# EQUIVARIANT STABLE SHEAVES AND TORIC GIT

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Abstract. For $(X, L)$ a polarized toric variety and $G \subset \mathrm{Aut}(X, L)$ a torus, denote by $Y$ the GIT quotient $X /\!\!/ G$. We define a family of fully faithful functors from the category of torus equivariant reflexive sheaves on $Y$ to the category of torus equivariant reflexive sheaves on $X$. We show, under a genericity assumption on $G$, that slope stability is preserved by these functors if and only if the pair $((X, L), G)$ satisfies a combinatorial criterion. As an application, when $(X, L)$ is a polarized toric orbifold of dimension $n$, we relate stable equivariant reflexive sheaves on $(X, L)$ to stable equivariant reflexive sheaves on certain $(n-1)$-dimensional weighted projective spaces.

## 1. Introduction

The construction of moduli spaces of projective varieties and vector bundles is a fundamental problem in algebraic geometry. Given a polarized variety $(X, L)$ or a vector bundle $\mathcal{E}$ on $(X, L)$, one considers various stability notions for $X$ and $\mathcal{E}$. In the presence of symmetries for $(X, L)$, that is given an algebraic action of a reductive Lie group $G$ on $(X, L)$, it is natural to ask whether these stability notions persist on the GIT quotient $Y$ of $(X, L)$ by $G$. By the Yau-Tian-Donaldson conjecture and the Kobayashi-Hitchin correspondence, the stability of $(X, L)$ or $\mathcal{E}$ can be related to the existence of a canonical metric on the underlying complex object, variety or bundle. From this differential geometrical point of view, $G$-orbits can detect curvature on $X$, and canonical metrics are not necessarily preserved under GIT quotients. As a motivating case, in [4], Futaki investigated GIT quotients of Fano varieties, giving a condition on the symplectic reduction of $X$ to be Kähler-Einstein. It is then natural to expect a relation between the stability of $X$, of the quotient $Y$, and the geometric properties of the representation $G \to \mathrm{Aut}(X, L)$. In this paper, we provide an example of such an interplay, by relating slope stability for reflexive sheaves on $X$ and $Y$ to a combinatorial criterion on the $G$-action, in the equivariant context of toric geometry (see also [5, 15] for related results).

A vector bundle, or more generally a torsion-free sheaf $\mathcal{E}$ on a complex projective variety $X$ is said to be slope stable with respect to an ample $\mathbb{R}$-divisor $\alpha \in N^1(X)_{\mathbb{R}}$ if for any subsheaf $\mathcal{F}$ with $0 < \mathrm{rk}\, \mathcal{F} < \mathrm{rk}\, \mathcal{E}$, the slope inequality holds

$$\mu_\alpha(\mathcal{F}) < \mu_\alpha(E),$$

where the slope is given by the intersection theoretical formula:

$$\mu_\alpha(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot \alpha^{n-1}}{\mathrm{rank}\mathcal{E}}.$$

The notion of slope stability originated in the construction of moduli spaces of sheaves [8]. Assume now that $(X, L)$ is a polarized toric variety over $\mathbb{C}$, that is

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endowed with an effective action of a complex torus $T$ with open and dense orbit. We further assume the toric varieties that we consider to come from fans, and in particular to be normal. We denote by $N$ the lattice of one-parameter subgroups of $T$, so that $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Consider $G \subset T$ a subtorus, given by a sublattice $N_0 \subset N$, that is $G = N_0 \otimes_{\mathbb{Z}} \mathbb{C}^*$. For any linearization $\gamma : T \to \mathrm{Aut}(L)$, we can form a toric variety obtained by GIT quotient $Y = X /\!\!/ G$. To avoid finite quotients, we will assume that $N_0$ is *saturated in* $N$, that is $N_0 = N \cap (N_0 \otimes \mathbb{R})$. We will further assume that the linearization $\gamma$ of $G$ on $L$ is *generic*, that is the stable and semistable loci coincide (see Section 3.1). Under these hypothesis, we build a family of fully faithful functors

$$\mathfrak{P}_{\mathbf{i}} : \mathfrak{Ref}^T(Y) \to \mathfrak{Ref}^T(X)$$

that embeds the category of torus equivariant reflexive sheaves on $Y$ into the category of torus equivariant reflexive sheaves on $X$ (Section 3.3). Given an ample class $\alpha \in N^1(Y)_{\mathbb{R}}$ on $Y$, we will say that such a functor $\mathfrak{P}$ preserves slope stability notions from $(Y, \alpha)$ to $(X, L)$ if an element $\mathcal{E} \in \mathfrak{Ref}^T(Y)$ is slope stable (resp. semistable, polystable) with respect to $\alpha$ if and only if $\mathfrak{P}(\mathcal{E})$ is slope stable (resp. semistable, polystable) with respect to $L$ (see Section 4.1 for the definition of these notions). Then our main result goes as follows:

**Theorem 1.1.** *Let $(X, L)$ be a polarized toric variety with torus $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Let $G = N_0 \otimes_{\mathbb{Z}} \mathbb{C}^*$ be a subtorus for a saturated sublattice $N_0 \subset N$. Let $\gamma : T \to \mathrm{Aut}(L)$ be a generic linearization of $G$, and denote by $Y$ the associated GIT quotient $X /\!\!/ G$. Then, the following statements are equivalent:*

i) *There is an ample class $\alpha \in N^1(Y)_{\mathbb{R}}$ on $Y$ such that the functors $\mathfrak{P}_{\mathbf{i}}$ preserve slope stability notions from $(Y, \alpha)$ to $(X, L)$.*
ii) *The pair $((X, L), (G, \gamma))$ satisfies the Minkowski condition*

$$\sum_{D \subset X^s} \deg_L(D)\, u_D = 0 \mod N_0 \otimes_{\mathbb{Z}} \mathbb{R} \tag{1}$$

*Moreover, there is at most one class $\alpha$ on $Y$ satisfying $(i)$ up to scale.*

In the statement of Theorem 1.1, the sum (1) is over the set of $T$-invariant divisors in the stable locus $X^s$ of the $G$-action and $u_D$ denotes the primitive generator of the ray associated to $D$ in the fan of $X$ (see Section 2).

When the moduli functors of stable reflexive sheaves on $(X, L)$ and $(Y, \alpha)$ are corepresentable, for example when $X$ and $Y$ are smooth and $\alpha$ is integral [11], and under hypothesis of Theorem 1.1, the functors $\mathfrak{P}_{\mathbf{i}}$ induce maps between connected components of the moduli spaces of stable reflexive sheaves on $(Y, \alpha)$ and $(X, L)$. We expect Theorem 1.1 to be the first step towards relating topological invariants between these moduli spaces, and will investigate these relations in future work.

Minkowski condition (1) is a very restrictive condition on $(G, \gamma)$. We will say that a subtorus $G \subset T$ is *compatible* with $(X, L)$ if there is a generic linearization $\gamma : T \to \mathrm{Aut}(L)$ for $G$ such that $((X, L), (G, \gamma))$ satisfies Minkowski condition. We obtain in Lemma 5.2 an explicit bound, depending on the dimension and the number of rays in the fan, on the number of compatible one-parameter subgroups for polarized complete toric varieties satisfying a mild hypothesis. Nevertheless, we manage to show the following:

**Proposition 1.2.** *Let $(X, L)$ be a polarized toric orbifold. Up to replacing $L$ by a sufficiently high power, there are at least $(n + 1)$ one-parameter subgroups of*

*$T$ compatible with $(X, L)$. The associated GIT quotients are weighted projective spaces.*

Compact weighted projective spaces are precisely the complete toric orbifolds of Picard rank 1, and as such are the simplest complete toric orbifolds. It is then interesting to be able to lift stable sheaves on this simpler objects to general toric orbifolds. By iterating the statement, we expect to obtain interesting information on the moduli spaces of equivariant reflexive sheaves on these varieties. What still lacks in this discussion is knowledge of the corepresentability of the moduli functors in such generality. We intend to study this and related questions in a sequel to this paper.

A fundamental theorem of Mehta and Ramanathan states that the restriction of a slope stable reflexive sheaf $\mathcal{E}$ on $X$ to a general complete intersection $Z \subset X$ of sufficiently high degree is again slope stable [13]. To the knowledge of the authors, there is no similar general statement for projections $\pi : X \to Y$, and our construction provides a result in this direction. More precisely, from Theorem 1.1, we deduce:

**Corollary 1.3.** *Let $Y$ be a complete toric variety, $V$ be a toric decomposable vector bundle on $Y$ and $X$ be the toric variety $X = \mathbb{P}(V^\vee)$, with projection map $\pi : X \to Y$. Let $L_Y$ be a polarization on $Y$ such that $L_X = \pi^* L_Y \otimes \mathcal{O}_X(1)$ is ample on $X$. Then, there exists a real ample class $\alpha \in N^1(Y)_\mathbb{R}$ such that an equivariant reflexive sheaf $\mathcal{E}$ on $(Y, \alpha)$ is slope stable if and only if $\pi^*\mathcal{E}$ is slope stable on $(X, L_X)$.*

*Remark* 1.4. In the setting of Corollary 1.3, we give examples where we can determine the class $\alpha$ on $Y$ (see Section 5.3). This class is not the one obtained from the GIT quotient of $(X, L_X)$, this being $L_Y$ in this case. A quick look at the examples coming from Corollary 1.3 suggests that in most cases, $\alpha$ will be different from $L_Y$. It would be interesting to obtain a general formula for $\alpha$ in terms of the geometric data $(X, L)$ and $G$, and in particular to understand if $\alpha$ is always rational or not.

The proof of Theorem 1.1 is divided in two main parts. The first one, in Section 3, is the construction of the functors $\mathfrak{P}_\mathbf{i}$. It is naturally associated to the study of the descent of reflexive equivariant sheaves on a toric variety $X$ under a generic toric GIT quotient $X \dashrightarrow Y$. Let us denote by $\iota : X^s \to X$ the inclusion of the stable locus and by $\pi : X^s \to Y$ the projection to the quotient. An equivariant sheaf $\mathcal{E}$ descends to $Y$ if there is a sheaf $\check{\mathcal{E}}$ on $Y$ such that $\pi^*\check{\mathcal{E}}$ is equivariantly isomorphic to $\iota^*\mathcal{E}$. In [16], Nevins gave a general criterion for the descent of a sheaf through a good quotient. In Section 3.2, we give a combinatorial criterion for the descent of reflexive equivariant sheaves under generic toric GIT quotients. We then build the functors $\mathfrak{P}_\mathbf{i}$, by extending reflexive equivariant sheaves pulled-back from $Y$ to $X^s$ across the unstable locus (see [9] for similar constructions). The elements in the images of these functors are described geometrically, and correspond precisely to the reflexive sheaves that descend to $Y$ and for which the slopes on $X$ and $Y$ will be comparable. The second part of the proof of Theorem 1.1, given in Section 4, gives a relation between slopes on $(X, L)$ and slopes on $(Y, \alpha)$. A combinatorial formula describes these slopes [3,7]. In this formula, there are contributions from the sheaves and from the polarizations. The functors $\mathfrak{P}_\mathbf{i}$ are precisely constructed so that the sheaf contributions can be compared on $X$ and $Y$. As for the polarization terms, they are related to the volumes of the facets of the associated polytopes. To be able to compare them through the quotient, we use a classical result of Minkowski stating precisely when the volumes of the facets of a polytope can be prescribed.

The organization of the paper is as follows. In Section 2 we gather standard facts about toric varieties and equivariant reflexive sheaves that will be used in the paper. In particular, we recall in Section 2.1 the classical correspondence between polytopes and polarized toric varieties, and describe its generalization to real ample divisors. In Section 2.2, we recall Klyachko's description of the category of equivariant reflexive sheaves on toric varieties. Along the way, we give a new and shorter proof for the combinatorial formula for the first Chern class of these objects, extending several earlier results to normal toric varieties (compare with [3, 7, 11]). Section 3 deals with the descent of reflexive sheaves under toric GIT. We start by recalling the necessary material of toric GIT in Section 3.1, then prove a descent criterion in Section 3.2, and last construct the pullback functors in Section 3.3. With this material at hand, we can prove Theorem 1.1 in Section 4. Together with Section 3, this forms the core of the paper. We first introduce the notions of slope stability in Section 4.1, and then recall a classical theorem of Minkowski in Section 4.2, to conclude with the proof of our main theorem. Finally, in Section 5 we study compatible actions and give applications of our result, proving Proposition 1.2 and Corollary 1.3.

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## 2. Equivariant reflexive sheaves on toric varieties

Throughout this paper, we consider toric varieties over the complex numbers. We recall the description of polarized toric varieties in terms of polytopes and the characterization of equivariant reflexive sheaves on toric varieties in terms of families of filtrations.

### 2.1. **Polarized toric varieties and polytopes.**

We refer to [1, Chapters 2, 3, 6] and [19] for this section. Let $N$ be a rank $n$ lattice, $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual with pairing $\langle \cdot, \cdot \rangle$. Then $N$ is the lattice of 1-parameter subgroups of a $n$-dimensional complex torus

$$T := N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*).$$

We set $N_{\mathbb{F}} = N \otimes_{\mathbb{Z}} \mathbb{F}$ and $M_{\mathbb{F}} = M \otimes_{\mathbb{Z}} \mathbb{F}$ for $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$.

Let $X = X(\Sigma)$ be a $n$-dimensional complete toric variety associated to a fan $\Sigma$, so that in particular $X$ is normal. Denote $\Sigma = \{\sigma_i, i \in I\}$, with $\sigma_i$ a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ for all $i \in I$. Denote also by $\Sigma(k)$ the set of $k$-dimensional cones in $\Sigma$. The variety $X$ is obtained by gluing affine charts $(U_\sigma)_{\sigma \in \Sigma}$, with

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]),$$

and $\mathbb{C}[S_\sigma]$ is the semi-group algebra of

$$S_\sigma = \sigma^\vee \cap M = \{m \in M; \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma\}.$$

There exists a bijective correspondence between cones $\sigma \in \Sigma$ and $T$-orbits $O(\sigma)$ in $X$. This satisfies, for $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim(\sigma)$ so to any $\rho \in \Sigma(1)$, there corresponds a $T$-invariant Weil divisor $D_\rho$ given by

$$D_\rho = \overline{O(\rho)} \tag{2}$$

where the closure is in both classical and Zariski topologies.

As $X$ is associated to the fan $\Sigma$, there is a bijective correspondence between torus-invariant ample divisors on $X$ and lattice polytopes $P \subset M_{\mathbb{R}}$ whose normal fan $\Sigma_P$ is equal to $\Sigma$ (see [1, Theorem 6.2.1]). Let $D$ be a $T$-invariant Cartier divisor on $X$. Recall that it is equal to a linear combination of the form

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho.$$

For each $\rho \in \Sigma(1)$ we denote by $u_\rho \in N$ the minimal generator of $\rho \cap N$. Assuming that $D$ is ample, we consider the associated polytope $P = P_D \subset M_{\mathbb{R}}$:

$$P = \{m \in M_{\mathbb{R}} \ ; \ \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\} \tag{3}$$

Note that if $D'$ is equivariant and linearly equivalent to $D$ then $P_{D'}$ is given by translation of $P_D$ in $M$ by some lattice element $m \in M$. In the same way, a lattice translation of the polytope $P_D$ corresponds to a different linearization of the action of $T$ on the line bundle $\mathcal{O}(D)$ (see Section 3.1).

As $P$ is ample, with normal fan equal to $\Sigma$, we have a correspondence between cones in $\Sigma$ and faces of $P$. For a face $Q$ in $P$, we denote by $\sigma_Q \in \Sigma$ (resp. by $O(Q)$) the associated cone (resp. the associated orbit). In particular, rays in $\Sigma$ corresponds to facets of $P$. For each $\rho \in \Sigma(1)$ the associated facet is

$$F = P \cap \{m \in M_{\mathbb{R}} \ ; \ \langle m, u_\rho \rangle = -a_\rho\}.$$

We will denote $u_\rho$ by $u_F$ and $a_\rho$ by $a_F$. We can also write

$$D = \sum_{F \prec P} a_F D_F,$$

where the sum is over all facets of $P$ and $D_F := D_{\rho_F}$. We will denote faces of $P$ of higher codimension by the letter $Q$ and vertices by the letter $v$. We use the relation $Q_1 \preccurlyeq Q_2$ to signify that $Q_1$ and $Q_2$ are faces of $P$, possibly equal to $P$ itself, and $Q_1 \subseteq Q_2$.

The correspondence between polarizations on $X$ and lattice polytopes with normal fan $\Sigma$ modulo lattice translations extends to *real* ample classes. We recall the definition of real ample divisors on a normal complex algebraic variety $X$ (see [12]). Let $\mathrm{Div}(X)$ denote the group of integral Cartier divisors on $X$. The Néron-Severi group is given by $N^1(X) = \mathrm{Div}(X)/\sim_{num}$ and we set $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\mathrm{WDiv}(X)$ the set of Weil divisors on $X$ and $\mathrm{WDiv}_{\mathbb{R}}(X)$ the vector space of real Weil divisors.

**Definition 2.1.** A class $\alpha \in N^1(X)_{\mathbb{R}}$ is *ample* if it is represented by a positive real linear combination of ample Cartier divisors.

We then have the following:

**Proposition 2.2.** *Let $X$ be a complete toric variety given by a fan $\Sigma$. Then there is a bijective correspondence between real ample classes on $X$ and real polytopes of the form*

$$P = \{m \in M_{\mathbb{R}}, \langle m, u_\rho \rangle \geq -a_\rho, \ \ \textit{for all } \rho \in \Sigma(1)\}$$

*for which the normal fan $\Sigma_P = \Sigma$, modulo real translations within $M_{\mathbb{R}}$.*

*Proof.* This statement between integral classes and lattice polytopes is standard (note that for complete toric varieties coming from fans, the real Picard group and the real Néron-Severi group coincide, see [1, Proposition 6.3.15]). The rational case follows by clearing denominators and scaling polytopes. We now prove the real case. Set $\mathrm{Pic}(X)$ and $\mathrm{Cl}(X)$ the Picard and class groups of $X$. From the exact sequence ( [1, Theorem 4.1.3]):

$$0 \longrightarrow M \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \xrightarrow{\pi} \mathrm{Cl}(X) \longrightarrow 0$$

we deduce the sequences of vector spaces, for $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$:

$$0 \longrightarrow M_{\mathbb{K}} \longrightarrow W_{\mathbb{K}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{K}} \longrightarrow 0$$

where we set $W = \pi^{-1}(\mathrm{Pic}(X))$, $W_{\mathbb{K}} = W \otimes_{\mathbb{Z}} \mathbb{K}$ and $\mathrm{Pic}(X)_{\mathbb{K}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{K}$.

Let $\alpha \in \mathrm{Pic}(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$ be an ample class. We can represent $\alpha$ by

$$D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho \in W_{\mathbb{R}}.$$

Define the set

$$P_\alpha = \{m \in M_{\mathbb{R}} \ ; \ \forall \rho \in \Sigma(1),\ \langle m, u_\rho\rangle \geq -a_\rho\}.$$

First observe that $P_\alpha$ is a polytope, rather than a polyhedron, since the fan $\Sigma$ is complete. By definition of ample real divisors, $D = \sum_{i=1}^N \alpha_i D_i$ for $D_i$ ample Cartier divisors and $\alpha_i$ positive real numbers. We approximate the values $\alpha_i \in \mathbb{R}$ by rational numbers $\beta_i \in \mathbb{Q}$ and write

$$D_\beta = \sum_{i=1}^N \beta_i D_i = \sum_{\rho\in\Sigma(1)} b_\rho D_\rho.$$

By the openness of the amplitude condition, we can assume that $D_\beta$ is an ample $\mathbb{Q}$-divisor, and hence the polytope

$$P_\beta = \{m \in M_{\mathbb{R}} \ ; \forall \rho \in \Sigma(1),\ \langle m, u_\rho\rangle \geq -b_\rho\}$$

has normal cone $\Sigma_{P_\beta} = \Sigma$. Furthermore, decomposing the divisors $D_i$ in the basis $\{D_\rho,\ \rho \in \Sigma(1)\}$, we see that the values $(b_\rho)$ vary continuously with respect to $(\beta_i)$, so can be made close enough to the $(a_\rho)$ to guarantee that $\Sigma_{P_\alpha} = \Sigma_{P_\beta} = \Sigma$. Note that as in the integral case, translations of $P_\alpha$ by elements of $M_{\mathbb{R}}$ correspond to different choices of representant of the class $\alpha$ in $W_{\mathbb{R}}$.

For the converse statement, we consider the polytope in $M_{\mathbb{R}}$

$$P = \{m \in M_{\mathbb{R}} \ ; \forall \rho \in \Sigma(1), \langle m, u_\rho\rangle \geq -a_\rho\}$$

for $a_\rho \in \mathbb{R}$, supposing that the normal fan of $P$ determined by the vectors $u_\rho$ is the fan of $X$. Then, the polytope $P$ determines the real Weil divisor $D_P = \sum_\rho a_\rho D_\rho \in \mathrm{WDiv}_{\mathbb{R}}(X)$. We show that $D_P$ lies in the space of real Cartier divisors, and is moreover ample. This is proven by induction on the number of $a_\rho$'s that are irrational. We list the rays in $\Sigma$ by $\rho_1, \ldots, \rho_d$ for $d = \#\Sigma(1)$. As noted above, the case where all $a_\rho$'s are rational is well-known. Suppose that, for fixed $k \geq 1$, $D = \sum_{i=1}^d a_{\rho_i} D_{\rho_i}$ defines a real ample class whenever its normal fan $\Sigma_P = \Sigma$ and $a_{\rho_i} \in \mathbb{Q}$ for all $i \geq k$. If $a_{\rho_i} \in \mathbb{Q}$ for $i \geq k+1$ then let $r_1, r_2$ be rational numbers for which $r_1 < a_{\rho_k} < r_2$ sufficiently close to $a_{\rho_k}$ that for any $s \in [r_1, r_2]$ the polytope defined by the inequalities

$$\begin{aligned} \langle m, u_{\rho_i} \rangle &\geq -a_{\rho_i}, \quad \text{for } i \neq k, \\ \langle m, u_{\rho_k} \rangle &\geq -s \end{aligned}$$

defines the same normal fan $\Sigma$. Then, for some $t \in [0, 1]$, we have $a_{\rho_k} = tr_1 + (1-t)r_2$ and

$$D = \sum_{i=1}^{d} a_{\rho_i} D_{\rho_i} = t \left( r_1 D_{\rho_k} + \sum_{i \neq k} a_{\rho_i} D_{\rho_i} \right) + (1-t) \left( r_2 D_{\rho_k} + \sum_{i \neq k} a_{\rho_i} D_{\rho_i} \right).$$

By the induction hypothesis, each of the two real divisors on the right hand side of the above equality is ample. By the convexity of the set of real ample classes, $D$ is ample. $\square$

*Remark* 2.3. We note that in the smooth case, a similar result can be given via symplectic geometry by using the correspondence between compact symplectic toric manifolds and Delzant polytopes up to translations.

2.2. **Equivariant reflexive sheaves.** We refer to the references [10,11,17] for this section. We consider a complete toric variety $X$ together with a polytope $P \subset M_{\mathbb{R}}$ associated to an ample class on $X$. Recall that a reflexive sheaf on $X$ is a coherent sheaf $\mathcal{E}$ that is canonically isomorphic to its double dual $\mathcal{E}^{\vee\vee}$. Klyachko gave a description of reflexive sheaves in terms of combinatorial data:

**Definition 2.4.** A family of filtrations $\mathbb{E}$ is the data of a finite dimensional vector space $E$ and for each facet $F$ of $P$, an increasing filtration $(E^F(i))_{i \in \mathbb{Z}}$ of $E$ such that $E^F(i) = \{0\}$ for $i \ll 0$ and $E^F(i) = E$ for some $i$. We will denote by $i_F$ the smallest $i \in \mathbb{Z}$ such that $E^F(i) \neq 0$.

*Remark* 2.5. Families of filtrations in [10] or [17] are labeled by the set of rays $\rho \in \Sigma(1)$. As $P$ is associated to an ample class, there is a $1:1$ correspondence between its facets and the rays of the fan of $X$, and we recover the usual definition. Note also that we are using increasing filtrations here, as in [17], rather than decreasing as in [10].

To a family of filtrations $\mathbb{E} := \{(E^F(i)) \subseteq E, F \prec P, i \in \mathbb{Z}\}$ we can assign a reflexive sheaf $\mathcal{E} := \mathfrak{K}(\mathbb{E})$ defined by

$$\Gamma(U_{\sigma_Q}, \mathcal{E}) := \bigoplus_{m \in M} \bigcap_{Q \preccurlyeq F} E^F(\langle m, u_F \rangle) \otimes \chi^m \tag{4}$$

for all proper faces $Q \prec P$, while $\Gamma(U_{\sigma_P}, \mathcal{E}) = E \otimes \mathbb{C}[M]$.

*Remark* 2.6. The conditions for a family of filtrations to define a locally-free sheaf are determined in [10].

The morphisms of families of filtrations are defined by:

**Definition 2.7.** A morphism between two families of filtrations $\mathbb{E}_1 = \{(E_1^F(i)) \subseteq E_1, F \prec P, i \in \mathbb{Z}\}$ and $\mathbb{E}_2 = \{(E_2^F(i)) \subset E_2, F \prec P, i \in \mathbb{Z}\}$ is a linear map $\phi : E_1 \to E_2$ preserving the filtrations, that is such that for all $F$ and all $i$, $\phi(E_1^F(i)) \subset E_2^F(i)$.

For a toric variety $Z$, and an ample polytope $P_Z$, we denote by:

i) $\mathfrak{Ref}^T(Z)$ the category of torus-equivariant reflexive sheaves on $Z$,

ii) $\mathfrak{Filt}(P_Z)$ the category of families of filtrations associated to $P_Z$.

From Klyachko and Perling [10,17], we obtain the following:

**Theorem 2.8** ( [10,17])**.** *The functor $\mathfrak{K}$ induces an equivalence of categories between the category $\mathfrak{Filt}(P)$ and the category $\mathfrak{Ref}^T(X)$.*

As the category of filtrations on a given finite dimensional vector space is abelian, we have:

**Corollary 2.9.** *The category $\mathfrak{Ref}^T(X)$ of reflexive equivariant sheaves on $X$ is an abelian category.*

We will need the combinatorial characterizations of equivariant reflexive subsheaves and of equivariant rank 1 reflexive sheaves. Let $\mathcal{E} = \mathfrak{K}(\mathbb{E})$ be an equivariant reflexive sheaf on $X$, given by a family of filtrations $\mathbb{E} = \{(E^F(i)) \subset E, F \prec P, i \in \mathbb{Z}\}$. For any vector subspace $W \subset E$, define a family of filtrations $\mathbb{E} \cap W$ by

$$\mathbb{E} \cap W = \{(W \cap E^F(i)) \subset W \cap E, F \prec P, i \in \mathbb{Z}\}.$$

Then, the sheaf $\mathcal{E}_W := \mathfrak{K}(\mathbb{E} \cap W)$ is an equivariant reflexive subsheaf of $\mathcal{E}$. Any equivariant reflexive subsheaf of $\mathcal{E}$ arises that way:

**Proposition 2.10.** ( [3, Cor. 3.0.2]) *Let $\mathcal{E} = \mathfrak{K}(\mathbb{E})$ be an equivariant reflexive sheaf on $X$. Let $\mathcal{F} \subset \mathcal{E}$ be an equivariant reflexive subsheaf of $\mathcal{E}$. Then, there is a unique vector subspace $W \subset E$ such that $\mathcal{F} = \mathfrak{K}(\mathbb{E} \cap W)$.*

As for rank 1 reflexive sheaves, from the definition we obtain:

**Proposition 2.11.** *Let $\mathcal{O}(-D)$ be the rank 1 reflexive sheaf associated to the invariant Weyl divisor $D = \sum_{F \prec P} a_F D_F$. Then, $\mathcal{O}(-D) = \mathfrak{K}(\mathbb{E}_D)$, where the family of filtrations $\mathbb{E}_D = \{(E^F(i)) \subset \mathbb{C}, F \prec P, i \in \mathbb{Z}\}$ satisfies*

$$E^F(i) = \left\{ \begin{array}{lll} 0 & \text{if} & i < a_F \\ \mathbb{C} & \text{if} & i \geq a_F. \end{array} \right.$$

We will also need the determinant and first Chern class of reflexive sheaves.

*Remark* 2.12. Let $A_k(X)$ be the $k$-th Chow group of $X$. This is the quotient of the free abelian group on $k$-dimensional subvarieties by rational equivalence. The first Chern class is the map $c_1 : \mathrm{Pic}(X) \to A_{n-1}(X)$ induced by the inclusion of Cartier divisors in Weil divisors. This defines a morphism $A_k(X) \to A_{k-1}(X)$ for each $k$ as follows. For $\mathcal{L}$ a line bundle and $V$ a $k$-dimensional subvariety on $X$, $\mathcal{L}|_V$ defines a Cartier divisor on $V$, hence a $(k-1)$-cycle on $X$.

As $X$ is normal, this definition extends to rank one reflexive sheaves since every rank-one reflexive sheaf is of the form $\mathcal{O}_X(D)$ for some Weil divisor $D$, and hence $c_1(\mathcal{L}) := [D] \in A_{n-1}(X)$. If $H$ is an ample line bundle on $X$, the degree of $\mathcal{L}$ is then given by

$$\deg_H(\mathcal{L}) = c_1(\mathcal{L}) \cdot H^{n-1} \in A_0(X) \cong \mathbb{Z}.$$

Recall the following:

**Definition 2.13.** If $\mathcal{E}$ is a torsion-free coherent sheaf on $X$, one defines the determinant of $\mathcal{E}$ to be the rank-one reflexive sheaf $\det(\mathcal{E}) = (\Lambda^{rank(\mathcal{E})}\mathcal{E})^{\vee\vee}$. Then, the first Chern class of $\mathcal{E}$ is $c_1(\mathcal{E}) := c_1(\det \mathcal{E})$.

To produce a combinatorial formula for the determinant of equivariant reflexive sheaves, we first need to introduce some notation:

**Definition 2.14.** Let $\mathcal{E} = \mathfrak{K}(\mathbb{E})$, with $\mathbb{E} = \{(E^F(i)) \subset E, F \prec P, i \in \mathbb{Z}\}$. We set, for all $F \prec P$ and all $i \in \mathbb{Z}$:

$$e^F(i) = \dim(E^F(i-1)) - \dim(E^F(i)). \tag{5}$$

We will refer to the integers $(e^F(i))_{F \prec P, i \in \mathbb{Z}}$ as the *dimension jumps* of $\mathbb{E}$ or $\mathcal{E}$.

Then we have:

**Proposition 2.15.** *Let $\mathcal{E} = \mathfrak{K}(\mathbb{E})$ be a rank $r$ equivariant reflexive sheaf on $X$, given by a family of filtrations $\mathbb{E} = \{(E^F(i)) \subset E, F \prec P, i \in \mathbb{Z}\}$. We define a family of filtrations $\det(\mathbb{E}) = \{(E^F_{\det}(i)) \subset \Lambda^r E, F \prec P, i \in \mathbb{Z}\}$ by setting, for all $F \prec P$,*

$$E^F_{\det}(i) = \begin{cases} 0 & \text{if} \quad i < i_F(\det \mathcal{E}) \\ \Lambda^r E & \text{if} \quad i \geq i_F(\det \mathcal{E}) \end{cases}$$

*where for all $F \prec P$,*

$$i_F(\det \mathcal{E}) = -\sum_{i \in \mathbb{Z}} i e^F(i).$$

*Then* $\det(\mathcal{E}) = \mathfrak{K}(\det(\mathbb{E}))$.

*Proof.* Note first that because $\mathfrak{Ref}^T(X)$ is abelian, $\Lambda^r \mathcal{E}$ is reflexive and $\det(\mathcal{E}) = \Lambda^r \mathcal{E}$. Then, the family of filtrations $\{(\Lambda^r E^F(i)), F \prec P, i \in \mathbb{Z}\}$ for $\Lambda^r \mathcal{E}$ satisfies:

$$\Lambda^r E^F(i) = \sum_{i_1, \ldots, i_r; \sum i_j = i} E^F(i_1) \wedge \ldots \wedge E^F(i_r).$$

Now, $\Lambda^r E$ is one dimensional, so

$$\Lambda^r E^F(i) = \begin{cases} 0 & \text{if} \quad i < \tilde{i}_F \\ \Lambda^r E & \text{if} \quad i \geq \tilde{i}_F \end{cases}$$

where $\tilde{i}_F$ is the smallest integer $l \in \mathbb{Z}$ such that there is a partition $i_1, \ldots, i_r$ of $l$ with $E^F(i_1) \wedge \ldots \wedge E^F(i_r) \neq \{0\}$. From the fact that $(E^F(i))$ forms a filtration of vector spaces, we deduce that $\tilde{i}_F$ must be the sum of the integers $i$ such that the dimension of $E^F(i)$ changes, counted with multiplicity. Hence $\tilde{i}_F = -\sum_{i \in \mathbb{Z}} i e^F(i)$, which ends the proof. $\square$

**Corollary 2.16.** *Let $\mathcal{E} = \mathfrak{K}(\mathbb{E})$ be a rank $r$ equivariant reflexive sheaf on $X$, given by a family of filtrations $\mathbb{E} = \{(E^F(i)) \subset E, F \prec P, i \in \mathbb{Z}\}$. The first Chern class of $\mathcal{E}$ is the class of the Weil divisor:*

$$c_1(\mathcal{E}) = \sum_{F \prec P} -i_F(\det \mathcal{E})\, D_F. \tag{6}$$

*where for all $F \prec P$,*

$$i_F(\det \mathcal{E}) = -\sum_{i \in \mathbb{Z}} i e^F(i).$$

*Remark* 2.17. In this paper, we restrict ourselves to reflexive sheaves. We expect that most of the results extend to equivariant torsion-free coherent sheaves, described in terms of families of multifiltrations [11, 17]. As the applications we have mind concern stable vector bundles, it is enough to consider the category of reflexive sheaves, where the results and proofs are simpler to express.

## 3. Descent of equivariant sheaves under toric GIT

In this section we study the descent of equivariant reflexive sheaves under toric GIT quotients. We denote by $X$ a projective toric variety, polarized by an equivariant line bundle $L$. We keep the notations of the previous section.

3.1. **Toric GIT.** We refer to [14, 19] for this section. We are interested in GIT quotients of $(X, L)$ by subtorus actions. Let $N_0$ be a sublattice of $N$ of rank $g$. We will assume that $N_0$ is saturated, that is $N_0 = (N_0 \otimes_{\mathbb{Z}} \mathbb{R}) \cap N$. The sublattice $N_0$ spans a $g$-dimensional subtorus $G = N_0 \otimes_{\mathbb{Z}} \mathbb{C}^*$ of $T$. We fix a linearization $\gamma$ of $T$ on $L$ and we will consider the GIT quotient with respect to the induced linearization of $G$. From Section 2.1, $(X, L)$ defines a family of lattice polytopes $\{P_D,\ \mathcal{O}(D) \sim L\}$, all equal up to translations by lattice elements. Then, the linearization $\gamma$ determines a unique polytope $P$ in this family such that

$$H^0(X, L) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m \tag{7}$$

is the weight decomposition of the $T$-action on the space of global sections induced by $\gamma$ (see [1, Proposition 4.3.3]). We have the following:

$$0 \to N_0 \to N \to N/N_0 \to 0,$$

and the associated dual sequence:

$$0 \to N_0^{\perp} \to M \to M/N_0^{\perp} \to 0.$$

Set $U = N_0^{\perp} \otimes_{\mathbb{Z}} \mathbb{R} \subset M_{\mathbb{R}}$. Then from [19, Proposition 3.2]:

**Proposition 3.1.** *The GIT quotient of $(X, L)$ by $G$ is the polarized toric variety $(Y, \check{L})$ described by the polytope $P_Y = U \cap P$. Moreover, its lattice is $N_Y = N/N_0$ with dual $N_0^{\perp}$.*

*Remark* 3.2. Note that the vertices of the polytope $P \cap U$ are not necessarily lattice points, so this polytope only induces a *rational* polarization $\check{L}$ on the variety $Y = X /\!\!/ G$.

*Remark* 3.3. Up to composition by a finite morphism, we can always reduce to the case $N_0$ saturated. To simplify the exposition, we will always make this assumption.

We can also describe the stable and semistable points :

**Proposition 3.4** ( [19], Lemma 3.3)**.** *The semistable and stable loci $X^{ss}$ and $X^s$ under the $G$ action on $(X, L)$ are each unions of $T$-orbits. More precisely, given a face $Q \preccurlyeq P$, the orbit $O(Q)$ is:*

- *semistable iff $Q \cap U \neq \varnothing$,*
- *stable iff $Q \cap U \neq \varnothing$ and the interior of $Q$ meets $U$ transversally.*

From this proposition, the set of faces of the polytope of $X^{ss} /\!\!/ G$ is

$$\{U \cap Q, Q \preccurlyeq P\}.$$

We will denote by $P^s$ (resp. $P^{ss}$ and $P^{us}$) the set of faces corresponding to stable orbits (resp. semistable orbits and unstable orbits). For technical reasons, we will need the following assumption on the action:

**Definition 3.5.** A pair of a subtorus $G \subset T$ and linearization $\gamma : T \to \mathrm{Aut}(L)$ will be called *generic* if the stable and semi-stable loci of the $G$ action on $(X, L)$ coincide and are not empty.

The nice fact about generic actions is the following:

**Lemma 3.6.** *Assume that $(G, \gamma)$ is generic. Then:*

i) *The points of $X^s$ have finite stabilizers under the $G$-action.*
ii) *There is a $1:1$-correspondence between facets of $P_Y$ and facets in $P^s$.*

*Proof.* The first point follows by definition of stable points. For the second point, any facet of $U \cap P$ must be of the form $U \cap Q$ for some face $Q$ of $P$. By assumption, $Q$ meets $U$ transversally. Moreover, $\dim(U \cap Q) = \dim(Y) - 1 = n - 1 - g$. This forces the dimension of $Q$ to be $n - 1$, hence the result. □

From now on, unless explicitly stated, we will assume that $(G, \gamma)$ is generic. We end this section with some definitions and lemmas relating the primitive normals to the facets of $P$ to those of $P_Y$. We consider the lattice projection, $\pi : N \to N_Y = N/N_0$.

**Definition 3.7.** For each facet $F \cap U$ of $P_Y$, set $\check{u}_F$ to be the primitive element in $N_Y$ defining $F \cap U$.

**Lemma 3.8.** *There is a unique element $b_F \in \mathbb{N}^*$ such that*

$$\check{u}_F = b_F^{-1} \pi(u_F).$$

*Proof.* Both elements lies on the ray $\rho_{F \cap U}$, and $\check{u}_F$ generates this ray. □

Let $F$ be facet in $P^s$. As $u_F$ is primitive, we can complete it into a basis $\mathcal{B}_F := \{u_F,\ u_i,\ i = 2 \dots n\}$ of $N$. We denote by $\mathcal{B}_F^* = \{m_F, m_i,\ i = 2 \dots n\}$ the dual basis of $M$, with obvious notations. In particular, note that $\{m_i,\ i = 2 \dots n\}$ is a basis for $u_F^\perp \cap M$. Similarly, we set $\check{\mathcal{B}}_F := \{\check{u}_F,\ \check{u}_i,\ i = 2 \dots n - g\}$ a basis for $N_Y$ with dual basis $\check{\mathcal{B}}_F^* := \{\check{m}_F,\ \check{m}_i,\ i = 2 \dots n - g\}$ and again $\{\check{m}_i,\ i = 2 \dots n - g\}$ is a basis for $\check{u}_F^\perp \cap N_Y$. From Lemma 3.8, we deduce:

**Lemma 3.9.** *The element $\check{m}_F \in N_0^\perp \subset M$ can be uniquely written as*

$$\check{m}_F = b_F m_F + m_F^\perp$$

*with $m_F^\perp \in u_F^\perp$.*

We conclude this section with the following observation:

*Remark* 3.10. If $X$ is smooth, it is easy to show in a local chart $U_F = \mathrm{Spec}(\mathbb{C}[\sigma_F^\vee \cap M])$ that $b_F$ is the order of the stabilizer of the orbit $\mathcal{O}(F)$ under the $G$-action.

3.2. **Descent criteria for reflexive equivariant sheaves.** We keep notations from the last section. We assume as before that $(X, L)$ is a toric variety of dimension $n$ and that $(G, \gamma)$ is generic. We consider $(Y, \check{L})$ the GIT quotient associated to the $G$ action on $(X, L)$. We want to compare equivariant reflexive sheaves on $X$ to equivariant reflexive sheaves on $Y$. We denote by $\iota : X^s \to X$ the inclusion and by $\pi : X^s \to Y$ the projection. We then introduce:

**Definition 3.11.** We say that a $G$-equivariant coherent sheaf $\mathcal{E}$ on $X$ descends to $Y$ if there is a coherent sheaf $\mathcal{F}$ on $Y$ such that $\pi^*\mathcal{F}$ is $G$-equivariantly isomorphic to $\iota^*\mathcal{E}$.

Let $\mathcal{E}$ be a $G$-equivariant coherent sheaf on $X$. For simplicity we will denote by $\mathcal{E}^s$ the restriction $\iota^*\mathcal{E}$. As explained for example in [16], $\mathcal{E}$ descends to $Y$ if and only if

$$\mathcal{E}^s \simeq \pi^*\pi_*^G\mathcal{E}^s$$

where $\pi_*^G$ is the invariant pushforward.

*Remark* 3.12. Note that the functors $\iota^*$, $\pi^*$ and $\pi_*^G$ preserve the torus equivariantness and reflexivity properties of coherent sheaves (for $\pi^*$, it follows from flatness of $\pi$).

We will give a description of the functors $\iota^*$, $\pi^*$ and $\pi_*^G$ for torus-equivariant reflexive sheaves in terms of families of filtrations. Making use of the equivalence of categories $\mathfrak{K}$, by abuse of notations, we will use the same symbols to design the associated functors on families of filtrations.

**Lemma 3.13.** *Let $\mathcal{E}$ be an equivariant reflexive sheaf on $X$. Assume that $\mathcal{E} = \mathfrak{K}(\mathbb{E})$, with $\mathbb{E} = \{(E^F(i)) \subset E, F \prec P, i \in \mathbb{Z}\}$. Then the restriction $\mathcal{E}^s := \iota^*\mathcal{E}$ is an equivariant reflexive sheaf on $X^s$ defined by the family of filtrations $\iota^*\mathbb{E} = \{(E^F(i)) \subset E, F \prec P^s, i \in \mathbb{Z}\}$.*

*Proof.* The proof follows from the fact that if $Q \prec P^s$, then for every facet $F$ that contains $Q$, $F \prec P^s$. Indeed, if $F$ contains $Q$, it intersects $U$ and thus lies in $P^{ss} = P^s$. Thus by definition of $\iota^*\mathbb{E}$, and from equation (4), we obtain the result. □

**Lemma 3.14.** *Let $\mathcal{E}$ be an equivariant reflexive sheaf on $X$. Assume that $\mathcal{E} = \mathfrak{K}(\mathbb{E})$, with $\mathbb{E} = \{(E^F(i)) \subset E, F \prec P, i \in \mathbb{Z}\}$. Then the invariant pushforward $\pi_*^G\mathcal{E}^s$ is an equivariant reflexive sheaf on $Y$ defined by the family of filtrations $\pi_*^G\mathbb{E}^s = \{(\check{E}^{\check{F}}(i)) \subset \check{E}, \check{F} \prec P_Y, i \in \mathbb{Z}\}$, where*

- *$\check{E} = E$,*
- *$\check{E}^{\check{F}}(i) = E^F(b_F i)$ for all $F \prec P^s$, where we set $\check{F} = F \cap U$.*

*Proof.* Let $F \prec P^s$. Then we have the GIT quotient projection

$$\pi : U_{\sigma_F} = \mathrm{Spec}(\mathbb{C}[\sigma_F^\vee \cap M]) \to U_{\sigma_{F\cap U}} = \mathrm{Spec}(\mathbb{C}[\sigma_{U\cap F}^\vee \cap N_0^\perp])$$

and by definition,

$$\Gamma(U_{\sigma_{F\cap U}}, \pi_*^G\mathcal{E}^s) = \bigoplus_{m\in N_0^\perp} \Gamma(U_{\sigma_F}, \mathcal{E}^s)_m.$$

Thus for all $m \in N_0^\perp$:

$$\Gamma(U_{\sigma_{F\cap U}}, \pi_*^G\mathcal{E}^s)_m = \Gamma(U_{\sigma_F}, \mathcal{E}^s)_m$$

that is

$$\check{E}^{U\cap F}(\langle m, \check{u}_F\rangle) \otimes \chi^m = E^F(\langle m, u_F\rangle) \otimes \chi^m.$$

Using Lemma 3.9 and the basis $\check{\mathcal{B}}_F^*$ and $\mathcal{B}_F^*$ to decompose $m$ we obtain the result. □

Similarly, we have

**Lemma 3.15.** *Let $\check{\mathcal{E}}$ be an equivariant reflexive sheaf on $Y$. Assume that $\check{\mathcal{E}} = \mathfrak{K}(\check{\mathbb{E}})$, with $\check{\mathbb{E}} = \{(\check{E}^{\check{F}}(i)) \subset \check{E}, \check{F} \prec P_Y, i \in \mathbb{Z}\}$. Then the pull-back $\pi^*\check{\mathcal{E}}$ is an equivariant reflexive sheaf on $X^s$ defined by the family of filtrations $\pi^*\check{\mathbb{E}} = \{(E^F(i)) \subset E, F \prec P^s, i \in \mathbb{Z}\}$ where*

- $E=\check{E}$,
- $E^F(i)=\check{E}^{F\cap U}(\lfloor \frac{i}{b_F}\rfloor)$ *for all* $F\prec P^s$.

*Proof.* By Lemma 3.14, we know that $E^F(i)=\check{E}^{U\cap F}(\lfloor \frac{i}{b_F}\rfloor)$ for $i\in b_F\mathbb{Z}$, and hence we also have $E=\check{E}$. In general, we have for $j,i\in\mathbb{Z}$:

$$\check{E}^{U\cap F}(j)\otimes\chi^{j\check{m}_F}=\Gamma(U_{\sigma_{F\cap U}},\check{\mathcal{E}})_{j\check{m}_F}$$

and

$$E^{U\cap F}(i)\otimes\chi^{im_F}=\Gamma(U_{\sigma_F},\pi^*\check{\mathcal{E}})_{im_F}.$$

Moreover, we have by definition

$$\Gamma(U_{\sigma_F},\pi^*\check{\mathcal{E}})=\Gamma(U_{\sigma_{F\cap U}},\check{\mathcal{E}})\otimes_{\mathbb{C}[\sigma^\vee_{U\cap F}\cap N_0^\perp]}\mathbb{C}[\sigma_F^\vee\cap M].$$

Note that $m\in M\cap\sigma_F^\vee$ if and only if $\langle m,u_F\rangle\geq 0$ and $m\in N_0^\perp\cap\sigma^\vee_{U\cap F}$ if and only if $\langle m,\check{u}_F\rangle\geq 0$. Thus, to prove the lemma, it is enough to show that if $im_F=m'+(im_F-m')$ with $m'\in N_0^\perp\cap\sigma^\vee_{U\cap F}$, and $(im_F-m')\in M\cap\sigma_F^\vee$ then $\langle m',\check{u}_F\rangle\leq\lfloor\frac{i}{b_F}\rfloor$ (note that we use the fact that the spaces $(E^F(i))$ form a filtration). Suppose then that we have such a decomposition $im_F=m'+(im_F-m')$ for $im_F$. Without loss of generality, we can assume that $m'=a\check{m}_F$ with $a=\langle m',\check{u}_F\rangle\in\mathbb{N}$. Then we have $\langle im_F-a\check{m}_F,u_F\rangle\geq 0$ and thus by Lemma 3.9 we obtain $i-ab_F\geq 0$. The result follows. □

From these lemmas we deduce a version of Nevins criterion for descent of reflexive equivariant sheaves on toric varieties [16]:

**Corollary 3.16.** *Let* $\mathcal{E}=\mathfrak{K}(\mathbb{E})$ *be a* $T$*-equivariant reflexive sheaf on* $X$ *with dimension jumps* $(e^F(i))$. *Then* $\mathcal{E}$ *descends to* $Y$ *if and only if for all facets* $F\prec P^s$, *for all* $i\in\mathbb{Z}$ *such that* $e^F(i)\neq 0$, $b_F$ *divides* $i$.

*Proof.* The proof follows from the fact that $\mathcal{E}$ descends if and only if

$$\iota^*\mathbb{E}\simeq\pi^*\pi_*^G\iota^*\mathbb{E}.$$

Using Lemmas 3.14 and 3.15, this is equivalent that for all $F\prec P^s$, for all $i\in\mathbb{Z}$,

$$E^F(i)=E^F\left(b_F\left\lfloor\frac{i}{b_F}\right\rfloor\right).$$

The result follows. □

For later use, we emphasize the following corollary:

**Corollary 3.17.** *Assume that* $\mathcal{E}$ *is an equivariant reflexive sheaf on* $X$ *that descends to* $Y$, *with* $\mathcal{E}=\mathfrak{K}(\mathbb{E})$. *Then for all* $W\subset E$, *the subsheaf* $\mathcal{E}_W\subset\mathcal{E}$ *descends to* $Y$.

3.3. **Pullback functors.** We preserve notations of the previous section, still assuming the pair $(G,\gamma)$ to be generic. In this section, we introduce pull-back functors from $\mathfrak{Ref}^T(Y)$ to $\mathfrak{Ref}^T(X)$. The images of these functors will contain precisely the equivariant reflexive sheaves on $X$ that descend to $Y$ and that are suitable to compare slope stability notions on $X$ and $Y$.

Let $\mathbf{D}^u:=\{D_F,\ F\prec P^{us}\}$ be the set of unstable $T$-equivariant divisors on $X$. For $\mathbf{i}:=(i_D)_{D\in\mathbf{D}^u}\in\mathbb{Z}^u$, we define a functor $\mathfrak{P}_{\mathbf{i}}$ from $\mathfrak{Ref}^T(Y)$ to $\mathfrak{Ref}^T(X)$. Using the functor $\mathfrak{K}$, it is enough to define the related functor from $\mathfrak{Filt}(P_Y)$ to $\mathfrak{Filt}(P)$. Let $\check{\mathbb{E}}\in\mathfrak{Filt}(P_Y)$, with $\check{\mathbb{E}}=\{(\check{E}^{\check{F}}(i))\subset\check{E},\check{F}\prec P_Y,i\in\mathbb{Z}\}$. Then we define $\mathfrak{P}_{\mathbf{i}}(\check{E})=\{(E^F(i))\subset E,F\prec P,i\in\mathbb{Z}\}$ by setting $E=\check{E}$ and for $F\prec P$ and $i\in\mathbb{Z}$:

- If $F \prec P^s$, then $E^F(i) = \check{E}^{U \cap F}(\lfloor \frac{i}{b_F} \rfloor)$.
- If $F \prec P^{us}$, then

$$E^F(i) = \begin{cases} 0 & \text{if} \quad i < i_{D_F} \\ E & \text{if} \quad i \geq i_{D_F} \end{cases}$$

It is straightforward to define the functors $\mathfrak{P}_{\mathbf{i}}$ on morphisms of families of filtrations. By abuse of notation, we denote by $\mathfrak{P}_{\mathbf{i}}$ the associated functors on $\mathfrak{Ref}^T(Y)$. We then have:

**Proposition 3.18.** *The functors $\mathfrak{P}_{\mathbf{i}}$ are faithful functors from the category $\mathfrak{Ref}^T(Y)$ to $\mathfrak{Ref}^T(X)$.*

*Remark* 3.19. One can extend these functors to equivariant torsion-free coherent sheaves, using their combinatorial description as in [17] or [11].

*Remark* 3.20. The functors $\mathfrak{P}_{\mathbf{i}}$ might not preserve the compatibility conditions for local freeness introduced by Klyachko [10]. One can show that if $X$ is smooth and complete of dimension $n \geq 3$, then the functors $\mathfrak{P}_{\mathbf{i}}$ preserve local freeness if and only if for any $Q \prec P^{us}$, there is an unstable facet $F \prec P^{us}$ that contains $Q$. Note that this condition is satisfied by the projective bundles examples (Section 5.3) but not by the GIT quotients of Section 5.2. We leave the proof of these facts to the interested reader.

We give a geometric interpretation of the images of the pull-back functors. First, we interpret the condition on unstable divisors with the following Lemma, whose proof is straightforward from the definitions.

**Lemma 3.21.** *Let $\mathcal{E}$ be a reflexive equivariant sheaf on $X$. Assume that $\mathcal{E} = \mathfrak{K}(\mathbb{E})$, with $\mathbb{E} = \{(E^F(i)) \subset E, F \prec P, i \in \mathbb{Z}\}$. Let $F$ be a facet of $P$ corresponding to an invariant divisor $D_F$ and let $i_F \in \mathbb{Z}$. Then the following are equivalent:*

i) *For all $i \in \mathbb{Z}$,*

$$E^F(i) = \begin{cases} 0 & \text{if} \quad i < i_F \\ E & \text{if} \quad i \geq i_F \end{cases} \tag{8}$$

ii) *For all $m \in M$, an element of $\Gamma(U_{\sigma_F} \setminus O(F), \mathcal{E})_m$ extends across the divisor $D_F$ to a section of $\Gamma(U_{\sigma_F}, \mathcal{E})$ if and only if $\langle m, u_F \rangle \geq i_F$.*

*Remark* 3.22. Condition $(ii)$ in Lemma 3.21 is equivalent to the vanishing of the quantities $(\Delta_j(k))$ introduced in [11, Proposition 3.20], for the index $j$ corresponding to $F$.

We deduce:

**Corollary 3.23.** *Let $\mathcal{E}$ be a reflexive equivariant sheaf on $X$, and let $\mathbf{i} \in \mathbb{Z}^u$. Then the following are equivalent:*

i) *The sheaf $\mathcal{E}$ lies in the image of $\mathfrak{P}_{\mathbf{i}}$.*
ii) *The sheaf $\mathcal{E}$ descends to $Y$ and satisfies the extension condition* (8) *for all unstable divisor $D_F$, with $i_F = i_{D_F}$.*

*Proof.* It follows from the definition of the functors and Lemmas 3.15 and 3.21 □

It will become clear in the next section that the extension condition (8) for unstable divisors is precisely the condition on $\mathfrak{P}_{\mathbf{i}}$ that enables us to compare the slopes of $\check{\mathcal{E}} \in \mathfrak{Ref}^T(Y)$ and of $\mathfrak{P}_{\mathbf{i}}(\check{\mathcal{E}})$ for all $\check{\mathcal{E}} \in \mathfrak{Ref}^T(Y)$. We finish this section with a more effective lemma that will be used in these comparisons:

**Lemma 3.24.** *Let $\check{\mathcal{E}} \in \mathfrak{Ref}^T(Y)$ and $\mathcal{E} = \mathfrak{P}_{\mathbf{i}}(\check{\mathcal{E}}) \in \mathfrak{Ref}^T(X)$. Let $(e^F(i))$ be the dimension jumps of $\mathcal{E}$ and let $(\check{e}^F(i))$ be the dimension jumps of $\check{\mathcal{E}}$. Then*

i) *For all $F \prec P^s$, $e^F(i) = 0$ if $i \neq 0 \mod b_F$ and $e^F(b_F i) = \check{e}^F(i)$.*
ii) *For all $F \prec P^{us}$, $e^F(i) = 0$ if $i \neq i_{D_F}$ and $e^F(i_{D_F}) = -\mathrm{rank}(\mathcal{E})$.*

*Proof.* It follows directly from the definition of the functors. □

## 4. Slopes under descent and the Minkowski condition

In this section, $(X, L)$ is a polarized toric variety with torus $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$, and $G = N_0 \otimes_{\mathbb{Z}} \mathbb{C}^*$ is a subtorus. We assume $N_0 = N \cap (N_0 \otimes \mathbb{R})$ and consider a linearization $\gamma$ such that $(G, \gamma)$ is generic. We will compare the slope stability notions on $X$ and $Y = X /\!\!/ G$. We will use the same notations of previous sections.

### 4.1. Some stability notions.

We recall now the notions of stabilities that we will consider, and state some propositions specific to the toric setting. We refer to [6, 8] for the stability notions (see in particular [6, Proposition 4.3] for the definition of slope on normal varieties) and to [12] for the definition of the intersection of a Weyl divisor with real Cartier divisors.

**Definition 4.1.** Let $\mathcal{E}$ be a torsion-free coherent sheaf on $X$. The *degree* of $\mathcal{E}$ with respect to an ample class $\alpha \in N^1(X)_{\mathbb{R}}$ is the real number obtained by intersection:

$$\deg_\alpha(\mathcal{E}) = c_1(\mathcal{E}) \cdot \alpha^{n-1}$$

and its *slope* with respect to $\alpha$ is given by

$$\mu_\alpha(\mathcal{E}) = \frac{\deg_\alpha(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

A torsion-free coherent sheaf $\mathcal{E}$ is said to be *$\mu$-semi-stable* or *slope semi-stable* with respect to $\alpha$ if for any proper coherent subsheaf of lower rank $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rk}\mathcal{F} < \mathrm{rk}\mathcal{E}$, one has

$$\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E}).$$

When strict inequality always holds, we say that $\mathcal{E}$ is *$\mu$-stable*. Finally, $\mathcal{E}$ is said to be $\mu$-polystable if it is the direct sum of $\mu$-stable subsheaves of the same slope.

In the toric setting, to check $\mu$-stability of a reflexive sheaf, it is enough to compare slopes for equivariant reflexive subsheaves. More precisely, using the description of equivariant reflexive subsheaves from [3] (see Proposition 2.10), we have:

**Proposition 4.2** ( [11], Proposition 4.13)**.** *Let $\mathcal{E} = \mathfrak{K}(\mathbb{E})$ be a $T$-equivariant reflexive sheaf on $X$. Then $\mathcal{E}$ is $\mu$-semi-stable (resp. $\mu$-stable) if and only if for all proper vector subspaces $W \subset E$, $\mu_L(\mathcal{E}_W) \leq \mu_L(\mathcal{E})$ (resp. $\mu_L(\mathcal{E}_W) < \mu_L(\mathcal{E})$), where $\mathcal{E}_W = \mathfrak{K}(W \cap \mathbb{E})$.*

Note that the proof of the above proposition is valid on normal toric varieties. Following [7, 11], we will use the following combinatorial formula for the slopes of equivariant reflexive sheaves (see also [7, Lemma 2.2]):

**Lemma 4.3.** *Let $\mathcal{E}$ be a $T$-equivariant reflexive sheaf with dimension jumps $(e^F(i))$. Then*

$$\mu_L(\mathcal{E}) = -\frac{1}{\mathrm{rank}(\mathcal{E})} \sum_{F \prec P} i_F(\det \mathcal{E}) \deg_L(D_F),$$

*where for all $F \prec P$,*

$$i_F(\det \mathcal{E}) = -\sum_{i\in\mathbb{Z}} i e^F(i).$$

*Proof.* It follows from the definition of slopes and Proposition 2.16. □

We will need to compare the data defining the slopes after descent:

**Lemma 4.4.** *Let $\mathbf{i} = (i_F)_{F\prec P^{us}} \in \mathbb{Z}^u$. Let $\check{\mathcal{E}}$ be a $T$-equivariant sheaf on $Y$. Then:*

i) *For all $F \prec P^s$, $i_F(\det \mathfrak{P}_{\mathbf{i}}(\check{\mathcal{E}})) = b_F i_{F\cap U}(\det \check{\mathcal{E}})$.*
ii) *For all $F \prec P^{us}$, $i_F(\det \mathfrak{P}_{\mathbf{i}}(\check{\mathcal{E}})) = i_F \operatorname{rank}(\mathcal{E})$*

*Proof.* The proof follows from the definition of $i_F(\det \mathcal{E})$ and Lemma 3.24. □

We will also need the degree for the pullbacks of the irreducible $T$-invariant divisors.

**Lemma 4.5.** *Let $F_1 \prec P^s$. Then for any $\mathbf{i} \in \mathbb{Z}^u$,*

$$\mathfrak{P}_{\mathbf{i}}(\mathcal{O}_Y(D_{F_1\cap U})) = \mathcal{O}_X(b_{F_1} D_{F_1} - \sum_{F\prec P^{us}} i_F D_F).$$

*Proof.* The sheaf $\mathcal{O}_Y(D_{F_1\cap U})$ is equivariant and reflexive. Combining Proposition 2.11 and the definition of the functor $\mathfrak{P}_{\mathbf{i}}$, $\mathfrak{P}_{\mathbf{i}}(\mathcal{O}_Y(D_{F_1\cap U}))$ is determined by the family of filtrations

$$L^{F_1}(i) = \begin{cases} 0 & i < -b_{F_1}, \\ \mathbb{C} & i \geq -b_{F_1}, \end{cases}$$

$$L^F(i) = \begin{cases} \begin{array}{ll} 0 & i < 0, \\ \mathbb{C} & i \geq 0, \end{array} & \text{if } F \neq F_1 \text{ is stable,} \\ \begin{array}{ll} 0 & i < i_F, \\ \mathbb{C} & i \geq i_F, \end{array} & \text{if } F \text{ is unstable.} \end{cases}$$

That is to say, $\mathfrak{P}_{\mathbf{i}}(\mathcal{O}_Y(D_{F_1\cap U})) = \mathcal{O}_X(b_{F_1} D_{F_1} - \sum_{F\prec P^{us}} i_F D_F)$. □

**Corollary 4.6.** *Let $F_1 \prec P^s$. Then for any $\mathbf{i} \in \mathbb{Z}^u$,*

$$\deg_L(\mathfrak{P}_{\mathbf{i}}(\mathcal{O}_Y(D_{F_1\cap U}))) = b_{F_1}\deg_L(D_{F_1}) - \sum_{F\prec P^{us}} i_F \deg_L(D_F).$$

4.2. **Comparison of slope stability via the Minkowski condition.** In this section, we prove Theorem 4.7 and Proposition 4.8, from which Theorem 1.1 follows. Recall from Sections 2.1 and 3.1 that the data of $(X, L)$ together with the linearization $\gamma : T \to \operatorname{Aut}(L)$ determines a polytope $P \subset M_{\mathbb{R}}$, and that facets $F$ in the stable locus $P^s$ of $P$ correspond to $T$-invariant stable divisors $D_F$.

**Theorem 4.7.** *Assume that Minkowski condition holds:*

$$\sum_{F\prec P^s} \deg_L(D_F)\, u_F = 0 \mod N_0 \otimes_{\mathbb{Z}} \mathbb{R}. \tag{9}$$

*Then there exists a unique ample class $\alpha \in N^1(Y)_{\mathbb{R}}$ such that for every $T$-equivariant reflexive sheaf $\check{\mathcal{E}}$ on $Y$, for any $\mathbf{i} = (i_F)_{F\prec P^{us}} \in \mathbb{Z}^u$, setting $\mathcal{E} = \mathfrak{P}_{\mathbf{i}}(\check{\mathcal{E}})$, we have*

$$\mu_L(\mathcal{E}) = \mu_\alpha(\check{\mathcal{E}}) - \sum_{F\prec P^s} i_F \deg_L(D_F). \tag{10}$$

*In this case, $\mathcal{E}$ is stable on $(X, L)$ if and only if $\check{\mathcal{E}}$ is stable on $(Y, \alpha)$.*

A converse of this statement is given in the following.

**Proposition 4.8.** *There exists an ample class $\alpha \in N^1(Y)_{\mathbb{R}}$ with respect to which the functors $\mathfrak{P}_{\mathbf{i}}$ from equivariant reflexive sheaves on $Y$ to equivariant reflexive sheaves on $X$ preserve each of the conditions of $\mu$-stability, $\mu$-semi-stability and $\mu$-polystability if and only if*

$$\sum_{F \prec P^s} \deg_P(D_F) u_F = 0 \mod N_0 \otimes_{\mathbb{Z}} \mathbb{R}.$$

We delay the proofs slightly in order to first present the classical Minkowski theorem. We recall that a lattice $M$ defines a measure $\nu$ on $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ as the pull-back of Haar measure on $M_{\mathbb{R}}/M$. It is determined by the properties

(a) $\nu$ is translation invariant,
(b) $\nu(M_{\mathbb{R}}/M) = 1$.

Let $\alpha \in N^1(X)_{\mathbb{R}}$ be an ample class, and denote by $P_\alpha$ the associated polytope. For any facet $F$ of $P_\alpha$, there is a unique $n-1$-form $\nu_F$, independent of $\alpha$, such that $\nu_F \wedge u_F = \nu$. If $P_\alpha$ is a lattice polytope, with

$$P_\alpha = \{m \in M_{\mathbb{R}}, \langle m, u_F \rangle \geq -a_F \text{ for all } F \prec P\},$$

the form $\nu_F$ corresponds to the measure defined by the lattice

$$M \cap \{m \in M_{\mathbb{R}}, \langle m, u_F \rangle = -a_F\}$$

in the affine span of $F$. We will denote the volume of the facet $F$ with respect to $\nu_F$ by $\operatorname{latvol}(F)$.

**Proposition 4.9** ( [2])**.** *For any facet $F$ of $P_\alpha$,*

$$\deg_\alpha(D_F) = \operatorname{latvol}(F).$$

*Proof.* From [2, Section 11], the equality holds for $\alpha$ integral. Since the two expressions in the equality are both homogeneous (of order $n-1$) and continuous, we also have $\deg_\alpha(D_F) = \operatorname{latvol}(F)$, whenever $\alpha$ is an ample $\mathbb{R}$-divisor. □

We have the elementary lemma:

**Lemma 4.10.** *Let $u_F \in N$ be the primitive element associated to a facet $F \prec P$. Let $g = \langle \cdot, \cdot \rangle$ be a positive definite inner product on $M_{\mathbb{R}}$ whose induced volume form coincides with $\nu$ and let $\tilde{u}_F = u_F/\|u_F\|$ be the normalized vector in $N_{\mathbb{R}}$. Then,*

$$\operatorname{latvol}(F)\, u_F = \operatorname{vol}_g(F)\, \tilde{u}_F.$$

From this observation we obtain the counterpart of a classical result of Minkowski, adapted to the case of lattice polytopes.

**Proposition 4.11.** *Let $u_1, \dots, u_r \in N$ be distinct primitive lattice elements that span the real vector space $N_{\mathbb{R}}$ and let $f_1, \dots, f_r > 0$ be positive real numbers. Then there exists a compact and convex polytope in $M_{\mathbb{R}}$ whose facets have inward normals the elements $(u_i)$ and lattice volumes the $(f_i)$ if and only if*

$$\sum_{i=1}^{r} f_i u_i = 0.$$

*Moreover, such a polytope is unique up to translation.*

*Proof.* This follows immediately from the classical result (see [18, p.455]) and the previous lemma. □

We deduce the following interesting corollary on intersection theory of toric varieties:

**Corollary 4.12.** *Let* $(\alpha_\rho)_{\rho\in\Sigma(1)} \in \mathbb{R}_{>0}^{\sharp\Sigma(1)}$. *Then, there exists an ample class* $\alpha \in N^1(X)_{\mathbb{R}}$ *such that for all facets* $F$ *in an associated polytope* $P_\alpha$, $\deg_\alpha(D_F) = \alpha_{\rho_F}$ *if and only if*

$$\sum_{F\prec P} \alpha_{\rho_F}\, u_F = 0$$

We can now give the proofs of Theorem 4.7 and Proposition 4.8.

*Proof of Theorem* 4.7. Let $\mathbf{i} \in \mathbb{Z}^u$. Let $\check{\mathcal{E}}$ be an equivariant reflexive sheaf on $Y$ and let $\mathcal{E} = \mathfrak{P}_{\mathbf{i}}(\check{\mathcal{E}})$ on $X$, both of rank $r$. Let $\alpha$ be an ample $\mathbb{R}$-divisor on $Y$ that determines a real polytope $\check{P} \subseteq \check{M}_{\mathbb{R}} = N_0^{\perp} \otimes_{\mathbb{Z}} \mathbb{R}$ whose facets $\check{F}$ have primitive normals $\check{u}_F = u_{F\cap U} \in N/N_0$ equal to those of $P_Y$. Then, from Lemma 4.3,

$$\mu_\alpha(\check{\mathcal{E}}) = \frac{-1}{r}\sum_{\check{F}\prec\check{P}} i_{\check{F}}(\det\check{\mathcal{E}})\deg_\alpha(D_{\check{F}}).$$

By the genericity assumption on $(G,\gamma)$, from Lemma 3.6, we deduce a bijective correspondence $F \leftrightarrow \check{F}$ between facets of $P^s$ and facets of $\check{P}$. Thus

$$\mu_\alpha(\check{\mathcal{E}}) = \frac{-1}{r}\sum_{F\prec P^s} i_{\check{F}}(\det\check{\mathcal{E}})\deg_\alpha(D_{\check{F}}),$$

whereas by Lemma 4.4,

$$\begin{aligned}
\mu_L(\mathfrak{P}_{\mathbf{i}}(\check{\mathcal{E}})) &= \mu_L(\mathcal{E}),\\
&= \frac{-1}{r}\sum_{F\prec P} i_F(\det\mathcal{E})\deg_L(D_F),\\
&= \frac{-1}{r}\sum_{F\prec P^s} i_{\check{F}}(\det\check{\mathcal{E}}) b_F \deg_L(D_F) - \sum_{F\prec P^{us}} i_F \deg_L(D_F).
\end{aligned}$$

Consider the right hand term of the last equality. We note two points:

(1) The second term $\sum_{F\prec P^{us}} i_F\deg_L(D_F)$ is independent of the reflexive sheaf $\check{\mathcal{E}}$. It depends only on the lattice volumes of the unstable facets $F \prec P^{us}$ and on the indices $i_F$ that determine the pull-back functors.
(2) The first term coincides with $\mu_\alpha(\check{\mathcal{E}})$ if for all $F \prec P^s$, $\deg_\alpha(D_{\check{F}}) = b_F\deg_L(D_F)$, which, by Proposition 4.9, is equivalent to $\mathrm{latvol}(\check{\mathrm{F}}) = b_F\deg_L(D_F)$.

By Proposition 4.11, we can chose the polytope $\check{P} \subseteq \check{M}_{\mathbb{R}}$ such that its facets satisfy $\mathrm{latvol}(\check{F}) = b_F\deg_L(D_F)$ if and only if

$$\sum_{\check{F}\prec\check{P}} b_F\deg_L(D_F)\check{u}_F = 0,$$

which, by Lemma 3.8, is to say

$$\sum_{F\prec P^s} \deg_L(D_F) u_F = 0 \mod N_0 \otimes_{\mathbb{Z}} \mathbb{R}.$$

Thus, the Minkowski condition implies the existence of the desired polytope and of $\alpha$.

Applying now (10) to the sheaf $\mathcal{O}(D_{\check{F}})$, using Corollary 4.6, we deduce that $\deg_\alpha D_{\check{F}} = b_F \deg_L(D_F)$ for all $\check{F} \subset \check{P}$. Then, the uniqueness statement of Theorem 4.7 follows from the equality $\mathrm{latvol}(\check{F}) = \deg_\alpha(D_{\check{F}})$ and unicity in Proposition 4.11.

As noted above in Proposition 2.10 (see [3,7]), to test for stability, it is sufficient to consider equivariant reflexive subsheaves of the form $\check{\mathcal{E}}_W = \mathfrak{K}(\mathbb{E} \cap W)$ where $W$ runs through vector subspaces of $E$. The slopes then satisfy

$$\mu_L(\mathcal{E}) - \mu_L(\mathcal{E}_W) = \mu_\alpha(\check{\mathcal{E}}) - \mu_\alpha(\check{\mathcal{E}}_W)$$

from which the equivalence of the stability conditions follows. □

*Proof of Proposition* 4.8 . The "if" statement is given in the previous theorem. We suppose that for some ample $\mathbb{R}$-divisor $\alpha$ on $Y$ the various stability notions are preserved by the pull-back functors. The ample divisor $\alpha$ defines a real polytope $\check{P} \subseteq \check{M}_{\mathbb{R}} = N_0^{\perp} \otimes_{\mathbb{Z}} \mathbb{R}$ whose facets $\check{F}$ have primitive normals $\check{u}_F = u_{F\cap U} \in N/N_0$.

We consider the functor $\mathfrak{P}_{\mathbf{i}}$ with $i_F = 0$ for all unstable facets $F \prec P^{us}$. For any two $F_1, F_2 \prec P^s$ consider the direct sum of rank one reflexive sheaves $\mathcal{O}_Y(d_1 D_{\check{F}_1}) \oplus \mathcal{O}_Y(d_2 D_{\check{F}_2})$ where $d_1 = b_{F_2}\deg_L(D_{F_2})$ and $d_2 = b_{F_1}\deg_L(D_{F_1})$. Then, using Lemma 4.5,

$$\mathfrak{P}_{\mathbf{i}}(\mathcal{O}_Y(d_1 D_{\check{F}_1}) \oplus \mathcal{O}_Y(d_2 D_{\check{F}_2})) = \mathcal{O}_X(d_1 b_{F_1} D_{F_1}) \oplus \mathcal{O}_X(d_2 b_{F_2} D_{F_2})$$

is the direct sum of rank-one reflexive sheaves of the same slope. By hypothesis, the initial sheaf must also be $\mu$-polystable so for any two stable facets of $P$,

$$\deg_\alpha(b_{F_2}\deg_L(D_{F_2}) D_{\check{F}_1}) = \deg_\alpha(b_{F_1}\deg_L(D_{F_1}) D_{\check{F}_2})$$

so the expression

$$c = b_F \frac{\deg_L(D_F)}{\deg_\alpha(D_{\check{F}})}$$

is independent of $F \prec P^s$. Then, $\mathrm{latvol}(\check{F}) = \deg_\alpha(D_{\check{F}})$ and by the lattice Minkowski theorem (Proposition 4.11),

$$\sum_{F \prec P^s} \deg_\alpha(D_{\check{F}}) \check{u}_F = 0.$$

That is, by Lemma 3.8,

$$\frac{1}{c} \sum_{F \prec P^s} b_F \deg_L(D_F)\, b_F^{-1} \pi(u_F) = 0$$

as desired. □

## 5. Applications

In this section, we investigate the Minkowski condition. We show that any $n$-dimensional polarized toric orbifold $(X, L)$ admits at least $n+1$ GIT quotients by $\mathbb{C}^*$-actions that satisfy the Minkowski condition. The associated quotients are weighted projective spaces. We also consider the case of projectivization of torus invariant bundles.

5.1. **Compatible one parameter subgroups.** Let $(X,L)$ be a polarized complete toric variety with torus $T = N\otimes_{\mathbb{Z}}\mathbb{C}^*$ and fan $\Sigma$. When considering subtori $G\subset T$, we will always assume that $G = N_0\otimes_{\mathbb{Z}}\mathbb{C}^*$, for a saturated sublattice $N_0$.

**Definition 5.1.** A subtorus $G\subset T$ is *compatible* with $(X,L)$ if there is a linearization $\gamma$ of $T$ on $L$ such that $(G,\gamma)$ is generic and $((X,L),(G,\gamma))$ satisfies Minkowski condition (9).

We will see that the Minkowski condition is very restrictive, in the sense that in most cases, for fixed $(X,L)$, there are few compatible one dimensional subtori. To see this, set $\mathrm{Div}^T_{\mathrm{ir}}(X) = \{D_\rho,\ \rho\in\Sigma(1)\}$ be the set of reduced and irreducible $T$-invariant divisors on $X$. Let $\mathbf{D}\subset \mathrm{Div}^T_{\mathrm{ir}}(X)$ be a non empty subset. Assume that there is a one parameter subgroup $G = N_0\otimes_{\mathbb{Z}}\mathbb{C}^*\subset T$ and a linearization $\gamma$ for $T$ on $L$ such that $(G,\gamma)$ is generic and the set of stable divisors for $(G,\gamma)$ is $\mathbf{D}$. Then, if Minkowski condition is satisfied, we have

$$\sum_{D_\rho\in\mathbf{D}} \deg_L(D_\rho)\, u_\rho = 0 \mod N_0\otimes_{\mathbb{Z}}\mathbb{R}$$

Consider the element of $N$:

$$u_{\mathbf{D}} := \sum_{D_\rho\in\mathbf{D}} \deg_L(D_\rho)\, u_\rho.$$

Assume that $u_{\mathbf{D}}$ is not zero. Then Minkowski condition implies that

$$N_0 = N\cap(\mathbb{R}\cdot u_{\mathbf{D}}),$$

so that in particular $G$ is entirely determined by $u_{\mathbf{D}}$ and thus by the set $\mathbf{D}$. So for a given $\mathbf{D}$ with $u_{\mathbf{D}}$ non-zero, there is at most one compatible one-parameter subgroup of $T$ with $\mathbf{D}$ as set of stable divisors. As the number of stable divisors is the number of faces of a given polytope in $M_{\mathbb{R}}$ that intersects the hyperplane $N_0^\perp\otimes_{\mathbb{Z}}\mathbb{R}$, in the generic situation, we must have $\sharp\mathbf{D}\in\{n,\dots,\sharp\Sigma(1)-1\}$. Thus we have proved:

**Lemma 5.2.** *Let $(X,L)$ be a $n$-dimensional complete polarized toric variety with fan $\Sigma$. Set $d=\sharp\Sigma(1)$ the number of rays of $\Sigma$. Assume that for all subset $\mathbf{D}\subsetneq \mathrm{Div}^T_{\mathrm{ir}}(X)$ with $\sharp\mathbf{D}\in\{n,\dots,d-1\}$, we have $u_{\mathbf{D}}\neq 0$. Then, the number of one parameter subgroups compatible with $(X,L)$ is bounded by $\sum_{k=n}^{d-1}\binom{d}{k}$.*

For example, the conditions of Lemma 5.2 are satisfied by Hirzebruch surfaces $\mathbb{F}_a$ for $a\geq 2$, but not by $\mathbb{P}^1\times\mathbb{P}^1$.

*Remark* 5.3. Note that if $X$ is a complete toric orbifold, then from the sequence

$$0\to M\to \bigoplus_{\rho\in\Sigma(1)}\mathbb{Z}\cdot D_\rho\to \mathrm{Cl}(X)\to 0$$

and from the fact that the Picard group as finite index in the class group $\mathrm{Cl}(X)$, we deduce that $\sharp\Sigma(1) = n+p$, where $p$ is the Picard rank of $X$ (see [1, Theorem 4.1.3 and Proposition 4.2.7]). This gives the bound $\sum_{k=n}^{n+p-1}\binom{n+p}{k}$ on the number of compatible one-parameter subgroups. In particular, for the complex projective space $\mathbb{CP}^n$, this bound equals $n+1$. Using Proposition 5.4, we see that for a given dimension, this bound is achieved.

5.2. **Quotients to weighted projective spaces.** Let $(X, L)$ be a $n$-dimensional polarized toric orbifold with torus $T$ and associated polytope $P$. We will show:

**Proposition 5.4.** *Up to replacing L by a sufficiently high power, there are at least $(n+1)$ one-parameter subgroups of $T$ compatible with $(X, L)$. The associated GIT quotients are weighted projective spaces.*

From this proposition, we deduce that one can obtain $\mu$-stable reflexive sheaves on any polarized toric orbifold from $\mu$-stable reflexive sheaves on weighted projective spaces.

*Remark* 5.5. As weighted projective spaces have Picard rank 1, the number of rays of their fans, and thus the number of facets of their polytopes, is the smallest possible for a given dimension (see remark 5.3). Thus, testing stability for reflexive sheaves is simpler on weighted projective spaces. We expect that these varieties could serve as simple bricks to study moduli spaces of equivariant reflexive sheaves on toric orbifolds, and will investigate in this direction in future work.

*Proof.* The argument is local at a vertex $v \in P$. As $X$ is an orbifold, its fan is simplicial, and thus the set $\{u_F,\ v \in F\}$ is a $\mathbb{Q}$-basis of $M_{\mathbb{Q}}$. Consider

$$\mathbf{D}_v := \{D_F,\ v \in F\}$$

and

$$u_{\mathbf{D}_v} = \sum_{D_F \in \mathbf{D}_v} \deg_L(D_F) u_F = \sum_{v \in F} \deg_L(D_F) u_F.$$

As all the degrees $\deg_L(D_F)$ are positive, it is clear that $u_{\mathbf{D}_v}$ is not zero. Let $N_0 = N \cap (\mathbb{R} \cdot u_{\mathbf{D}_v})$ and let $G = N_0 \otimes_{\mathbb{Z}} \mathbb{C}^*$. Then, by construction of $u_{\mathbf{D}_v}$, up to dilation and translations of $P$, we can assure that $P$ intersects $N_0^{\perp}$ transversally and precisely along the faces $F$ that contains $v$. Thus, up to scaling $L$ and choosing an appropriate linearization, the set of stable divisors for the $G$ action is $\mathbf{D}_v$. By construction, Minkowski condition is satisfied and $G$ is compatible with $(X, L)$. The associated quotient is a weighted projective space as the number of stable facets is $n$, giving $n$ rays for the $(n-1)$-dimensional toric GIT quotient. To conclude, there are at least $n+1$ vertices in the polytope, and thus at least $n+1$ such quotients, hence the result. □

*Remark* 5.6. From Lemma 5.2 and Proposition 5.4, we deduce that the compatible one parameter subgroups for $(\mathbb{CP}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ are given by the $\mathbb{C}^*$-actions

$$\lambda \cdot [z_0, \ldots, z_n] = [z_0, \ldots, \lambda\, z_i, \ldots, z_n],\ i \in \{0, \ldots, n\},$$

and the associated quotients are all isomorphic to $\mathbb{CP}^{n-1}$.

5.3. **Projectivization of toric vector bundles.** As a motivating example, we consider the variety given as the projectivization of a torus invariant vector bundle. We refer to [12, Appendix A] and [1, Section 7.3] for the construction and classical results. Let $Y$ be a complete projective toric variety with fan $\Sigma_Y$. For $i = 1, \ldots, r$, let $D_i = \sum_{\rho \in \Sigma_Y(1)} a_{i\rho} D_\rho$ be invariant divisors on $Y$ and let $V_{\mathcal{F}}$ be the vector bundle associated to the locally free sheaf

$$\mathcal{F} = \mathcal{O}_Y \oplus \mathcal{O}_Y(D_1) \oplus \cdots \oplus \mathcal{O}_Y(D_r).$$

Let $X = \mathbb{P}(V_{\mathcal{F}}^{\vee})$ be the projectivization of the dual of $V_{\mathcal{F}}$, with $\pi : X \to Y$ the projection to the base $Y$. Then $X$ is a projective toric variety, with torus $T_X =$

$T_Y \times (\mathbb{C}^*)^r$. Denote by $N_X, N_Y$ the lattices of $X$ and $Y$, and by $M_X$ and $M_Y$ their duals. Then $N_X = N_Y \times \mathbb{Z}^r$ and we have exact sequences

$$0 \longrightarrow \mathbb{Z}^r \longrightarrow N_Y \times \mathbb{Z}^r \longrightarrow N_Y \longrightarrow 0,$$
$$0 \longrightarrow M_Y \longrightarrow M_Y \times (\mathbb{Z}^r)^* \longrightarrow (\mathbb{Z}^r)^* \longrightarrow 0.$$

Let $L_Y$ be an ample line bundle on $Y$ and let $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}(V_{\mathcal{F}})}(1)$ be the Serre line bundle on $X$. Then, up to replacing $L_Y$ by a sufficiently high power, $L_X = \pi^* L_Y \otimes \mathcal{O}_X(1)$ is an ample line bundle on $X$. Assume now that $L_Y = \mathcal{O}(D_Y)$ for an ample divisor on $Y$

$$D_Y = \sum_{\rho \in \Sigma_Y(1)} b_\rho D_\rho,$$

with associated lattice polytope

$$P_Y = \{m \in (M_Y)_\mathbb{R} \,;\, \langle m, u_\rho \rangle \geq -b_\rho \ \text{ for all } \rho \in \Sigma_Y(1)\}.$$

To determine a polytope $P_X \subseteq (M_Y)_\mathbb{R} \times \mathbb{R}^r$ associated to $(X, L_X)$, we first describe the invariant divisors of $X$. They are of two types:

$$\mathrm{Div}^T_{ir}(X) = \{\pi^{-1}(D_\rho), \text{ for } \rho \in \Sigma_Y(1)\} \cup \{\{s_i = 0\} \text{ for } i = 0, 1, \ldots, r\}$$

where the $\{s_i = 0\}$ are the relative hyperplane sections associated to the line subbundles of $V_\mathcal{E}^\vee$. Set $\hat{D}_\rho = \pi^{-1}(D_\rho)$ for all $\rho \in \Sigma_Y(1)$ and $D_{s_i} := \{s_i = 0\}$ for all $i \in \{0, \ldots, r\}$. Then $L_X = \mathcal{O}(\hat{D})$ with

$$\hat{D} = \sum_{\rho \in \Sigma_Y(1)} b_\rho \hat{D}_\rho + \sum_{i=0}^{r} D_{s_i}$$

and associated polytope $P_X$ defined by $(m_y, m_x) \in (M_Y)_\mathbb{R} \times (\mathbb{R}^r)^*$ is in $P_X$ if and only if

$$\langle m_x, e_i \rangle \geq -1, \text{ for all } i = 0, \ldots, r, \text{ and}$$
$$\langle (m_y, m_x), u_\rho + \sum_{i=1}^{r} a_{i\rho} e_i \rangle \geq -b_\rho, \text{ for all } \rho \in \Sigma_Y(1),$$

where $\{e_i, i = 1, \ldots, r\}$ is the standard basis of $\mathbb{R}^r$ and $e_0 = -(e_1 + \ldots + e_r)$.

We will perform a GIT quotient of $X$ by the torus $G = N_0 \otimes_\mathbb{Z} \mathbb{C}^*$, where $N_0 = \{0_Y\} \times \mathbb{Z}^r$. Consider the linearization $\gamma_y$ for $T_Y$ on $L_Y$ giving the polytope $P_Y$ (recall Section 3.1, equation (7)). Consider the standard linearization $\gamma_x$ of $(\mathbb{C}^*)^r$ on $\mathcal{O}_{\mathbb{CP}^r}(1)$ associated to the polytope

$$\{\langle m_x, e_i \rangle \geq -1, \text{ for all } i = 0, \ldots, r\} \subset (\mathbb{R}^r)^*.$$

Combining the two give the linearization $\gamma = \gamma_y \times \gamma_x$ of $T_X$ on $L_X$ associated to $P_X$. Then, setting $U = N_0^\perp \otimes_\mathbb{Z} \mathbb{R}$, we have $U = (M_Y)_\mathbb{R} \times \{0\}$. In particular, the only facets of $P_X$ that intersect $U$ are those given by equations $\langle m_y, u_\rho \rangle + \sum_i a_{i\rho} \langle m_x, e_i \rangle = -b_\rho$ for some $\rho \in \Sigma_Y(1)$ and the polytope $P_X \cap U$ is $P_Y \times \{0\}$. That is, the GIT quotient of $(X, L_X)$ by $G$ is $(Y, L_Y)$. We also note that $(G, \gamma)$ is generic.

**Proposition 5.7.** *The polarized toric variety $(X, L_X)$, with toric subgroup $G = (\mathbb{C}^*)^r \subseteq T_X$ and linearization $\gamma$ satisfies the Minkowski condition of Theorem* 4.7.

*Proof.* The line bundle $L_X$ determines a polytope in $(M_X)_{\mathbb{R}}$ whose normal fan equals the fan of $X$. By Corollary 4.12, we see that

$$\sum_{F \prec P_X} \deg_{L_X}(D_F)\, u_F = 0,$$

$$\text{and so} \quad \sum_{F \prec P_X^s} \deg_{L_X}(D_F)\, u_F = -\sum_{F \prec P_X^{us}} \deg_{L_X}(D_F)\, u_F.$$

Note that the unstable divisors are precisely the $D_{s_i}$ for $i \in \{0, \ldots r\}$. Moreover, the normal elements $u_F$, for $F \prec P_X^{us}$, are given by the basis elements $e_i \in \mathbb{R}^r$ together with $e_0$. Then, $\pi(e_i) = 0 \in N_Y = N_X/\mathbb{Z}^r$ and hence

$$\sum_{F \prec P_X^s} \deg_{L_X}(D_F)\, u_F = 0 \mod N_0 \otimes_{\mathbb{Z}} \mathbb{R}$$

as desired. □

The GIT quotient map $p : X^s \to Y$ coincides with the restriction $\pi|_{X^s}$ so it makes sense to compare two methods of pulling back sheaves from $Y$ to $X$. Given a $T$-equivariant reflexive sheaf $\mathcal{E}$ on $Y$, for each $\mathbf{i} \in \mathbb{Z}^{r+1}$, we have invariant reflexive sheaves $\pi^*\mathcal{E}$ and $\mathfrak{P}_{\mathbf{i}}(\mathcal{E})$ on $X$.

**Proposition 5.8.** *Taking $\mathbf{i} = 0$, if $\check{\mathcal{E}}$ is a $T$-equivariant reflexive sheaf on $Y$, we have*

$$\pi^*\check{\mathcal{E}} = \mathfrak{P}_0(\check{\mathcal{E}}).$$

*Proof.* Suppose that $\check{\mathcal{E}} = \mathfrak{K}(\check{\mathbb{E}})$ where $\check{\mathbb{E}} = \{(E^{\check{F}}(i)) \subset E\,;\, \check{F} \prec P_Y,\, i \in \mathbb{Z}\}$. Then, $\pi^*\check{\mathcal{E}}$ and $\mathfrak{P}_0(\check{\mathcal{E}})$ are determined up to isomorphism by their respective filtrations. On one hand, $\mathfrak{P}_0(\check{\mathbb{E}}) = \{(E^F(i)) \subset E\,;\, F \prec P_X,\, i \in \mathbb{Z}\}$ is given by the filtration

$$E^F(i) = \begin{cases} E^{F\cap U}(i), & F \prec P^s,\, i \in \mathbb{Z}, \\ \begin{cases} 0 & i < 0, \\ E & i \geq 0, \end{cases} & F \prec P^{us}. \end{cases}$$

On the other hand, by Lemma 3.15, for all $F \prec P_X^s$, observing that $b_F = 1$, we have $(\pi^*\check{\mathbb{E}})^F(i) = E^{F\cap U}(i)$. Let $F_i \prec P_X^{us}$ be the unstable facet corresponding to $D_{s_i}$. The morphism $\pi : X \to Y$ corresponds to a lattice homomorphism $\bar{\pi} : N_X \to N_Y$. From the fact that $\bar{\pi}$ sends the normal generator $u_{F_i} = e_i$ of the ray $\rho_{F_i}$ to $0 \in N_Y$, we deduce that the image of the affine chart $\pi(U_{\sigma_{F_i}})$ is $T_Y \subseteq Y$. Hence,

$$\Gamma(U_{\sigma_{F_i}}, \pi^*\check{\mathcal{E}}) = \Gamma(T_Y, \check{\mathcal{E}}) \otimes_{\mathbb{C}[M_Y]} \mathbb{C}[\rho_{F_i}^\vee \cap M_X] = E \otimes \mathbb{C}[\rho_{F_i}^\vee \cap M_X]$$

from which it follows that

$$(\pi^*\check{\mathbb{E}})^{F_i}(j) = \begin{cases} 0 & \text{for } j < 0, \\ E & \text{for } j \geq 0, \end{cases}$$

allowing us to conclude that $\pi^*\mathcal{E} = \mathfrak{P}_0(\mathcal{E})$. □

From Propositions 5.7 and 5.8, together with Theorem 4.7, we obtain

**Proposition 5.9.** *There exists an ample class $\alpha$ on $Y$ such that an equivariant reflexive sheaf $\mathcal{E}$ on $Y$ is $\mu$-stable with respect to $\alpha$ if and only if $\pi^*\mathcal{E}$ is $\mu$-stable on $X$ with respect to $L_X$.*

*Remark* 5.10. This construction could be considered an intermediate step within a *Bott tower*, as considered in [3].

We conclude with a special case where the ample class $\alpha$ can be computed explicitly, that is for $Y$ of dimension 2 and $V_{\mathcal{F}}$ of rank 3. It seems unlikely that a similar method could be used to compute the ample class $\alpha$ for higher dimensions and rank. Nevertheless, the examples below show that in general, $\alpha$ is different from the polarization induced by the GIT quotient.

**Lemma 5.11.** *With previous notations, assume in addition that* $Y$ *is an orbifold of dimension* 2 *and that* $V_{\mathcal{F}}$ *is of rank* 3. *Then the class* $\alpha$ *from Proposition* 5.9 *is equal to* $c_1(L_Y^3 \otimes \det(V_{\mathcal{F}}))$, *up to scale.*

*Proof.* As in that setting $Y$ and $X$ are orbifolds, we can make use of Poincaré duality to compute intersections, and degrees. From the proof of Theorem 4.7, we see that $\alpha$ is the ample class that satisfies for all $\rho \in \Sigma_Y(1)$,

$$\deg_\alpha(D_\rho) = \deg_{L_X}(\hat{D}_\rho).$$

We compute the later using that $H^2(X, \mathbb{Z})$ is the algebra over $H^2(Y, \mathbb{Z})$ generated by the class $\xi := c_1(\mathcal{O}_X(1))$, with relation $\xi^3 = \pi^* c_1(V_{\mathcal{F}}) \cdot \xi^2 - \pi^* c_2(V_{\mathcal{F}}) \cdot \xi + \pi^* c_3(V_{\mathcal{F}})$. Taking into account the dimensions, and the rank of $V_{\mathcal{F}}$, a direct computation gives

$$\deg_{L_X}(\hat{D}_\rho) = c_1(L_X)^3 \cdot \pi^* c_1(\mathcal{O}(D_\rho)) = \pi^*(3\, c_1(L_Y) + c_1(V_{\mathcal{F}})) \cdot \pi^* c_1(\mathcal{O}(D_\rho)) \cdot \xi^2.$$

The result follows from Whitney's formula. □

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