorphic to the Burnside ring $A(G)$. When G is finite, $A(G)$ is defined to be the Grothendieck ring associated to the semi-ring of finite G-sets, and it is the free Abelian group with one generator $[G/H]$ for each conjugacy class of subgroups of G. When G is a general compact Lie group, $A(G)$ is more complicated to define, but it turns out to be a free Abelian group, usually of infinite rank, with one basis element $[G/H]$ for each conjugacy class of subgroups H such that $W_G H$ is finite.

Eilenberg-MacLane G-spaces and Postnikov towers

The homotopy groups $\pi_n^H(X)$ of a G-space X are related as H varies, and we must take all of them into account to develop obstruction theory. Let $\mathcal{O}$ denote the orbit category of G-spaces G/H and G-maps between them, and let $h\mathcal{O}$ be its homotopy category. By our first description of homotopy groups, we see that the definition $\underline{\pi}_n(X)(G/H) = \pi_n^H(X)$ gives a set-valued contravariant functor on $h\mathcal{O}$; it is group-valued if $n = 1$ and Abelian group-valued if $n \geq 2$. An Eilenberg-MacLane G-space $K(\underline{\pi}, n)$ associated to such a contravariant functor $\underline{\pi}$ on $h\mathcal{O}$ is a G-space such that $\underline{\pi}_n(K(\underline{\pi}, n)) = \underline{\pi}$ and all other homotopy groups of $K(\underline{\pi}, n)$ are zero. Either by killing homotopy groups or by use of a bar construction [20], one sees that Eilenberg-MacLane G-spaces exist for all $\underline{\pi}$ and n.

Recall that a space X is simple if it is path connected and if $\pi_1(X)$ is Abelian and acts trivially on $\pi_n(X)$ for $n \geq 2$. More generally, X is nilpotent if it is path connected and if $\pi_1(X)$ is nilpotent and acts nilpotently on $\pi_n(X)$ for $n \geq 2$. A G-space X is is said to be simple or nilpotent if each X^H is simple or nilpotent. Exactly as in the nonequivariant situation, simple G-spaces are weakly equivalent to inverse limits of simple Postnikov towers and nilpotent G-spaces are weakly equivalent to inverse limits of nilpotent Postnikov towers.

Ordinary cohomology theory; localization and completion

We define a "coefficient system" M to be a contravariant Abelian group-valued functor on $h\mathcal{O}$. There are associated cohomology theories on pairs of G-spaces, denoted $H_G^*(X, A; M)$. They satisfy and are characterized by the equivariant versions of the usual axioms: homotopy, excision, exactness, wedge, weak equivalence, and dimension; the last states that

$$H_G^*(G/H; M) \cong M(G/H),$$

functorially on $h\mathcal{O}$. This is a manifestation of the philosophy that orbits play the role of points. There are also homology theories, denoted $H_*^G(X, A; N)$, but these must be defined using covariant functors $N : h\mathcal{O} \longrightarrow \mathcal{A}\lfloor$.

By the weak equivalence axiom, it suffices to define these theories on G-CW pairs. The cohomology of such a pair (X, A) is the reduced cohomology of X/A, so it suffices to deal with G-CW complexes X. These have cellular chain coefficient systems that are specified by

$$\underline{C}_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H; \mathbb{Z});$$

the differential d_n is the connecting homomorphism of the triple

$$((X^n)^H, (X^{n-1})^H, (X^{n-2})^H).$$

The homology and cohomology groups of X are then calculated from chain and cochain complexes of Abelian groups given by

$$\underline{C}_*(X) \otimes_{h\mathcal{O}} N \quad \text{and} \quad \mathrm{Hom}_{h\mathcal{O}}(\underline{C}_*(X), M).$$

Here $\mathrm{Hom}_{h\mathcal{O}}(\underline{C}_n(X), M)$ is the group of natural transformations $\underline{C}_n(X) \longrightarrow M$, and the tensor product over $h\mathcal{O}$ is described categorically as a coend of functors.

Alternatively, for based G-CW complexes X, one has the equivalent description of reduced cohomology as

$$\tilde{H}^n(X; M) = [X, K(M,n)]_G.$$

¿From here, it is an exercise to transcribe classical obstruction theory to the equivariant context. This was first done by Bredon [11], who introduced these cohomology theories.

One can localize or complete nilpotent G-spaces at a set of primes. One first works out the construction on $K(\underline{\pi}, n)$'s, and then proceeds by induction up the Postnikov tower. See [55, 57]. When G is finite, one can algebraicize equivariant rational homotopy theory, by analogy with the nonequivariant theory. See [63]. Bredon cohomology is the basic tool in these papers.

While the theory we have described looks just like nonequivariant theory, we emphasize that it behaves very differently calculationally. For example, a central calculational theorem in nonequivariant homotopy theory states that the rationalization of a connected Hopf space splits, up to homotopy, as a product of Eilenberg-Mac Lane spaces. The equivariant analogue is false [64].

2. The equivariant stable homotopy category

The entire foundational framework described in [22] works equally well in the presence of a compact Lie group G acting on all objects in sight. We here run through the equivariant version of [22], with emphasis on the new equivariant phenomena that appear. From both the theoretical and calculational standpoint, the main new feature is that the equivariant analogs of spheres are the spheres associated to representations of G, so that there is a rich interplay between the homotopy theory and representation theory of G. The original sources for most of this material are the rather encyclopedic [42] and the nonequivariantly written [22]; a more leisurely and readable exposition will appear in [53].

By a G-universe U, we understand a countably infinite dimensional real inner product space with an action of G through linear isometries. We require that U be the sum of countably many copies of each of a set of representations of G and that it contain a trivial representation and thus a copy of $\mathbb{R}^\infty$. We say that U is complete if it contains a copy of every irreducible representation of G. At the opposite extreme, we say that U is G-fixed if $U^G = U$. When G is finite, the sum of countably many copies of the regular representation $\mathbb{R}\mathbb{G}$ gives a canonical complete universe. We refer to a finite dimensional sub G-inner product space of U as an indexing space.

A G-spectrum indexed on U consists of a based G-space EV for each indexing space V in U together with a transitive system of based G-homeomorphisms

$$\tilde{\sigma} : EV \xrightarrow{\cong} \Omega^{W-V} EW$$

for $V \subset W$. Here $\Omega^V X = F(S^V, X)$ and $W - V$ is the orthogonal complement of V in W. A map of G-spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : EV \to E'V$ which commute with the respective structure maps.

We obtain the category $G\mathcal{S} = \mathcal{GSU}$ of G-spectra indexed on U. Dropping the requirement that the maps $\tilde{\sigma}_{V,W}$ be homeomorphisms, we obtain the notion of a G-prespectrum and the category $G\mathcal{P} = \mathcal{GPU}$ of G-prespectra indexed on U. The forgetful functor $\ell : G\mathcal{S} \longrightarrow \mathcal{GP}$ has a left adjoint L. When the structure maps $\tilde{\sigma}$ are inclusions, $(LE)(V)$ is just the union of the G-spaces $\Omega^{W-V} EW$ for $V \subset W$. We write $\sigma : \Sigma^{W-V} EV \longrightarrow EW$ for the adjoint structure maps.

Examples 2.1. Let X be a based G-space. The suspension G-prespectrum $\Pi^\infty X$ has Vth space $\Sigma^V X$, and the suspension G-spectrum of X is $\Sigma^\infty X = L\Pi^\infty X$. Let $QX = \cup \Omega^V \Sigma^V X$, where the union is taken over the indexing spaces $V \subset U$; a more accurate notation would be $Q_U X$. Then $(\Sigma^\infty X)(V) = Q(\Sigma^V X)$. The functor Σ^∞ is left adjoint to the zeroth space functor. More generally, for an indexing space $Z \subset U$, let $\Pi^\infty_Z X$ have Vth space $\Sigma^{V-Z} X$ if $Z \subset V$ and a point otherwise and define $\Sigma^\infty_Z X = L\Pi^\infty_Z X$. The "shift desuspension" functor Σ^∞_Z is left adjoint to the Zth space functor from G-spectra to G-spaces.

For a G-space X and G-spectrum E, we define G-spectra $E \wedge X$ and $F(X, E)$ exactly as in the non-equivariant situation. There result homeomorphisms

$$G\mathcal{S}(\mathcal{E} \wedge \mathcal{X}, \mathcal{E}') \cong \mathcal{GT}(\mathcal{X}, \mathcal{S}(\mathcal{E}, \mathcal{E}')) \cong \mathcal{GS}(\mathcal{E}, \mathcal{F}(\mathcal{X}, \mathcal{E}')),$$

where $G\mathcal{T}$ is the category of based G-spaces.

Proposition 2.2. *The category $G\mathcal{S}$ is complete and cocomplete.*

A homotopy between maps $E \longrightarrow F$ of G-spectra is a map $E \wedge I_+ \longrightarrow F$. Let $[E, F]_G$ denote the set of homotopy classes of maps $E \longrightarrow F$. For example, if X and Y are based G-spaces and X is compact,then

$$[\Sigma^\infty X, \Sigma^\infty Y]_G \cong \operatorname{colim}_V [\Sigma^V X, \Sigma^V Y]_G.$$

Fix a copy of $\mathbb{R}^\infty$ in U and write $\Sigma^\infty_n = \Sigma^\infty_{\mathbb{R}^n}$. For $n \geq 0$, the sphere G-spectrum S^n is $\Sigma^\infty S^n$. For $n > 0$, the sphere G-spectrum S^{-n} is $\Sigma^\infty_n S^0$. We shall often write S_G rather than S^0 for the zero sphere G-spectrum. Remembering that orbits are the analogues of points, we think of the G-spectra $G/H_+ \wedge S^n$ as generalized spheres. Define the homotopy groups of a G-spectrum E by

$$\pi^H_n(E) = [G/H_+ \wedge S^n, E]_G.$$

A map $f : E \longrightarrow F$ of G-spectra is said to be a weak equivalence if $f_* : \pi_*^H(E) \longrightarrow \pi_*^H(F)$ is an isomorphism for all H. Here serious equivariant considerations enter for the first time.

Theorem 2.3. *A map $f : E \longrightarrow F$ of G-spectra is a weak equivalence if and only if $f_V : EV \longrightarrow FV$ is a weak equivalence of G-spaces for all indexing spaces $V \subset U$.*

This is obvious when the universe U is trivial, but it is far from obvious in general. To see that a weak equivalence of G-spectra is a spacewise weak equivalence, one sets up an inductive scheme and uses the fact that spheres S^V are triangulable as G-CW complexes [42, I.7.12]

The equivariant stable homotopy category $\bar{h}G\mathcal{S}$ is constructed from the homotopy category $hG\mathcal{S}$ of G-spectra by adjoining formal inverses to the weak equivalences, a process that is made rigorous by G-CW approximation. The theory of G-CW spectra is developed by taking the sphere G-spectra as the domains of attaching maps of cells $G/H_+ \wedge CS^n$, where $CE = E \wedge I$ [42, I§5]. This works just as well equivariantly as nonequivariantly, and we arrive at the following theorems.

Theorem 2.4 (Whitehead)**.** *If E is a G-CW spectrum and $f : F \longrightarrow F'$ is a weak equivalence of G-spectra, then $f_* : [E, F]_G \longrightarrow [E, F']_G$ is an isomorphism. Therefore a weak equivalence between G-CW spectra is a homotopy equivalence.*

Theorem 2.5 (Cellular approximation)**.** *Let A be a subcomplex of a G-CW spectrum E, let F be a G-CW spectrum, and let $f : E \longrightarrow F$ be a map whose restriction to A is cellular. Then f is homotopic relative to A to a cellular map. Therefore any map $E \longrightarrow F$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

Theorem 2.6 (Approximation by G-CW spectra)**.** *For a G-spectrum E, there is a G-CW spectrum ΓE and a weak equivalence $\gamma : \Gamma E \longrightarrow E$. On the homotopy category $hG\mathcal{S}$, Γ is a functor such that γ is natural.*

Thus the stable category $\bar{h}G\mathcal{S}$ is equivalent to the homotopy category of G-CW spectra. As in the nonequivariant context, we have special kinds of G-prespectra that lead to a category of G-spectra on which the smash product has good homotopical properties. Of course, we define cofibrations of G-spaces via the homotopy extension property in the category of G-spaces. For example, X is G-LEC if its diagonal map is a G-cofibration.

Definition 2.7. A G-prespectrum D is said to be

- (i) Σ-cofibrant if each $\sigma : \Sigma^{W-V} DV \to DW$ is a based G-cofibration.
- (iii) G-CW if it is Σ-cofibrant and each DV is G-LEC and has the homotopy type of a G-CW complex.

A G-spectrum E is said to be Σ-cofibrant if it is isomorphic to LD for some Σ-cofibrant G-prespectrum D; E is said to be tame if it is of the homotopy type of a Σ-cofibrant G-spectrum.

There is no sensible counterpart to the nonequivariant notion of a strict CW prespectrum for general compact Lie groups, and any such notion is clumsy at best even for finite groups. The next few results are restated from [22]. Their proofs are the same equivariantly as non-equivariantly.

Theorem 2.8. *If D is a G-CW prespectrum, then LD has the homotopy type of a G-CW spectrum. If E is a G-CW spectrum, then each space EV has the homotopy type of a G-CW complex and E is homotopy equivalent to LD for some G-CW prespectrum D. Thus a G-spectrum has the homotopy type of a G-CW spectrum if and only if it has the homotopy type of LD for some G-CW prespectrum D.*

In particular, G-spectra of the homotopy types of G-CW spectra are tame.

Proposition 2.9. *If $E = LD$, where D is a Σ-cofibrant G-prespectrum, then*

$$E \cong \operatorname{colim}_V \ \Sigma^\infty_V DV,$$

where the colimit is computed as the prespectrum level colimit of the maps

$$\Sigma^\infty_W \sigma : \Sigma^\infty_V DV \cong \Sigma^\infty_W \Sigma^{W-V} DV \longrightarrow \Sigma^\infty_W DW.$$

That is, the prespectrum level colimit is a G-spectrum that is isomorphic to E. The maps of the colimit system are shift desuspensions of based G-cofibrations.

Proposition 2.10. *There is a functor $K : G\mathcal{PU} \longrightarrow \mathcal{GPU}$ such that KD is Σ-cofibrant for any G-prespectrum D, and there is a natural spacewise weak equivalence of G-prespectra $KD \longrightarrow D$. On G-spectra E, define $KE = LK\ell E$. Then there is a natural weak equivalence of G-spectra $KE \longrightarrow E$.*

For G-universes U and U', there is an associative and commutative smash product

$$G\mathcal{SU} \times \mathcal{GSU}' \to \mathcal{GS}(\mathcal{U} \oplus \mathcal{U}').$$

It is obtained by applying the spectrification functor L to the prespectrum level definition

$$(E \wedge E')(V \oplus V') = EV \wedge E'V'.$$

We internalize by use of twisted half-smash products. For G-universes U and U', we have a G-space $\mathcal{I}(\mathcal{U},\mathcal{U}')$ of linear isometries $U \longrightarrow U'$, with G acting by conjugation. For a G-map $\alpha : A \to \mathcal{I}(\mathcal{U},\mathcal{U}')$, the twisted half-smash product assigns a G-spectrum $A \ltimes E$ indexed on U' to a G-spectrum E indexed on U. While the following result is proven the same way equivariantly as nonequivariantly, it has different content: for a given $V \subset U$, there may well be no $V' \subset U'$ that is isomorphic to V.

Proposition 2.11. *For a G-map $A \longrightarrow \mathcal{I}(\mathcal{U},\mathcal{U}')$ and an isomorphism $V \cong V'$, where $V \subset U$ and $V' \subset U'$, there is an isomorphism of G-spectra*

$$A \ltimes \Sigma^\infty_V X \cong A_+ \wedge \Sigma^\infty_{V'} X$$

that is natural in G-spaces A over $\mathcal{I}(\mathcal{U},\mathcal{U}')$ and based G-spaces X.

Propositions 2.9 and 2.11 easily imply the following fundamental technical result.

Theorem 2.12. *Let $E \in G\mathcal{S}\mathcal{U}$ be tame and let A be a G-space over $\mathcal{I}(\mathcal{U},\mathcal{U}')$, where the universe U' contains a copy of every indexing space $V \subset U$. If $\phi : A' \longrightarrow A$ is a homotopy equivalence, then $\phi \ltimes \mathrm{id} : A' \ltimes E \longrightarrow A \ltimes E$ is a homotopy equivalence.*

If A is a G-CW complex and E is a G-CW spectrum, then $A \ltimes E$ is a G-CW spectrum when G is finite and has the homotopy type of a G-CW spectrum in general, hence this has the following consequence.

Corollary 2.13. *Let $E \in G\mathcal{S}\mathcal{U}$ have the homotopy type of a G-CW spectrum and let A be a G-space over $\mathcal{I}(\mathcal{U},\mathcal{U}')$ that has the homotopy type of a G-CW complex. Then $A \ltimes E$ has the homotopy type of a G-CW spectrum.*

We define the equivariant linear isometries operad $\mathcal{L}$ by letting $\mathcal{L}(|)$ be the G-space $\mathcal{I}(\mathcal{U}^|,\mathcal{U})$, exactly as in [22, 2.4]. A G-linear isometry $f : U^j \to U$ defines a G-map $\{*\} \longrightarrow \mathcal{L}(|)$ and thus a functor f_* that sends G-spectra indexed on U^j to G-spectra indexed on U. Applied to a j-fold external smash product $E_1 \wedge \cdots \wedge E_j$, there results an internal smash product $f_*(E_1 \wedge \cdots \wedge E_j)$.

Theorem 2.14. *Let $G\mathcal{S}_\sqcup \subset \mathcal{GS}$ be the full subcategory of tame G-spectra and let $hG\mathcal{S}_\sqcup$ be its homotopy category. On $G\mathcal{S}_\sqcup$, the internal smash products $f_*(E \wedge E')$ determined by varying $f : U^2 \to U$ are canonically homotopy equivalent, and $hG\mathcal{S}_\sqcup$ is symmetric monoidal under the internal smash product. For based G-spaces X and tame G-spectra E, there is a natural homotopy equivalence $E \wedge X \simeq f_*(E \wedge \Sigma^\infty X)$.*

We can define $\Sigma^V E = E \wedge S^V$ for any representation V. This functor is left adjoint to the loop functor Ω^V given by $\Omega^V E = F(S^V, E)$. For $V \subset U$, *and only for such* V, we also have the shift desuspension functor Σ^∞_V and therefore a $(-V)$-sphere $S^{-V} = \Sigma^\infty_V S^0$. Now the proof of [22, 2.6] applies to show that we have arrived at a stable situation *relative to* U.

Theorem 2.15. *For $V \subset U$, the suspension functor $\Sigma^V : hG\mathcal{S}_\sqcup \longrightarrow \langle\mathcal{GS}_\sqcup$ is an equivalence of categories with inverse given by smashing with S^{-V}. A cofibre* *sequence* $E \xrightarrow{f} E' \longrightarrow Cf$ *in* $G\mathcal{S}_{\sqcup}$ *gives rise to a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_q^H(E) \longrightarrow \pi_q^H(E') \longrightarrow \pi_q^H(Cf) \longrightarrow \pi_{q-1}^H(E) \longrightarrow \cdots.$$

¿From here, the theory of $\mathbb{L}$-spectra, S-modules, S-algebras, and modules over S-algebras that was summarized in [22, §§3-7] applies verbatim equivariantly, with one striking exception: duality theory only works when one restricts to cell R-modules that are built up out of sphere R-modules $G/H_+ \wedge S_R^n$ such that G/H embeds as a sub G-space of U. We shall focus on commutative S_G-algebras later, but we must first explain the exception just noted, along with various other matters where considerations of equivariance are central to the theory.

3. Homology and cohomology theories and fixed point spectra

In the previous section, the G-universe U was arbitrary, and we saw that the formal development of the stable category $\bar{h}G\mathcal{S}\mathcal{U}$ worked quite generally. However, there is very different content to the theory depending on the choice of universe. We focus attention on a complete G-universe U and its fixed point universe U^G. We call G-spectra indexed on U^G "naive G-spectra" since these are just spectra with G-action in the most naive sense. Examples include nonequivariant spectra regarded as G-spectra with trivial action. Genuine G-spectra are those indexed on U, and we refer to them simply as G-spectra. Their structure encodes the relationship between homotopy theory and representation theory that is essential for duality theory and most other aspects of equivariant stable homotopy theory.

$RO(G)$-graded homology and cohomology

Some of this relationship is encoded in the notion of an $RO(G)$-graded cohomology theory, which will play a significant role in our discussion of completion theorems. To be precise about this, one must remember that virtual representations are formal differences of *isomorphism classes* of orthogonal G-modules; we refer the interested reader to [53] for details and just give the idea here. For a virtual representation $\nu = W - V$, we can form the sphere G-spectrum $S^\nu = \Sigma^W S^{-V}$. We then define the homology and cohomology groups represented by a G-spectrum E by

$$E_\nu^G(X) = [S^\nu, E \wedge X]_G \tag{3.1}$$

and

$$E^\nu_G(X) = [S^{-\nu} \wedge X, E]_G = [S^{-\nu}, F(X, E)]_G. \tag{3.2}$$

If we think just about the $\mathbb{Z}$-graded part of a cohomology theory on G-spaces, then $RO(G)$-gradability amounts to the same thing as naturality with respect to stable G-maps.

Underlying nonequivariant spectra

To relate such theories to nonequivariant theories, let $i : U^G \longrightarrow U$ be the inclusion. We have the forgetful functor $i^* : G\mathcal{SU} \longrightarrow \mathcal{GSU}^{\mathcal{G}}$ specified by $i^*E(V) = E(i(V))$ for $V \subset U^G$; that is, we forget about the indexing spaces with non-trivial G-action. The "underlying nonequivariant spectrum" of E is i^*E with its action by G ignored. Recall that i^* has a left adjoint $i_* : G\mathcal{SU}^{\mathcal{G}} \longrightarrow \mathcal{GSU}$ that builds in non-trivial representations. Using an obvious notation to distinguish suspension spectrum functors, we have $i_*\Sigma^\infty_{U^G} X \cong \Sigma^\infty_U X$. These change of universe functors play a critical role in relating equivariant and nonequivariant phenomena. Since, with G-actions ignored, the universes are isomorphic, the following result is intuitively obvious.

Lemma 3.3. *For $D \in G\mathcal{SU}^{\mathcal{G}}$, the unit G-map $\eta : D \longrightarrow i^*i_*D$ of the (i_*, i^*) adjunction is a nonequivariant equivalence. For $E \in G\mathcal{SU}$, the counit G-map $\varepsilon : i_*i^*E \longrightarrow E$ is a nonequivariant equivalence.*

Fixed point spectra and homology and cohomology

We define the fixed point spectrum D^G of a naive G-spectrum D by passing to fixed points spacewise, $D^G(V) = (DV)^G$. This functor is right adjoint to the forgetful functor from naive G-spectra to spectra (compare Lemma 1.1):

$$G\mathcal{SU}^{\mathcal{G}}(\mathcal{C}, \mathcal{D}) \cong \mathcal{SU}^{\mathcal{G}}(\mathcal{C}, \mathcal{D}^{\mathcal{G}}) \text{ for } \mathcal{C} \in \mathcal{SU}^{\mathcal{G}} \text{ and } \mathcal{D} \in \mathcal{GSU}^{\mathcal{G}}. \tag{3.4}$$

It is essential that G act trivially on the universe to obtain well-defined structural homeomorphisms on D^G. For $E \in G\mathcal{SU}$, we define $E^G = (i^*E)^G$. Composing the (i_*, i^*)-adjunction with (3.4), we obtain

$$G\mathcal{SU}(\rangle_*\mathcal{C}, \mathcal{E}) \cong \mathcal{SU}^{\mathcal{G}}(\mathcal{C}, \mathcal{E}^{\mathcal{G}}) \text{ for } \mathcal{C} \in \mathcal{SU}^{\mathcal{G}} \quad \text{and} \quad \mathcal{E} \in \mathcal{GSU}. \tag{3.5}$$

The sphere G-spectra $G/H_+ \wedge S^n$ in $G\mathcal{SU}$ are obtained by applying i_* to the corresponding sphere G-spectra in $G\mathcal{SU}^{\mathcal{G}}$. When we restrict (3.1) and (3.2) to integer gradings and take $H = G$, we see that (3.5) implies

$$E^G_n(X) \cong \pi_n((E \wedge X)^G) \tag{3.6}$$

and

$$E^n_G(X) \cong \pi_{-n}(F(X, E)^G). \tag{3.7}$$

Exactly as in (3.7), naive G-spectra D represent $\mathbb{Z}$-graded cohomology theories on naive G-spectra, or on G-spaces. In sharp contrast, we cannot represent interesting homology theories on G-spaces X in the form $\pi_*((D \wedge X)^G)$ for a naive G-spectrum

D: smash products of naive G-spectra commute with fixed points, hence such theories vanish on X/X^G. For genuine G-spectra, there is a well-behaved natural map

$$E^G \wedge (E')^G \longrightarrow (E \wedge E')^G, \tag{3.8}$$

but, even when E' is replaced by a G-space, it is not an equivalence. Similarly, there is a natural map

$$\Sigma^\infty(X^G) \longrightarrow (\Sigma^\infty X)^G, \tag{3.9}$$

which, by Theorem 3.10 below, is the inclusion of a wedge summand but not an equivalence. Again, the fixed point spectra of free G-spectra are non-trivial. We shall shortly define a different G-fixed point functor that commutes with smash products and the suspension spectrum functor and which is trivial on free G-spectra.

Fixed point spectra of suspension G-spectra

Because the suspension functor from G-spaces to genuine G-spectra builds in homotopical information from representations, the fixed point spectra of suspension G-spectra are richer structures than one might guess. The following important result of tom Dieck [18] (see also [42, V§11]), gives a precise description.

Theorem 3.10. *For based G-CW complexes X, there is a natural equivalence*

$$(\Sigma^\infty X)^G \simeq \bigvee_{(H)} \Sigma^\infty(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} X^H),$$

where $WH = NH/H$ and $Ad(WH)$ is its adjoint representation; the sum runs over all conjugacy classes of subgroups H.

Quotient spectra and free G-spectra

Quotient spectra D/G of naive G-spectra are constructed by first passing to orbits spacewise on the prespectrum level and then applying the functor L from prespectra to spectra. This orbit spectrum functor is left adjoint to the forgetful functor to spectra:

(3.11)

$$\mathcal{SU}^{\mathcal{G}}(\mathcal{D}/\mathcal{G}, \mathcal{C}) \cong \mathcal{GSU}^{\mathcal{G}}(\mathcal{D}, \mathcal{C}) \text{ for } \mathcal{C} \in \mathcal{SU}^{\mathcal{G}} \text{ and } \mathcal{D} \in \mathcal{GSU}^{\mathcal{G}}.$$

Commuting left adjoints, we see that $(\Sigma^\infty X)/G \cong \Sigma^\infty(X/G)$. There is no useful quotient functor on genuine G-spectra in general, but there is a suitable substitute for free G-spectra.

Recall that a based G-space is said to be free if it is free away from its G-fixed basepoint. A G-spectrum, either naive or genuine, is said to be free if it is equivalent to a G-CW spectrum built up out of free cells $G_+ \wedge CS^n$. The functors

$$\Sigma^\infty : \mathcal{T} \longrightarrow \mathcal{GSU}^{\mathcal{G}} \quad \text{and} \quad \rangle_* : \mathcal{GSU}^{\mathcal{G}} \longrightarrow \mathcal{GSU}$$

carry free G-spaces to free naive G-spectra and free naive G-spectra to free G-spectra. In all three categories, X is homotopy equivalent to a free object if and only if the canonical G-map $EG_+ \wedge X \longrightarrow X$ is an equivalence. A free G-spectrum E is equivalent to i_*D for a free naive G-spectrum D, unique up to equivalence; the orbit spectrum D/G is the appropriate substitute for E/G. A useful mnemonic slogan is that "free G-spectra live in the G-fixed universe". For free naive G-spectra D, it is clear that $D^G = *$. However, this is false for free genuine G-spectra. For example, Theorem 3.10 specializes to give that $(\Sigma^\infty X)^G \simeq (\Sigma^{Ad(G)} X)/G$ if X is a free G-space. Thus the fixed point functor on free G-spectra has the character of a quotient.

More generally, for a family $\mathcal{F}$, we say that a G-spectrum E is $\mathcal{F}$-free, or is an $\mathcal{F}$-spectrum, if E is equivalent to a G-CW spectrum all of whose cells are of orbit type in $\mathcal{F}$. Thus free G-spectra are $\{1\}$-free. We say that a map $f : D \longrightarrow E$ is an $\mathcal{F}$-equivalence if $f^H : D^H \longrightarrow E^H$ is an equivalence for all $H \in \mathcal{F}$ or, equivalently by the Whitehead theorem, if f is an H-equivalence for all $H \in \mathcal{F}$.

Split G-spectra

It is fundamental to the passage back and forth between equivariant and nonequivariant phenomena to calculate the equivariant cohomology of free G-spectra in terms of the nonequivariant cohomology of orbit spectra. To explain this, we require the subtle and important notion of a "split G-spectrum".

Definition 3.12. A naive G-spectrum D is said to be split if there is a nonequivariant map of spectra $\zeta : D \longrightarrow D^G$ whose composite with the inclusion of D^G in D is homotopic to the identity. A genuine G-spectrum E is said to be split if i^*E is split.

The K-theory G-spectra K_G and KO_G are split. Intuitively, the splitting is obtained by giving nonequivariant bundles trivial G-action. Similarly, equivariant Thom spectra are split. The naive Eilenberg-MacLane G-spectrum HM that represents Bredon cohomology with coefficients in M is split if and only if the restriction map $M(G/G) \longrightarrow M(G/1)$ is a split epimorphism; this implies that G acts trivially on $M(G/1)$, which is usually not the case. The suspension G-spectrum $\Sigma^\infty X$ of a G-space X is split if and only if X is stably a retract up to homotopy of X^G, which again is usually not the case. In particular, however, the sphere G-spectrum $S = \Sigma^\infty S^0$ is split. The following consequence of Lemma 3.3 gives more examples.

Lemma 3.13. *If $D \in G\mathcal{SU}^{\mathcal{G}}$ is split, then $i_*D \in G\mathcal{SU}$ is also split. In particular, i_*D is split if D is a nonequivariant spectrum regarded as a naive G-spectrum with trivial action.*

The notion of a split G-spectrum is defined in nonequivariant terms, but it admits the following equivariant interpretation.

Lemma 3.14. *If E is a G-spectrum with underlying nonequivariant spectrum D, then E is split if and only if there is a map of G-spectra $i_*D \longrightarrow E$ that is a nonequivariant equivalence.*

Theorem 3.15. *If E is a split G-spectrum and X is a free naive G-spectrum, then there are natural isomorphisms*

$$E^G_n(i_*X) \cong E_n((\Sigma^{Ad(G)}X)/G) \quad \text{and} \quad E_G^n(i_*X) \cong E^n(X/G),$$

where $Ad(G)$ is the adjoint representation of G and E_* and E^* denote the theories represented by the underlying nonequivariant spectrum of E.

The cohomology isomorphism holds by inductive reduction to the case $X = G_+$. The homology isomorphism is deeper, and we shall say a bit more about it later.

Geometric fixed point spectra

There is a "geometric" fixed-point functor

$$\Phi^G : G\mathcal{S}\mathcal{U} \longrightarrow \mathcal{S}\mathcal{U}^{\mathcal{G}}$$

that enjoys the properties

$$\Sigma^\infty(X^G) \simeq \Phi^G(\Sigma^\infty X) \tag{3.16}$$

and

$$\Phi^G(E) \wedge \Phi^G(E') \simeq \Phi^G(E \wedge E'). \tag{3.17}$$

It is trivial on free G-spectra and, more generally, on $\mathcal{P}$-spectra, where $\mathcal{P}$ is the family of proper subgroups of G. Recall that, for a family $\mathcal{F}$, $\tilde{E}\mathcal{F}$ is the cofibre of the natural map $EG_+ \longrightarrow S^0$. We define

$$\Phi^G(E) = (E \wedge \tilde{E}\mathcal{P})^{\mathcal{G}}, \tag{3.18}$$

where $\mathcal{P}$ is the family of proper subgroups of G. Here $E \wedge \tilde{E}\mathcal{P}$ is H-trivial for all $H \in \mathcal{P}$. The isomorphism (3.16) is clear from Theorem 3.10.

We call Φ^G the "geometric" fixed point functor because its properties are like those of the space level G-fixed point functor and because it corresponds to the direct prespectrum level construction that one is likely to think of first. Restricting to finite groups G for simplicity and indexing G-prespectra on multiples of the regular representation, we can define a prespectrum level fixed point functor Φ^G by $(\Phi^G D)(\mathbb{R}^{\ltimes}) = (\mathbb{D}(\ltimes\mathbb{R}\mathbb{G}))^{\mathbb{G}}$. If D is tame, then $(\Phi^G)(LD)$ is equivalent to $L\Phi^G D$. Therefore, if we start with a G-spectrum E, then $\Phi^G(E)$ is equivalent to

$L\Phi^G(K\ell E)$, where K is the cylinder functor. This alternative description leads to the proof of (3.17). It also leads to a proof that

(3.19) $$[E, F \wedge \tilde{E}\mathcal{P}]_{\mathcal{G}} \cong [\Phi^{\mathcal{G}}(\mathcal{E}), \Phi^{\mathcal{G}}(\mathcal{F})] \text{ for } G\text{-spectra } E \text{ and } F.$$

Euler classes and a calculational example

As an illuminating example of the use of $RO(G)$-grading to allow descriptions invisible to the $\mathbb{Z}$-graded part of a theory, we record how to compute $E^G_*(X \wedge \tilde{E}\mathcal{P})$ in terms of $E^G_*(X)$ for a ring G-spectrum E and any G-spectrum X. When $X = S$, it specializes to a calculation of

$$E^G_*(\tilde{E}\mathcal{P}) = \pi_*(\Phi^{\mathcal{G}}\mathcal{E}).$$

The example may look esoteric, but it is at the heart of the completion theorems that we will discuss later. We use the Euler classes of representations, which appear ubiquitously in equivariant theory. For a representation V, we define the Euler class $\chi_V \in E^G_{-V} = E^V_G(S^0)$ to be the image of $1 \in E^0_G(S^0) \cong E^V_G(S^V)$ under $e(V)^*$, where $e(V) : S^0 \longrightarrow S^V$ sends the basepoint to the point at ∞ and the non-basepoint to 0.

Proposition 3.20. *Let E be a ring G-spectrum and X be any G-spectrum. Then $E^G_*(X \wedge \tilde{E}\mathcal{P})$ is isomorphic to the localization of the E^G_*-module $E^G_*(X)$ obtained by inverting the Euler classes of all representations V such that $V^G = \{0\}$.*

Proof. A check of fixed points, using the cofibrations $S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$, shows that we obtain a model for $\tilde{E}\mathcal{P}$ by taking the colimit Y of the spaces S^V as V ranges over the indexing spaces $V \subset U$ such that $V^G = \{0\}$. The point is that if H is a proper subgroup of G, then $V^H \neq \{0\}$ for all sufficiently large V, so that $Y^H \simeq *$. Therefore

$$E^G_\nu(X \wedge \tilde{E}\mathcal{P}) \cong \operatorname{colim}_{\mathcal{V}}\, \mathcal{E}^{\mathcal{G}}_{-\nu}(\mathcal{X} \wedge \mathcal{S}^{\mathcal{V}}) \cong \operatorname{colim}_{\mathcal{V}}\, \mathcal{E}^{\mathcal{G}}_{\nu-\mathcal{V}}(\mathcal{X}).$$

Since the colimit is taken over iterated products with χ_V, it coincides algebraically with the cited localization. □

4. Change of groups and duality theory

So far, we have discussed the relationship between G-spectra and 1-spectra, where 1 is the trivial group. We must consider other subgroups and quotient groups of G.

Induced and coinduced G-spectra

First, consider a subgroup H. Since any representation of NH is a summand in a restriction of a representation of G and since a WH-representation is just an H-fixed NH-representation, the H-fixed point space U^H of our given complete G-universe U is a complete WH-universe. We define

$$E^H = (i^*E)^H, \quad i : U^H \subset U. \tag{4.1}$$

This gives a functor $G\mathcal{SU} \longrightarrow (\mathcal{WH})\mathcal{SU}^{\mathcal{H}}$. For $D \in (NH)\mathcal{SU}^{\mathcal{H}}$, the orbit spectrum D/H is also a WH-spectrum.

Exactly as on the space level, we have induced and coinduced G-spectra generated by an H-spectrum $D \in H\mathcal{SU}$. These are denoted by

$$G \ltimes_H D \quad \text{and} \quad F_H[G, D).$$

The "twisted" notation $\ltimes$ is used because there is a little twist in the definitions to take account of the action of G on indexing spaces. As on the space level, these functors are left and right adjoint to the forgetful functor $G\mathcal{SU} \longrightarrow \mathcal{HSU}$: for $D \in H\mathcal{SU}$ and $E \in G\mathcal{SU}$, we have

$$G\mathcal{SU}(\mathcal{G} \ltimes_{\mathcal{H}} \mathcal{D}, \mathcal{E}) \cong \mathcal{HSU}(\mathcal{D}, \mathcal{E}) \tag{4.2}$$

and

$$H\mathcal{SU}(\mathcal{E}, \mathcal{D}) \cong \mathcal{GSU}(\mathcal{E}, \mathcal{F}_{\mathcal{H}}[\mathcal{G}, \mathcal{D})). \tag{4.3}$$

Again, as on the space level, for a G-spectrum E, we have

$$G \ltimes_H E \cong (G/H)_+ \wedge E \tag{4.4}$$

and

$$F_H[G, E) \cong F(G/H_+, E). \tag{4.5}$$

We can now deduce as on the space level that

$$\pi_n^H(E) \equiv [G/H_+ \wedge S^n, E]_G \cong [S^n, E]_H \cong \pi_n(E^H). \tag{4.6}$$

We also have a geometric H-fixed point functor Φ^H. It is obtained by regarding G-spectra as NH-spectra and setting

$$\Phi^H(E) = (E \wedge \tilde{E}\mathcal{F}[\mathcal{H}])^{\mathcal{H}},$$

where $\mathcal{F}[\mathcal{H}]$ is the family of subgroups of NH that do not contain H. Again, $\Phi^H E$ is an NH-spectrum indexed on U^H. While the Whitehead theorem appeared originally as a statement about homotopy groups and thus about the genuine fixed point functors, it implies a version in terms of the Φ-fixed point functors.

Theorem 4.7. *Let $f : E \longrightarrow F$ be a map between G-CW spectra. Then the following statements are equivalent.*

(i) *f is a G-homotopy equivalence.*
(ii) *Each f^H is a nonequivariant homotopy equivalence.*
(iii) *Each $\Phi^H f$ is a nonequivariant homotopy equivalence.*

Subgroups and the Wirthmüller isomorphism

In cohomology, the isomorphism (4.2) gives

$$E^*_G(G \ltimes_H D) \cong E^*_H(D). \tag{4.8}$$

We shall not be precise, but we can interpret this in terms of $RO(G)$ and $RO(H)$ graded cohomology theories. The isomorphism (4.3) does not have such a convenient interpretation as it stands. However, there is an important change of groups result, called the Wirthmüller isomorphism, which in its most conceptual form is given by a calculation of the functor $F_H[G, D)$. It leads to the following homological complement of (4.8). Let $L(H)$ be the tangent H-representation at the identity coset of G/H. Then

$$E^G_*(G \ltimes_H D) \cong E^H_*(\Sigma^{L(H)} D). \tag{4.9}$$

Theorem 4.10 (Generalized Wirthmüller isomorphism)**.** *For H-spectra D, there is a natural equivalence of G-spectra*

$$F_H[G, \Sigma^{L(H)} D) \xrightarrow{\simeq} G \ltimes_H D.$$

Therefore, for G-spectra E,

$$[E, \Sigma^{L(H)} D]_H \cong [E, G \ltimes_H D]_G.$$

The last isomorphism complements the isomorphism from (4.2):

$$[G \ltimes_H D, E]_G \cong [D, E]_H. \tag{4.11}$$

We deduce (4.8) by replacing E in (4.9) by a sphere, replacing D by $E \wedge D$, and using the generalization

$$G \ltimes_H (D \wedge E) \cong (G \ltimes_H D) \wedge E$$

of (4.4).

Quotient groups and the Adams isomorphism

Now let N be a normal subgroup of G with quotient group J. In practice, one is often thinking of a quotient map $NH \longrightarrow WH$ rather than $G \longrightarrow J$. There is an analogue of the Wirthmüller isomorphism, called the Adams isomorphism, that compares orbit and fixed-point spectra. It involves the change of universe functors associated to the inclusion $i : U^N \longrightarrow U$ and requires restriction to N-free G-spectra. We emphasize that U^N is not a complete G-universe. We have generalizations of the adjunctions (3.4) and (3.11): for $D \in J\mathcal{SU}^{\mathcal{N}}$ and $E \in G\mathcal{SU}^{\mathcal{N}}$,

$$G\mathcal{SU}^{\mathcal{N}}(\mathcal{D}, \mathcal{E}) \cong \mathcal{JSU}^{\mathcal{N}}(\mathcal{D}, \mathcal{E}^{\mathcal{N}}) \tag{4.12}$$

and

$$J\mathcal{SU}^{\mathcal{N}}(\mathcal{E}/\mathcal{N}, \mathcal{D}) \cong \mathcal{GSU}^{\mathcal{N}}(\mathcal{E}, \mathcal{D}). \tag{4.13}$$

Here we suppress notation for the pullback functor $J\mathcal{SU}^{\mathcal{N}} \longrightarrow \mathcal{GSU}^{\mathcal{N}}$. An N-free G-spectrum E indexed on U is equivalent to i_*D for an N-free G-spectrum D indexed on U^N, and D is unique up to equivalence. Thus our slogan that "free G-spectra live in the G-fixed universe" generalizes to the slogan that "N-free G-spectra live in the N-fixed universe". This gives force to the following version of (4.12). It compares maps of J-spectra indexed on U^N with maps of G-spectra indexed on U.

Theorem 4.14. *Let $J = G/N$. For N-free G-spectra E indexed on U^N and J-spectra D indexed on U^N,*

$$[E/N, D]_J \cong [i_*E, i_*D]_G.$$

The conjugation action of G on N gives rise to an action of G on the tangent space of N at e; we call this representation $Ad(N)$, or $Ad(N;G)$. The following result complements the previous one, but is considerably deeper. When $N = G$, it is the heart of the proof of the homology isomorphism of Theorem 3.15.

Theorem 4.15 (Generalized Adams isomorphism)**.** *Let $J = G/N$. For N-free G-spectra $E \in G\mathcal{SU}^{\mathcal{N}}$, there is a natural equivalence of J-spectra*

$$E/N \xrightarrow{\simeq} (\Sigma^{-Ad(N)} i_*E)^N.$$

Therefore, for $D \in J\mathcal{SU}^{\mathcal{N}}$,

$$[D, E/N]_J \cong [i_*D, \Sigma^{-Ad(N)} i_*E]_G.$$

The last two results admit homological and cohomological interpretations, like those of Theorem 3.15, that are based on a generalization of the notion of a split G-spectrum. We shall not go into that here; see [42, Ch.II].

Spanier-Whitehead and Atiyah duality

Recall that the dual of a G-space or G-spectrum X is $DX = F(X, S)$. This is defined for any universe, but we observe the striking fact that if we work over U^G, then the sphere S has trivial G-action and $F(X, S) = F(X/G, S)$; in particular, the dual of every orbit G/H_+ is S. We must therefore work in the complete universe U to give useful content to the formal theory of duality, and the first thing we must do is to identify the duals of orbits. In fact, this identification is the real content of the Wirthmüller isomorphism, which implies that

$$D(G/H_+) \simeq G \ltimes_H S^{-L(H)}. \tag{4.16}$$

In particular, orbits are self-dual if G is finite.

It follows that finite G-CW spectra are strongly dualizable, and the Spanier-Whitehead duality theorem is a formal consequence.

Theorem 4.17 (Spanier-Whitehead duality). *If X is a wedge summand of a finite G-CW spectrum and E is any G-spectrum, then*

$$\nu : DX \wedge E \xrightarrow{\simeq} F(X, E)$$

is an isomorphism in $\bar{h}G\mathcal{SU}$. Therefore, for any virtual representation ν,

$$E^G_\nu(DX) \cong E_G^{-\nu}(X).$$

By developing a space level analysis of how to identify dual G-spectra, one can generalize the identification of duals of orbits to an identification of the duals of smooth G-manifolds. Working on the space level, one has a notion of V-duality between spaces X and Y. It involves evaluation and coevaluation maps $Y \wedge X \longrightarrow S^V$ and $S^V \longrightarrow X \wedge Y$ and implies that $\Sigma^{-V}\Sigma^\infty Y$ is dual to $\Sigma^\infty X$.

Theorem 4.18 (Atiyah duality). *If M is a smooth closed G-manifold embedded in a representation V with normal bundle ν, then M_+ is V-dual to the Thom complex $T\nu$. If M is a smooth compact G-manifold with boundary ∂M, $V = V' \oplus \mathbb{R}$, and $(M, \partial M)$ is properly embedded in $(V' \times [0,\infty), V' \times \{0\})$ with normal bundles ν' of ∂M in V' and ν of M in V, then $M/\partial M$ is V-dual to $T\nu$, M_+ is V-dual to $T\nu/T\nu'$, and the cofibration sequence*

$$T\nu' \longrightarrow T\nu \longrightarrow T\nu/T\nu' \longrightarrow \Sigma T\nu'$$

is V-dual to the cofibration sequence

$$\Sigma(\partial M)_+ \longleftarrow M/\partial M \longleftarrow M_+ \longleftarrow (\partial M)_+.$$

We display the coevaluation map $\eta : S^V \longrightarrow M_+ \wedge T\nu$ explicitly in the closed case. By the equivariant tubular neighborhood theorem, we may extend the embedding of M in V to an embedding of the normal bundle ν and apply the Pontryagin-Thom construction to obtain a map $t : S^V \longrightarrow T\nu$. The diagonal map of the total space of ν induces the Thom diagonal $\Delta : T\nu \longrightarrow M_+ \wedge T\nu$, and η is just the composite $\Delta \circ t$.

Specializing to $M = G/H$, we have

$$\tau = G \times_H L(H) \quad \text{and} \quad T\tau = G_+ \wedge_H S^{L(H)}.$$

If G/H is embedded in V with normal bundle ν and W is the orthogonal complement to $L(H)$ in the fiber over the identity coset, then $\nu = G \times_H W$ and therefore $\Sigma^\infty_V T\nu \simeq G \ltimes_H S^{-L(H)}$. Observe that we have a composite map

$$S^V \xrightarrow{t} T\nu \longrightarrow T(\nu \oplus \tau) \cong G/H_+ \wedge S^V. \tag{4.19}$$

This is called the "transfer map" associated to the projection $G/H \longrightarrow *$.

We can deduce equivariant versions of the Poincaré and Lefschetz duality theorems by combining Spanier-Whitehead duality, Atiyah duality, and the Thom isomorphism. However, the results are more subtle and less algebraically tractable than their nonequivariant analogs because G-manifolds are not homogeneous: they look locally like $G \times_H W$ for a subgroup H and H-representation W, which means that there is generally no natural "dimension" in which the orientation class or fundamental class of a manifold should lie. We refer the reader to [42, Ch.III] for discussion.

5. Mackey functors, $K(M,n)$'s, and $RO(G)$-graded cohomology

We have considered the ordinary cohomology $H^*_G(X;M)$ of a G-space X with coefficients in a coefficient system M. We can construct an additive category $\mathbb{Z}[\eqsim\mathcal{O}]$ from the homotopy category $h\mathcal{O}$ of orbits by applying the free Abelian group functor. The resulting category is isomorphic to the full subcategory of naive orbit spectra $\Sigma^\infty G/H_+$ in the stable homotopy category $\bar{h}G\mathcal{S}\mathcal{U}^{\mathcal{G}}$ of naive G-spectra. Clearly, a coefficient system is the same thing as an additive contravariant functor $\mathbb{Z}[\eqsim\mathcal{O}] \longrightarrow \mathcal{A}\lfloor$. Just as nonequivariantly, we can construct naive Eilenberg-Mac Lane G-spectra $HM = K(M,0)$ associated to coefficient systems M and so extend our cohomology theories on G-spaces to cohomology theories on naive G-spectra.

It is natural to ask when these cohomology theories can be extended to $RO(G)$-graded cohomology theories on genuine G-spectra. The answer is suggested by the previous paragraph. Define $h\mathcal{O}\mathcal{S}$ to be the full subcategory of orbit spectra $\Sigma^\infty G/H_+$ in the stable homotopy category $\bar{h}G\mathcal{S}\mathcal{U}$ of genuine G-spectra. Define a Mackey functor to be an additive contravariant functor $M : \bar{h}\mathcal{O}\mathcal{S} \longrightarrow \mathcal{A}\lfloor$; we abbreviate $M(G/H) = M(\Sigma^\infty G/H_+)$. This is the appropriate definition for general compact Lie groups, but we shall describe an equivalent algebraic definition when G is finite. It turns out that the cohomology theory $H^*_G(\cdot, M)$ can be extended to an $RO(G)$-graded theory if and only if the coefficient system M extends to a Mackey functor [40].

The idea can be made clear by use of the transfer map (4.18). If $H^*_G(\cdot;M)$ is $RO(G)$-gradable, then, for based G-spaces X, the transfer map induces homomorphisms

$$
\begin{array}{c}
\tilde{H}^n_G(G/H_+ \wedge X;M) \cong \tilde{H}^{n+V}_G(\Sigma^V(G/H_+ \wedge X);M) \\
\Big\downarrow \\
\tilde{H}^n_G(X;M) \cong \tilde{H}^{n+V}_G(\Sigma^V X;M).
\end{array}
\tag{5.1}
$$

Taking $n = 0$ and $X = S^0$, we obtain a transfer homomorphism

$$M(G/H) \longrightarrow M(G/G).$$

An elaboration of this argument shows that the coefficient system M must extend to a Mackey functor.

Algebraic description of Mackey functors

For finite groups G, calculational analysis of the category $h\mathcal{OS}$ leads to an algebraic translation of our topological definition. Let $\mathcal{F}$ denote the category of finite G-sets and G-maps and let $h\mathcal{FS}$ be the full subcategory of the stable category whose objects are the $\Sigma^\infty X_+$ for finite G-sets X. Then $h\mathcal{OS}$ embeds as a full subcategory of $h\mathcal{FS}$, and every object of $h\mathcal{FS}$ is a finite wedge of objects of $h\mathcal{OS}$. Since an additive functor necessarily preserves any finite direct sums in its domain, it is clear that an additive contravariant functor $h\mathcal{OS} \longrightarrow \mathcal{A}\lfloor$ determines and is determined by an additive contravariant functor $h\mathcal{FS} \longrightarrow \mathcal{A}\lfloor$. In turn, an additive contravariant functor $h\mathcal{FS} \longrightarrow \mathcal{A}\lfloor$ determines and is determined by a Mackey functor in the algebraic sense defined by Dress [19]. Precisely, such a Mackey functor M consists of a contravariant functor M^* and a covariant functor M_* from finite G-sets to Abelian groups. These functors have the same object function, denoted M, and M converts disjoint unions to direct sums. Write $M^*\alpha = \alpha^*$ and $M_*\alpha = \alpha_*$. For pullback diagrams of finite G-sets

$$\begin{array}{ccc} P & \xrightarrow{\delta} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow[\beta]{} & Z, \end{array}$$

it is required that $\alpha^* \circ \beta_* = \delta_* \circ \gamma^*$. For an additive contravariant functor $M : h\mathcal{FS} \longrightarrow \mathcal{A}\lfloor$, the maps induced by the projections $G/H \longrightarrow G/K$ for $H \subset K$ and the corresponding transfer maps specify the contravariant and covariant parts of the corresponding algebraic Mackey functor, and conversely. The algebraic notion has applications to many areas of mathematics in which finite group actions are studied.

In the compact Lie case it is hard to prove that an algebraically defined coefficient system extends to a Mackey functor, but there is one important example.

Proposition 5.2. *Let G be any compact Lie group. There is a unique Mackey functor $\underline{\mathbb{Z}} : h\mathcal{OS} \longrightarrow \mathcal{A}\lfloor$ such that the underlying coefficient system of $\underline{\mathbb{Z}}$ is constant at $\mathbb{Z}$ and the homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ induced by the transfer map $\Sigma^\infty G/K_+ \longrightarrow \Sigma^\infty G/H_+$ associated to an inclusion $H \subset K$ is multiplication by the Euler characteristic $\chi(K/H)$.*

Construction of $RO(G)$-graded cohomology theories and $K(M,0)$'s

Returning to our original problem of constructing an $RO(G)$-graded ordinary cohomology theory and thinking on the spectrum level, we see that we want to construct a genuine Eilenberg-MacLane G-spectrum $HM = K(M,0)$. It is clear that the coefficient system $M = \underline{\pi}_0(HM)$ must be a Mackey functor since, by our homotopical definition of Mackey functors, the homotopy group system $\underline{\pi}_n(E)$ must be a Mackey functor for any G-spectrum E. The following result was first proven in [40].

Theorem 5.3. *For a Mackey functor M, there is an Eilenberg-MacLane G-spectrum $HM = K(M,0)$, unique up to isomorphism in $\bar{h}G\mathcal{S}$. For Mackey functors M and M', $[HM, HM']_G$ is the group of maps of Mackey functors $M \longrightarrow M'$.*

We prove this by constructing a $\mathbb{Z}$-graded cohomology theory on G-spectra. By Brown's representability theorem, its degree zero part can be represented. The representing G-spectrum is our HM, and, since it is a genuine G-spectrum, it must of course represent an $RO(G)$-graded theory. The details that we use to construct the desired cohomology theories are virtually identical to those that we used to construct ordinary theories in the first place.

We start with G-CW spectra X. They have skeletal filtrations, and we define a Mackey-functor valued cellular chain complex by setting

$$\underline{C}_n(X) = \underline{\pi}_n(X^n/X^{n-1}). \tag{5.4}$$

Of course, X^n/X^{n-1} is a wedge of n-sphere G-spectra $G/H_+ \wedge S^n$, and the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}) specifies the required differential. For a Mackey functor M, we define

$$C^n_G(X; M) = \mathrm{Hom}_{\mathcal{M}}(\underline{C}_n(X), M), \quad \text{with} \quad \delta = \mathrm{Hom}_{\mathcal{M}}(d, \mathrm{Id}). \tag{5.5}$$

Then $C^*_G(X; M)$ is a cochain complex of Abelian groups. We denote its cohomology by $H^*_G(X; M)$. The evident cellular versions of the homotopy, excision, exactness, and wedge axioms admit exactly the same derivations as on the space level, and we use G-CW approximation to extend from G-CW spectra to general G-spectra: we have a $\mathbb{Z}$-graded cohomology theory on $G\mathcal{S}\mathcal{U}$. It satisfies the dimension axiom

$$H^*_G(\Sigma^\infty G/H_+; M) = H^0_G(\Sigma^\infty G/H_+; M) = M(G/H), \tag{5.6}$$

and these isomorphisms give an isomorphism of Mackey functors. The zeroth term is represented by a G-spectrum HM, and we read off its homotopy groups from (5.5):

$$\underline{\pi}_0(HM) = M \quad \text{and} \quad \underline{\pi}_n(HM) = 0 \text{ if } \ n \neq 0.$$

The uniqueness of HM is evident, and the calculation of $[HM, HM']_G$ follows easily from the functoriality in M of the theories $H^*_G(X; M)$.

We should observe that spectrum level obstruction theory works exactly as on the space level, modulo connectivity assumptions to ensure that one has a dimension in which to start inductions.

For G-spaces X, we have now given two meanings to the notation $H^*_G(X; M)$: we can regard our Mackey functor as a coefficient system and take the ordinary cohomology of X as in §1, or we can take our newly constructed cohomology. We know by the axiomatic characterization of ordinary cohomology that these must in fact be isomorphic, but it is instructive to check this directly. At least after a single suspension, we can approximate any G-space by a weakly equivalent based G-CW complex, with based attaching maps. The functor Σ^∞ takes based G-CW complexes to G-CW spectra, and we find that the space level and spectrum level chain complexes are isomorphic. Alternatively, we can check on the represented level:

$$[\Sigma^\infty X, \Sigma^n HM]_G \cong [X, \Omega^\infty \Sigma^n HM]_G \cong [X, K(M,n)]_G.$$

The Conner conjecture

Lest this all seem too abstract, let us us retrieve a direct and important space level consequence of this machinery, namely the Conner conjecture.

Theorem 5.7 (The Conner conjecture)**.** *Let X be a finite dimensional G-space with finitely many orbit types, where G is any compact Lie group, and let A be any Abelian group. If $\tilde{H}^*(X;A) = 0$, then $\tilde{H}^*(X/G;A) = 0$.*

This was first proven by Oliver [60], using Čech cohomology and wholly different techniques. It was known early on that the conjecture would hold if one could construct a suitable transfer map. It is now easy to do so [40].

Theorem 5.8. *Let X be a G-space and $\pi : X/H \longrightarrow X/G$ be the projection, where $H \subseteq G$. For any $n \geq 0$ and any Abelian group A, there is a natural transfer homomorphism*

$$\tau : H^n(X/H; A) \longrightarrow H^n(X/G; A)$$

such that $\tau \circ \pi^*$ is multiplication by the Euler characteristic $\chi(G/H)$.

Proof. Tensoring the Mackey functor $\underline{\mathbb{Z}}$ of Proposition 5.2 with A, we obtain a Mackey functor $\underline{A}$ whose underlying coefficient system is constant at A. The map $\underline{A}(G/H) \longrightarrow \underline{A}(G/G)$ associated to the stable transfer map $G/G_+ \longrightarrow G/H_+$ is multiplication by $\chi(G/H)$. By the axiomatization, the ordinary G-cohomology of a G-space X with coefficients in a constant coefficient system is isomorphic to the ordinary nonequivariant cohomology of its orbit space X/G:

$$H^n_G(X;\underline{A}) \cong H^n(X/G;A) \quad \text{and} \quad H^n_G(G/H\times X;\underline{A}) \cong H^n_H(X;\underline{A}|H) \cong H^n(X/H;A).$$

Taking $M = \underline{A}$, (5.1) displays the required transfer map. □

How does the Conner conjecture follow? Conner [15] proved it when G is a finite extension of a torus, the methods being induction and use of Smith theory: one proves that both X^G and X/G are A-acyclic. For example, the result for a torus reduces immediately to the result for a circle. Here the "finitely many orbit types" hypothesis implies that $X^G = X^C$ for C cyclic of large enough order, so that we are in the realm where classical Smith theory can be applied. Assuming that the result holds when G is a finite extension of a torus, let N be the normalizer of a maximal torus in G. Then N is a finite extension of a torus and $\chi(G/N) = 1$. The composite

$$\tau \circ \pi^* : \tilde{H}^n(X/G; A) \longrightarrow \tilde{H}^n(X/N; A) \longrightarrow \tilde{H}^n(X/G; A)$$

is the identity, and that's all there is to it.

The rational equivariant stable category

Exactly as for simple spaces and for spectra, we can use our Eilenberg-Mac Lane G-spectra to show that any G-spectrum can be approximated as the homotopy inverse limit of a Postnikov tower constructed out of $K(M,n)$'s and k-invariants, where $K(M,n) = \Sigma^n HM$. For finite groups, the k-invariants vanish rationally.

Theorem 5.9. *Let G be finite. Then, for rational G-spectra E, there is a natural equivalence $E \xrightarrow{\simeq} \prod K(\underline{\pi}_n(E), n)$.*

Counterexamples of Triantafillou [64] show that, unless G is cyclic of prime power order, the conclusion is false for naive G-spectra. A counterexample of Haeberly [34] shows that the conclusion is also false for genuine G-spectra when G is the circle group, the rationalization of KU_G furnishing a counterexample.

The proof of Theorem 5.9 depends on two facts, one algebraic and one topological. Assume that G is finite.

Proposition 5.10. *All objects are projective and injective in the Abelian category of rational Mackey functors.*

The analogue for coefficient systems is false, and so is the analogue for general compact Lie groups. One of us has recently studied what does happen for compact Lie groups [27]. The following result is easy for finite groups and false for compact Lie groups, as we see from Theorem 3.10.

Proposition 5.11. *For $H \subseteq G$ and $n \neq 0$, $\underline{\pi}_n(G/H_+) \otimes \mathbb{Q} = \nvdash$.*

Let $\mathcal{M} = \mathcal{M}[\mathcal{G}]$ denote the Abelian category of Mackey functors over G. For G-spectra E and F, there is an evident natural map

$$\theta : [E, F]_G \longrightarrow \prod \mathrm{Hom}_{\mathcal{M}}(\underline{\pi}_n(E), \underline{\pi}_n(F)).$$

Let F be rational. By the previous result and the Yoneda lemma, θ is an isomorphism when $E = \Sigma^\infty G/H_+$ for any H. Clearly, we can extend θ to a graded map

$$\theta : F^q_G(E) = [E, F]^q_G = [\Sigma^{-q} E, F]_G \longrightarrow \prod \mathrm{Hom}_{\mathcal{M}}(\underline{\pi}_n(\Sigma^{-q} E), \underline{\pi}_n(F)).$$

It is still an isomorphism when E is an orbit. We obtain the same groups if we replace E and the Mackey functors $\underline{\pi}_n(\Sigma^{-q} E)$ by their rationalizations. Since the

Mackey functors $\underline{\pi}_n(F)$ are injective, the right hand side is a cohomology theory on G-spectra E. Clearly θ is a map of cohomology theories and this already implies the following result. With $F = \prod K(\underline{\pi}_n(E), n)$, Theorem 5.9 is a direct consequence.

Theorem 5.12. *Let G be finite. If F is rational, then θ is a natural isomorphism.*

This classifies rational G-spectra and one can go on to classify maps between them and so obtain a complete algebraization of the rational equivariant stable category. We refer the reader to [30, App A].

6. Philosophy of localization and completion theorems

We shall work with reduced homology and cohomology theories in the rest of this article.

It is natural to want to know about the homology and cohomology of classifying spaces, as invariants of groups, as homes of characteristic classes, and as groups of bordism classes of G-manifolds.

One reason that it is difficult to calculate $k^*(BG_+)$ or $k_*(BG_+)$ is that BG_+ is an infinite complex. The conventional approach to calculation is based on the skeletal filtration of BG_+, which gives rise to Atiyah-Hirzebruch spectral sequences. One problem with this approach is that ordinary cohomology is not the most natural way to look at BG, and much of its good behaviour when viewed by other cohomology theories is invisible to ordinary cohomology.

An attractive alternative is to consider equivariant forms of k-theory. We shall say that $k^*_G(\cdot)$ is an equivariant form of $k^*(\cdot)$ if it is represented by a split G-spectrum k_G whose underlying spectrum k represents $k^*(\cdot)$. This means in particular that there is a map $k^* \longrightarrow k^*_G$ and also that for any free G-spectrum X there is a natural isomorphism $k^*_G(X) = k^*(X/G)$.

Typically, there will be many equivariant versions of $k^*(\cdot)$, and some will serve our purposes better than others. Perhaps the most obvious version is i_*k, but that is usually not the most useful version. We suppose that one particular version has been chosen in the following discussion. For example, the nicest equivariant form of topological K-theory is the Atiyah-Segal equivariant K-theory defined using equivariant bundles [62].

The point of thinking equivariantly is that

$$k^*_G(EG_+) = k^*(BG_+) \quad \text{and} \quad k^G_*(EG_+) = k_*(EG_+ \wedge_G S^{Ad(G)}),$$

so that we have moved the problem into the equivariant world: we have to understand the homology and cohomology of free G-spectra, and we may hope to do so in general, allowing effective use of finite G-CW complexes to obtain information about our infinite G-CW complex EG. To carry out this idea, we introduce a parameter G-space X. By introducing equivariance, we have made available the comparison map

$$\pi^* : k_G^*(X) \longrightarrow k_G^*(EG_+ \wedge X) = k^*(EG_+ \wedge_G X),$$

induced by the projection $\pi : EG_+ \longrightarrow S^0$. It is appropriate to think of X as finite, so that the domain is easily calculated, whilst the codomain is the cohomology of an infinite complex. The motivating case $X = S^0$ gives the map

$$\pi^* : k_G^* \longrightarrow k_G^*(EG_+) \cong k^*(BG_+).$$

It is only slightly over-optimistic to hope that this is an isomorphism, as we now explain.

To obtain some algebraic control, we assume that $k^*(\cdot)$ and $k_G^*(\cdot)$ are ring theories, and that the splitting map is a ring map. More generally, we assume given module theories $m^*(\cdot)$ and $m_G^*(\cdot)$ over $k^*(\cdot)$ and $k_G^*(\cdot)$, with suitable splitting maps. Then all groups $m_G^*(X)$ are modules over the coefficient ring k_G^*. It turns out that the ideal theoretic geometry of the k_G^*-module $m_G^*(X)$ is the controlling structure. We discussed the algebra that we have in mind in the previous article [31].

Consider the augmentation ideal

$$J = \ker\left(\mathrm{res}_1^G : \mathrm{k}_G^* = \mathrm{k}_G^*(\mathrm{S}^0) \longrightarrow \mathrm{k}_*^G(\mathrm{G}_+) \cong \mathrm{k}^*\right),$$

which by definition acts as zero on $k_G^*(G_+)$ and therefore on $m_G^*(G_+)$. Since any free G-spectrum is constructed from cells $S^n \wedge G_+$ it follows that a power of J acts as zero on $m_G^*(X)$ whenever X is finite and free. We emphasize that we are thinking about *$\mathbb{Z}$-graded, but $RO(G)$-gradable,* equivariant cohomology theories. If we allowed $RO(G)$-grading in our definition of J, the discussion would still make sense, but the results would often be trivial to prove and useless in practice.

Now observe that EG_+ is a direct limit of finite free complexes and consider its cohomology. If there are no $\lim^1$ problems, $m_G^*(EG_+)$ is an inverse limit of J-nilpotent modules, and therefore the nicest answer we could hope to have is that π^* is completion, so that

$$m_G^*(EG_+ \wedge X) = (m_G^*(X))^\wedge_J.$$

However the algebra has already warned us against this: the topology guarantees that the left hand side is an exact functor of X, whereas the right hand side is only known to be exact when k_G^* is Noetherian and $m_G^*(X)$ is finitely generated. The solution is to replace J-completion by the associated functor on the derived category: this will be exact in a suitable sense and its homology groups will be calculated by left derived functors of completion. We gave the relevant descriptions of derived functors in [31].

Nicest Possible Answer 6.1. *For any G-spectrum X, $m_G^*(EG_+ \wedge X)$ is the 'homotopy J-completion' of the k_G^*-module $m_G^*(X)$ and hence there is a spectral sequence*

$$E_2^{*,*} = H_*^J(m_G^*(X)) \Longrightarrow m_G^*(EG_+ \wedge X).$$

If this nicest possible answer is the correct answer we say that the completion theorem *holds for $m_G^*(\cdot)$.*

Now consider the situation in homology. In any case, $m_*^G(EG_+)$ is a direct limit of J-nilpotent modules. The nicest functor of this form is the J-power torsion functor, but we saw in the previous article that this is rarely exact, and so even in the best cases we need to take derived functors into account.

Nicest Possible Answer 6.2. *For any G-spectrum X, $m_*^G(EG_+ \wedge X)$ is the 'homotopy J-power torsion' of the m_*^G-module $m_G^*(X)$ and hence there is a spectral sequence*

$$E^2_{*,*} = H_J^*(m_*^G(X)) \Longrightarrow m_*^G(EG_+ \wedge X).$$

If this nicest possible answer is the correct answer we say that the localization theorem *holds for $m_*^G(\cdot)$.*

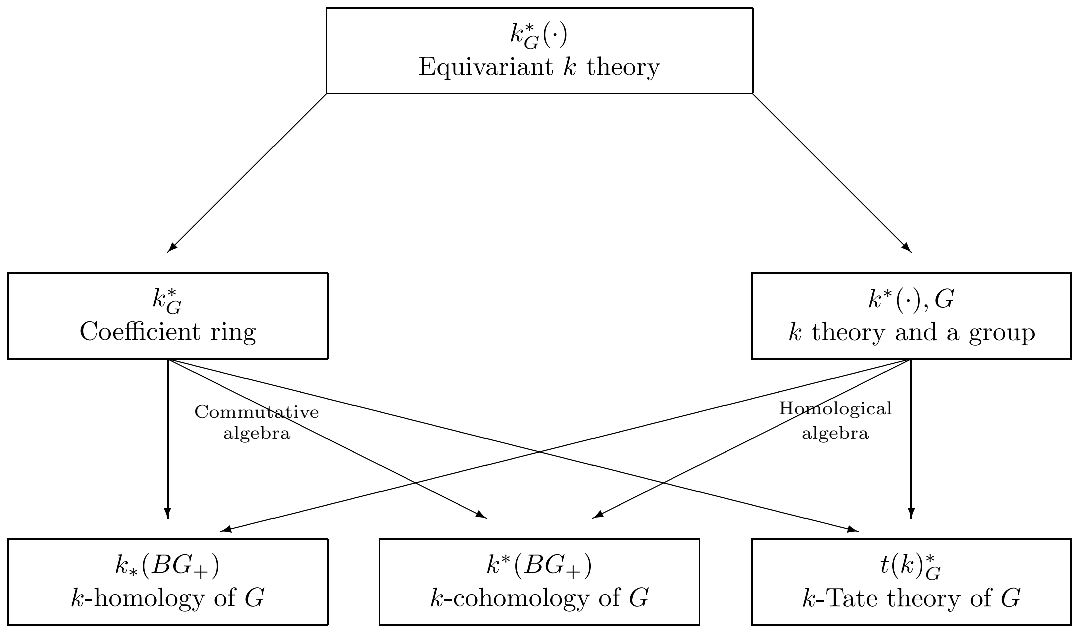
One of us used to call this a 'local cohomology theorem' [24]. We shall explain in the next section why we now understand it to be a 'localization theorem'. We shall also recall what we mean by 'homotopy J-completion' and 'homotopy J-power torsion' and describe how one can hope to prove that theories $m_G^*(\cdot)$ and $m_*^G(\cdot)$ enjoy such good behaviour. However, the statements about spectral sequences are perfectly clear as they stand; the initial terms of the spectral sequences are local homology and local cohomology groups, respectively, as defined in [31, §1].

The entire discussion just given applies equally well to the calculation of $m_G^*(E\mathcal{F}_+)$ and $m_*^G(E\mathcal{F}_+)$ for an arbitrary family $\mathcal{F}$, provided that the ideal J is replaced by

$$J\mathcal{F} = \bigcap_{H \in \mathcal{F}} \ker(k_G^* \longrightarrow k_H^*).$$

This case cannot usually be reduced to a non-equivariant statement, but it often has its own applications. For example, it leads to calculations of the cohomology and homology of equivariant classifying spaces and thus to determinations of equivariant characteristic classes.

We consider the alternative methods of calculation available to us in the following schematic diagram, restricting attention to our given ring theory $k_G^*(\cdot)$.



In this picture, the conventional (Atiyah-Hirzebruch) homological algebra route takes as input the non-equivariant k-theory together with the group structure on G; it results in a calculation of infinite homological dimension and with infinitely many extension problems. Where it applies, the more favourable route through commutative algebra takes as input the equivariant augmented coefficient ring $k_G^* \longrightarrow k^*$; the calculation usually has finite homological dimension, and in favourable cases the spectral sequences collapse and there are no extension problems.

There is an undefined term here, namely the Tate theory $t(k)_G^*$ [30]. It fits into a long exact sequence whose other two terms are $k_*(BG_+)$ and $k^*(BG_+)$. Returning to the context of module theories and remembering that every theory is a module theory over stable cohomotopy, we have the following remarkable relationship between our two Nicest Possible Answers.

Theorem 6.3. *Let G be finite and let J be the augmentation ideal of the Burnside ring $A(G)$. Regard a G-spectrum m_G as a module over the sphere G-spectrum S_G and recall that $A(G) \cong \pi_0^G(S_G)$. The localization theorem for the calculation of $m_*(BG_+)$ is true if and only if the completion theorem for the calculation of $m^*(BG_+)$ is true and $t(m)_G^*$ is rational.*

The Tate theory is relatively easy to compute. It is a direct consequence of Theorem 3.10 that the Tate theory $t(S)^*_G$ is not rational, so that one cannot hope to prove the localization theorem in stable homotopy, although the completion theorem is true in stable cohomotopy. We shall say no more about the Tate theory here, referring the interested reader to [30].

7. How to prove localization and completion theorems

We now outline a strategy for proving that the Nicest Possible Answer applies in both homology and cohomology [24]. One limitation of the method is obvious: it cannot apply to theories like stable homotopy.

The calculational restriction that we will shortly place on our homology theory and that will rule out stable homotopy is that the theory should have Thom isomorphisms for complex representations V:

$$R^G_*(S^V \wedge X) \cong R^G_*(S^{|V|} \wedge X) \tag{7.1}$$

as R^G_*-modules, where $|V|$ denotes the real dimension of V. The point is that localization theorems are often automatic, by arguments like the proof of Proposition 3.20, if one grades over the representation ring. Thom isomorphisms allow us to reinterpret that result in terms of integer grading.

There are two further assumptions. The first is fundamental to the general strategy: we assume that we are working in the category of modules over a commutative S_G-algebra R_G with underlying nonequivariant commutative S-algebra R. (Remember that commutative S_G-algebras are essentially the same things as E_∞ ring G-spectra.) We have switched notation from k to R to emphasize this assumption. Without it, we cannot make the constructions we need except under very favourable circumstances.

The second is made solely to simplify the exposition: we assume that the ring R^*_G is Noetherian. If this is not the case, the outline of the argument is the same but its implementation is considerably more complicated since one must use topological arguments to show that the relevant ideals can be replaced by finitely generated ones; at present, these arguments only apply to the trivial family $\mathcal{F} = \{\infty\}$.

The idea of the proofs is to model the algebra in topology; the model is so chosen that formal arguments imply that constructions on isotropy types are directly related to constructions on ideals in commutative rings. The necessary topological constructions are described in [31, §3].

We restrict attention to the augmentation ideal

$$J = \ker(res^G_1 : R^G_* \to R_*)$$

and consider the canonical map

$$\kappa' : EG_+ \wedge K(J) \to S^0 \wedge K(J)$$

of R_G-modules. The module $K(J) = \Gamma_J(R_G)$ encodes homotopical J-power torsion. By our Noetherian assumption, we may take $J = (\beta_1, \ldots, \beta_n)$. Then $K(J)$ is the smash product over R_G of the fibers $K(\beta_i)$ of the localizations $R_G \longrightarrow (R_G)[1/\beta_i]$. Since the β_i are trivial as nonequivariant maps, we have the following observation.

Lemma 7.2. *The natural map $K(J) \to R_G$ is a non-equivariant equivalence.*

Thus $EG_+ \wedge K(J) \simeq EG_+ \wedge R_G$ and κ' induces a map of R_G-modules

$$\boxed{\kappa : EG_+ \wedge R_G \longrightarrow K(J)}. \tag{7.3}$$

When G is finite, the homotopy groups of $R_G \wedge EG_+$ are $R^G_*(EG_+)$. More generally, we consider an R_G-module M_G with underlying nonequivariant R-module M, and we have

$$(EG_+ \wedge R_G) \wedge_{R_G} M_G \simeq EG_+ \wedge M_G \quad \text{and} \quad F_{R_G}(EG_+ \wedge R_G, M_G) \simeq F(EG_+, M_G).$$

Recall the definitions

$$\Gamma_J(M_G) = K(J) \wedge_{R_G} M_G \quad \text{and} \quad (M_G)^\wedge_J = F_{R_G}(K(J), M_G).$$

The homotopy groups of these modules may be calculated by the spectral sequences [31, 3.2 and 3.3]. Clearly the map κ induces maps

$$EG_+ \wedge M_G \longrightarrow \Gamma_J(M_G) \quad \text{and} \quad (M_G)^\wedge_J \longrightarrow F(EG_+, M_G),$$

and these maps are equivalences if κ is an equivalence. Therefore, if we can prove that κ is a homotopy equivalence, we can deduce the spectral sequences of the Nicest Possible Answers for both $M^G_*(EG_+)$ and $M_G^*(EG_+)$ for all R_G-modules M_G. Given a G-spectrum X, we can replace M_G by $X \wedge M_G$ and $F(X, M_G)$ and so arrive at the the Nicest Possible Answers as stated in (6.1) and (6.2).

We pause to describe the role of localization away from J. We have the cofibre sequence

$$K(J) \longrightarrow R_G \longrightarrow \check{C}(J).$$

Smashing over R_G with M_G, recalling that $M_G[J^{-1}] = \check{C}(J) \wedge_{R_G} M_G$, and using a standard comparison of cofibre sequences argument in the category of R_G-modules, we obtain a map of cofibre sequences

$$\begin{array}{ccccc} EG_+ \wedge M_G & \longrightarrow & M_G & \longrightarrow & \tilde{E}G \wedge M_G \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \Gamma_J(M_G) & \longrightarrow & M_G & \longrightarrow & M_G[J^{-1}]. \end{array}$$

Clearly the left arrow is an equivalence if and only if the right arrow is an equivalence. This should be interpreted as stating that the 'topological' localization of

M_G away from its free part is equivalent to the 'algebraic' localization of M_G away from J. This is why we call our Nicest Possible Answer in homology a localization theorem. The parallel with the completion theorem, which states that the 'algebraic' completion $M_J^\wedge$ is equivalent to the 'topological' completion $F(EG_+, M_G)$ of M_G at its free part, is now apparent.

The strategy for proving that the map κ of (7.3) is an equivalence is an inductive scheme. To set it up, we need to know that if we restrict κ to a subgroup H, we obtain an analogous map of H-spectra. We have

$$K(\beta, \dots, \beta_n)\,|_H = K(\beta_1|_H, \dots, \beta_n|_H);$$

the latter is defined with respect to $R_H = R_G\,|_H$. That is, if we write J_G instead of J, as we shall often do to clarify inductive arguments,

$$\Gamma_{J_G}(R_G)\,|_H \simeq \Gamma_{\mathrm{res(J_G)}}(R_H),$$

where res : $\mathrm{R_*^G} \longrightarrow \mathrm{R_*^H}$ is restriction. It is rarely the case (even for cohomotopy, when one is looking at Burnside rings) that $\mathrm{res(J_G) = J_H}$, but these ideals do have the same radical.

Theorem 7.4. *Assume that G is finite and each R_*^H is Noetherian. Then*

$$\sqrt{\mathrm{res(J_G)}} = \sqrt{J_H}$$

for all subgroups $H \subseteq G$.

We therefore have the equivalence of H-spectra

$$\Gamma_{J_G}(R_G)|_H \simeq \Gamma_{J_H} R_H.$$

Sketch proof of Theorem 7.4. For theories such as cohomotopy and K-theory, where we understand all primes of R_*^G, this can be verified algebraically.

In general, if G acts freely on a product of spheres, one may check that J_G is the radical of the ideal generated by all Euler classes and deduce the result. This covers the case when G is a p-group, and general finite groups can then be dealt with by transfer. □

The argument just sketched requires considerable elaboration, and it can be the main technical obstruction to the implementation of our strategy when we work more generally with compact Lie groups and non-Noetherian coefficient rings.

Theorem 7.5 (Localization and completion theorem)**.** *Assume that G is finite and each R_*^H is Noetherian. If all of the theories $R_*^H(\cdot)$ admit Thom isomorphisms (7.1), then the map of R_G-modules*

$$\kappa : EG_+ \wedge R_G \xrightarrow{\simeq} K(J)$$

is an equivalence. Therefore, for any R_G-module M_G and any G-spectrum X, there are spectral sequences

$$E^2_{*,*} = H^*_J(R^G_*; M^G_*(X)) \Rightarrow M^G_*(EG_+ \wedge X)$$

and

$$E_2^{*,*} = H_*^J(R_G^*; M_G^*(X)) \Rightarrow M_G^*(EG_+ \wedge X).$$

Proof. [Sketch] Write J_G instead of J, and observe from the original construction of κ' that the cofibre of κ is $\tilde{E}G \wedge K(J_G)$. We must prove that this is contractible.

We proceed by induction on the order of the group. By Theorem 7.4, we have

$$(\tilde{E}G \wedge K(J_G))\,|\,_H \simeq \tilde{E}H \wedge K(J_H),$$

and so our inductive assumption implies that

$$G/H_+ \wedge \tilde{E}G \wedge K(J_G) \simeq *$$

for all proper subgroups $H \subset G$.

We now use the idea in Proposition 3.20 and its proof. We take $\tilde{E}\mathcal{P} = \mathrm{colim}_{\mathcal{V}}\, \mathcal{S}^{\mathcal{V}}$, where the colimit is taken over indexing G-spaces $V \subset U$ such that $V^G = \{0\}$. With G finite, we may restrict attention to copies of the reduced regular representation of G. Since $(\tilde{E}\mathcal{P})^{\mathcal{G}} = \mathcal{S}'$, $\tilde{E}\mathcal{P}/\mathcal{S}'$ is triangulable as a G-CW complex whose cells are of the form $G/H_+ \wedge S^n$ with H proper. Therefore

$$\tilde{E}\mathcal{P}/\mathcal{S}' \wedge \tilde{\mathcal{E}}\mathcal{G} \wedge \mathcal{K}(\mathcal{J}_{\mathcal{G}}) \simeq *$$

by the inductive assumption, hence

$$\tilde{E}G \wedge K(J_G) \simeq \tilde{E}\mathcal{P} \wedge \tilde{\mathcal{E}}\mathcal{G} \wedge \mathcal{K}(\mathcal{J}_{\mathcal{G}}).$$

Since $\tilde{E}\mathcal{P} \wedge \mathcal{S}' \longrightarrow \tilde{\mathcal{E}}\mathcal{P} \wedge \tilde{\mathcal{E}}\mathcal{G}$ is an equivalence, we have established the following useful reduction.

Lemma 7.6. (Carlsson's reduction) *It suffices to show that $\tilde{E}\mathcal{P} \wedge \mathcal{K}(\mathcal{J}_{\mathcal{G}}) \simeq *$.* □

Now recall that we have Euler classes $\chi_V \in R^G_{-V}(S^0)$ obtained by applying $e(V)^*$, $e(V) : S^0 \longrightarrow S^V$, to the unit $1 \in E^0(S^0) \cong E^V(S^V)$. At this point, our Thom isomorphisms (7.1) come into play, allowing us to move these Euler classes into integer gradings. Thus let $\chi(V) \in R^G_{-|V|}$ be the image of χ_V under the Thom isomorphism. When $V \neq \{0\}$, $e(V)$ is non-equivariantly null homotopic and therefore $\chi(V)$ is in J_G. Via the Thom isomorphism, Proposition 3.20 implies that, for any G-spectrum X, $\pi^G_*(\tilde{E}\mathcal{P} \wedge \mathcal{X})$ is the localization of $\pi^G_*(X)$ obtained by inverting the Euler classes $\chi(V)$. Here we may restrict everything to lie in integer gradings. With $X = K(J_G)$, the localization is zero since the $\chi(V)$ are in J_G [31, 1.1]. From the spectral sequence [31, 3.2], we see that

$$\pi^G_*(\tilde{E}\mathcal{P} \wedge \mathcal{K}(\mathcal{J}_{\mathcal{G}})) = \prime.$$

Since $\tilde{E}\mathcal{P}$ is H-equivariantly contractible for all proper subgroups H, this shows that $\tilde{E}\mathcal{P} \wedge \mathcal{K}(\mathcal{J}_{\mathcal{G}}) \simeq *$, as required. □

8. Examples of localization and completion theorems

The discussion in the previous section was very general. In this section we consider a number of important special cases in a little more detail. In each case, we give some history, state precise theorems, discuss their import, and comment on wrinkles in their proofs. We refer the reader to [53] for precise descriptions of the representing G-spectra and more extended discussions of these results and their proofs.

8.1. **K-theory.** Historically this was the beginning of the whole subject. Atiyah [5] first proved the completion theorem for finite groups, by the conventional homological algebra route. Full use of equivariance appeared in the 1969 paper of Atiyah and Segal [8], which gave the completion theorem for compact Lie groups in essentially the following form. Let I be the augmentation ideal of the representation ring $R(G)$.

Theorem 8.1 (Atiyah-Segal)**.** *If G is a compact Lie group and X is a finite G-CW complex, then*

$$K^*_G(X)^{\wedge}_I \cong K^*_G(EG_+ \wedge X). \quad \square$$

Their proof, like any other, depends fundamentally on the equivariant Bott periodicity theorem, which provides Thom isomorphisms via isomorphisms

$$K_G(\Sigma^V X) \cong K_G(X)$$

for complex representations V. The coefficient ring is $K^*_G = K^0_G[\beta, \beta^{-1}]$, and $K^0_G = R(G)$. Since nonequivariant K-theory is also periodic, the augmentation ideal is $J = I[\beta, \beta^{-1}]$, and the completion theorem is therefore stated using I. The ring $R(G)$ is Noetherian [61], and Theorem 7.4 holds for all compact Lie groups G.

Atiyah and Segal used an inductive scheme in which they first proved the result for a torus, then used holomorphic induction to deduce it for a unitary group, and finally deduced the general case from the case of unitary groups. A geodesic route from Bott periodicity to the conclusion, basically a cohomological precursor of the homological argument sketched in the previous section, is given in [2]. That paper also gives the generalization of the result to arbitrary families of subgroups in G. A remarkable application of that generalization has been given by McClure [57]: restriction to finite subgroups detects equivariant K-theory.

Theorem 8.2 (McClure)**.** *For a compact Lie group G and a finite G-CW complex X, restriction to finite subgroups F specifies a monomorphism*

$$K^*_G(X) \longrightarrow \prod K^*_F(X). \quad \square$$

It is not known that K_G is a commutative S_G-algebra in general, although recent work shows that this does hold when G is finite [23]. Therefore the techniques of the previous section do not apply in general. The arguments in [8] and [2] prove the isomorphism of Theorem 8.1 directly in cohomology. The trick that recovers enough exactness to make this work is to study pro-group valued cohomology theories.

A pro-group is just an inverse system of (Abelian) groups. There is an Abelian category of pro-groups, and the inverse limit functor is exact in that category. For a cohomology theory k_G^* on G-CW complexes, one obtains a pro-group valued theory $\boldsymbol{k}_G^*$ by letting $\boldsymbol{k}_G^*(X)$ be the system $\{k_G^*(X_\alpha)\}$, where X_α runs through the finite subcomplexes of X. Working with pro-groups has an important bonus: for a finite G-CW complex X, the system $\{K_G^*(X)/I^n\}$ clearly satisfies the Mittag-Leffler condition. One proves that this system is pro-isomorphic to the system $\boldsymbol{k}_G^*(EG_+ \wedge X)$, and one is entitled to conclude that

$$K_G^*(EG_+ \wedge X) \cong \lim_n K_G^*(EG_+^n \wedge X).$$

That is, the relevant $\lim^1$ term vanishes.

Various people have deduced calculations of the K-homology of classifying spaces for finite groups using suitable universal coefficient theorems, but the use of local cohomology and the proof via the localization theorem were first given in [24].

Theorem 8.3. *If G is finite, then the localization and completion theorems hold for equivariant K-theory. Therefore, for any G-spectrum X, there are short exact sequences*

$$0 \longrightarrow H_I^1(K_*^G(\Sigma X)) \longrightarrow K_*^G(EG_+ \wedge X) \longrightarrow H_I^0(K_*^G(X)) \longrightarrow 0$$

and

$$0 \longrightarrow L_1^I K_G^*(\Sigma X) \longrightarrow K_G^*(EG_+ \wedge X) \longrightarrow L_0^I K_G^*(X) \longrightarrow 0. \quad \square$$

In [24], the strategy of the previous section was applied to K_G regarded as an S_G-module: we have the permutation representation homomorphism $A(G) \longrightarrow R(G)$, and the completion of an $R(G)$-module at the augmentation ideal of $R(G)$ is isomorphic to its completion at the augmentation ideal of $A(G)$ [28, 4.5]. Using the new result that K_G is a commutative S_G-algebra when G is finite, the strategy can now be applied directly: Theorem 8.3 is an application of Theorem 7.5. The collapse of the relevant spectral sequnces to short exact sequences results from the fact that $A(G)$ and $R(G)$ have Krull dimension 1 when G is finite.

There is an alternative strategy. In view of Theorems 6.3 and 8.1, one can prove Theorem 8.2 by proving directly that the Tate theory $t(K)_G^*$ is rational. This approach is carried out in [30]. It has the bonus that the topology carries out the commutative algebra of calculating the local cohomology groups, leading to the following succinct conclusion. Let $\mathbb{CG}$ be the regular representation of G; the ideal it generates in $R(G)$ is a free abelian group of rank 1, and the composite $I \longrightarrow R(G) \longrightarrow R(G)/(\mathbb{CG})$ is an isomorphism.

Theorem 8.4. *Let G be finite. Then $K_0(BG) \cong \mathbb{Z}$, with generator the image of $\mathbb{CG}$, and*

$$K_1(BG) \cong (R(G)/(\mathbb{CG}))^{\wedge}_{\mathbb{I}} \otimes (\mathbb{Q}/\mathbb{Z}).$$

When G is a p-group, I-adic and p-adic completion agree on $I \cong R(G)/(\mathbb{CG})$, and explicit calculations in both K-homology and K-cohomology are easily obtained.

For general compact Lie groups, these strategies all fail: we do not know that K_G is a commutative S_G-algebra, and the alternative based on use of S_G fails since $A(G)$ has Krull dimension 1 and is non-Noetherian in general, whereas $R(G)$ is Noetherian but has Krull dimension $\mathrm{rank}(G) + 1$ [61]. The localization theorem is not known to hold in general.

8.2. **Bordism.** The case of bordism is the greatest success of the method outlined in Section 7. The correct equivariant form of bordism to use is tom Dieck's homotopical equivariant bordism [16]. A completion theorem for the calculation of $MU^*(BG)$ for Abelian compact Lie groups was proven by Löffler [46, 47] soon after the Atiyah-Segal completion theorem appeared, but there was no further progress until quite recently.

It is easy to describe the representing G-spectrum MU_G. Consider the usual model for the prespectrum with associated spectrum MU. The spaces comprising it are the Thom complexes of the Grassmannian models for universal vector bundles. Now carry out the construction using indexing spaces in a complete G-universe. The Vth space is defined using $|V|$-dimensional subspaces of the appropriate Grassmannian and therefore, up to G-homeomorphism, depends only on the dimension of V. This fact leads to the Thom isomorphisms required by our general strategy. Moreover, the explicit construction leads to a quick proof that the Thom G-spectrum MU_G is in fact a commutative S_G-algebra. Our general strategy applies [32].

Theorem 8.5 (Greenlees-May)**.** *Let G be finite. Then the localization and completion theorems hold for any module M_G over MU_G. Thus there are equivalences*

$$M_G \wedge EG_+ \simeq \Gamma_J(M_G) \quad \textit{and} \quad F(EG_+, M_G) \simeq (M_G)^{\wedge}_J$$

and, for any G-spectrum X, there are spectral sequences

$$E^2_{*,*} = H^*_J(MU^G_*; M^G_*(X)) \Rightarrow M^G_*(EG_+ \wedge X)$$

and

$$E_2^{*,*} = H_*^J(MU_G^*; M_G^*(X)) \Rightarrow M_G^*(EG_+ \wedge X). \quad \square$$

We have several comments on this theorem, beginning with comments on its proof. An immediate difficulty is that MU^*_G is certainly not Noetherian. Furthermore, we have no good reason to think that the augmentation ideal $J \subset MU^*_G$ is finitely generated unless G is abelian. We modify our strategy accordingly, proving the theorem for any sufficiently large finitely generated subideal of J. By definition, the stated constructions based on J mean the relevant constructions based on such a sufficiently large subideal. When G is a p-group, the arguments of the previous section apply to ideals generated by a finite number of Euler classes. Rather elaborate multiplicative transfer and double coset formula arguments allow us to deduce the result for general finite groups using ideals that are generated by the transfers of the Euler classes from all p-Sylow subgroups and finitely many more elements. We expect that the result for an arbitrary compact Lie group can be proved by similar methods, but we do not yet see how to use these methods to give the result for arbitrary families.

Next we comment on the meaning of the theorem. Its most striking feature is its generality. The methods explained in [22, §12] apply to give equivariant forms of all of the important modules over MU, such as ku, K, BP, $BP\langle n\rangle$, $E(n)$, $P(n)$, $B(n)$, $k(n)$ and $K(n)$. The equivariant and nonequivariant constructions are so closely related that we can deduce MU_G-ring spectrum structures on the equivariant spectra from the MU-ring spectra structures on the nonequivariant spectra. There are a variety of nonequivariant calculations of the homology and cohomology of classifying spaces with coefficients in one or another of these spectra in the literature, and our theorem gives a common framework for all such calculations.

We should comment on the specific case of connective K-theory. Here it is known that the completion theorem is false for connective equivariant K-theory: $ku^*(BG_+)$ is not a completion of ku^*_G at its augmentation ideal. However the theorem is consistent, since the equivariant form of ku constructed by the methods of [22, §12] is not the connective cover of equivariant K-theory. Indeed connective equivariant K-theory does not have Thom isomorphisms and is therefore not a module over MU_G.

We should also note that the coefficient ring MU^*_G is only known in the abelian case, and even then only in a rather awkward algebraic form. On the other hand, $M^*(BG_+)$ is known in a good many other cases, and in reasonably attractive form. Thus the theorem does not at present give a useful way of calculating $M^*(BG_+)$. However, there are several ways that it might be used for calculational purposes. For example, in favourable cases, such as $M = MU$ for Abelian groups G, one can work backwards to deduce that M^*_G is tame, in the sense that its local homology is its completion concentrated in degree zero. The local cohomology of M^*_G is then the same as that of its completion [31, 2.7], hence one can hope to calculate its local cohomology as well and to use this information to study $M_*(BG_+)$. The point is that, nonequivariantly, the calculation of homology is often substantially more difficult than the calculation of cohomology. Again, if M is an MU-algebra, then one can use invariance under change of base [31, 1.3] to calculate the local cohomology and local homology over M_G^*; it sometimes turns out that M_G^* is a ring of small Krull dimension, and this gives vanishing theorems that make calculation more feasible.

These comments are speculative: the theorem is too recent to have been assimilated calculationally. Certainly it renews interest in the connection through cobordism between algebraic and geometric topology.

8.3. **Cohomotopy.** Soon after the Atiyah-Segal theorem was proved, Segal conjectured that the analogous result would hold for stable cohomotopy, at least in degree 0. In simplest terms, the idea is that the Burnside ring $A(G)$ plays a role in equviariant cohomotopy analogous to the role that $R(G)$ plays in equivariant K-theory and should therefore play an analogous role in the calculation of the nonequivariant cohomotopy groups of classifying spaces.

We restrict attention to finite groups G. Then the elements of positive degree in the homotopy ring π_*^G are nilpotent, so that it is natural to take its degree zero part $\pi_0^G \cong A(G)$ as our base ring; A(G) is Noetherian, and we let I denote its augmentation ideal $\ker(A(G) \longrightarrow \mathbb{Z})$. Theorem 7.4 applies.

Segal's original conjecture was simply that $A(G)^\wedge_I \cong \pi^0(BG_+)$. However, it quickly became apparent that, to prove the conjecture, it would be essential to extend it to a statement concerning the entire graded module $\pi^*(BG_+)$. In view of Theorem 3.10, we have enough information to formulate the conjecture in entirely nonequivariant terms [41], but it was the equivariant formulation that led to a proof.

In accordance with our philosophy we make a spectrum level statement and take the algebraic statement as a corollary, although the proofs proceed the opposite way.

Theorem 8.6 (Carlsson)**.** *For any finite group G and any G-spectrum X there is an equivalence of G-spectra*

$$(DX)^\wedge_I \xrightarrow{\simeq} D(EG_+ \wedge X).$$

If X is finite, then

$$\pi_G^*(X)^\wedge_I \cong \pi_G^*(EG_+ \wedge X);$$

in general, there is a short exact sequence

$$0 \longrightarrow L_1^I \pi_G^*(\Sigma X) \longrightarrow \pi_G^*(EG_+ \wedge X) \longrightarrow L_0^I \pi_G^*(X) \longrightarrow 0. \quad \square$$

We have already remarked that the localization theorem for stable homotopy fails and that cohomotopy does not have Thom isomorphisms. Therefore the strategy of proof must be quite different from that presented in Section 7. We first note that the generality of our statement is misleading: it was observed in [28, 4.1] that the statement for general X is a direct consequence of the statement for $X = S_G$. One reason for working on the G-spectrum level is to allow such deductions.

Taking $X = S_G$, it suffices to prove that, after completion, the map $\varepsilon : S_G \longrightarrow D(EG_+)$ induced by the projection $EG_+ \longrightarrow S^0$ induces an isomorphism on homotopy groups. Proceeding by induction on the order of G and using Theorem 7.4, we may assume that the homotopy groups π_*^H for proper subgroups H are mapped isomorphically, so that we need only consider the groups π_*^G. As with the Atiyah-Segal theorem, we think cohomologically and control exactness by working with pro-groups. We find that it suffices to show that ε induces an isomorphism of pro-groups

$$\{\pi_G^*(S^0)/I^n\} \xrightarrow{\cong} \boldsymbol{\pi}_G^*(EG_+).$$

At this point, a useful piece of algebra comes into play. In the context of Mackey functors, there is a general framework for proving induction theorems, due to Dress [19]. An induction theorem for I-adically complete Mackey functors was proven in [54], and it directly reduces the problem at hand to the study of p-groups and p-adic completion. A more sophisticated reduction process, developed in [3], shows that the generalization of the Segal conjecture to arbitrary families of subgroups of G also reduces to this same special case.

This reduces the problem to what Carlsson actually proved [13]. Fix a p-group G, assume the theorem for all proper subgroups of G, and write $\pi_G^*(X)$ and $[X, Y]_G^*$ for the pro-group valued, p-adically completed, versions of these groups, where p-adic completion is understood in the pro-group sense. We replace G-spaces by their suspension G-spectra without change of notation. What Carlsson proved is that

$$\pi_G^*(S^0) \xrightarrow{\cong} \pi_G^*(EG_+)$$

is a pro-isomorphism.

A first reduction (see Lemma 7.6) shows that it suffices to prove that $\pi_G^*(\tilde{E}\mathcal{P}) = [\tilde{\mathcal{E}}\mathcal{P}, \mathcal{S}']_{\mathcal{G}}^*$ is pro-zero. The cofibre sequence $EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G$ gives rise to a long exact sequence

$$\cdots \longrightarrow [\tilde{E}\mathcal{P}, \mathcal{E}\mathcal{G}_+]_{\mathcal{G}}^{\text{II}} \longrightarrow [\tilde{\mathcal{E}}\mathcal{P}, \mathcal{S}']_{\mathcal{G}}^{\text{II}} \longrightarrow [\tilde{\mathcal{E}}\mathcal{P}, \tilde{\mathcal{E}}\mathcal{G}]_{\mathcal{G}}^{\text{II}} \xrightarrow{\delta} [\tilde{\mathcal{E}}\mathcal{P}, \mathcal{E}\mathcal{G}_+]_{\mathcal{G}}^{\text{II}+\infty} \longrightarrow \cdots .$$

The $\tilde{E}G$ terms carry the singular part of the problem; the EG_+ terms carry the free part. It turns out that if G is *not* elementary Abelian, then both $[\tilde{E}\mathcal{P}, \mathcal{E}\mathcal{G}_+]_{\mathcal{G}}^*$ and $[\tilde{E}\mathcal{P}, \tilde{\mathcal{E}}\mathcal{G}]_{\mathcal{G}}^*$ are pro-zero. This is not true when G is elementary abelian, but then the connecting homomorphism δ is a pro-isomorphism.

The calculation of the groups $[\tilde{E}\mathcal{P}, \tilde{\mathcal{E}}\mathcal{G}]^*_{\mathcal{G}}$ involves a functorial filtered approximation with easily understood subquotients of the singular subspace SX of a G-space X. Here SX consists of the elements of X with non-trivial isotropy groups; it is relevant since, on the space level,

$$[X, \tilde{E}G \wedge Y]_G \cong [SX, Y]_G.$$

A modification of Carlsson's original approximation given in [14] shows that SX depends only on the fixed point spaces X^E for elementary Abelian subgroups E of G, and this analysis reduces the vanishing of the $[\tilde{E}\mathcal{P}, \tilde{\mathcal{E}}\mathcal{G}]^*_{\mathcal{G}}$ when G is not elementary Abelian to direct application of the induction hypothesis.

Recall the description of $\tilde{E}\mathcal{P}$ as the union $\cup S^{nV}$, where V is the reduced regular representation of G. One can describe $[S^{nV}, EG_+]^*_G$ as the homotopy groups of a nonequivariant Thom spectrum BG^{-nV} (see [52]) and so translate the calculation of the free part to a nonequivariant problem that can be attacked by use of an inverse limit of Adams spectral sequences. The vanishing of $[\tilde{E}\mathcal{P}, \mathcal{E}\mathcal{G}_+]^*_{\mathcal{G}}$ when G is not elementary abelian is an Euler class argument: a theorem of Quillen implies that $\chi(V) \in H^*(BG; \mathbb{F}_p)$ is nilpotent if G is not elementary Abelian, and this implies that the E_2 term of the relevant inverse limit of Adams spectral sequences is zero.

When G is elementary Abelian, it turns out that all of the work in the calculation of $[\tilde{E}\mathcal{P}, \mathcal{E}\mathcal{G}_+]^*_{\mathcal{G}}$ lies in the calculation of the E_2 term of the relevant inverse limit of Adams spectral sequences. When G is $\mathbb{Z}_2$ or $\mathbb{Z}_p$, the calculation is due to Lin [44, 45] and Gunawardena [33], respectively, and they were the first to prove the Segal conjecture in these cases. For general elementary Abelian p-groups, the calculation is due to Adams, Gunawardena, and Miller [4]. While these authors were the first to prove the elementary Abelian case of the Segal conjecture, they didn't publish their argument, which started from the nonequivariant formulation of the conjecture. A simpler proof within Carlsson's context was given in [14], which showed that the connecting homomorphism δ is an isomorphism by comparing it to the corresponding connecting homomorphism for a theory, Borel cohomology, for which the completion theorem holds tautologously.

The Segal conjecture has been given a number of substantial generalizations, such as those of [40, 3, 56]. The situation for general compact Lie groups is still only partially understood; Lee and Minami have given a good survey [43]. One direction of application has been the calculation of stable maps between classifying spaces. The Segal conjecture has the following implication [40, 51], which reduces the calculation to pure algebra.

Let G and Π be finite groups and let $A(G, \Pi)$ be the Grothendieck group of Π-free finite $(G \times \Pi)$-sets. Observe that $A(G, \Pi)$ is an $A(G)$-module.

Theorem 8.7. *There is a canonical isomorphism*

$$A(G,\Pi)^{\wedge}_{I} \cong [\Sigma^\infty BG_+, \Sigma^\infty B\Pi_+]. \quad \square$$

Many authors have studied the relevant algebra [59, 48, 35, 10, 65], which is now well understood. One can obtain an analog with Π allowed to be compact Lie [56], and even with G and Π both allowed to be compact Lie [58].

8.4. **The cohomology of groups.** We have emphasized the use of ideas and methods from commutative algebra in equivariant stable homotopy theory. We close with a remark on equivariant cohomology which shows that ideas and methods from equivariant stable homotopy theory can have interesting things to say about algebra.

The best known equivariant cohomology theory is simply the ordinary cohomology of the Borel construction:

$$H^*_G(X) = H^*(EG_+ \wedge_G X; k),$$

where we take k to be a field. The coefficient ring is the cohomology ring $H^*_G(S^0) = H^*(G)$ of the group G, and the augmentation ideal J consists of the elements of positive degree. Of course, this theory can be defined algebraically in terms of chain complexes. As far as completion theorems are concerned, this case has been ignored since $H^*_G(X)$ is obviously complete for the J-adic topology and the completion theorem is true trivially, by virtue of the equivalence $EG_+ \wedge EG_+ \simeq EG_+$.

However, once one has formulated the localization theorem, it is easy to give a proof along the lines sketched above, using either topology or algebra. We give an algebraic statement proven in [26].

Theorem 8.8. *For any finite group G and any bounded below chain complex M of kG modules there is a spectral sequence with cohomologically graded differentials*

$$E_2^{p,q} = H_J^{p,q}(H^*(G;M)) \Longrightarrow H_{-(p+q)}(G;M). \quad \square$$

It would be perverse to attempt to use the theorem to calculate $H_*(G;M)$, but if we consider the case when the coefficient ring is Cohen-Macaulay, so that the only non-vanishing local cohomology group occurs for $d = \dim H^*(G)$, we see that the theorem for $M = k$ states that

$$H_n(G) = H_J^{d,-n-d}(H^*(G)).$$

In particular, using that $H_*(G)$ is the k-dual of $H^*(G)$, this duality theorem implies that the ring $H^*(G)$ is also Gorenstein, which is a theorem originally proven by Benson and Carlson [9].

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